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Inner Product Spaces and Hilbert Space

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بِسم اللهِ الرَحمِنِ الرَحِيم

"قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ

الْعَلِيمُ الْحَكِيم"

صدق الله العلي العظيم

سورة البقرة الاية (٣٢)

Dedication

This Work is Dedicated

To all my beloved family

Thank for your endless love, sacrifices, prayers

supports and advice

Ahmed

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Ahmed

Abstract

We study a special class of Banach spaces, namely Hilbert spaces, in which the presence of a so-called "inner product" allows us to define angles between elements. In particular, we can introduce the geometric concept of orthogonality. This has far-reaching consequences

Contents

Introduction

In the preceding chapters, we discussed normed linear spaces and Banach spaces. These spaces has linear properties as well as metric properties. Although the norm on a linear space generalizes the elementary concept of the length of a vector, but the main geometric concept other than the length of a vector is the angle between two vectors, In this chapter, we take the opportunity to study linear spaces having an inner product, a generalization of the usual dot product on finite dimensional linear spaces. The concept of an inner product in a linear space leads to an inner product space and a complete inner product space which is called a Hilbert space. The theory of Hilbert Spaces does not deal with angles in general. Most interestingly, it helps us to introduce an idea of perpendicularity for two vectors and the geometry deals in various fundamental aspects with Euclidean geometry.

The basics of the theory of Hilbert spaces was given by in 1912 by the work of German mathematician D. Hilbert (1862 -1943) on integral equations. However, an axiomatic basis of the theory was given by famous mathematician J. Von Neumann (1903 -1957). However, Hilbert spaces are the simplest type of infinite dimensional Banach spaces to tackle a remarkable role in functional analysis.

1.Inner Product Spaces

Definition 1.1 : Let, X be a linear space over a field of complex numbers. If for every pair $(x, y) \in X \times X$ there corresponds a scalar denoted by $\langle x, y \rangle$ called inner product of x and y of X such that the following properties hold.
(IP.1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ where $(x, y) \in X \times X$ and denotes the conjugate of the complex number.
(IP.2) for all $\alpha \in \mathbb{C}$,

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \forall (x, y) \in X \times X.$$

(IP.3) for all $x, y, z \in X$

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$$

(IP.4) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = \theta$.
Then $(X, \langle\rangle.) iscalledaninnerproductspaceorpre - Hilbertspace.$
Remark 1.2: The following properties hold in an inner product space.
Let X be an inner product space then,
(i) for all $\alpha, \beta \in \mathbb{C}, \langle \alpha x + \beta y, z) = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \ \forall x, y, z \in X$.
(ii) for all $\alpha \in \mathbb{C} \langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle \ \forall x, y \in X$
(iii) for all $\alpha, \beta \in \mathbb{C} \ \forall x, y, z \in X$.

$$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle.$$

Remark 1.3: Inner product induces a norm. For this we proceed as follows:
Proof: Let X be an inner product space. Take $x \in X$
Define $\| x \| = +\sqrt{\langle x, x \rangle}$ (1)
Now, $\| x \| \geq 0$ as $\langle x, x \rangle \geq 0$ by (IP.4).
Also, $\| x \| = 0$ iff $x = \theta$ by (IP.4)
Take $\alpha \in \mathbb{C}$ so,

$$\begin{aligned} \| \alpha x \|^2 &= \langle \alpha x, \alpha x \rangle, \alpha \in X \\ &= \alpha \bar{\alpha} \langle x, x \rangle, x \in X \\ &= |\alpha|^2 \| x \|^2, \ x \in X \\ \text{so } \| \alpha x \| &== |\alpha| \| x \|, \ x \in X \end{aligned}$$

2. Cauchy Schwarz Inequality

To prove the triangle inequality we first state and prove Cauchy Schwarz Inequality

$$|\langle x, y\rangle| \leq \| x \| \| y \| \quad \forall x, y \in X \tag{2}$$

Proof: If $y = \theta_X$ then the result follows trivially.
Let $y \neq \theta_X$. Then for every scalars $\lambda \in \mathbb{C}$,

$$\langle x + \lambda y, x + \lambda y\rangle \geq 0$$
$$\Rightarrow \langle x, x\rangle + \langle x, \lambda y\rangle + \langle \lambda y, x\rangle + \langle \lambda y, \lambda y\rangle \geq 0 \text{ (By (IP.3))}$$
$$\Rightarrow \langle x, x\rangle + \bar{\lambda}\langle x, y\rangle + \lambda\langle y, x\rangle + \| \lambda y \|^2 \geq 0$$
$$\Rightarrow \| x \|^2 + \bar{\lambda}\langle x, y\rangle + \lambda\langle \overline{x, y}\rangle + |\lambda|^2 \| y \|^2 \geq 0$$

