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Lagrange's Equations

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بسم الله الرحمن الرحيم

{وَمَا أُوتِيتُمْ مِنَ الْعِلْمِ إِلَّا قَلِيلًا}

صدق الله العلي العظيم

الإسراء - آية (٨٥)

إقرار الاستاذ المشرف

اقر ان البحث المرسوم(Lagrange's Equation)

والذي تقدم به الطالب في كلية التربية للعلوم الصرفة / قسم رياضيات

(علي حسين ياسر) تحت اشرافي وبذلك ارشحه للمناقشة

توقيع التدريسي:

الاسم :

التاريخ:

Dedication

To whom do I prefer it over myself, and why not; You sacrificed

for me

She spared no effort to keep me happy

(My beloved mother).

We walk the paths of life, and those who control our minds remain in every path we take

His kind face, and good deeds.

He did not skimp on me all his life

(My dear father) .

To my friends, and all those who stood by me and helped me with whatever they had, in many ways

I present to you this research, and I hope it will be to your satisfaction.

Acknowledgments

I thank God first and foremost for the great grace He has given to me. Thanks to my dear parents who have not stopped making all their efforts from the moment I was born to these blessed moments. Thanks to all those who advised me, guided me, contributed to my education and I thank in particular my dear supervisor,

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support and guidance with advice, education, correction and follow-up. I also thank the members of the distinguished debate committee, all friends and those who accompanied me during my study trip.

CONTENTS

Content	Page

Figures

Abstract

Lagrangian mechanics is a restatement of classical mechanics introduced by Joseph Louis Lagrangian, in Lagrangian mechanics, the trajectory of a body is derived by finding the path that minimizes work, which is an integral of a quantity we call Lagrangian over time, Lagrangian for classical mechanics is the difference between Kinetic energy and potential energy. This topic greatly simplifies many physical problems, for example a small ball in a ring. If we calculate this issue on the basis of Newtonian mechanics, we will get a complex set of equations that will take into account the forces that the vortex affects the ball at every moment

In the first chapter we talked about Introduction to Lagrange Equations and their Applications

As mentioned in the second chapter The simplest optimization problem

Lagrange's equation is a temporal version of a first-order condition that characterizes it Choice of an optimal equation (expected) marginal costs and marginal benefits. Many economic problems are problems of dynamic optimization in which choices are made associated with the passage of time. Any marginal, temporary, and feasible change in behavior has marginal benefits equal to Marginal costs now and in the future. Assuming the original problem satisfies a bit regularity conditions, the Lagrange equation is a necessary but not sufficient condition for optimum. This differential equation or difference is the law of motion of the economy model variables, and as such are useful (in part) for characterizing theory. Implications for the model for optimal dynamic behavior.

In the calculus of inequalities and classical mechanics, the Euler Lagrange equations are a system of second-order ordinary differential equations whose solutions are points invariants of a given functional procedure. The equations were discovered in the 1750s by Swiss mathematician Leonhard Euler and Italian mathematician Joseph Louis Lagrange. Since the differentiable function is invariant at its local extrema , the Lagrange equation is useful for solving optimization problems in which, given some function, one searches for a minimizing or maximizing function. This is similar to Fermat's theorem of calculus, which states that at any point at which a differentiable function reaches a local maximum, its derivative is zero. [1]

In Lagrangian mechanics, according to Hamilton's principle of invariant action, the evolution of a physical system is described by solutions of the Lagrange equation of system action. In this context, Lagrange's equations are usually called Lagrange's equations. In classical mechanics, is equivalent to Newton's laws of motion. In fact, the Lagrange equations will produce the same equations as Newton's laws. This is particularly useful when analyzing systems whose force vectors are particularly complex. It has the advantage that it takes the same form in any generalized coordinate system, and is better suited to generalizations. In classical field theory, a similar equation exists to calculate field dynamics. [2].

This research consists of two chapters. The first chapter talks about basic definitions and historical background of Lagrange equation, as well as the applications of Lagrange equation and its derivation.

The second chapter talks about the simplest optimization problem by using Lagrange equation with some algorithms and examples.

