The Republic of Iraq

Ministry of Higher Education and sciences research

University of Babylon

Faculty of Education Sciences Exchange

Search address: Linear Operator and Linear Functionals

Research presented by the student (Baqir Jafar Sajit)

Faculty of education sciences of drainage, which is part of the requirements to obtain a bachelor's degree

By (Baqir Jafar Sajit)

Supervising by (Rehab Amer Kamel)

2023 1444

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ ﴾

صدق الله العظيم

[المجادلة: ١١]

الشكر و التقدير

قال تعالى:

{ رب أوزعني أن أشكر نعمتك الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ
لِي فِي ذُرِّيَّتِي إِنِّي تُبْتُ إِلَيْكَ
وَإِنِّي مِنَ الْمُسْلِمِينَ}

قال رسول الله محمد (صلى الله عليه واله وسلم) (من لا يشكر الناس لا يشكر الله)

نتقدم بالشكر الجزيل لكل من وقف بجانبنا وشجعنا حتى إتمام هذا البحث ونخص بالذكر دكتورتنا الفاضلة

د.(رحاب عامر كامل)

التي أشرفت على هذا البحث ولم تأل جهدا في مساعدتنا فكانت عوناً لنا وسند. الى كل من وقف بجانبنا وشجعنا واخذ بيدنا حتى رأى هذا البحث النور

الإهداء

إلى صاحب السيرة العطرة، والفكر المُستنير؛

فلقد كان له الفضل الأوَّل في بلوغي التعليم العالي

(والدي الحبيب)، أطال الله في عُمره.

إلى من وضعتني على طريق الحياة، وجعلتني رابط الجأش،

وراعتني حتى صرت كبيرًا

(أمي الغالية)، اطال الله في عمرها.

إلى إخوتي؛ من كان لهم بالغ الأثر في كثير من العقبات والصعاب.

إلى جميع أساتذتي الكرام؛ ممن لم يتوانوا في مد يد العون لي

أُهدي إليكم بحثي في.............

Contents

Abstrcat

We study in this research the concept of linear operators and some important definitions ,facts about it . Also we consider the relation between linear operators and functionals ,we offer some examples about linear functionals

1. Section one

(Introduction)

1.Introduction

In mathematics,an operator is generally a mapping or function that acts on elements of a space to produce elements of another space (possibly and sometimes required to be the same space). There is no general definition of an *operator*, but the term is often used in place of *function* when the domain is a set of functions or other structured objects. Also, the domain of an operator is often difficult to be explicitly characterized (for example in the case of an integral operator), and may be extended to related objects (an operator that acts on functions may act also on differential equations whose solutions are functions that satisfy the equation).

The most basic operators are linear maps, which act on vector spaces. Linear operators refer to linear maps whose domain and range are the same space, for example $\mathbb{R}^n$ to $\mathbb{R}^n$. Such operators often preserve properties, such as continuity. For example, differentiation and indefinite integration are linear operators; operators that are built from them are called differential operators, integral operators or integro-differential operators. Operator is also used for denoting the symbol of a mathematical operation. This is related with the meaning of "operator" in computer programming.

The purpose of this first set of research about linear operator theory is to provide the basics regarding the mathematical key features of operators from definitions and basic result about it.

2.section two

(Basic definition and results)

2. Basic definition and results

Definition 2.1 (Vector space) [1]:-

A vector space (or linear space) over a field K is a nonempty set X of elements x, y, ... (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of K.

Vector addition associates with every ordered pair (x, y) of vectors a vector x + y, called the sum of x and y, in such a way that the following properties hold.! Vector addition is commutative and associative, that is, for all vectors we have

$$x+y=y+x \quad \text{.............}(1)$$

$$x+(y+z)=(x+y)+z; \quad \text{..............}(2)$$

furthermore, there exists a vector 0, called the zero vector, and for every vector x there exists a vector -x, such that for all vectors we

Have

$$x+0=x \quad \text{............}(3)$$

$$x+(-x)=0 \quad \text{....................}(4)$$

Multiplication by scalars associates with every vector x and scalar α a vector ax (also written xα), called the product of α and x, in such a way that for all vectors x, y and scalars α,β we have

$$\alpha(\beta x)=(\alpha\beta)x \quad \text{............}(5)$$

$$1x=x$$

and the distributive laws

$$\alpha(x+y)=\alpha x+\alpha y \text{ ....(6)}$$

$$(\alpha + \beta)x = \alpha x + \beta x \text{...............(7)}$$

Note 2.2 :-

From the definition we see that vector addition is a mapping $X \times X \longrightarrow X$, whereas multiplication by scalars is a mapping $KxX \longrightarrow X$. K is called the scalar field (or coefficient field) of the vector space X, and X is called a real vector space if K = R (the field of real numbers), and a complex vector space if K = C (the field of complex numbers).

