وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

ℓ^2 Space and Hilbert Bases

بحث تخرج مقدم الى مجلس قسم الرياضيات وهي جزء من متطلبات نيل شهادة البكالوريوس في كلية التربية للعلوم الصرفة /قسم الرياضيات

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بسم الله الرحمن الرحيم

(وَأَن لَّيۡسَ لِلۡإِنسَٰنِ إِلَّا مَا سَعَىٰ (39) وَأَنَّ
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أَضۡحَكَ وَأَبۡكَىٰ (43) وَأَنَّهُۥ هُوَ أَمَاتَ وَأَحۡيَا (44))

صدق الله العلي العظيم

سورة النجم

الاهداء

الى الينبوع الذي لا يمل العطاء ، الى رمز الحب وبلسم الشفاء (والدتي العزيزة) ،الى من كلت اناملة ليقدم لنا لحظة سعادة الى من سعى وشقى لانعم بالراحة (والدي العزيز).
الى الفاضلة مشرفة البحث الأستاذة (**أ.م. ميادة علي كريم**) المحترمة.
الى من علمونا حروفاً من ذهب وكلمات من درر وعبارات من اسمى واجلى عبارات العلم الى من صاغوا لنا علمهم حروفاً ومن فكرهم منارة تنير لنا سيرة العلم والنجاح الى استاذتنا الكرام.

طالبة البحث

شكر وتقدير

اللهم لك الحمد والشكر كما ينبغي لجلال وجهك وعظيم سلطانك حمداً لا ينفد أوله ولا ينقطع آخره ...... والصلاة والسلام على خاتم المرسلين رسول الرحمة ونبي الامة محمد المصطفى وعلى أهل بيت النبوة، باب النجاة من الضلال.....

أتقدم بالشكر الجزيل وعظيم الامتنان الى مشرفة البحث الأستاذة (أ.م. ميادة علي كريم) لتفضلها الكريم بالإشراف على هذا البحث وتكرمها بالنصائح والتوجيه ومساعدتها لي طيلة فترة الدراسة متمنية لها كل التوفيق.

وأتقدم بكافة الاحترام والتقدير الى عمادة كلية التربية للعلوم الصرفة / جامعة بابل وجميع الأساتذة والعاملين الاكارم في قسم الرياضيات.
والى زملائي الذين قدموا يد العون في إتمام هذا البحث ليرتقي الى المستوى المطلوب.

طالبة البحث

Abstract

In this research we present some basic concepts and important definitions about Hilbert space and ℓ^2 space, as well as some theorems that explain the properties of these spaces.

1.Introduction

Hilbert spaces are the closest generalization to infinite dimensional spaces of the Euclidean spaces. These notes were written for students wishing a basic introduction to Hilbert space theory but who have no knowledge of Banach spaces. First, we consider a normed space and we see that if the space is finite dimensional all the norms defined on it generate the same topology, so that the convergence of sequences does not depend on the norm that is used. Later we consider linear transformations defined in a normed space and we see that all of them are continuous if the space is finite dimensional. Continuous linear transformations are called bounded operators and, by introducing a norm in the space of all bounded operators, we convert it into a Banach space under certain conditions. Once we have clear the concepts of convergence of a sequence and completeness of the space, we define Hilbert spaces and consider some of their properties. Mainly we focus on the Pythagorean theorem, the Cauchy-Schwarz inequality, the Parallelogram Identity and in the introduction of the concept of basis for a Hilbert space. Furthermore, we show that every Hilbert space of dimension n is isomorphic to C^n and that every separable Hilbert space is isomorphic toℓ^2, the space of all square summable sequences. One of the main theorems related to Hilbert spaces is the Riesz Representation Theorem, which characterizes the continuous linear transformations defined in a Hilbert space with values in C.

2. Hilbert Spaces

Recall that any inner product space V has an associated norm defined by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Thus, an inner product space can be viewed as a special kind of normed vector space. In particular, every inner product space V has a metric defined by

$$d(v, w) = \|v - w\| = \sqrt{(v - w, v - w)}.$$

Definition 2.1.