Take $\lambda = -\dfrac{\langle x, y\rangle}{\langle y, y\rangle}$

So, $\| x \|^2 - \dfrac{\langle \overline{x, y}\rangle\langle x, y\rangle}{\| y \|^2} - \dfrac{\langle x, y\rangle\langle \overline{x, y}\rangle}{\| y \|^2} + \dfrac{|\langle x, y\rangle|^2}{\| y \|^4} \| y \|^2 \geq 0.$

So, $\| x \|^2 - \dfrac{|\langle x, y\rangle|^2}{\| y \|^2} - \dfrac{|\langle x, y\rangle|^2}{\| y \|^2} + \dfrac{|\langle x, y\rangle|^2}{\| y \|^2} \geq 0$

So $\quad \| x \|^2 - \dfrac{|\langle x, y\rangle|^2}{\| y \|^2} \geq 0$

$\Rightarrow \quad \| x \|^2 \| y \|^2 \geq |\langle x, y\rangle|^2$

$\Rightarrow \quad \| x \| \| y \| \geq |\langle x, y\rangle|$

Sometimes this inequality is abbreviated as C-S inequality. We see that equality sign will hold if and only if in above derivation $\langle x + \lambda y, x + \lambda y\rangle = 0 \Rightarrow \| x + \lambda y \|^2 = 0 \Rightarrow x + \lambda y = \theta_X$, i.e x and y are linearly dependent.
We shall now prove triangle inequality for norm. Now $\forall x, y \in X$.

Now, $\| x+y \|^2 = \langle x+y, x+y \rangle$
$= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle$
$= \| x \|^2 + \| y \|^2 + \langle x,y \rangle + \langle y,x \rangle$
So, $\| x+y \|^2 = |\| x \|^2 + \| y \|^2 + \langle x,y \rangle + \langle y,x \rangle|$
$\leq \| x \|^2 + \| y \|^2 + |\langle x,y \rangle + \langle y,x \rangle|$
$\leq \| x \|^2 + \| y \|^2 + 2 \| x \| \| y \|$
$= (\| x \| + \| y \|)^2$

So, $\| x+y \| \leq \| x \| + \| y \|$.

Hence, inner product induces a norm and consequently every inner product space is a normed linear space.

Remark 2.1 : So every inner product space is a metric space and the metric induced by inner product is defined as follows: for all $x, y \in X$ define $d: X \times X \to \mathbb{R}$ by

$$d(x,y) = \| x-y \| = +\sqrt{\langle x-y, x-y \rangle} \tag{3}$$

Theorem 2.2: Every inner product function is a continuous function. (Equivalently, if $f: X \times X \to \mathbb{C}$ defined by $f(x,y) = \langle x,y \rangle, \forall x,y \in X$ then f is continuous).

Proof:

Let X be an inner product space. Define $f: X \times X \to \mathbb{C}$ by $(x,y) = \langle x,y \rangle$, $\forall x,y \in X$. Now take $\{x_n\}$ and $\{y_n\}$ be a sequence in X such that $x_n \to x$ as

$n \to \infty$ and $y_n \to y$ as $n \to \infty$
So, $\|x_n - x\| \to 0$ as $n \to \infty$ and $\|y_n - y\| \to 0$ as $n \to \infty$.
As, $x_n \to x$ as $n \to \infty$ then, $\|x_n\| \to \| x \|$ as $n \to \infty$.
So, $\{\|x_n\|\}$ are bounded. So, there exists a constant $M > 0$ such that $\|x_n\| \leq M$, $\forall n$
Now, $|\langle x_n, y_n \rangle - \langle x,y \rangle|$

$= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$
$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$
$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle$
$\leq \|x_n\|\|y_n - y\| + \|x_n - x\| \| y \|$ [By C-S inequality(2)]
$\leq M\|y_n - y\| + \|x_n - x\| \| y \|$
$\to 0$ as $n \to \infty$

i.e $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$, implying that $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$. So, f is continuous.