Example 2.3 [4]:-

(1) Space R^n. This is of n-dimensional Euclidean space R^n it is obtained if we take the set of all ordered n-tuples of real numbers, written'

$$X=(\xi_1, \dots , \xi_n) \qquad y=(\eta_1, \dots , \eta_n) \text{ .... (8)}$$

etc,and we now see that this is a real vector space with the two algebraic operations defined in the usal fashion

$$x+y=(\xi_1+\eta_1 ,\dots\dots,\xi_n+\eta_n)$$

$$\alpha x=(\alpha\xi_1,\dots\dots\dots,\alpha\xi_n) \qquad \alpha \in R$$

(2) Function space C[a, b]. As a set X we take the set of all real-valued functions x, y, ... which are functions of an independent real variable t and are defined and continuous on a given closed interval J = [a, b]. Choosing the metric defined

$$d(x,y) = \max_{t\in j} |x(t) - y(t)| \ldots\ldots\ldots\ldots\ldots (9)$$

where max denotes the maximum,the set of all these functions forms a real leder space with the algebraic operations defined in the usaluay

$$(x+y)(t)=x(t)+y(t)$$

$$(\alpha x)(t)=\alpha x(t), \qquad \alpha \in R.$$

Definition ٢.4 [3]:-

A subspace of a vector space X is a nonempty subset Y of X such that for all Y1, Y2 ∈Y and all scalars α, β we have $\alpha\, y_1 + \beta y_2 \in Y$. Hence Y is itself a vector space, the two algebraic operations being those induced from X. A special subspace of X is the improper subspace Y = X. Every other subspace of X (≠{0}) is called proper. Another special subspace of any vector space X is Y={0}.

Definition ٢.5 [2]:-

A linear combination of vectors X1. ... , Xm of a vector space X is an expression of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m \ldots\ldots (10)$$

where the coefficients $\alpha_1 \ldots, \alpha_m$ are any scalars. For any nonempty subset M⊂ X the set of all linear combinations of vectors of M is called the span of M, written span M.

Obviously, this is a subspace Yof X, and we say that Y is spanned or generated by M.

Definition ٢.6

(Linear independence, linear dependence):- Linear independence and dependence of a given set M of vectors

X_1. ... , Xr ($r \geqq 1$) in a vector space X are defined by means of the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r \text{ ....(11)}$$

Where $\alpha_1 \cdots, \alpha_r$ are scalars. Clearly, equation (11) holds for $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$. If this is the only r-tuple of scalars for which (11) holds, the set M is said to be linearly independent. M is said to be linearly dependent if M is not linearly independent, that is, if (11) also holds for some r-tuple of scalars, not all zero. An arbitrary subset M of X is said to be linearly independent if every nonempty finite subset of M is linearly independent. M is said to he linearly dependent if M is not linearly independent. •

Definition ٢.7

(Finite and infinite dimensional vector spaces):- A vector space X is said to be finite dimensional if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of n+ 1 or more vectors of X is linearly dependent. n is called the dimension of X, written n = dim X. By definition, X = {0} is finite dimensional and dim X = 0. If X is not finite dimensional, it is said to be infinite dimensional.

Theorem 2.8 (Dimension of a subspace). Let X be an n-dimensional vector space. Then any proper subspace Y of X has dimension less than n-

Proof.

If n = 0, then X = {0} and has no proper subspace. If dim Y=0, then Y={0}, and X≠Y implies dim X≧1. Clearly, dim Y ≦ dim X

= n. If dim Y were n, then Y would have a basis of n elements, which would also be a basis for X since dim X = n, so that X = Y. This shows that any linearly independent set of vectors in Y must have fewer than n elements, and dim Y < n.

Definition ٢.9 [5]:-

(Normed space, Banach space):- A normed space X is a vector space with a norm defined on it, A Banach space is a complete normed space (complete in the metric defined by the norm; see (1), below). Here a norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by

$$\|x\| \qquad \text{(read "norm of x")}$$

and which has the properties

(N1) $\|x\| \geqq 0$

(N2) $\|x\| = 0 \quad \Leftrightarrow \quad x = 0$

(N3) $\|\alpha x\| = |\alpha| \, \|x\|$

(N4) $\|x+y\| \leqq \|x\| + \|y\|$ ....(12)

(Triangle inequality);

here x and y are arbitrary vectors in X and α is any scalar. A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \qquad (x, y \in X) \text{...........(13)}$$

and is called the metric induced by the norm. The normed space just defined is denoted by $(X, \|\cdot\|)$ or simply by X. •

Examples 2.10 [2]:-

(1) Euclidean space R^n and unitary space c^n. These spaces were defined in. They are Banach spaces with norm defined by

$$\|x\|=\left(\sum_{j=1}^{n}|\xi_j|^2\right)^{1/2}=\sqrt{|\xi_1|^2+\cdots+|\xi_n|^2}.$$

..............................(14)

(2) Space l^p, Hilbert sequence space l^2, Hölder and Minkowski inequalities for sums. Let $p \geqq 1$ be a fixed real number. By definition, each element in the space l^p is a sequence $x = (\xi_j) = (\xi_1, \xi_2, \cdots)$ of numbers such that $|\xi_1|^p + |\xi_2|^p + \cdots$ converges; thus

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty \ (p \geqq 1, \text{fixed}) \ \ldots\ldots\ldots\ldots\ldots (15)$$

and the metric is defined by
(2)

$$d(x,y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{1/p} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

where y = (η_j) and $\sum |\eta_j|^p < \infty$.

gt is Banach space with norm

give by $\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$

this norm induces the metric in ......(16)

$$d(x, y) = \| x - y \| = \left(\sum_{j=1}^{\infty} \left| \xi_j - \eta_j \right|^p \right)^{1/p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Lemma 2.11 (Translation invariance) [4]. A metric d induced by a norm on a normed space X satisfies .

$$d(x+a, y+a)=d(x, y)$$

$$d(\alpha x,\alpha y)=|\alpha|\, d(x,y)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

for all x, y, a $\in$ X and every scalar α.