A Hilbert space is an inner product space whose associated metric is complete.

That is, a Hilbert space is an inner product space that is also a Banach space. For example, R^n is a Hilbert space under the usual dot product:

$$\langle v, w\rangle = v.w = v_1 w_1 + \cdots + v_n w_n.$$

More generally, a finite-dimensional inner product space is a Hilbert space. The following theorem provides examples of infinite-dimensional Hilbert spaces.

Theorem 2.2.

For any measure space (X, μ), the associated L^2-space L^2(X) forms a Hilbert space under the inner product

$$\langle f, g\rangle = \int_x f g d\mu.$$

Proof: The norm associated to the given inner product is the L^2-norm:

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_x f^2 g d\mu} = \|f\|_2.$$

We have already proven that L^2(X) in complete with respect to this norm, and hence L^2(X) is a Hilbert space.

Corollary 2.3.

The space ℓ^2 of all square-summable sequences is a Hilbert space under the inner product

$$\langle V, W\rangle = \sum_{n\in N} V_n\, w_n$$

Proposition 2.4.

Let $\{v_n\}$ be a sequence of orthogonal vectors in a Hilbert space. Then the series

$$\sum_{n=1}^{\infty} v_n$$

converges if and only if

$$\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$$

Proof: Let s_n be the sequence of partial sums for the given series. By the Pythagorean theorem,

$$\left\|s_i - s_j\right\|^2 = \left\|\sum_{n=i+1}^{j} v_n\right\|^2 = \sum_{n=i+1}^{j} \|v_n\|^2$$

for all $i \leq j$. It follows that$\{s_n\}$ is a Cauchy sequence if and only if $\sum_{n=1}^{\infty}\|v_n\|^2$ converges.

We wish to apply this proposition to linear combinations of orthonormal vectors. First, recall that a sequence $\{u_n\}$ of vectors in an inner product space is called orthonormal if

$$\langle u_i, u_j \rangle = \begin{cases} 1 & if\ i = j, \\ 0 & if\ i \neq j, \end{cases}$$

for all i and j.

Corollary 2.5.

Let $\{u_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let $\{ a_n\}$ be a sequence of real numbers. Then the series

$$\sum_{n=1}^{\infty} a_n u_n$$

converges if and only if the sequence $\{a_n\}$ lies in ℓ^2.

In general, if $\{a_n\}$ is an ℓ^2 sequence, then the sum

$$\sum_{n=1}^{\infty} a_n u_n$$

is called a ℓ^2 -linear combination of the vectors $\{u_n\}$. By the previous corollary, every ℓ^2-linear combination orthonormal vectors in a Hilbert space converges.

Proposition 2.6.

Let $\{u_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$v = \sum_{n=1}^{\infty} a_n u_n \ and\ w = \sum_{n=1}^{\infty} b_n u_n$$

be ℓ^2-linear combinations of the vectors $\{u_n\}$. Then

$$\langle v, w \rangle = \sum_{n=1}^{\infty} a_n u_n.$$

Proof : Let $s_N = \sum_{n=1}^{N} a_n u_n$ and $t_N = \sum_{n=1}^{N} = b_n u_n$, and note that $s_N \to v$ and $t_N \to w$ as $N \to \infty$. Since the inner product $\langle v, w\rangle$ is a continuous function, it follows that

$$\langle v, w\rangle = \lim_{N\to\infty} \langle s_N, t_N\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{\infty} a_n b_n$$

In the case where v = w, this gives the following.

Corollary 2.7.

Let $\{u_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$v = \sum_{n=1}^{\infty} a_n u_n$$

be an ℓ^2-linear combination of these vectors. Then

$$\|v\| = \sqrt{\sum_{n=1}^{\infty} a_n^2}$$

Corollary 2.8.