Theorem 2.3: (Parallelogram Law): Let X be an inner product space and let $x, y \in X$. Then,

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)$$

Proof:

$$\begin{aligned} \| x + y \|^2 &= \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \| x \|^2 + \| y \|^2 + \langle x, y\rangle + \langle y, x\rangle \qquad (4) \\ \text{and } \| x - y \|^2 &= \langle x - y, x - y\rangle = \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle \\ &= \| x \|^2 + \| y \|^2 - \langle x, y\rangle - \langle y, x\rangle \qquad (5) \end{aligned}$$

Adding (4) and (5) we get

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)$$

Theorem2.4 (Polarization Identity): Let X be an inner product space, let $x, y \in X$. Then

$$\langle x, y\rangle = \frac{1}{4}[\| x + y \|^2 - \| x - y \|^2 + i \| x - iy \|^2 - i \| x - iy \|^2] \qquad (6)$$

Proof:

Now, $\| x + y \|^2 = \| x \|^2 + \| y \|^2 + \langle x, y\rangle + \langle y, x\rangle$ (7)

$$\| x - y \|^2 = \| x \|^2 + \| y \|^2 - \langle x, y\rangle - \langle y, x\rangle \qquad (8)$$

Replacing y by iy in (7) and (8)

$$\begin{aligned} \| x+iy \|^2 &= \| x \|^2 + \| iy \|^2 + \langle x, iy\rangle + \langle iy, x\rangle \\ &= \| x \|^2 + \| y \|^2 - i\langle x, y\rangle + i\langle y, x\rangle \end{aligned} \tag{9}$$

$$\begin{aligned} \| x-iy \|^2 &= \| x \|^2 + \| iy \|^2 - \langle x, iy\rangle - \langle iy, x\rangle \\ &= \| x \|^2 + \| y \|^2 + i\langle x, y\rangle - i\langle y, x\rangle \end{aligned} \tag{10}$$

$(7) - (8) + i(9) - i(10)$, we get (6). Hence the result

Theorem ٢.٥: Let X be an inner product space. Then
(i) Every Cauchy sequence is bounded
(ii) If $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X then $\{\langle x_n, y_n\rangle\}$ is also a Cauchy sequence in $\mathbb{C}$ and hence convergence in $\mathbb{C}$.

Proof:

(i) Let, $\{x_n\}$ be a Cauchy sequences in X. Then for $\varepsilon = 1$ there exists a positive integer N such that, $\|x_n - x_m\| < 1$, whenever, $n, m \geq N$. In particular, $\|x_n - x_N\| < 1$, whenever $n \geq N$.
Now, $\|x_n\| \leq \|x_n - x_N\| + \|x_N\| < 1 + \|x_N\| \forall n \geq N$.
Let, $M = \max\{\|x_1\|, \|x_2\|, \cdots, \|x_{N-1}\|, \|x_N\| + 1\}$ so, $\|x_n\| \leq M \ \forall n$ so, $\{x_n\}$ is bounded.
(ii) Let $\{x_n\}, \{y_n\}$ be two Cauchy sequences in X.
So, $\|x_n - x_m\| \to 0$ as $n, m \to \infty$ and $\|y_n - y_m\| \to 0$ as $n, m \to \infty$.
Also, by (i) $\|x_n\| \leq M$ for all n and for some $M > 0$. Similarly $\|y_n\| \leq K$, for some $K > 0$ and $\forall n$.

$$\begin{aligned} \text{Now,} \quad & |\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| \\ = \quad & |\langle x_n, y_n\rangle - \langle x_n, y_m\rangle + \langle x_n, y_m\rangle - \langle x_m, y_m\rangle| \\ = \quad & |\langle x_n, y_n - y_m\rangle + \langle x_n - x_m, y_m\rangle| \\ \leq \quad & |\langle x_n, y_n - y_m\rangle| + |\langle x_n - x_m, y_n\rangle| \end{aligned}$$

$$\begin{aligned} \leq \quad & \|x_n\|\|y_n - y_m\| + \|x_n - x_m\|\|y_n\| \text{ (By C-S inequality)} \\ \leq \quad & M\|y_n - y_m\| + K\|x_n + x_m\| \\ & \to 0 \ \text{ as } n, m \to \infty \end{aligned}$$

So, $\{\langle x_n, y_n\rangle\}$ is a Cauchy sequences of scalars in $\mathbb{C}$. As $\mathbb{C}$ is complete, $\{\langle x_n, y_n\rangle\}$ is convergent in $\mathbb{C}$.

3. Hilbert space

Definition 3.1: A complete inner product space is called a Hilbert space i.e. an inner product space X which is complete with respect to a metric $d: X \times X \to \mathbb{R}$ induced by the inner product $\langle , \rangle on X \times$ i.e. $d(x,y) = \langle x-y, x-y \rangle^{1/2} \forall x, y \in X$.