Proof. We have

$$d(x+a, y+a)=\|x+a-(y+a)\|=\|x-y\|=d(x, y)$$

and

$$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha|\, \|x - y\| = |\alpha|\, d(x, y).$$

Lemma 2.12 (Linear combinations)[4]:- Let $\{x_1,\ldots\ldots\ldots,x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number c >0 such that for every choice of scalars

$\alpha_1\ ,\ldots\ldots\ldots, \alpha_n$ we have

$$\| \alpha_1 x_1 + \ldots\ldots\ldots + \alpha_n x_n \| \geqq c(|\alpha_1| + \ldots\ldots + |\alpha_n|) \qquad (c > 0)$$

3.section three

(linear operators)

3. linear operators

In calculus we consider the real line R and real-valued functions on R (or on a subset of R).obviously, any such function is a mapping of its domain into R. In functional analysis we consider more general spaces, such as metric spaces and normed spaces, and mappings of these spaces. In the case of vector spaces and, in particular, normed spaces, a mapping is called an operator. Of special interest are operators which "preserve" the two alge-braic operations of vector space, in the sense of the following definition.

Definition 3.1 [5]:-

(Linear operator):- A linear operator T is an operator such that

(1) the domain D(T) of T is a vector space and the range R(T) lies in a vector space over the same field,

(2) for all x, y ∈ D(T) and scalars α,

$$T(x+y) = Tx + TY \qquad \text{...........(20)}$$

$$T(\alpha x) = \alpha Tx$$

Note 3.2:-

Observe the notation; we write Tx instead of T(x); this simplifica-tion is standard in functional analysis. Furthermore, D(T) denotes the domain of T.

$R(T)$ denotes the range of T.

$N(T)$ denotes the null space of T.

Definition 3.3 [1]:-

null space of T is the set of all $x \in D(T)$ such that $T(x) = 0$.
(Another word for null space is "kernel.")

Note 3.4:-

we should also say somenthing about the use of arrows in con-nection with operators. Let $D(T) \subset X$ and $R(T) \subset Y$, where X and Y are vector spaces, both real or both complex. Then T is an operator from (or mapping of) D(T) onto R(T), written

$$T: D(T) \longrightarrow R(T),$$

Clearly (20),is equivalent to

$$T(\alpha x+\beta y)=\alpha Tx+\beta Ty \quad \ldots\ldots\ldots\ldots\ldots(21)$$

By taking $\alpha = 0$ we obtain the following formula which we shall need many times in our work:

$$T0=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Example 3.5 [3]:-

(1) Identity operator. The identity operator Ix: X →X is defined by $Ix^X = x$ for all $x \in X$. We also write simply I for Ix; thus, Ix = x.

(2) Zero operator. The zero operator 0: X →Y is defined by 0x.= 0 for all $x \in X$.

(3) Differentiation. Let X be the vector space of all polynomials on [a, b]. We may define a linear operator T on X by setting

$$Tx(t) = x'(t) \qquad \text{.....................(23)}$$

for every $x \in X$, where the prime denotes differentiation with respect to t. This operator T maps X onto itself.

(4) Integration. A linear operator T from C[a, b] into itself can be defined by

$$Tx(t) = \int_a^t x(T)dT \qquad t \in [a, b].$$

(5) Multiplication by t. Another linear operator from C[a, b] into itself is defined by

$$Tx(t) = tx(t). \qquad \text{..................(24)}$$

(6) **Elementary vector algebra.** The cross product with one factor kept fixed defines a linear operator $T_1 : R^3 \to R^3$. Similarly, the dot product with one fixed factor defines a linear operator $T_2 : R^3 \to$ R, say,

$$T_2 x = x \cdot a = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3 \qquad \text{..................(25)}$$

where $a = (\alpha_j) \in R^3$ is fixed.

(7) **Matrices.** A real matrix $A = (\alpha_{jk})$ with r rows and n columns defines an operator $T: R^n \to R^r$ by means of

$$y = Ax$$

where $x = (\xi_j)$ has n components and $y = (\eta_J)$ has r components and both vectors are written as column vectors because of the usual convention of matrix multiplication; writing y = Ax out, we have

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ . \\ . \\ . \\ \eta_r \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \alpha_{r1} & . & \dots & \alpha_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \\ . \\ . \\ . \\ \xi_n \end{bmatrix}.$$

T is linear because matrix multiplication is a linear operation

Theorem 3.6 [1](Range and null space) . Let T be a linear operator. Then:

(a) The range R(T) is a vector space.

(b) If dim D(T)=$n<\infty$, then dimR(T)$\leqq$n.

(c) The null space N(T) is a vector space.