Let $\{u_n\}$ be an orthonormal sequence of vectors in a Hilbert space, and let

$$v = \sum_{n=1}^{\infty} a_n u_n$$

be an ℓ^2-linear combination of these vectors. Then for all n ∈ N,

$a_n = \langle u_n, v\rangle$.

Proof: Given an n ∈ N, we can write $u_n = \sum_{k=1}^{\infty} b_k\, u_k$, where b_n = 1 and b_k = 0

for all $k \neq n$. By the inner product formula, it follows that

$$\langle u_n, v\rangle = \sum_{k=1}^{\infty} a_k\, b_k = a_n$$

In general, we say that a vector v is in the ℓ^2-span of $\{u_n\}$ if v can be expressed as an ℓ^2 −linear combination of the vectors $\{u_n\}$. According to the previous corollary, any vector v in the ℓ^2-span of $\{u_n\}$ can be written as

$$v = \sum_{n=1}^{\infty} \langle u_n, v\rangle u_n$$

It follows that

$$\|v\| = \sqrt{\sum_{n=1}^{\infty} \langle u_n, v\rangle^2}$$

and

$$\langle v, w\rangle = \sum_{n=1}^{\infty} \langle u_n, v\rangle \langle u_n, w\rangle$$

for any two vectors v and w in the ℓ^2-span of $\{u_n\}$.

3. Projections

Definition 3.1.

Let V be an inner product space, let S be a linear subspace of V, and let $v \in V$. A vector $p \in S$ is called the projection of v onto S if

$$\langle s, v - p \rangle = 0$$

for all $s \in S$

It is easy to see that the projection **p** of **v** onto **S**, if it exists, must be unique. In particular, if p_1 and p_2 are two possible projections, then

$$\|P_1 - P_2\|^2 = (P_1 - P_2, P_1 - P_2) = (P_1 - P_2, V - P_2) - (P_1 - P_2, V - P_1),$$

and both of the inner products on the right are zero since $P_1 - P_2 \in S$.

It is always possible to project onto a finite-dimensional subspace.

Proposition 3.2.

Let **V** be an inner product space, let **S** be a finite-dimensional subspace of **V**, and let $\{\mathrm{u}_1, \dots, \mathrm{u}_\mathrm{n}\}$ be an orthonormal basis for S. Then for any $\mathrm{v} \in \mathrm{V}$, the vector

$$P = \sum_{k=1}^{n} \langle u_k , v \rangle u_k$$

is the projection of v onto S.

Proof : Observe that $\langle u_k, p \rangle = \langle u_k, v \rangle$for each k, and hence $\langle u_k, v - p \rangle = 0$ for each k. By linearity, it follows that $\langle s,\ v - p \rangle$= 0 for all $s \in S$, and hence **p** is the projection of **v** onto **S**.

Lemma 3.3.

Let V be a Hilbert space, let $\{u_n\}$ be an orthonormal sequence in V, and let $v \in V$. Then

$$\sum_{n=1}^{\infty} \langle u_n, v\rangle^2 \leq \|v\|^2$$

Proof: Let $N \in N$, and let

$$p_N = \sum_{n=1}^{N} \langle u_n, v\rangle u_n$$

be the projection of **v** onto Span $\{u_1, \ldots, u_N\}$. Then $\langle p_N, v - p_N\rangle = 0$, so by the Pythagorean theorem

$$\|V\|^2 = \|p_N\|^2 + \|v - p_N\|^2 \geq \|p_N\|^2 = \textstyle\sum_{n=1}^{N} \langle u_n, v\rangle^2.$$

This holds for all $N \in N$, so the desired inequality follows.

Proposition 3.4.

Let *V* be a Hilbert space, and let $\{u_n\}$ be an orthonormal sequence of vectors in *V*. Then for any $v \in V$, the sequence $\{\langle u_n, v\rangle\}$ is ℓ^2, and the vector

$$p = \sum_{n=1}^{\infty} \langle u_n, v\rangle u_n$$

is the projection of ***v*** onto the ℓ^2-span of $\{u_n\}$.

