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GAMMA , BETA AND BESSEL FUNCTIONS

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بسم الله الرحمن الرحيم

وَمَا خَلَقْنَا السَّمَاوَاتِ وَالْأَرْضَ وَمَا بَيْنَهُمَا
لَاعِبِينَ * مَا خَلَقْنَاهُمَا إِلَّا بِالْحَقِّ وَلَكِنَّ
أَكْثَرَهُمْ لَا يَعْلَمُونَ

صدق الله العظيم

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الإهداء

اهدي تخرجي الى من كان سبب وجودي بالحياة إلى من لهما الفضل فيها إلى من أوصاني الرحمن بهما برا واحسانا إلى من جنة الله تحت قدميها إلى من قدمت عمرها لي كي تراني هنا إلى من سعا وعانا من أجلي

إلى امي وابي

وإلى الأستاذة الفاضلة أ . سحر محسن والى كل شخص ساهم ووقف الى جانبي في تكملة مسيرتي الدراسية

شكر وتقدير

اتقدم بخالص شكري وامتناني إلى عمادة كلية
التربية للعلوم الصرفة ، كما أتقدم بخالص
الامتنان إلى أساتذتي الكرام
وبالأخص الأستاذة الفاضلة (أ .سحر محسن)
للمساعدة السديدة والملاحظات الدقيقة التي
لولاها لما اكتمل البحث
كما اشكر زملائي وزميلاتي للأيام الجميلة التي
قضيناها معا
إلى كل من ساعدني في معلومة أو نصيحة لكم
مني كل الحب والتقدير

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Abstract

Many linear differential equations having variable coefficients cannot be solved by usual methods and we need to employ series solution method to find their solutions in terms of infinite convergent series.

The most popular functions in mathematics which recently called special functions, are Beta , gamma and Bessel. It is known that, Beta is a two-variable function, whereas Gamma is a single variable function. Such functions have been used to solve many problems in physics and mathematics.

This work based on some basic concepts related with Beta, Gamma and Bessel functions as well as the connection between all the three functions. Their properties and illustrative examples are represented also.

Introduction

Finding solutions of differential equations has been a problem in pure mathematics since the invention of calculus by Newton and Leibniz in the 17th century. Besides this, these equations are used in some other disciplines such as engineering, biology, economics and physics. In fact, because there are advanced aspects in mathematics that need another theory elementary functions to calculate difficult integrals or solve differential equations, integral equations, so this need translates into existence for special functions such as gamma, beta, and Bessel. Such functions are also important in the fields of applied sciences.

Gamma function, is a generalization of the factorial function to non-integral values, introduced by the Swiss mathematician Leonhard Euler in the 18th century.

The common method for determining the value of n! is naturally recursive, found by multiplying *1*2*3*... *(n-2)*(n-1)*n,* though this is terribly inefficient for large n. So, in the early 18th century, the question was posed: As the definition for the nth triangle number can be explicitly found, is there an explicit way to determine the value of $\boldsymbol{n!}$ which uses elementary algebraic operations? In 1729, Euler proved no such way exists, though he posited an integral formula for $\boldsymbol{n!}$. Later, Legendre would change the notation of Euler's original formula into that of the Gamma function that we use today.

In mathematics, ***Beta function***, also known as an ***Euler integral*** of the first type, it is a special function that given by:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dx$$

Euler and Legendre studied this function, and Jacques Binet gave it this name. the symbol β is one of the capital letters in Greek writing .

Whereas, ***Bessel functions*** are solutions of a particular differential equation, called Bessel's equation. In the late 17th century, the Italian mathematician Jacopo Riccati studied what we known nowadays.

This crucial function is involved in many applications such as :

The Oscillatory motion of a hanging chain •
Euler's theory of a circular membrane
The studies of planetary motion •
The propagation of waves •
The Elasticity •
The fluid motion.
The potential theory •
Cylindrical and spherical waves
Theory of plane waves •

In this research, some important aspects have been introduced about these special functions, A relationship has been defined for Beta, Gamma and Bessel Functions. Various properties of Beta, Gamma, and Bessel's Functions have been used to simplify the Bessel's summation down to a single term by building the relationship between Beta, Gamma and itself. Illustrative examples also presented, thus through this work, we will know the significant role and large purpose of these functions in Mathematics and Physics.

Chapter One

Gamma , Beta and Bessel Functions

1.1. Introduction

Due to the need to discover an easiest way to solve many difficulties problems (equations), special functions such as Gamma, Beta and Bessel function appeared.