Proof. (a) Wetake any y_1, $y_2 \in$ R(T) and show that $\alpha y_1 + \beta y_2 \in R(T)$ for any scalars α, β. Since y_1, $y_2 \in R(T)$, we have $y_1 = Tx_1$,$y_2 = Tx_2$ for some x_1, $x_2 \in D(T)$, and $\alpha x_1 + \beta x_2 \in D(T)$ because D(T) is a vector space. The linearity of T yields

$$T(\alpha x_1+\beta x_2)=\alpha Tx_1+\beta Tx_2=\alpha y_1+\beta y_2.$$

Hence $\alpha y1+\beta y_2 \in R(T)$. Since $y_1,y_2 \in R(T)$ were arbitrary and so were the scalars, this proves that R(T) is a vector space.

(b) We choose n+1 elements $y_1,\cdots,y_{n+1}$ of R(T) in an arbitrary fashion. Then we have $y_1=Tx_1,\cdots,y_{n+1}=T_{n+1}$ for some $x_1,\cdots,x_{n+1}$ in D(T). Since dim D(T)=n, this set $\{x_1,\cdots,x_{n+1}\}$ must be linearly dependent. Hence

$$\alpha_1 x_1+\ldots\ldots+\alpha_{n+1}x_{n+1}=0$$

for some scalars $\alpha_1,\cdots,\alpha_{n+1}$, not all zero. Since T is linear and T0=0, application of T on both sides gives

$$T(\alpha_1 x_1+\ldots..+\alpha_{n+1}x_{n+1})=\alpha_1 y_1+\ldots\ldots+\alpha_{n+1}y_{n+1}=0.$$

This shows that $\{y_l, \ldots, y_{n+l}\}$ is a linearly dependent set because the α_j 's are not all zero. Remembering that this subset of R(T) was chosen in an arbitrary fashion, we conclude that R(T) has no linearly independent subsets of n+1 or more elements. By the definition this means that dim R(T)≦n.has no linearly independent subsets of n+1 or more elements. By the definition this means that dim R(T)≦n.

$$Tx_1=Tx_2$$

(c) We take any $x_1,x_2 \in N(T)$. Then $Tx_1 = Tx_2= 0$. Since

T is linear, for any scalars α, β we have

$$T(\alpha x_1+\beta x_2)=\alpha Tx_1+\beta Tx_2=0.$$

This shows that $\alpha X_l + \beta X_2 \in N(T)$. Hence N(T) is a vector space. •

4. section four

(Additional Results about Linear Operators)

4. Additional Results about Linear Operators.

Definition 4.1 [1]:-

injective or one-to-one Let us turn to the inverse of a linear operator. We first remember

that a mapping $T: D(T) \longrightarrow Y$ is said to be injective or one-to-one if

different points in the domain have different images, that is, if for any $x_1, x_2 \in D(T)$

$$x_1 \neq x_2 \quad \Rightarrow \quad Tx_1 \neq Tx_2 ;$$

equivalently,

..............................(26)

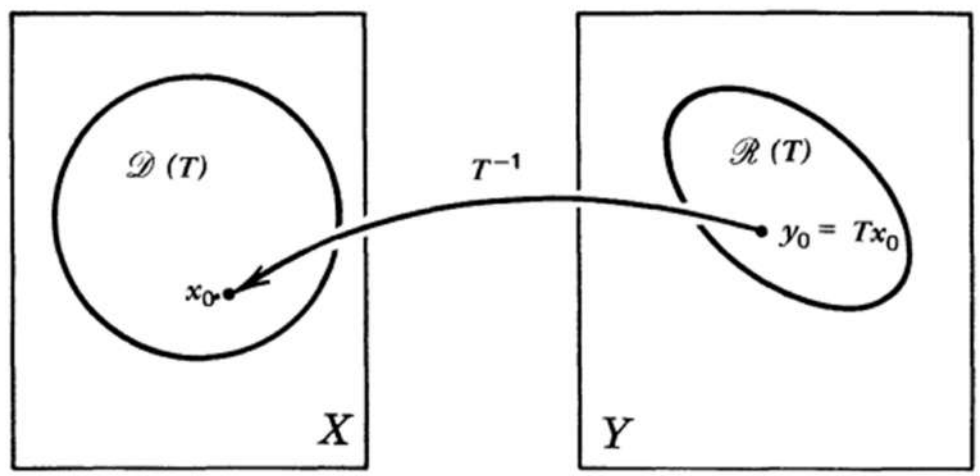
$$Tx1 = Tx2 \quad \Rightarrow \quad x1 = x2$$

In this case there exists the mapping

$$T^{-1}: R(T) \to D(T) \quad \text{................(27)}$$

$$y_0 \mapsto x_0 \qquad\qquad (y_0 = Tx_0)$$

which maps every $y_0 \in R(T)$ onto that $x_0 \in D(T)$ for which $Tx_0 = y_0$.



Figer(1) (the mapping T^{-1} is called the inverse[6] of T.)

Form(27), we clearly have

$$T^{-1}Tx = x \quad \text{for all } x \in D(T)$$
$$TT^{-1}y = y \quad \text{for all } y \in \mathcal{R}(T)$$

Note 4.2:-

In connection with linear operators on vector spaces the situation

is as follows. The inverse of a linear operator exists if and only if the null space of the operator consists of the zero vector only.