Proof: Bessel's inequality shows that the sequence $\{\langle u_n, v\rangle\}$ is ℓ^2, and thus the sum for p converges. By the coefficient formula (Corollary 2.8), we have that

$$\langle u_n, p\rangle = \langle u_n, v\rangle$$

for all $n \in N$, and hence $\langle u_n, v - p\rangle = 0$ for all $n \in N$. By the continuity of $\langle -, -\rangle$, it follows that $\langle s, v - p\rangle = 0$ for any s in the ℓ^2-span of $\{u_n\}$, and hence p is the projection of **v** onto this subspace.

4. Hilbert Bases

Definition 4.1.

Let V be a Hilbert space, and let $\{u_n\}$ be an orthonormal sequence of vectors in V. We say that $\{u_n\}$ is a Hilbert basis for $\boldsymbol{V}$ if for every $v \in V$ there exists a sequence $\{a_n\}$ in ℓ^2 so that

$$v = \sum_{n=1}^{\infty} a_n u_n.$$

That is, $\{u_n\}$ is a Hilbert basis for V if every vector in V is in the ℓ^2-span of $\{u_n\}$. For convenience, we are requiring all Hilbert bases to be countably infinite, but in the more general theory of Hilbert spaces a Hilbert basis may have any cardinality. Note that a Hilbert basis $\{u_n\}$ for V is not actually a basis for V in the sense of linear algebra. In particular, if $\{a_n\}$ is any ℓ^2 sequence with infinitely many nonzero terms, then the vector

$$\sum_{n=1}^{\infty} a_n\, u_n$$

cannot be expressed as a finite linear combination of Hilbert basis vectors. Of course, it is clearly much more useful to allow ℓ^2 $-$linear combinations, and in the context of Hilbert spaces, it is common to use the word "basis" to mean Hilbert basis, while a standard linear-algebra-type basis is referred to as a Hamel basis.

Example 4.1. The Standard Basis for ℓ^2

Consider the following orthonormal sequence in ℓ^2:

e_1 =(1,0,0,0,...), e_2 =(0,1,0,0,...), e_3 = (0,0,1,0,...), …

If v = $(v_1, v_2, \ldots .)$ is a vector in ℓ^2, it is easy to show that

$$v = \sum_{n=1}^{\infty} v_n e_n,$$

and therefore $\{e_n\}$ is a Hilbert basis for ℓ^2.

This example is in some sense quite general, as shown by the following proposition.

Proposition 4.2.

Let V be a Hilbert space, and suppose that V has a Hilbert basis $\{u_n\}$. Then there exists an isometric isomorphism T: $\ell^2 \to V$ such that T(e_n) = u_n for each n

Proof: Define a function T: $\ell^2 \to V$ by

$$T(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} a_n u_n$$

Clearly T is linear. Note also that T is a bijection, with inverse given by

$$T^{-1}(v) = (\langle u_1, v\rangle, \langle u_2, v\rangle, \ldots),$$

and hence T is a linear isomorphism. Finally, we have

$$\|T(a_1, a_2, \ldots)\| = \|\sum_{n=1}^{\infty} a_n u_n\| = \sqrt{\sum_{n=1}^{\infty} a_n^2} = \|(a_1, a_2, \ldots)\|_2$$

for all $(a_1, a_2, \ldots) \in \ell^2$, so T is isometric.

Proposition 4.3.

Let V be a Hilbert space, and let $\{u_n\}$ be an orthonormal sequence of vectors in V. Then the following are equivalent:

1. The sequence $\{u_n\}$ is a Hilbert basis for V.

2. The set of all finite linear combinations of elements of $\{u_n\}$ is dense in V.

3. For every nonzero $v \in V$, there exists an $n \in N$ so that $(u_n, v) \neq 0$.

Proof: Let S be the set of all finite linear combinations of elements of $\{u_n\}$, i.e., the linear span of $\{u_n\}$. We prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

(1) ⇒ (2) Suppose that $\{u_n\}$ is a Hilbert basis, and let $v \in V$. Then

$$v = \sum_{n=1}^{\infty} a_n u_n$$

for some ℓ^2 sequence $\{a_n\}$. Then **v** is the limit of the sequence of partial sums

$$s_N = \sum_{n=1}^{N} a_n u_n$$

so **v** lies in the closure of S.