CHAPTER ONE

Introduction to Lagrange Equations

and their Applications

1.1. Historical Background

Lagrange function is the function from which you can get everything you want from the physical system, as they are like the wave function in quantum mechanics. Lagrangian equations are based on the least action principle, which means that among all the paths that a moving body can take, the path that the body takes from point a to point b is the path that makes the integration of L from a to b as large as possible.[3]

The Euler-Lagrange equation was developed in the 1750s by Euler and Lagrange in connection with their study of the isochronicZ problem. This is the problem of determining a curve along which a weighted particle will fall to a fixed point in a fixed period of time, regardless of the starting point.

Lagrange solved this problem in 1755 and sent the solution to Euler. Both developed the Lagrangian method and applied it to mechanics, resulting in the formulation of Lagrangian mechanics. Their correspondence eventually led to the calculus of variances, a term coined by Euler himself In 1766. [4].

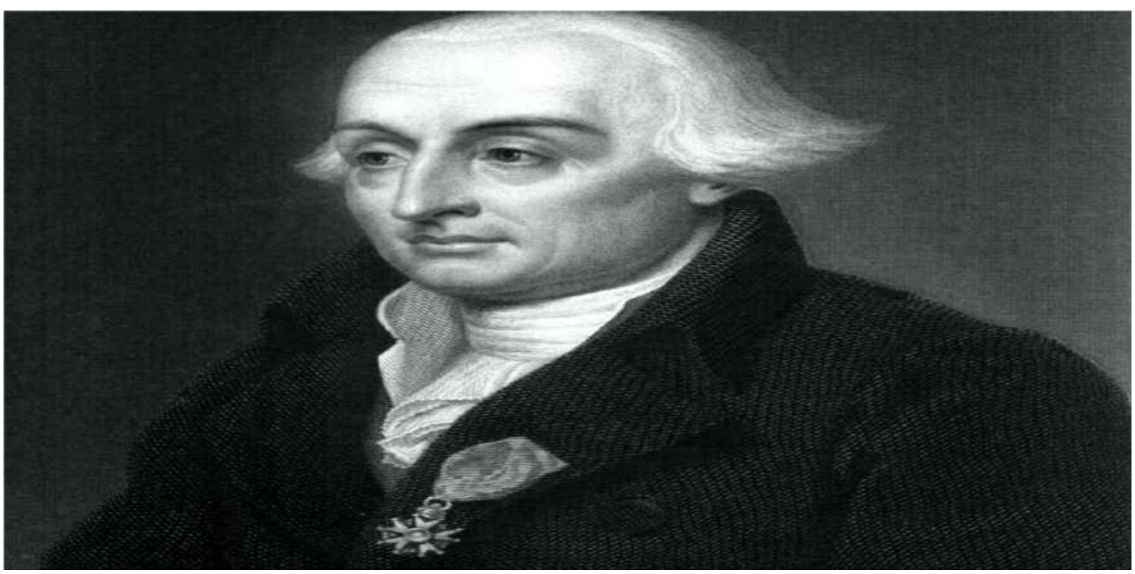

Figure 3.1 Joseph-Louis Lagrange (1736-1813)

Joseph-Louis Lagrange (born January 25, 1736 in Turin, Piedmont-Sardinia; died April 10) 1813 in Paris) was an Italian Enlightenment Era mathematician and astronomer. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

In 1766, on the recommendation of Euler and d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed. For over twenty years, producing volumes of his work and winning several prizes from the French Academy of Sciences. Lagrange's treatise on analytical mechanics written in Berlin and first published in 1788, offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century. In 1787, at the age of 51, he moved from Berlin to Paris and became a member of the French Academy. He remained in France until the end of his life. He was significantly involved in the decimalization in revolutionary

Figure 1. 4 Leonhard Euler (1707-1783)

France, became the first professor of analysis at the Ecole Polytechnique upon its opening in 1794, founding member of the Bureau des Longitudes and Senator in 1799. Leonhard Euler was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. [4],[5].