Theorem 4.3 (Inverse operator). Let X, Y be vector spaces, both real or both complex. Let $T: D(T) \longrightarrow Y$ be a linear operator with

Domain D(T) ⊂ X and rang R(T) ⊂ Y
(a) The inverse $T^{-1}: \mathrm{R}(T) \longrightarrow \mathcal{D}(T)$ exists if and only if

$$Tx=0 \qquad \Longrightarrow \qquad x=0$$

(b) If T^{-1} exists, it is a linear operator.
(c) If $\dim D\,(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}\,(T) = \dim D(T)$.
Proof.

(a) Suppose that $Tx = 0$ implies $x = 0$. Let $Tx_1 = Tx_2$. Since T is linear,

$$T(x_1-x_2)=Tx_1 -Tx_2=0,$$

so that $x_1 - x_2 = 0$ by the hypothesis. . Hence $Tx_1 = Tx_2$ implies $x_1 = x_2$, and T^{-1}exists by(26). . Conversely, if T −1 exists, then (26) holds. From 26 with x 2=0 and (23) we obtain

$$T\, x_1=T\,0=0 \qquad \Longrightarrow \qquad x\,1=0.$$

This completes the proof of (a).

(b) We assume that T^{-1} exists and show that T^{-1} is linear. The domain of T^{-1} is $\mathcal{R}(T)$ and is a vector space by Theorem 3.6 (a) . We consider any $x_1, x_2 \in D(T)$ and

$$y_1 = Tx_1 \qquad \text{and} \qquad y_2 = Tx_2.$$

Then

$$x_1 = T^{-1}y_1 \qquad \text{and} \qquad x_2 = T^{-1}y_2.$$

T is linear, so that for any scalars α and β we have

$$\alpha y_1 + \beta y_2 = \alpha T x_1 + \beta T x_2 = T(\alpha x_1 + \beta x_2).$$

Since $x_j = T^{-1}y_j$, this implies

$$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1} y_1 + \beta T^{-1} y_2$$

and proves that T^{-1} is linear.

(c) We have $\dim \mathcal{R}(T) \leqq \dim D(T)$ by Theorem 3.6(b), and $\dim D(T) \leq \dim R(T)$) by the same theorem applied to T^{-1} .

Lemma 4.4 [4] (Inverse of product). Let $T: X \longrightarrow Y$ and $S: Y \longrightarrow Z$ Z be bijective linear operators, where X, Y, Z are vector spaces (see Fig. inverse of product). Then the inverse $(ST)^{-1}: Z \longrightarrow X$ of the product (the compos-ite) ST exists, and

$$(ST)^{-1} = T^{-1}S^{-1}, \dots\dots\dots\dots\dots\dots (28)$$

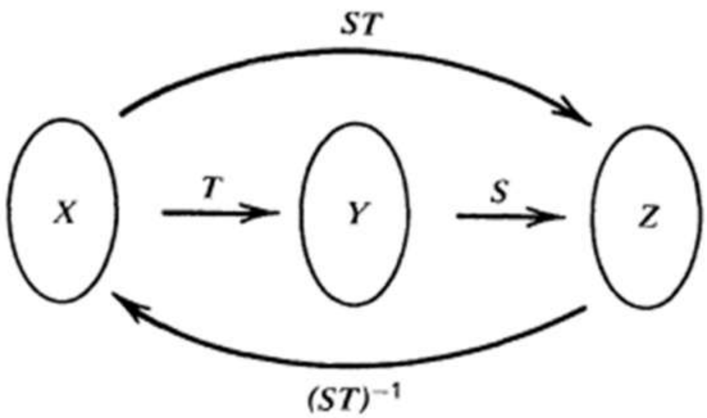
Proof.

The operator $ST: X \longrightarrow Z$ is bijective, so that $(ST)^{-1}$ exists. We thus have

$$ST(ST)^{-1} = I_Z$$

where I_Z is the identity operator on Z. Applying S^{-1} and using $S^{-1}S = I_Y$ (the identity operator on Y), we obtain

$$S^{-1}ST(ST\ ^{-1} = T(ST\ ^{-1} = S^{-1}I_Z = S^{-1}.$$



Figer(2) (inverse product).

Applying T^{-1} and using $T^{-1}T = I_X$, we obtain the desired result

$$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1}.$$

This completes the proof.

5. section five

(Bounded and Continuous Linear Operators)

5 .Bounded and Continuous Linear Operators

The reader may have noticed that in the whole last section we did not make any use of norms. We shall now again take norms into account, in the following basic definition.

Definition5.1 [5]:-

(Bounded linear operator). Let X and Y be normed spaces and $T: D(T) \longrightarrow Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$,

$$\| Tx \| \leqq c \| x \| \quad ....(29)$$

Note 5.2: –

(1) In (29) the norm on the left is that on Y, and the norm on the right is that on X. For simplicity we have denoted both norms by the same symbol $\|\cdot\|_{\mathrm{r}}$. without danger of confusion. Formula(29) shows that a bounded linear operator maps bounded sets in $D(T)$ onto bounded sets in Y. This motivates the term "bounded operator."