(2) ⇒ (3) Suppose that S is dense in V, and let $\boldsymbol{v}$ be a nonzero vector in V. Let $\{s_n\}$ be a sequence in S that converges to **v**. Then there exists an n$\in N$ so that $\|s_n - v\| < \|v\|$, and it follows that

$$\langle s_n , v\rangle = \frac{\|s_n\|^2 + \|v\|^2 - \|s_n - v\|^2}{2} > \frac{\|s_n\|^2}{2} \geq 0.$$

But since $s_n \in S$, we know that $s_n \in \text{Span}\,\{u_1, \dots, u_k\}$ for some k ∈ N, and it follows that $\langle u_i, v\rangle \neq 0$ for some $i \leq k$.

(3) ⇒ (1) Suppose that condition (3) holds, let $v \in V$, and let

$$p = \sum_{n=1}^{\infty} \langle u_n, v \rangle u_n$$

be the projection of **v** onto the ℓ^2 −span of $\{u_n\}$ (by Proposition 3.4). Then $\langle u_n, p - v \rangle = 0$ for all $n \in N$, so by condition (3) the vector ***p-v*** must be zero. Then **v=p**, so **v** lies in the ℓ^2-span of $\{u_n\}$, which proves that $\{u_n\}$ is a Hilbert basis.

5. Fourier Series

The theory of Hilbert spaces lets us to provide a nice theory for Fourier series on the interval $[-\pi, \pi]$. We begin with the following theorem.

Theorem 5.1.

For any closed interval $[a, b] \subseteq R$, the continuous functions on [a, b] are dense in L^2([a,b]).

It follows that any closed subset of L^2 ([a, b]) that contains the continnous functions must be all of L^2 ([a, b]).

Theorem 5.2.

The sequence

$$\frac{1}{\sqrt{2\pi}}, \frac{cosx}{\sqrt{\pi}}, \frac{sinx}{\sqrt{\pi}}, \frac{cos2x}{\sqrt{\pi}}, \frac{sin2x}{\sqrt{\pi}}, \frac{cos3x}{\sqrt{\pi}}, \frac{sin3x}{\sqrt{\pi}}$$

is a Hilbert basis for $L^2([-\pi, \pi])$.

Proof: It is easy to check that the given functions are orthonormal. Let *S* be the set of all finite linear combinations of the basis elements, i.e., the

set of all finite trigonometric polynomials. By Proposition 4.3, it suffices to prove that *S* is dense in $L^2([-\pi, \pi])$.

Let C(T) be the set of all continuous functions f on $[-\pi, \pi] \to \boldsymbol{R}$ for which $f(-\pi) = f(\pi)$. Every function in $C(T)$ is the uniform limit (and hence the L^2 limit) of trigonometric polynomials, so the closure of *S* continues *C(T).* But clearly every continuous function on [a, b] is the L^2 limit of functions in *C(T),* and hence the closure of *S* contains every continuous function. By Theorem 5.1, we conclude that the closure of *S* is all of L^2 $([-\pi, \pi])$.

In general, an orthogonal system for [a,b] if the sequence $\{f_n/\|f_n\|_2\}$ of normalizations is a Hilbert basis for $L^2([a, b])$. According to the above theorem, the sequence

$$1, cosx, sinx, cos2x, sin2x, cos3x, sin3x, \quad ...$$

is a complete orthogonal system for the interval $[-\pi, \pi]$.

Definition5 .3.