He also introduced much of the modern mathematical terminology and notation, Particularly for mathematical analysis, such as the notion of a mathematical function. He is also Renowned for his work in mechanics, fluid dynamics, optics, astronomy, and music theory. Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He is also one of the most prolific mathematicians; His collected works fill 60-80 quart volumes. He spent most of his adulthood life in St. Petersburg, Russia, and in Berlin, Prussia.

1.2. Lagrange Equation Applications

Joseph Louis Lagrange, the mathematician, sent a message containing a new analytical method for studying boundary problems. Euler announced that the technique developed in this study extends the beginning of the emergence of the calculus of changes from his researches: "Calculation of changes and their application in mechanics", "A study on the nature of sound and its propagation", "Integral calculus", and "Theory of the motion of Jupiter and Saturn".[6]

"Lagrangian mechanics is a reformulation of classical mechanics, by Joseph Louis Lagrange. In Lagrangian mechanics: the trajectory of a body is calculated by solving the Lagrangian equation in one of its two forms, which differ from each other in how they deal with physical constraints. But what are physical constraints?

Limitations in physics are things that the physical system cannot overcome. For example, imagine that a cube is being pulled off a curve by the force of its own gravity. The cube should then stay in contact with the curve.

The Lagrange equation of the first kind treats the constraints as equations by themselves. For example: There is an equation that determines the height of the cube as it slides, thus ensuring that it stays on the curved surface. As for the Lagrange equation in its second form, it deals with the restrictions by defining the allowed coordinates in a way that corresponds to them. That is, there are only certain coordinates that the cube can occupy that ensure that the cube stays on the curved surface.”

1.3. Lagrangian Mechanics

Lagrangian mechanics uses the energies in the system. The central quantity of Lagrangian mechanics is the Lagrangian, a function which summarizes the dynamics of the entire system. Overall, the Lagrangian has units of energy, but no single expression for all physical systems. Any function which generates the correct equations of motion, in agreement with physical laws, can be taken as a Lagrangian. It is nevertheless possible to construct general expressions for large classes of applications. The non-relativistic Lagrangian for a system of particles in the absence of a magnetic field is given by[7] .

$$L = T - V \quad T = \frac{1}{2}\sum_{k=1}^{n} m_k {v_k}^2$$

is the total kinetic energy of the system, equaling the sum Σ of the kinetic energies of the particles,[8] and V is the potential energy of the system. Kinetic energy is the energy of the system's motion, and ${v_k}^2 = v_k * v_k$ is the magnitude squared of velocity, equivalent to the dot product of the velocity with itself. The kinetic energy is a function only of the velocities v_k , not the positions r_k nor time t, so $T = T(v_1, v_2, \dots)$

1.4. Derivation of the equation

The solutions of Lagrange equations are called critical curves. The Lagrange equation is in general a second order differential equation, but in some special cases, it can be reduced to a first order differential equation or where its solution can be obtained entirely by evaluating integrals

$$(X, L)nXL = L(t, q, u)q \in X, unXL: Rt \, * \, TX \to \, R, TXX$$

In this chapter, we will give necessary conditions for an extremum of a function of the type with various types of boundary conditions. The necessary condition is in the form of a differential equation that the extremal curve should satisfy, and

Let $(X, L\,)$ be a mechanical system with n degrees of freedom. Here X is the configuration space and $L = (t, q\,, u)$ the Lagrangian, i.e. a smooth real-valued function such that $q \, \in \, X$, and u is an n-dimensional "vector of speed". (For those familiar with differential geometry, X is a smooth manifold, and

$L: R_t \, * \, TX \, \to \, R$, where TX is the tangent bundle of X). Let$\mathcal{P}(a, b, \boldsymbol{x_a}, \boldsymbol{x_b})$be the set of smooth paths $\boldsymbol{q}: [\boldsymbol{a}\,, \boldsymbol{b}] \to \boldsymbol{X}$ for which $\boldsymbol{q}(\boldsymbol{a}) = \boldsymbol{x_a}$ and $\boldsymbol{q}(\boldsymbol{b}) = \boldsymbol{x_b}$ .The action functional $S: \mathcal{P}(a, b, \boldsymbol{x_a}, \boldsymbol{x_b}) \to R$ is defined by

$$S[q] = \int_a^b L\big(t, q(t), q'(t)\big)dt.$$

The path is a fixed point for if and only if $q \in \mathcal{P}(a, b, \boldsymbol{x_a}, \boldsymbol{x_b})S$

$$\frac{\partial L}{\partial q^i}\,\big(t, q(t), q'(t)\big) - \frac{d}{dt}\frac{\partial L}{\partial q'^i}\,\big(t, q(t), q'(t)\big) = 0 \quad , i = 1, \dots ., n.$$

Here, time is derivative from,$'\boldsymbol{q}(\boldsymbol{t})$ $\boldsymbol{q}(\boldsymbol{t})$.