Theorem 3.2 : A Banach space X is a Hilbert space if and only if parallelogram law holds in it.

Proof:

We know that every Hilbert space X is a Banach space where parallelogram law holds in it.

Conversely suppose that X is a Banach space where parallelogram law holds. Without loss of generality we can assume a function $\langle , \rangle whoserangeis$. For all $x, y \in X$.

Define $\langle \rangle : , X \times X \to \mathbb{R}$ by

$$\langle x, y \rangle = \frac{1}{4}[\| x+y \|^2 - \| x-y \|^2] \tag{11}$$

i.e. for real inner product space we start with (9.1.11) and sometimes we write $R\langle x, y \rangle = \frac{1}{4}[\| x+y \|^2 - \| x-y \|^2]$, $\forall x, y \in X$. Clearly $\langle x, y \rangle = \langle \overline{y, x} \rangle$ as $\langle x, y \rangle$ is real. Also $\langle x, x \rangle \geq 0 \; \forall x \in X$ and $\langle x, x \rangle = 0$ iff $x = \underline{0}$. So, (IP.1) and (IP.4) holds.

Now, for $u, v, w \in X$

$$\| u+v+w \|^2 + \| u+v-w \|^2 = 2(\| u+v \|^2 + \| w \|^2) \tag{12}$$

$$\| u-v+w \|^2 + \| u-v-w \|^2 = 2(\| u-v \|^2 + \| w \|^2) \tag{13}$$

By $(12) - (13)$ we get

$$\begin{aligned} & \| u+v+w \|^2 + \| u+v-w \|^2 - \| u-v+w \|^2 + \| u-v-w \|^2 \\ & = 2(\| u+v \|^2 - \| u-v \|^2) \\ \Longrightarrow \quad & 4[\langle u+w, v \rangle + \langle u-w, v \rangle] = 2.4\langle u, v \rangle \\ \Longrightarrow \quad & \langle u+w, v \rangle + \langle u-w, v \rangle = 2\langle u, v \rangle \end{aligned} \tag{14}$$

Put, $u = w$ (15)

$$\langle 2u, v \rangle = 2\langle u, v \rangle$$

Again put, $x_1 = u + w, x_2 = u - w, x_3 = v$ then from (14) we get

$$\begin{aligned} \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle &= 2\langle u, v \rangle = \langle 2u, v \rangle \text{ (by (15))} \\ &= \langle x_1 + x_2, x_3 \rangle \end{aligned} \tag{16}$$

So, (IP.3) holds.
From (16) for $x_1, x_2, x_3, x_4 \in X$,

$$\langle x_1 + x_2 + x_3, x_4 \rangle = \langle x_1 + x_2, x_4 \rangle + \langle x_3, x_4 \rangle = \langle x_1, x_4 \rangle + \langle x_2, x_4 \rangle + \langle x_3, x_4 \rangle$$

Put $x_1 = x_2 = x_3 = x, x_4 = y$ i.e. $\langle 3x, y \rangle = 3\langle x, y \rangle$.
So, by Principle of Mathematical Induction for any positive integer n

$$\langle nx, y \rangle = n\langle x, y \rangle \; \forall x, y \in X \tag{17}$$

Now,
$$\begin{aligned} \langle -x, y \rangle &= \frac{1}{4}[\| -x + y \|^2 - \| -x - y \|^2] \\ &= \frac{1}{4}[-\| x + y \|^2 + \| y - x \|^2] = -\langle x, y \rangle \end{aligned} \tag{18}$$

Take, $n = -m (m > 0)$

$$\begin{aligned} \langle nx, y \rangle = \langle -m \;\; , y \rangle &= \langle m(-x), y \rangle \\ &= m\langle -x, y \rangle \text{ (by (9.1.17))} \\ &= -m\langle x, y \rangle \text{ (by (9.1.18))} \\ &= n\langle x, y \rangle \end{aligned}$$

So (17) is also true for any negative integer. Thus $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, when λ is either positive integer or a negative integer.