Each one of these functions has been defined in different ways, and the most important of these definitions are represented in this chapter.

1.2. Gamma Function

The Gamma function was first introduced by the mathematician Leon-hard Euler (1707 – 1783) in his goal to generalize the factorial to non - integer values . Later , because of its great importance , it was studied by other mathematicians like Adrien - Marie Legendre (1752 – 1833), carl- Friedrich Gauss (1777 – 1855), Christoph Gudermann (1798 – 1852), Joseph Liouville (1809 – 1882), Karl Weierstrass (1815 – 1897), and Charles Hermit (1822 – 1901) as well as many others .

1.2.1. Definition : (Euler .1730)

Let $x > 0$, $\Gamma\,(x) = \int_0^1 (-\log(t)^{x-1})\, dt$. Since $\Gamma\,(1) = \int_0^\infty e^{-t} dt = 1$.

And for x > 0 , an integration by parts yields:

$$\Gamma(x+1) = \int_0^\infty t^x\, e^{-t} dt = \left[-t^x e^{-t}\right]_0^\infty + x \int_0^\infty t^{x-1}\, e^{-t} dt = x\,\Gamma\,(x),$$

the relation $\Gamma(x+1) = x\Gamma\,(x)$ is the important equation. For integral values the functional equation becomes $\Gamma(n+1) = n!$.

1.2.2. Some Special Values of $\Gamma(x)$:

* $\Gamma\left(1/2\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = 2\int_0^\infty e^{-u^2}\,du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}\,.$
* $\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5\ldots(2n-1)}{2^n}\sqrt{\pi}.$
* $\Gamma\left(n+\frac{1}{3}\right) = \frac{1.4.7\ldots(3n-2)}{3^n}\,\Gamma\left(\frac{1}{3}\right).$
* $\Gamma\left(n+\frac{1}{4}\right) = \frac{1.5.9\ldots(4n-3)}{4^n}\,\Gamma\left(\frac{1}{4}\right).$
* $\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n}{1.3.5\ldots(2n-1)}\sqrt{\pi}\,.$
* $\frac{1}{\Gamma(X)\Gamma(1-X)} = \mathrm{X}\prod_{p=1}^{\infty}\left(1-\frac{x^2}{p^2}\right),$ $\quad$ *where* $\sin(\pi x) = \pi\mathrm{X}\prod_{p=1}^{\infty}\left(1-\frac{x^2}{p^2}\right).$
* Γ (x) Γ (1-x) $= \frac{\pi}{\sin \pi x}$.

To prove $\Gamma\left(1/2\right) = \sqrt{\boldsymbol{\pi}}$:

Proof :

$\because\ \Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$, so replace

$x^n\ by\ y \quad \Longrightarrow \quad nx^{n-1}dx = dy\,, \quad x = y^{\frac{1}{n}}$. Thus

$$\Gamma(n) = \int_0^\infty \frac{x^n}{x}e^{-x}dx = \int_0^\infty \frac{y}{y^{\frac{1}{n}}}e^{-y^{\frac{1}{n}}}\frac{dy}{nx^{n-1}}$$

$$\Gamma(n) = \int_0^\infty y^{\frac{n-1}{n}}e^{-y^{\frac{1}{n}}}\frac{dy}{nx^{n-1}} = \int_0^\infty y^{\frac{n-1}{n}}e^{-y^{\frac{1}{n}}}\frac{dy}{ny^{\frac{n-1}{n}}} \quad = \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}}dy$$

$$if\ \ n = \frac{1}{2} \quad \Longrightarrow \Gamma\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}dy = 2\left(\frac{1}{2}\sqrt{\pi}\right) = \sqrt{\pi}.$$

$$\Longrightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

1.2.3. Illustrative Examples for Gamma Function

Example 1: Find the exact value of $\Gamma = (\frac{3}{2})$.

Sol.:

$\because\ \Gamma(n) = n\Gamma(n) \Longrightarrow \Gamma(n) = (n+1) = n\Gamma(n)$. Thus for $n = \frac{3}{2}$, we have:

$$\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$$

Example 2: Find the exact value of $\Gamma\left(\frac{-1}{2}\right)$.

Sol.:

$$\Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(1 + \left(\frac{-1}{2}\right)\right)}{\frac{-1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-1}{2}} = = -2\sqrt{\pi}.$$

Example 3: Find the exact value of $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$.