1.4.1. Derivation of the one-dimensional Lagrange equation

The derivation of the one-dimensional Euler–Lagrange equation is one of the classic proofs in mathematics. It relies on the fundamental lemma of calculus of

$$J = \int_a^b L(x, f(x), f'(x))dx.$$

variations. We wish to find a function $\boldsymbol{f}$ which satisfies the boundary conditions $\boldsymbol{f(a) = A, f(b) = B}$, and which extremizes the functional

We assume that $\boldsymbol{L}$ is twice continuously differentiable.[9] A weaker assumption can be used, but the proof becomes more difficult. If $\boldsymbol{f}$ extremizes the functional subject to the boundary conditions, then any slight perturbation of $\boldsymbol{f}$ that preserves the boundary values must either increase $\boldsymbol{J}$ (if $\boldsymbol{f}$ is a minimizer) or decrease $\boldsymbol{J}$ (if $\boldsymbol{f}$ is a maximizer).

Let $\boldsymbol{g_\varepsilon(x) = f(x) + \varepsilon\eta(x)}$ be the result of such a perturbation $\boldsymbol{\varepsilon\eta(x)}$ of $\boldsymbol{f}$, where $\boldsymbol{\varepsilon}$ is small and $\boldsymbol{\eta(x)}$ is a differentiable function satisfying

$\boldsymbol{\eta(a) = \eta(b) = 0}$. Then define

$$J_\varepsilon = \int_a^b L(g_\varepsilon(x), g'_\varepsilon(x))dx = \int_a^b L_\varepsilon\ dx)$$

Where $L_\varepsilon = L(x, g_\varepsilon(x), g'_\varepsilon(x))$

We now wish to calculate the total derivative of $J\varepsilon$ with respect *to* ε

$$\frac{dJ_\varepsilon}{d_\varepsilon} = \frac{d}{d_\varepsilon}\int_a^b L_\varepsilon dx = \int_a^b \frac{dL_\varepsilon}{d_\varepsilon}\ dx\,.$$

It follows from the total derivative that

$$\left(\frac{dL_\varepsilon}{d_\varepsilon}\right) = \left(\frac{dx}{d_\varepsilon}\right)\left(\frac{\partial L_\varepsilon}{\partial x}\right) + \left(\frac{dg_\varepsilon}{d\varepsilon}\right)\left(\frac{\partial L_\varepsilon}{\partial g_\varepsilon}\right) + \left(\frac{dg'_\varepsilon}{d_\varepsilon}\right)\left(\frac{\partial L_\varepsilon}{\partial\, g'_{\ \varepsilon}}\right)$$

$$= \left(\frac{dg_\varepsilon}{d_\varepsilon}\right)\left(\frac{\partial L_\varepsilon}{\partial g_\varepsilon}\right) + \left(\frac{dg_{\prime\varepsilon}}{d_\varepsilon}\right)\left(\frac{\partial L_\varepsilon}{\partial g'_\varepsilon}\right)$$

$$n(x)\frac{\partial L_\varepsilon}{\partial g_\varepsilon} + n'(x)\left(\frac{\partial L_\varepsilon}{\partial g'_\varepsilon}\right)$$

The second line comes from the fact that $\boldsymbol{x}$ doesn't depend on $\boldsymbol{\varepsilon}$ i.e $\frac{dx}{d\varepsilon} = 0$So

$$\frac{dJ_\varepsilon}{d_\varepsilon} = \int_a^b \left[\eta(x)\frac{\partial L_\varepsilon}{\partial g_\varepsilon} + \eta'(x)\frac{\partial L_\varepsilon}{\partial g'_{\ \varepsilon_\varepsilon}}\ dx\right].$$