Take $\lambda = p/q$ = a rational number where, $\gcd(p, q) = 1$ and p and q are integer.

$$\begin{aligned} \text{Now } \langle \lambda x, y \rangle &= \left\langle \frac{p}{q} x, y \right\rangle \\ \text{So } q\langle \lambda x, y \rangle &= q \left\langle \frac{p}{q} x, y \right\rangle = \langle p \;\; , y \rangle = p\langle x, y \rangle \\ \Longrightarrow \langle \lambda x, y \rangle &= \frac{p}{q} \langle x, y \rangle \\ \Longrightarrow \left\langle \frac{p}{q} x, y \right\rangle &= \frac{p}{q} \langle x, y \rangle \end{aligned} \tag{19}$$

Let, λ be any real. So there exists a sequence of rationals $\{r_n\}$ such that $r_n \to \lambda$ as $n \to \infty$. So, $\langle r_n x, y \rangle = r_n \langle x, y \rangle \to \lambda \langle x. y \rangle$ as $n \to \infty$

Now, $|\langle r_n x, y\rangle - \langle \lambda x, y\rangle|$
$= |\langle (r_n - \lambda)x, y\rangle| \leq \|(r_n - \lambda)x\| \parallel y \parallel$ (by C-S inequality (2))
$= |r_n - \lambda| \parallel x \parallel \parallel y \parallel \to 0$ as $n \to \infty$
$\langle r_n x, y\rangle \to \langle \lambda_x, y\rangle$ as $n \to \infty$. So, $\langle \lambda_x, y\rangle = \lambda\langle x, y\rangle \forall x, y \in X$. So, (IP.2) holds.
So, X is an inner product space with respect to (11) consequently, X is a Hilbert space.
Note: For a complete inner product space we start with $Rl\langle x, y\rangle = \frac{1}{4}[\parallel x + y \parallel^2 - \parallel x - y \parallel^2]$, $\text{Im}\langle x, y\rangle = \frac{1}{4}[\parallel x + iy \parallel^2 - \parallel x - iy \parallel^2], \forall x, y \in X$ and the proof is similar to the proof of real inner product space. (Readers can verify it)

Example 3.3: The Euclidean space $\mathbb{R}^n$ is a Hilbert space.
Solution: Let $x = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ and $y = (\eta_1, \eta_2, \cdots, \eta_n)$ be two element of $\mathbb{R}^n$. We define the inner product of x and y by $(x, y) = \zeta_1\eta_1 + \zeta_2\eta_2 + \cdots + \zeta_n\eta_n$. Then $\parallel x \parallel = \sqrt{\langle x, x\rangle} = (\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_n^2)^{\frac{1}{2}}$. It may be easily verified that all the inner product axioms are satisfied in $\mathbb{R}^n$ and the Euclidean metric d is obtained by $(x, y) = \parallel x - y \parallel = \langle x - y, x - y\rangle^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n}(\zeta_i - \eta_i)^2}$. With respect to this metric we can at once see that $\mathbb{R}^n$ is complete so as to make $\mathbb{R}^n$, a Hilbert space.

Example 3.4: The Euclidean space $\mathbb{C}^n$ is a Hilbert space.
Solution: Let $x = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ and $y = (\eta_1, \eta_2, \cdots, \eta_n)$ be two elements of $\mathbb{C}^n$. We define the inner product of x and y by

$$\langle x, y\rangle = \zeta_1\overline{\eta_1} + \zeta_2\overline{\eta_2} + \cdots + \zeta_n\overline{\eta_n}.$$

Then $\parallel x \parallel = \sqrt{\langle x, x\rangle} = (|\zeta_1|^2 + |\zeta_2|^2 + \cdots + |\zeta_n|^2)^{\frac{1}{2}}$. It may be easily verified that all the inner product axioms are satisfied in $\mathbb{C}^n$ and the Euclidean metric d is obtained by $d(x, y) = \parallel x - y \parallel = \langle x - y, x - y\rangle^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n}(\zeta_i - \eta_i)^2}$. With respect to this metric we can at once see that $\mathbb{C}^n$ is complete so as to make $\mathbb{C}^n$, a Hilbert space.

Example 3.5: The space l_2 is a Hilbert space.
Solution: Let $x = \{\zeta_i\}$ and $y = \{\eta_i\}$ be elements of l_2. We define the inner product of x and y by $\langle x, y \rangle = \sum_{i=1}^{\infty} \zeta_i \overline{\eta_i}$ convergence of the series on the right hand side follows from the fact that $x \in l_2$ and $|\zeta_i \overline{\eta_i}| \leq \frac{|\zeta_i|^2}{2} + \frac{|\eta_i|^2}{2}$.