Let $f : [-\pi, \pi] \to R$ be an L^2 function. Then the Fourier coefficients of f are defined as follows:

$$a = \frac{\langle f, 1\rangle}{2\pi} = \frac{1}{2\pi}\int_{[-\pi,\pi]} f\, dm,$$

$$b_n = \frac{\langle f, \cos nx\rangle}{\pi} = \frac{1}{\pi}\int_{[-\pi,\pi]} f(x) cosnx\, dm(x),$$

$$c_n = \frac{\langle f, \sin nx\rangle}{\pi} = \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\, sin\, nx\, dm(x).$$

Note that the Fourier coefficients are the coefficients for the functions

$$1, cosx, sinx, cos2x, sin2x, cos3x, sin3x, \quad ...$$

which are not unit vectors. The actual coefficients of the Hilbert basis vectors are

$$a\sqrt{2\pi}, \quad \{b_n\sqrt{\pi}\}, \ and \ \{c_n\sqrt{\pi}\}.$$

Corollary 5.4.

Let f: $[-\pi, \pi] \to R$ be an L^2 function with Fourier coefficients a, $\{b_n\}$ and $\{c_n\}$. Then $\{b_n\}$ and $\{c_n\}$ are ℓ^2 sequences, and the Fourier series

$$a + \sum_{n=1}^{\infty}(b_n \cos nx + c_n \sin nx)$$

converges to f in L^2.

Proof: This follows from Theorem 5.2 and the coefficient formula (Corollary 2.7).

Corollary 5.5.

Let $f : [-\pi, \pi] \to R$ be an L^2 function with Fourier coefficients a, $\{b_n\}$, $\{c_n\}$, and let $g : [-\pi, \pi] \to R$ be an L^2 function with Fourier coefficients A, $\{B_n\}$, and $\{C_n\}$. Then

$$\frac{1}{\pi} \int_{[-\pi, \pi]} fg \, dm = 2aA + \sum_{n=1}^{\infty} (b_n B_N + c_n C_n)$$

Proof: By the inner product formula (Proposition 5), we have

$$\langle f, g \rangle = (a\sqrt{2\pi})(A\sqrt{2\pi} + \sum_{n=1}^{\infty} ((b_n\sqrt{\pi})(B_n\sqrt{\pi}) + (c_n\sqrt{\pi})(B_n\sqrt{\pi})),$$

and dividing through by π gives the desired formula.

Corollary 5.6.

If $a < b$, then $L^2([a, b])$ and ℓ^2 are isometrically isomorphic.

Proof: Since

$$\frac{1}{\sqrt{2\pi}}, \frac{cosx}{\sqrt{\pi}}, \frac{sinx}{\sqrt{\pi}}, \frac{cos2x}{\sqrt{\pi}}, \frac{sin2x}{\sqrt{\pi}}, \frac{cos3x}{\sqrt{\pi}}, \frac{sin3x}{\sqrt{\pi}}, \quad \ldots$$

is a Hilbert basis for $L^2([-\pi, \pi])$, it follows from Proposition 11 that the linear transformation $T: \ell^2 \to L^2([-\pi, \pi])$ defined by

$$T(a_1, a_2, a_3 \ldots) = \frac{a_1}{\sqrt{2\pi}}, \frac{a_2 cosx}{\sqrt{\pi}}, \frac{a_3 sinx}{\sqrt{\pi}}, \frac{a_4 cos2x}{\sqrt{\pi}}, \frac{a_5 sin2x}{\sqrt{\pi}} + \cdots$$

is an isometric isomorphism.

6. Other Orthogonal Systems

The Fourier basis is not the only Hilbert basis for $L^2([a,b])$. Indeed, many such families of orthogonal functions are known. In this section, we derive an orthonormal sequence of polynomials that is a Hilbert basis for $L^2([a,b])$.