When $\boldsymbol{\varepsilon = 0}$ we have $\boldsymbol{g_\varepsilon = f},\ \boldsymbol{L_\varepsilon = L(x\,, f(x), f'(x))}$ and $\boldsymbol{J_\varepsilon}$ has an extremum value, so that

$$\frac{dJ_\varepsilon}{d_\varepsilon}\Big|_{\varepsilon=0}\ \int_a^b \left[\eta(x)\ \frac{\partial L}{\partial f} + \eta'(x)\ \frac{\partial L}{\partial f'}\, dx\right] = 0.$$

The next step is to use integration by parts on the second term of the integrand, yielding $\int_a^b \left[\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'}\right]\ \eta(x)dx + \left[\eta(x)\frac{\partial L}{\partial f'}\right]_a^b = 0.$

Using the boundary conditions $\boldsymbol{\eta\,(a) = \eta\,(b) = 0}$,

$$\int_a^b \left[\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'}\right]\ \eta'(x)dx = 0.$$

Applying the fundamental lemma of calculus of variations now yields the Euler–Lagrange equation

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0.$$

1.4.2. Alternative derivation of the one-dimensional Lagrange equation

Given a functional $J = \int_a^b L(t, y(t), y'(t))dt$ on $C^1([a, b])$ with the boundary conditions $y(a) = A \; and \; y(b) = B,$ we proceed by approximating the extremal curve by a polygonal line with n segments and passing to the limit as the number of segments grows arbitrarily large. Divide the interval $[a, b]$ into n equal segments with endpoints

$\boldsymbol{t_0} = \boldsymbol{a}, \boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_n} = \boldsymbol{b}$ and let $\Delta t = t_k - t_{k-1}$ ather than $\boldsymbol{a}$ smooth function $\boldsymbol{y}(\boldsymbol{t})$ we consider the polygonal line with vertices $(\boldsymbol{t_0}, \boldsymbol{y_0}), \ldots, (\boldsymbol{t_n}, \boldsymbol{y_n}),$ where $\boldsymbol{y_0} = \boldsymbol{A} \; and \; \boldsymbol{y_n} = \boldsymbol{B}$. Accordingly, our functional becomes a real function of $\boldsymbol{n} - \boldsymbol{1}$ variables given by

$$J(y_1, \ldots y_n - 1) \approx \sum_{k=0}^{n-1} L\left(t_k, y_k \frac{y_k + 1 - y_k}{\Delta t}\right) \Delta t.$$

Extremals of this new functional defined on the discrete points $(\boldsymbol{t_0}, \ldots \boldsymbol{t_n})$ correspond to points where $\frac{\partial J(y_1, \ldots, y_n)}{\partial y_m} = 0.$

Evaluating this partial derivative gives

$$\frac{\partial J}{\partial y_m} = L_y\left(t_m, y_m, \frac{y_m + 1 - y_m}{\Delta t}\right)\Delta t + L_{y'}\,(t_{m-1, y \; m-1,} \quad \frac{y_m - y_m - 1}{\Delta t})$$
$$- L_{y'}(t_m, y_m, \frac{y_m + 1 - y_m}{\Delta t})$$

Dividing the above equation by Δt gives

$\frac{\partial J}{\partial y_{m\Delta t}} = L_y\,(t_m, y_m, \frac{y_m + 1 - y_m}{\Delta t})$ -
$\frac{1}{\Delta t}\left[L_{y'}\left(t_m, y_m, \frac{y_m+1-y_m}{\Delta t}\right) - L_{y'}\,(t_{m-1,\; y_{m-1,}} \quad \frac{y_m - y_m - 1}{\Delta t})\right],$

and taking the limit as $\Delta t \rightarrow 0$ of the right –hand side of this expression yields

$$Ly - \frac{d}{dt} L_{y\prime} = 0 .$$

The left hand side of the previous equation is the functional derivative $\delta J/\delta yJ$. of the function J. A necessary condition for a differentiable functional to have an extremum on some function is that its functional derivative at that function vanishes, which is granted by the last equation.