Then $\| x \| = \sqrt{\langle x, x \rangle} = (\sum_{i=1}^{\infty} \zeta_i)^{\frac{1}{2}}$. It can be easily shown that all the inner product axioms (IP.1) -(IP.4) are satisfied in l_2. The metric d of l_2 is defined by $(x, y) = \| x - y \| = \langle x - y, x - y \rangle^{\frac{1}{2}} = (\sum_{i=1}^{n} |\zeta_i - \eta_i|^2)^{\frac{1}{2}}$. With respect to this metric we can at once see that l_2 is complete so as to make l_2 a Hilbert space.

But for $1 \leq p < \infty$, $l_p(p \neq 2$ is not a Hilbert space. It can be shown by the following example.

Example 3.6: For $1 \leq p < \infty, l_p(p \neq 2)$ is not an inner product space and hence not a Hilbert space.

Solution: Let $x = (1,1,0,0, \cdots) \in l_p$ and $y = (1, -1,0,0, \cdots) \in l_p$. Then $\| x \| = \| y \| = 2^{\frac{1}{p}}$ and $\| x + y \| = \| x - y \| = 2$. Now we see that if $p \neq 2$, the parallelogram law does not hold.

Hence $1 \leq p < \infty$ $l_p(p \neq 2)$ is not an inner product space and consequently it is not a Hilbert space.

Example 3.7 : The space $c[a, b]$ of all real valued continuous in the closed interval $[a, b]$ is not an inner product space with respect to sup norm and hence not a Hilbert space.

Solution: Here the norm defined by $\| x \| = \sup_{a \leq t \leq b} |x(t)|$. Take $x(t) = 1, \forall t \in [a, b]$ and $y(t) = \frac{t-a}{b-a}, \forall t \in [a, b]$. Then $\| x \| = 1, \| y \| = 1, \| x + y \| = 2, \| x - y \| = 1$. By simple calculations we see that parallelogram law does not hold in it. Hence $c[a, b]$ is not a Hilbert space.

Example 3.8 : The space $L_2[a, b]$, the space of all square integrable functions over $[a, b]$ is a Hilbert space.

Solution: Define the inner product on $L_2[a,b]$ by
$\langle x,y\rangle = \int_a^b |x(t)\overline{y(t)}|dt, \forall x,y \in L_2[a,b]$ and the norm on $L_2[a,b]$ is given by
$\| x \| = \sqrt{\int_z^b |x(t)|^2 dt}$. Also with respect
to this norm it can be shown that $L_2[a,b]$ is complete with respect to a metric defined by

$$d(x,y) = \left[\int_a^b |x(t)-y(t)|^2 dt\right]^{\frac{1}{2}}$$

So $L_2[a,b]$ is a Hilbert space.

4. Orthogonal Projection

Lemma 4.1: Let X be an inner product space. Then (.|.) : X×X K is continuous.
Proof. Let $x_n \to x$ and $y_n \to y$. Then $M := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} |(x_n \mid y_n) - (x \mid y)| &\leq |(x_n \mid y_n - y)| + |(x_n - x \mid y)| \\ &\leq M\|y_n - y\| + \| y \| \, \|x_n - x\| \to 0 \end{aligned}$$

In this section we show that in a Hilbert space one can project onto any closed subspace. In other words, for any point there exists a unique best approximation in any given closed subspace. For this the geometric properties arising from the inner product are crucial.

Definition 4.2: Let $(X, (\cdot|\cdot))$ be an inner product space. We say that two vectors $x, y \in X$ are orthogonal (and write $\perp y$) if $(x \mid y) = 0$. Given a subset $S \subset X$, the annihilator $S^\perp$ of S is defined by

$$S^\perp := \{y \in X : y \perp x \text{ for all } x \in S\}$$

If S is a subspace, then $S^\perp$ is also called the orthogonal complement of S.
In an inner product space the following fundamental and classic geometric identity holds.

Lemma 4.3 (Pythagoras): Let X be an inner product space. If $x \perp y$, then

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$

Proof.

$$\| x + y \|^2 = \| x \|^2 + 2\text{Re}(x \mid y) + \| y \|^2 = \| x \|^2 + \| y \|^2.$$

Proposition 4.4: Let X be an inner product space and $S \subset X$.
(a) $S^\perp$ is a closed, linear subspace of X.
(b) $\overline{\text{span}} S \subset (S^\perp)^\perp$.
(c) $\overline{\text{span}} S \cap S^\perp = \{0\}$.