Sol.:

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1 - \frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}\pi.$$

1.3. Beta Function

Beta function $\beta(m,n)$is the name that used by Legendre, Whittaker and Watson (1990) for the beta integral (also called the Eulerian integral of the first kind) . It is defined by :

$$\beta_m(n) = \beta(m,n) = \int_0^1 t^{m-1}\,(1-t)^{n-1} dt.$$

The solution of this integral is the Beta function $\beta(n+1\,, m+1\,)$. The Beta function evolves Gamma function as :

$$\beta(n\,,m) = \frac{\Gamma\,(n)\Gamma\,\,(m)}{\Gamma\,(n\,,m)} = \frac{(n-1)!\,(m-1)}{(n+m-1)!}\,.$$

1.3.1. Some Properties of Beta Function

$(i)\quad \beta(m,n) = \beta(n,m).$

$(ii)\quad \beta(m,n-1) = \dfrac{n}{m}\beta(m-1,n).$

$(iii)\quad \beta(m,n-1) = \beta(m,n) - \beta(m-1,n).$

$(iv)\quad \displaystyle\int_0^{\frac{x}{2}} sin^{2m}\theta d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, m-\frac{1}{2}\right).$

$(v)\quad \displaystyle\int_0^{\frac{x}{2}} sin^{2m+1}\theta d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, m-1\right).$

To prove $\beta(m,n) = \beta(n,m)$:

Proof:

Since we have $\beta(m,n) = \int_0^1 t^{m-1}\,(1-t)^{n-1}dt$. Using the substitution *t = 1 –s* to get :

$$\beta(m,n) = \int_0^1 t^{m-1}\,(1-t)^{n-1}dt = \int_0^1 (1-s)^{m-1}\,s^{n-1}ds$$
$$= \int_0^1 s^{n-1}(1-s)^{m-1}\,ds$$
$$= \beta(n,m).$$

1.3.2. Illustrative Examples for Beta Function

Example 1: Find $\beta(7,9)$. 2. $\beta\left(\frac{1}{3},\frac{2}{3}\right)$.

Sol.:

$$\beta(7,9)=\frac{\Gamma(7)\Gamma(9)}{\Gamma(16)}=\frac{6!\Gamma(9)}{15.14.13.12.11.10.9\Gamma(9)}=\frac{6.5.4.3.2.1}{15.14.13.12.11.10.9}$$

$$=\frac{1}{15.7.13.11.3}.$$

Example 2 : Find $\beta\left(\frac{1}{3},\frac{2}{3}\right)=\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}$.

Sol.:

Using $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$, for $x=\frac{1}{3}$:

$$\therefore\ \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi}{\sin\frac{\pi}{}}=\frac{\pi}{\sqrt{3}}=\frac{2\pi}{\sqrt{3}}.$$

1.4. Bessel Function

Bessel function, also called ***cylinder function***, it is one of the most important ordinary differential equation in applied mathematics is Bessel's equation :

$$x^2y''+xy'+(x^2-v^2)y=0$$

Where parameter v is a given real number which positive or zero . It has a solution of the from : $y(x)=\sum_{r=0}^{\infty}a_r x^{m+r}$.

From first and second derivatives into Bessel's equation , we obtain :

$$\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r}+\sum_{r=0}^{\infty}(m+r)a_r x^{m+r}+$$
$$\sum_{r=0}^{\infty}a_r x^{m+r+2}-v^2\sum_{r=0}^{\infty}a_r x^{m+r}=0$$

We equate the sum of the coefficients of x^{m+r} to zero . Note that this power x^{m+r} corresponds to $m = r$ in the first , second , and fourth series , and $m = r - 2$ in the third series . Hence for $m = 0$ and $r = 1$, the third series does not contribute $m \geq 0$. For $r = 2,3,4$.....all four series contribute , so that we get a general for all these r . We find :

$m(m-1)a_0 + ma_0 - v^2a_0 = 0 \qquad (r = 0)$

$(m+1)ma_1 + (m+1)a_1 - v^2a_1 = 0 \qquad (r = 1)$

$(m+r)(m+r-1)a_r + (m+r)a_r + a_{r-2} - v^2a_r = 0 \quad (r = 2,3,\ldots).$

We obtain the indicial equation by dropping a_0 :

$(m+v)\,(m-v) = 0$. The roots are $m_1 = v(\geq 0)$ *and* $m_2 = -v$.