1.5. Example

One standard example is to find the real-valued function $y\,(\,x\,)$ on the interval

$[\,a\,,b\,]$, such as $y\,(\,a\,) = c \; and \; y\,(\,b\,) = d$, where the length of the path along the curve traced by y is as short as possible.

$$S = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + y'^2} \; dx$$

Land 's integral function is $L(x, y, y') = \sqrt{1 + y'^2}$. The partial derivatives of L are: $\frac{\partial L(x,y,y')}{\partial y'} = \frac{y'}{\sqrt{1+y\prime^2}} \; and \; \frac{\partial L(x,y,\acute{y})}{\partial y} = 0.$

Substituting these into the Euler-Lagrange equation ,we get

$$\frac{d}{dx} \frac{y'(x)}{\sqrt{1 + \left(y'(x)\right)^2}} = 0.$$

$$\left(\frac{y'(x)}{\sqrt{1 + \left(y'(x)\right)^2}} \right) = C = constant \; y'(x) = \left(\frac{c}{\sqrt{\;1 - C^2}} \right) =: A \;\rightarrow\; y(x) = Ax + B$$

That is, the function must have a constant first derivative, and therefore its graph is a straight line.

CHAPTER TWO

Optimization Problem

2.1. The simplest optimization problem

The simplest optimization problem can be formulated as follows:

Let $F(\alpha, \beta, \gamma\ )\ be\ a$ function with continuous first and second partial derivatives with respect to

(α, β, γ). Then and $x\ \in\ C^1[a;\ b]$ such that $x(a)\ =\ y_a\ and\ x(b)\ =\ y_b$, and which is an extremum for the function

$$I(x) = \int_a^b F(x(t), x'(t), t)dt.$$

In other words, the simplest optimization problem consists of finding an extremum of a function where the class of admissible curves comprises all smooth curves joining two fixed points; see Figure 2.1. We will apply the necessary condition for an extremum to the solve the simplest optimization problem described above.

Let $C^k\ [a, b]$ denote the set of continuous functions defined on the interval $a\ \leq\ x \leq\ b$ which have their first $k-$derivatives also continuous on $a \leq\ x \leq\ b.$

The proof to follow requires the integrand $F(x, y, y')$ to be twice differentiable with respect to each argument. What's more, the methods that we use in this module to solve problems in the calculus of variations will only find those solutions which are in $C^2[a, b]$. More advanced techniques [10] are designed to overcome this last restriction. This isn't just a technicality: discontinuous extremal functions are very important in optimal control problems, which arise in engineering applications.

Theorem2.1.1. Let $S\ =\ \{x\ \in\ C^1[a;\ b]\ |x(a)\ =\ y_a\ and\ x(b)\ =\ y_b\}$, and *let* $I\ :\ S\ \rightarrow\ R\ 0\ be\ a$ function of the form $I\ (x) \int_a^b F(x(t), x'(t), t)dt.$

If I has an extremum at $x_0\ \in\ S$, then x_0 satisfies the Lagrange equation:

$$\left(\frac{\partial F}{\partial a}\right)(x_0(t), x'_0\,(t), t) - \left(\frac{d}{dt}\right)\left(\frac{\partial F}{\partial \beta}(x_0(t), x'_{0(t)}, t)\right) = 0, \quad t\epsilon[a, b].$$

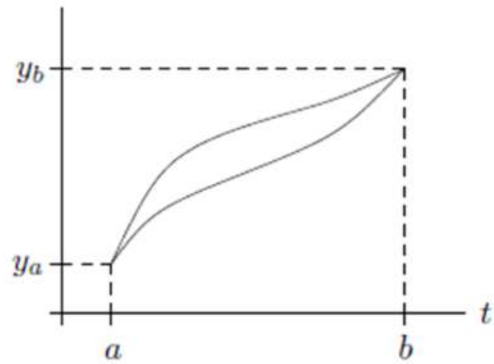

Figure 2. 1 (Possible paths joining the two fixed points $(\boldsymbol{a};\ \boldsymbol{y_a})$ $\boldsymbol{and}$ $(\boldsymbol{b};\ \boldsymbol{y_b})$)