Proof:

(a) If $x, y \in S^{\perp}$ and $\lambda \in \mathbb{K}$, then, for $z \in S$, we have $(\lambda x + y \mid z) = \lambda(x \mid z) + (y \mid z) = 0$, hence $\lambda x + y \in S^{\perp}$. This shows that $S^{\perp}$ is a linear subspace. If (x_n) is a sequence in $S^{\perp}$ which converges to x, then we infer from Lemma 4.1 that $(x \mid z) = \lim(x_n \mid z) = 0$ for all $z \in S$.
(b) By (a), $(S^{\perp})^{\perp}$ is a closed linear subspace which contains S. Thus $\overline{\text{span}} S \subset (S^{\perp})^{\perp}$
(c) If $x \in \overline{\text{span}} S \cap S^{\perp}$, then, by (b), $x \in S^{\perp} \cap (S^{\perp})^{\perp}$ and hence $x \perp x$. But this means $(x \mid x) = 0$. By (IP1) it follows that $x = 0$.

We now come to the main result of this section.

Theorem 4.5: Let $(X, (\cdot|\cdot))$ be a Hilbert space and $K \subset X$ be a closed linear subspace. Then for every $x \in X$, there exists a unique element $P_K x$ of K such that

$$\|P_K x - x\| = \min\{\| y - x \| : y \in K\}$$

Proof:

Let $d := \inf\{\| y - x \| : y \in K\}$. By the definition of the infimum, there exists a sequence (y_n) in K with $\|y_n - x\| \to d$. Applying the parallelogram identity 4.8 to the vectors $x - y_n$ and $x - y_m$, we obtain

$$\begin{aligned} 2\left(\|x - y_n\|^2 + \|x - y_m\|^2\right) &= \|(x - y_n) + (x - y_m)\|^2 + \|x - y_n - (x - y_m)\|^2 \\ &= \|2x - y_n - y_m\|^2 + \|y_n - y_m\|^2 \\ &= 4\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 + \|y_n - y_m\|^2. \end{aligned}$$

Since $z_{nm} := \frac{1}{2}(y_n + y_m) \in K$, we have $\|x - z_{nm}\|^2 \geq d^2$ and thus

$$\|y_n - y_m\|^2 \leq 2\left(\|x - y_n\|^2 + \|x - y_m\|^2\right) - 4d^2$$

By the choice of the sequence (y_n), the right-hand side of this equation converges to 0 as $n, m \to \infty$, proving that (y_n) is a Cauchy sequence. Since X is complete,

(y_n) converges to some vector $P_K x$. Since K is closed, $P_K x \in K$. We have thus proved existence.

As for uniqueness, if $\| z - x \| = \min\{\| y - x \| : y \in K\}$, then, by the parallelogram identity,

$$2\|x - P_K x\|^2 + 2 \| x - z \|^2 = 4 \left\| x - \frac{1}{2}(P_K x + z) \right\|^2 + \|z - P_K x\|^2$$

and thus

$$4d^2 = 4 \left\| x - \frac{1}{2}(P_K x + z) \right\|^2 + \|P_K x - z\|^2 \geq 4d^2 + \|P_K x - z\|^2$$

proving that $\|P_K x - z\| = 0$; hence $P_K x = z$.

Definition 4.6: The map $P_K : X \to X$ from Theorem 4.5 is called the orthogonal projection onto K.

We now collect some properties of P_K.

Proposition 4.7: Let X be a Hilbert space, K be a closed subspace of X and P_K be the orthogonal projection onto K.

(a) For all $x, y \in X$, we have $P_K x = y$ if and only if $y \in K$ and $x - y \in K^\perp$.

(b) P_K is a bounded linear operator on X.

(c) $P_K^2 = P_K$ and $(P_K x \mid y) = (x \mid P_K y)$ for all $x, y \in X$.

Proof:

(a) If $y \in K$ and $x - y \in K^\perp$, then for every $z \in K$ we have $y - z \in K$ and thus $x - y \perp y - z$. By Pythagoras,

$$\| x - z \|^2 = \| x - y \|^2 + \| y - z \|^2 \geq \| x - y \|^2.$$

Thus $\| x - y \| = \min\{\| x - z \| : z \in K\}$, proving that $P_K x = y$.