Consider the sequence of functions

$$1, \qquad x, \qquad x^2, \qquad x^3, \qquad \ldots$$

on the interval [-1,1]. These functions are not a Hilbert basis for $L^2([-1,1])$, since they are not orthonormal. However, it is possible to use these functions to make a Hilbert basis of polynomials via the Gram-Schmidt process. We start by making the constant function 1 into a unit vector:

$$p_0(x) = \frac{1}{\|1\|_2} = \frac{1}{\sqrt{2}}$$

The function x is already orthogonal to p_0 on the interval [-1,1], so we normalize x as well:

$$p_1(x) = \frac{x}{\|x\|_2} = x\sqrt{\frac{3}{2}}.$$

Now we want a quadratic polynomial orthogonal to p_0 and p_1. The function x^2 is already orthogonal to p_1, but not to p_0. However, if we subtract from x^2 the projection of x^2 onto p_0, then we get a quadratic polynomial orthogonal to p_0.

$$x^2 - \langle p_0 \,, x^2 \rangle p_0(x) = x^2 - \frac{1}{3}$$

Normalizing gives:

$$p_2(x) = \frac{3\sqrt{5}}{2\sqrt{2}} \left(x^2 - \frac{1}{3}\right)$$

Continuing in this fashion, we obtain an orthonormal sequence $\{p_n\}$ of polynomials, where each p_n is obtained from x^2 by subtracting the projections of x^2 onto$p_0, \ldots, p_{n-1}$ and then normalizing.

Definition 6.1.

The normalized Legendre polynomials are the sequence of polynomial functions $p_n: [-1,1] \to R$ defined recursively by $p_0(\mathrm{x}) = {}^{1}/_{\sqrt{2}}$ and

$$p_n(x) = c_n \left(x^n - \sum_{k=0}^{n-1} \langle p_k, x^n \rangle p_k(x) \right)$$

for n ≥ 1, where the constant $c_n > 0$ is chosen so that $\|p_n\|_2 = 1$.

Theorem 6.2.

The sequence $p_0 p_1 p_2 \ldots$ of normalized Legendre polynomials is a Hilbert basis for $L^2([-1,1])$.

Proof: Let S be the linear span of $p_0 p_1 p_2 \ldots$ Since

$$x^n = \frac{p_n(x)}{c_n} \sum_{k=0}^{n-1} \langle p_k, x^n \rangle p_k(x),$$

The subspace S contains each x^n, and hence contains all polynomials. By the Weier-strass approximation theorem, every continuous function on [-1, 1] is a uniform limit (and hence the L^2 limit) of a sequence of polynomials. It follows that the closure of S contains all the continuous functions, and hence contains all L^2 functions by Theorem 5.1.

Definition 6.3.

Let B^3 denote the closed unit ball on R^3, and let S^2 denote the unit sphere. The Dirichlet problem on B^3 can be stated as follows:

Given a continuous function $f: S^2 \to R$, find a harmonic function $F: B^3 \to R$ that agrees with f on S^2.

Since we are working on the ball, it makes sense to use spherical coordinates $(p, \theta, \emptyset)$, which are defined by the formulas

$$x = pcas\theta sin\emptyset, \qquad y = psin\theta sin\emptyset, \qquad z = pcas\emptyset.$$

Using spherical coordinates, one family of solutions to Laplace's equation on the ball can be written as follows:

$$F(p, \theta, \emptyset) = p^n p_n(cos\emptyset)$$

where p_n is the nth Legendre polynomial. These solutions are all axially symmetric about the z-axis, meaning that they have no explicit dependence on θ.

Since the Legendre polynomials are a Hilbert basis, we can use these solutions to solve the Dirichlet problem for any axially symmetric function $f: S^2 \to R$. All we do is write f as the sum of a Legendre series.

$$f(\theta, \emptyset) = \sum_{n=0}^{\infty} a_n p_n(cos\emptyset),$$

and then the corresponding harmonic function F will be defined by the formula

$$F(p, \theta, \emptyset) = \sum_{n=0}^{\infty} a_n p^n p_n(cos\emptyset).$$

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