(2) What is the smallest possible c such that (29) still holds for all nonzero $x \in D(T)$? [We can leave out $x = 0$ since $Tx = 0$ for $x = 0$ By division,

$$\frac{\|Tx\|}{\|x\|} \leqq c \qquad (x \neq 0)$$

and this shows that c must be at least as big as the supremum of the expression on the left taken over $\mathrm{D}(T) - \{0\}$. Hence the answer to our question is that the smallest possible c in $\| Tx \| \leqq c \| x \|$ is that supremum. This quantity is denoted by $\| T \|$; thus

$$\| T \| = \sup_{\substack{x \in \mathrm{D(T)} \\ x \neq 0}} \frac{\| Tx \|}{\| x \|}. \quad \ldots\ldots\ldots\ldots\ldots(30)$$

$\| T \|$ is called the norm of the operator T. If $D(T) = \{0\}$, we define $\| T \| = 0$; in this (relatively uninteresting) case, $T = 0$ since $T0 = 0$

Note that (29) with $c = \| T \|$ is

$$\| Tx \| \leqq \| T \| \| x \|. \ldots\ldots\ldots\ldots\ldots(31)$$

Lemma 5.3 [4](Norm). Let T be a bounded linear operator as defined in 5.1 . Then:

(a) An altemative formula for the norm of T is

$$\| T \| = \sup_{\substack{x \in D(T) \\ \| x \| = 1}} \| Tx \| \cdots \ldots\ldots\ldots\ldots\ldots(32)$$

(b) The norm defined by (2) satisfies (Normed space Definition (2.9))

Proof. (a) We write $\| x \| = a$ and set $y = (1/a)x$, where $x \neq 0$. Then $\| y \| = \| x \| / a = 1$, and since T is linear, (30) gives

$$\| T \| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{1}{a} \| Tx \| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\| T\left(\frac{1}{a} x\right) \right\| = \sup_{\substack{y \in D(T) \\ \|y\| = 1}} \| Ty \|$$

Writing x for y on the right, we have (32).

(b) (N1) is obvious, and so is $0 \| = 0$. From $\| T \| = 0$ we have $Tx = 0$ for all $x \in D(T)$, so that $T = 0$. Hence (N2) holds. Furthermore, (N3) is obtained from

$$\sup_{\|x\|=1} \| \alpha Tx \| = \sup_{\|x\|=1} |\alpha| \, \| Tx \| = |\alpha| \sup_{\|x\|=1} \| Tx \|$$

where $x \in D(T)$. Finally, (N4) follows from

$$\sup_{\|x\|=1} \|(T_1 + T_2)x\| = \sup_{\|x\|=1} \|T_1 x + T_2 x\| \leq \sup_{\|x\|=1} \|T_1 x\| + \sup_{\|x\|=1} \|T_2 x\|$$

here, $x \in D(T)$.

Examples 5.4 [2]:-

(1)Identity operator. The identity operator $I: X \longrightarrow X$ on a normed space $X \neq \{0\}$ is bounded and has norm $\| I \| = 1$.

(2) Zero operator. The zero operator $0: X \longrightarrow Y$ on a normed space X is bounded and has norm $\| 0 \| = 0$.

(3) Differentiation operator. Let X be the normed space of all polynomials on $J = [0,1]$ with norm given $\| x \| = \max|x(t)|, t \in J$. A differentiation operator T is defined on X by

$$Tx(t) = x'(t)$$

where the prime denotes differentiation with respect to t. This operator is linear but not bounded. Indeed, Iet $x_n(t) = t^n$, where $n \in \mathbf{N}$. Then $\|x_n\| = 1$ and

$$Tx_n(t) = x_n{}'(t) = nt^{n-1}$$

so that $\|Tx_n\| = n$ and $\|Tx_n\|/\|{}_{x_n}\| = n$. Since $n \in \mathbf{N}$ is arbitrary, this shows that there is no fixed number c such that

$\|Tx_n\|/\|x_n\| \leqq c$. From this and (29) we conclude that T is not bounded.

(4) Integral operator. We can define an integral operator $T : C[0,1] \to C[0,1]$ by

$$y = Tx \quad \text{where} \quad y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau.$$

Here k is a given function, which is called the kernel of T and is assumed to be continuous on the closed square $G = J \times J$ in the t tT-plane, where $J = [0,1]$. This operator is linear.
T is bounded.
To prove this, we first note that the continuity of k on the closed square implies that k is bounded, say, $|k(t,\tau)| \leqq k_0$ for all $(t,\tau) \in G$, where k_0 is a real number. Furthermore,

$$|x(t)| \le \max_{t\in J}|x(t)| = \| x \|.$$

Hence

$$\begin{aligned} \| y \| = \| Tx \| &= \max_{t\in J}\left|\int_0^1 k(t,\tau)x(\tau)d\tau\right| \\ &\le \max_{t\in J}\int_0^1 |k(t,\tau)||x(\tau)|d\tau \\ &\le k_0 \| x \|. \end{aligned}$$

The result is $\| Tx \| \leq k_0 \| x \|$. T is bounded.