Coefficient Recursion for $m = m_1$. For $m = v$, reduces to $(2\,m + 1\,)\,a_1 = 0$. Hence $a_1 = 0$ since $m \geq 0$ substituting $m = v$ and combining the three terms containing a_r gives simply : $(\,m + 2v\,)\,ma_r + a_{r-2} = 0$.

Since $a_1 = 0$ and $v \geq 0$, it follows that $a_3 = 0\,, a_5 = 0$, ….. . Hence we have to deal only with even-number coefficients a_5 with $r = 2m$. For $r = 2m,$ becomes $(\,2m + 2v\,)2m\,a_{2m} + a_{2m-2} = 0$.

Solving for a_{2m} gives the recursion formula $a_{2m} = -\frac{1}{2^2m(v+m)}\,a_{2m-2}$, m = 1,2,…. We can now determine $a_2\,, a_4$ …….. successively :

$$a_2 = -\frac{a_0}{2^2(v+1)}$$

$$a_4 = -\frac{a_2}{2^2 2(v+2)} = \frac{a_0}{2^4 2!\,(v+1)(v+2)}\ .$$

And so on , and in general ,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v+1)(v+2)\ldots\ldots(v+m)}\,, \qquad m == 1,2,3,4,\ldots\ldots$$

Bessel function $\boldsymbol{J_n(X)}$ for integer $\boldsymbol{v} = n$ is :

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(n+1)(n+2)\ldots\ldots(n+m)}\,, \qquad m = 1,2,3,4,\ldots\ldots$$

a_0 is still arbitrary, so that the series with these coefficients would contain this arbitrary factor . We have make thus $a_0 = \frac{1}{2^n n!}$. Thus:

$$a_{2m} = x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+n} m!\,(n+m)!} \quad , m = 1,2, \ldots \ldots$$

By inserting these coefficients and remembering that $a_1 = 0 \,, a_3 = 0$, ….. we obtain a particular solution of Bessel´s equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \square!\,(n+m)!} \quad (n \geq 0).$$

$J_n(x)$ is called the Bessel function of the first kind of order n . The series converges for all x , Hence $J_n(x)$ is defined for all x .

Note : Bessel function $J_0(x)$ and $J_1(x)$ are :

$$J_0\,(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \frac{x^6}{2^6(3!)^2} + \cdots ..$$

$$J_1\,(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!\,(m+1)!} = \frac{1}{2} - \frac{x^3}{2^3 1!\,2!} + \frac{x^5}{2^5 2!\,3!} - \frac{x^7}{2^7 3!\,4!} + \cdots ..$$

1.4.1. Properties of Bessel Function

In this section , we introduced some properties of Bessel function , with illustrative examples :

1. $\int_0^{\infty} \frac{x^c}{c^x} dx$.

2. $\int_0^{1} \frac{1}{\sqrt{1-x^4}} dx$.

3. $J_0(0) = 1 \,, J_n(0) = 0 \,, n \neq 0.$

4. $\frac{d}{dx}[x^v J_v(X)] = x^v J_{v-1}(x).$

5. $\dfrac{d}{dx}[x^v J_v(ax)] = ax^v J_{v+1}(ax).$

6. $\dfrac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x).$

7. $J_v^{'}(x) + \dfrac{v}{t} J_v(x) = J_{v-1}(x).$

8. $(\dfrac{d}{xdx})^m [x^v J_v(x)] = x^{v-m} J_{v-m}(X). \quad m = 1,2,3, \ldots.$

To prove $\frac{d}{dx}[x^v J_v(X)] = x^v J_{v-1}(x).$

Proof:

$$\because\ J_v(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(v+r)!}\left(\frac{x}{2}\right)^{v+2r} \Longrightarrow x^v J_v(x) = \sum \frac{(-1)^r}{r!\,(v+r)!}\,\frac{2^{2v+2r}}{2^{v+2r}}$$

$$\Longrightarrow \qquad \frac{d}{dx}[x^v J_v(x)] = \sum \frac{(-1)^r 2(v+r) x^{2x+2r-1}}{r!\,(v+r)(v+r-1)!\,2^{v+2r}}$$

$$= x^v \sum \frac{(-1)^r}{r!\,(r+v-1)!}\left(\frac{x}{2}\right)^{2r+(v-1)}$$

$$= x^v J_{v-1}(x)$$

1.4.2. Illustrative Examples of Bessel Function

Example 1: Calculate $J_{\frac{1}{2}}(x)$.