Conversely, if $P_K x = y$, then clearly $y \in K$. Assume that $x - y \notin K^\perp$. Then there exists $z \in K \setminus \{0\}$ with $(x - y \mid z) \neq 0$. We may assume that $(x - y \mid z) = 1$ (otherwise, we divide z by $\overline{(x - y \mid z)}$). Then, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \| x - y - \lambda z \|^2 &= \| x - y \|^2 - 2 \operatorname{Re} \lambda (x - y \mid z) + \lambda^2 \| z \|^2 \\ &= \| x - y \|^2 - 2\lambda + \lambda^2 \| z \|^2 \end{aligned}$$

The latter is strictly less than $\| x - y \|^2$ for small $\lambda > 0$, for example if $\lambda^2 \| z \|^2 < 2\lambda$, i.e. $\lambda < 2 \| z \|^{-2}$. Hence we find an element in K, namely $y + \| z \|^{-2} z$, for example, which is closer to x than to y. But then $y \neq P_K x$.

(b) Let $x, y \in X$ and $\lambda \in \mathbb{K}$. By (a), $x - P_K x, y - P_K y \in K^\perp$. Since $K^\perp$ is a subspace by Proposition 4.4, $\lambda x - \lambda P_K x + y - P_{Ky} = (\lambda x + y) - (\lambda P_K x + P_K y) \in K^\perp$. Since $\lambda P_K x + P_K y \in K$, it follows from (a) that $P_K(\lambda x + y) = \lambda P_K x + P_K y$, i.e. P_K is linear. As for the boundedness, observe that $x = P_K x + (x - P_K x)$ where $P_K x \perp x - P_K x$ by (a). Thus, by Pythagoras,

$$\| x \|^2 = \|P_K x\|^2 + \|x - P_K x\|^2 \geq \|P_K x\|^2,$$

proving the boundedness of P_K.
(c) $P_K x \in K$ and $0 = p_K x - P_K x \in K^\perp$. Hence, by (a), $P_K P_K x = P_K x$.
For the second part, observe that

$$(P_K x \mid y) = (P_K x \mid P_K y) + (P_K x \mid y - P_K y) = (P_K x \mid P_K y)$$

since $y - P_K y \in K^\perp$ and $P_K x \in K$ by (a). Similarly, one sees that $(x \mid P_K y) = (P_K x \mid P_K y)$
We can now refine Proposition 4.4 for linear subspaces.

Corollary 4.8: If X is a Hilbert space and K is a linear subspace of X then $\bar{K} = (K^\perp)^\perp$

Proof:

We have seen already that $\bar{K} \subset (K^\perp)^\perp$. Now let $y \in (K^\perp)^\perp$. Then $y = P_K y + (I - P_K)y =: y_1 + y_2$. Thus $\|y_2\|^2 = (y_2 \mid y_2) = (y_2 \mid y) - (y_2 \mid y_1) = 0$, since $y_2 \in \bar{K}^\perp = K^\perp$ and $y \in (K^\perp)^\perp$ and $y_1 \in \bar{K}$. It follows that $y_2 = 0$, hence $y = y_1 \in \bar{K}$. This shows $(K^\perp)^\perp \subset \bar{K}$.

An important consequence of Theorem 4.5 is the following result, which shows that in a Hilbert space all bounded linear functionals can be expressed in a specific way in terms of the inner product.

Theorem 4.9 (Fréchet-Riesz): Let X be a Hilbert space. Then $\varphi \in X^*$ if and only if there exists a $y \in X$ such that $\varphi(x) = (x \mid y)$ for all $x \in X$.

Proof:

If $\varphi(x) = (x \mid y)$, then φ is continuous as a consequence of Lemma 4.1.
Conversely, let $\varphi \in X^*$ be given. Then $K := \ker \varphi$ is a closed subspace of X. If $K = X$, pick $y = 0$. If $K \neq X$, there exists an $x_0 \in X$ with $\varphi(x_0) \neq 0$. Put $z = x_0 - P_K x_0$. Since $x_0 \notin K$, we have $z \neq 0$ and may thus define $w = \| z \|^{-1} z$. Then $\| w \| = 1$ and $w \in K^\perp$. In particular, $\varphi(w) \neq 0$.
Now for $x \in X$, we have $\varphi(x) = \frac{\phi(x)}{\phi(w)} \varphi(w)$. Define $\lambda := \frac{\varphi(x)}{\varphi(w)}$. Then, by linearity, $\varphi(x - \lambda w) = 0$ and thus $x - \lambda w \in K$. Put $y := \overline{\varphi(w)} w$. Then

$$\begin{aligned} (x \mid y) &= \varphi(w)(x \mid w) \\ &= \varphi(w)((x - \lambda w \mid w) + (\lambda w \mid w)) \\ &= \varphi(w)\lambda \| w \|^2 = \varphi(x). \end{aligned}$$

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