Theorem 5.5 [1](Finite dimension). If a normed space X is finite dimensional, then every linear operator on X is bounded.

Proof. Let $\dim X = n$ and $\{e_1, \cdots, e_n\}$ a basis for X. We take any

$X = \sum \xi_j e_j$ and consider any linear operator T on X. Since T is linear,

$$\| Tx \| = \left\| \sum \xi_j T e_j \right| \leqq \sum \left| \xi_j \right| \| T e_j \| \leqq \max_k \| T e_k \| \sum \left| \xi_j \right|$$

(summations from 1 to n). To the last sum we apply Lemma 2.12 with $\alpha_i = \xi_i$ and $x_i = e_r$. Then we obtain

$$\sum |\xi_i| \leqq \frac{1}{c} |\sum \epsilon_i e_i| = \frac{1}{c} \| x \|$$

Together,

$$\| Tx \| \leq \gamma \mid x \| \quad \text{where} \quad \gamma = \frac{1}{c} \max_k \| T e_k \|$$

T is bounded

Definition 5.6 [3]:-

(continuous mapping) :-We shall now consider important general properties of bounded linear operators.

Let $T: D(T) \longrightarrow Y$ be any operator, not necessarily linear, where $D(T) \subset X$ and X and Y are normed spaces. By Def (5.6)the operator T is continuous at an $x_0 \in D(T)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|Tx - Tx_0\| < \varepsilon \text{ for all } x \in D(T) \text{ satisfying } \|x - x_0\| < \delta.$$

T is continuous if T is continuous at every $x \in D(T)$.

Theorem 5.7 [5] (Continuity and boundedness).

Let $T: \mathrm{D}(T) \longrightarrow Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces. Then:

(a) T is continuous if and only if T is bounded.
(b) If T is continuous at a single point, it is continuous.

Proof.

(a) For $T = 0$ the statement is trivial. Let $T \neq 0$. Then $\| T \| \neq 0$. We assume T to be bounded and consider any $x_0 \in D(T)$, Let any $\varepsilon > 0$ be given. Then, since T is linear, for every $x \in D(T)$ such that

$$\|x - x_0\| < \delta \text{ where } \delta = \frac{\varepsilon}{\| T \|}$$

we obtain

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leqq \| T \| \, \|x - x_0\| < \| T \| \, \delta = \varepsilon.$$

Since $x_0 \in D(T)$ was arbitrary, this shows that T is continuous. Conversely, assume that T is continuous at an arbitrary $x_0 \in D(T)$. Then, given any $\varepsilon > 0$, there is a $\delta > 0$ such that
(33) $\|Tx - Tx_0\| \leqq \varepsilon$ for all $x \in D(T)$ satisfying $\|x - x_0\| \leq \delta$.
We now take any $y \neq 0$ in $D(T)$ and set

$$x = x_0 + \frac{\delta}{\| y \|} y. \text{ Then } x - x_0 = \frac{\delta}{\| y \|} y$$

Hence $\|x - x_0\| = \delta$, so that we may use (33). Since T is linear, we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| = \left\|T\left(\frac{\delta}{\|y\|}y\right)\right\| = \frac{\delta}{\|y\|} \parallel Ty \parallel$$

and (33) implies

$$\frac{\delta}{\parallel y \parallel} \parallel Ty \parallel \leq \varepsilon. \text{ Thus } \parallel Ty \parallel \leqslant \frac{\varepsilon}{\delta} \parallel y \parallel$$

This can be written $\parallel Ty \parallel \leq c \parallel y \parallel$, where $c = \varepsilon/\delta$, and shows that T is bounded.
(b) Continuity of T at a point implies boundedness of T by the second part of the proof of (a), which in turn implies continuity of T by (a).

Corollary 5.8 [2] (Continuity, null space). Let T be a bounded linear operator. Then:
(a) $x_n \longrightarrow x[$ where $x_n, x \in D(T)]$ implies $Tx_n \longrightarrow Tx$.
(b) The null space $\mathcal{N}(T)$ is closed.
Proof.

(a) follows from (22) because, as $n \longrightarrow \infty$,

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \parallel T \parallel \|x_n - x\| \longrightarrow 0$$

(b) For every $x \in \overline{\mathcal{N}(T)}$ there is a sequence (x_n) in $\mathcal{N}(T)$ such that $x_n \longrightarrow x$.Hence $Tx_n \longrightarrow Tx$ by part (a) of this Corollary. Also $Tx = 0$ since $Tx_{\mathrm{m}} = 0$, so that $x \in \mathcal{N}(T)$. Since $x \in \overline{\mathcal{N}(T)}$ was arbitrary, $\mathcal{N}(T)$ is closed.

Definition 5.9 [1]:-

Two operators T_1 and T_2 are defined to be equal, written

$$T_1 = T_2$$

if they have the same domain $D(T_1) = D(T_2)$ and if $T_1 x = T_2 x$ for all $x \in D(T_1) = D(T_2)$.

The restriction of an operator $T: D(T) \longrightarrow Y$ to a subset $B \subset D(T)$ is denoted by

$$T|_B$$

and is the operator defined by

$$T|_B: B \longrightarrow Y,\ T|_B x = Tx \text{ for all } x \in B$$

Theorem 5.10 (Closure, closed set). Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure as defined in the previous section.