Sol.:

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma\left(\frac{3}{2}+r\right)} \left(\frac{x}{2}\right)^{2r+\frac{1}{2}}$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} - \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{1!\Gamma\left(\frac{5}{2}\right)} + \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{2!\Gamma\left(\frac{7}{2}\right)} - \cdots$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}} - \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{1!.\frac{1}{2}.\frac{1}{2}\sqrt{\pi}} + \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{2!.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}} - \cdots = \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}}\, o\, \frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}} \sin x.$$

Example 2: Calculate $J_{\frac{-1}{2}}(x)$.

Sol.:

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma\left(\frac{1}{2}+r\right)} \left(\frac{x}{2}\right)^{2r+\frac{1}{2}}$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} - \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{1!\,\Gamma\left(\frac{3}{2}\right)} + \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{2!\,\Gamma\left(\frac{5}{2}\right)} - \cdots$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] = \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\sqrt{\pi}}\, o\, \frac{\cos x}{x} = \sqrt{\frac{2}{\pi x}} \cos x$$

Chapter Two

Relationship Between Gamma, Beta and Bessel Functions

2.1. Introduction

A relationship has now been defined for Beta, Gamma and Bessel functions. Various properties of Beta, Gamma, and Bessel's functions have been used to simplify the Bessel's summation down to a single term by building the relationship between Beta, Gamma and itself. In this chapter, we focus on the crucial role for this relationship.

2.2. Relation Between Gamma and Beta Functions

The relationship between Gamma function and the Beta function can be derived as :

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u}u^{x-1}du \int_0^\infty e^{-v}v^{y-1}dv.$$

Now taking $u = a^2, v = b^2 \ \ , so$

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty e^{-a^2}a^{2x+1}\,\mathrm{da} \int_0^\infty e^{-b^2}b^{2y+1}db$$

$$\Gamma(x)\Gamma(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(a^2+b^2)}a^{2x-1}b^{2y-1}dadb.$$

Transforming to polar coordinates with $a = r\cos\theta \ \ , b = \sin\theta$, we get:

$$\Gamma(x)\Gamma(y) = \int_0^{2\pi}\int_0^\infty e^{-r^2}(r\, cos\theta)^{2x-1}(r\, sin\theta)^{2y-1}rdrd\theta$$

$$\Gamma(\mathrm{x})\Gamma(\mathrm{y}) = \int_0^\infty e^{-r^2}r^{2x+2y-2}rdr\int_0^{2\pi}(cos\theta)^{2x-1}(sin\theta)^{2y-1}\,d\theta$$

Which means that : $\Gamma(\mathrm{x})\Gamma(\mathrm{y}) = (x+y)\beta(x,y).$

Hence , re- writing the arguments with the usual from of Beta function ,we get :

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Example 1: *Find* $\beta\left(\frac{1}{2},\frac{1}{2}\right)$.

Sol.:

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\frac{\sqrt{\pi}\sqrt{\pi}}{\Gamma(1)}=\pi$$

2.3. Relation Between Bessel and Beta - Gamma Functions

It is clear that a relation between Bessel and Gamma function from definition of Bessel's function :

$$J_{n+1}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r+1}}{2^{n+2r+1}r!\,\Gamma\,(n+r+2)}$$

or

$$J_{n+1}(x)=\frac{x^{n+1}}{2^{n+1}}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{2r}r!\,\Gamma\,(n+r+2)}$$

Since we know that ,

$$\beta(n+1,r+1)=\frac{\Gamma\,(n+1)\Gamma\,(r+1)}{\Gamma\,(n+r+2)}$$

So we can drive the relation between the three functions as following:

$$J_{n+1}(x)=\frac{x^{n+1}}{2^{n+1}}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}\beta(n+1,r+1)}{2^{2r}r!\,\Gamma\,(n+1)\Gamma\,(r+1)}$$

$$J_{n+1}(x)=\frac{x^{n+1}}{2^{n+1}}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}\beta(n+1,r+1)}{2^{2r}r!\,n!\,r!}$$

$$J_{n+1}(x)=\frac{x^{n+1}}{2^{n+1}n!}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}\beta(n+1,r+1)}{2^{2r}r!\,r!}$$

where ,

$\beta(n+1,r+1)=\int_0^1 x^n(1-x)^r d_x$. Using integration by parts to get:

$\beta(n+1,r+1) = x^n \int_0^1 (1-x)^r\, dx - \int_0^1 nx^{n-1} \int_0^1 (1-x)^r\, dx.$

Which gives : $\beta(n+1,r+1) = \frac{x^n}{r+1} - n\int_0^1 \frac{x^{n-1}d_x}{r+1}$. On solving the integral ,

$$\beta(n+1,r+1) = \frac{x^n-1}{r+1}.$$

substituting it back into the summation , we obtain:

$$J_{n+1}(x) = \frac{x^{n+1}(x^n-1)}{2^{n+1}n!}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{2r}(r+1)r!\,r!}$$

Multiplying and dividing by 2 / x :

$$J_{n+1}(x) = \frac{x^{n+1}(x^n-1)}{2^{n+1}n!}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r+1}2}{2^{2r+1}\Gamma\,(r+1+1)r!\,x}$$

$$or \qquad J_{n+1}(x) = \frac{x^n(x^n-1)}{2^n n!}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r+1}}{2^{2r+1}\Gamma\,(r+1+1)r!}$$

$$Thus\ we\ get: \qquad J_{n+1}(x) = \frac{x^n(x^n-1)}{2^n n!}\,J_1(x)\,.$$

2.4. Various Examples

Example 1 : Evaluate $\int_0^\infty \frac{x^c}{c^x} dx$.

Sol. :

$$\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty e^{-x logc} x^c dx$$

$$= \int_0^\infty e^{-t} \left(\frac{t}{logc}\right)^c \frac{dt}{logc}$$

$but\ x \log c = t$, so it is equal to

$$\frac{1}{(logc)^{c+1}} \int_0^\infty e^{-t}\, t^c dt \ \ (\,by\ the\ Gamm\ function\ definition\,)$$

$$\int_0^\infty \frac{x^c}{c^x} dx \ = \frac{1}{(logc)^{c+1}}\ \Gamma\,(c+1).$$

Example 2 : Find $s = \sum_{n=1}^{\infty} \frac{1}{n.\binom{2n}{n}}$.

Sol. :

S= $\sum_{n=1}^{\infty} \beta(n+1\,, n\,)\, n \ = \ \sum_{n=1}^{\infty} \int_0^1 t^n (\,1-t)^{n-1} dt$.

Given the absolute convergence of the integrand , we can switch $\sum$ and $\int$.

$S = \int_0^1 \sum_{n=1}^{\infty} t^n (1-t\,)^{n-1}\ dt$.

Using the sum of geometric progressions , we get

$$S = \int_0^1 \frac{t}{t^2 - t + 1}\, dt = \frac{\sqrt{3}}{9}\, \pi\, .$$

Example 3 : Evaluate $\int_0^b t\, J_0(a\,t\,)\,dt$.

Sol. :

$$\int_0^{\infty} t\,J_0(at)dt = \int_0^{\infty} \frac{1}{a}\frac{d}{dt}[t\,J_1(at)]\,dt = \frac{1}{a}[tJ_1\,(at)]\Big|\begin{matrix} b \\ t=0 \end{matrix} = \frac{b}{a}J_1(ab).$$

Now , comparing with the standard Beta function:

$$B(p\,,q\,) = \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt$$

So , we can say that p =11 and q = 10 .

Using the factorial formula of Beta function :

$$B(p,q) = (p-1)!\,(q-1)!\,/\,(p+q-1)!$$

Here , $p! = p.(p-1).(p-2)\,\dots\dots 3.2.1.$

$$B(p,q) = {}^{(10!\,.\,9!)}\!/_{20!} = 0.0000005413\,.$$

Example 4 : Find the exact value of $\int_0^{\infty} x^3 e^{\frac{-1}{2}x^2}\,dx$.

Sol. :

$$\int_0^{\infty} x^3 e^{\frac{-1}{2}x^2}\,dx = \int_0^{\infty} x^3 e^{-u}(\frac{du}{x})$$

$$\int_0^{\infty} x^2\,e^{-u}\;\;du = \int_0^{\infty} 2u\,e^{-u}du$$

$$\Gamma\,(\alpha) = \int_0^{\infty} x^{\alpha-1}\,e^{-x}\,dx$$

$$\Gamma(x) = \int_0^1 u^{2-1}\,e^{-u}\;\;du = 2\,\Gamma\,(2) = 2\,.\,1! = 2\,.$$

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