Theorem 2.2.2. If $I(Y)$ is an extremum of the functional

$$I(y) = \int_a^b F(x, y, y')dx$$

defined on all functions $y \in C^2[a,\ b]$ such that $y(a)\ =\ A,\ y(b)\ =\ B,$
then $Y(x)$ satisfies the second order ordinary differential equation

$$\left(\frac{d}{dx}\right)\left(\frac{\partial F}{\partial y'}\right) - \left(\frac{\partial F}{\partial y}\right) = 0.$$

The equation above is the Lagrange equation, or sometimes just Lagrange's equation.

Warning We might be wondering what $\boldsymbol{\frac{\partial F}{\partial y'}}$ is suppose to mean: how can we differentiate with respect to a derivative? Think of it like this: F is given to you as a function of three variables, say $F(u,\ v,\ w)$, and when we evaluate the functional I we plug in $x,\ y(x),\ y'(x)$ for $u,\ v,\ w$ and then integrate. The derivative $\frac{\partial F}{\partial y'}$ is just the partial derivative of F with respect to its second variable v. In other words, to find $\frac{\partial F}{\partial y'}$, just pretend y' *is* a variable.

Equally, there's an important difference between $\frac{dF}{dx}$ and $\frac{\partial F}{\partial x}$. The former is the derivative of F with respect to x, taking into account the fact that $y = y(x)$ *and* $y' = y'(x)$ are functions of x too. The latter is the partial derivative of F with respect to its first variable, so it's found by differentiating F with respect to x and pretending that y and y' are just variables and do not depend on x.

Example2.1.3. Let $S = \{x \in C^1[0;\ 1]\ j\ x(0) = 0\ and\ x(1) = 1\}$. Consider the function $I : S \to R$ given by

$$I(x) = \int_0^1 (\frac{d}{dt}x(t) - 1)^2 dt$$

We wish to find $x_0 \in S$ that minimizes I. We proceed as follows:

Step 1. We have $F(\alpha, \beta, \gamma) = (\beta - 1)^2$, and so by

$\frac{\partial F}{\partial a} = 0\ and = 2(\beta - 1)$.

Step 2. The Lagrange equation (2.2) is now given by

$$0 - (\frac{d}{dt})(2(x_0'(t) - 1)) = 0\ for\ all\ t \in [0,1].$$

Step 3. Integrating , we obtain $2(x_0'(t) - 1) = C$, for some constant C, and so

$$x_0' = \frac{\boldsymbol{c}}{\mathbf{2}} + 1 =: A.$$

Integrating again, we have $x_0(t) = At + B$, where A and B are suitable constants

Step 4. The constants $A \, and \, B$ can be determined by using that fact that $x_0 \in S$, and so $x_0(0) = 0$ and $x_0(a) = 1$. Thus we have

$$A0 + B = 0;$$

$$A1 + B = 1;$$

which $yield \, A = 1 \, and \, B = 0$. So the unique solution x_0 of the Lagrange equation in S is $x_0(t) = t, t \in [0; 1]$; see

Figure 2.2.

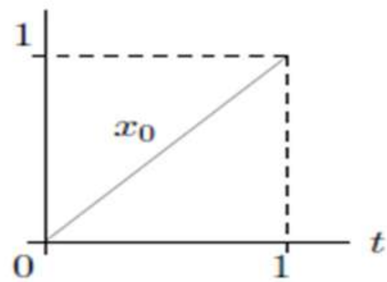

Figure 2. 2 (Minimizer for I.)

Now we argue that the solution x_0 indeed minimizes I. Since $(x`(t)\text{-}1)^2\ ) \geq 0$ for all $t \in [0; 1]$, it follows that $I(x) > 0 \, for \, all \, x \in C^1[0; 1]$. But

$$I(x_0) = \int_0^1 (x_0'(t) - 1)^2 dt = \int_0^1 (1 - 1)^2 dt = \int_0^1 0dt = 0$$

As $I(x) > 0 = I(x_0)$ for all $x \in S$, it follows that x_0 minimizes I.

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