Then:

(a) $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that

$x_n \longrightarrow x$

(b) M is closed if and only if the situation $x_n \in$ M, x $\longrightarrow$ x implies

that $x \in M$

Theorem 5.11 [3](Bounded linear extension). Let

$$T: \mathrm{D}(T) \longrightarrow Y$$

be a bounded linear operator, where $D(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension

$$\tilde{T}: \overline{D(T)} \longrightarrow Y$$

where $\tilde{T}$ is a bounded linear operator of norm $\| \tilde{T} \| = \| T \|$

6.section six

(Linear Functionals)

6.Linear Functionals

A functional is an operator whose range lies on the real line R or in the complex plane C. And functional analysis was initially the analysis of functionals. The latter appear so frequently that special notations are used.

We denote functionals by lowercase letters f, g, h, ... , the domain of f by $D(f)$, the range by $\mathrm{R}(f)$ and the value of f at an $x \in \mathrm{D}(f)$ by $f(x)$, with parentheses.

Functionals are operators, so that previous definitions apply. We shall need in particular the following two definitions because most of the functionals to be considered will be linear and bounded.

Definition 6.1 [4] (Linear functional):- A linear functional f is a linear operator with domain in a vector space X and range in the scalar field K of X; thus,

$$f : D(f) \longrightarrow K$$

where $K = \mathbf{R}$ if X is real and $K = \mathbf{C}$ if X is complex.

Definition 6.2 [3] (Bounded linear functional):- A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $D(f)$ lies. Thus there exists a real number c such that for all $x \in D(f)$,

$$|f(x)| \leqq c \parallel x \parallel. \quad \ldots\ldots\ldots\ldots\ldots (34)$$

Furthermore, the norm of f is

$$\| f \| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$$

………… ……………………(35)

$$\| f \| = \sup_{\substack{x \in D(f) \\ \|x\|=1}} |f(x)|.$$

Formula (31) in implies

$$|f(x)| \leq \| f \| \| x \|, \ldots\ldots\ldots\ldots (36)$$

Theorem 6.3 [1] (Continuity and boundedness). A linear functional f with domain D(f) in a normed space is continuous if and only if f is bounded.

Examples 6.4 [1]:-
(1) Norm. The norm $\|\cdot\|: X \longrightarrow \mathbf{R}$ on a normed space $(X, \|\cdot\|)$ is a functional on X which is not linear.

(2) Dot product. The familiar dot product with one factor kept fixed defines a functional $f: \mathbf{R}^3 \longrightarrow \mathbf{R}$ by means of

$$f(x) = x \cdot a = \xi_1 \alpha_1 + \xi_2 \alpha_2 + \xi_3 \alpha_3$$

where $a = (\alpha_j) \in \mathbf{R}^3$ is fixed.
f is linear. f is bounded. In fact,

$$|f(x)| = |x + a| \leq \| x \| \| a \|$$

so that $\| f \| \leqq \| a \|$ follows from (35) if we take the supremum over all x of norm one. On the other hand, by taking $x = a$ and using (36) we obtain

$$\| f \| \geq \frac{|f(a)|}{\| a \|} = \frac{\| a \|^2}{\| a \|} = \| a \|$$

Hence the norm of f is $\| f \| = \| a \|$.

(3) Space $C[a,b]$. Another practically important functional on $C[a,b]$ is obtained if we choose a fixed $t_0 \in J = [a,b]$ and set

$$f_1(x) = x(t_0) \qquad x \in C[a,b].$$

f_1 is linear. f_1 is bounded and has norm $\|f_1\| = 1$. In fact, we have

$$|f_1(x)| = |x(t_0)| \leqq \| x \| ,$$

and this implies $\|f_1\| \leq 1$ by (35). On the other hand, for $x_0 = 1$ we have $\|x_0\| = 1$ and obtain from (36)

$$\|f_1\| \geqq |f_1(x_0)| = 1.$$

(4) Space l^2. We can obtain a linear functional f on the Hilbert space l^2 by choosing a fixed $a = (\alpha_j) \in l^2$ and setting

$$f(x) = \sum_{j=1}^{\infty} \xi_i \alpha_{\mathrm{j}}$$

where $x = (\xi_j) \in l^2$. This series converges absolutely and f is bounded, since the Cauchy-Schwarz inequality gives (summation over j from 1 to x)

$$|f(x)| = \left|\sum \xi_j \alpha_i\right| \leq \sum |\xi_i \alpha_j| \leqq \sqrt{\sum |\xi_i|^2} \sqrt{\sum |\alpha_i|^2} = \| x \| \| a \|.$$

References

[1] Conway J (1990) A course in functional analysis. Springer, New York.

[2] Kreyszig E (1978) Introductory functional analysis with applications. Wiley, New York.

[3] Muscat J (2014) Functional Analysis. An Introduction to Metric Spaces ,Hilbert Spaces, and Banach Algebras. Springer International Publishing Switzerland.

[4] Rudin W (1991) Functional analysis. McGraw-Hill, New York.

[5] Schechter M (2002) Principles of functional analysis. AMS Bookstore, Boston.