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Research Title

Pre- Hilbert and Hilbert Spaces

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بسم الله الرحمن الرحيم
{ نَرْفَعُ دَرَجَاتٍ مِّن نَّشَاء وَفَوْقَ كُلِّ
ذِي عِلْمٍ عَلِيمٌ }
صَدَقَ اللهُ العَليّ العَظِيم
سورة يوسف الآية : 76

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الشكر والتقدير

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وخاتماً اسأل الله العلي القدير ان يكون هذا العمل خالصاً لوجه، وأن يجعله علماً نافعاً، ويسهل لي به طريقاً الى الجنة.

الباحث

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X	Vector Space	15
L	Linear map	2

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Abstract

In this research, we study the concept of linear space, Inner product space, and Hilbert space, and find out results about them.

We also study we dealt with the inner product space in terms of its definition, examples, and some theories related to it, as well as its relationship to the Hilbert space.

We also define Hilbert space and dealt with examples and the most important results related to it.

1. introduction

In mathematics, Hilbert spaces (named after David Hilbert) allow generalizing the methods of linear algebra and calculus from (finite– dimensional) Euclidean vector spaces to spaces that may be infinite– dimensional. Hilbert spaces arise naturally and frequently in mathematics and physics, typically as function spaces. formally, a Hilbert space is a vector space equipped with an inner product that defines a distance function for which the spaces is a complete metric space. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert , Erhard Schmidt , and Frigyes Riesz . They are indispensable tools in the theories of partial differential equations , quantum mechanics, fourier analysis (which includes applications to signal processing and heat transfer), John Van Neumann coined the term Hilbert spaces for the abstract concept that underlies money of these diverse applications . The success of Hilbert spaces methods us hered in a very fruitful era for functional analysis . Aeart from the classical Euclidean vector spaces , examples of Helbert spaces include spaces of square – integrable functions , spaces of sequences , sobolev spaces consisting of generalized functions , and Hardy spaces of holomorphic functions.

Geometric intuition plays an important role in many aspects of Hilbert space theory. Exact analys of the Pythagorean theorem and parallelogram law hald in a Hilbert space. At a deeper level, perpendicular projection on to a linear subspace or a subspace or a subspace (the analys of " dropping the altitude " of a triangle) plays a significant role in optimization proplems and other aspects of the theory. An element of a Hilbert space can be uniquely specified by its coordinates with respect to an orthonormal basis, in analogy with cortesian coordinates in classical geometry when this basis is countably infinite, it allows identifying the Hilbert space with the space of the infinite sequences that are square– summable. The latter space is often in the older literature referred to as the Hilbert space.

2- Important definition and results .

Definition 2.1 (Linear space):[1]

Let (F, + , -) be a field whose elements are called scalars. Let L is a nonempty set whose elements are called vectors . Then L is a linear space (or a vectors space) over the field F, if

(1) addition : There is a binary operation + on L called addition (not usual addition) such that (L , +) is a commutative group .

(2) scalar multiplication: $\propto \, . \, \mathcal{X} \in L \quad \forall \, \mathcal{X} \in L \, , \, \forall \propto \in F.$

(3) The scalar multiplication and addition satisfy

(i) $\propto . (\mathcal{X} + y) = \propto . \mathcal{X} + \propto . y \quad \forall \, \mathcal{X} , y \in L \quad , \forall \propto \in F$

(ii) $(\propto + \beta) . \mathcal{X} = \propto . \mathcal{X} + \beta . \mathcal{X} \quad \forall \, \mathcal{X} \in L \quad \forall \propto , \beta \in F$

(iii) $(\propto . \beta) . \mathcal{X} = \propto . (\beta . \mathcal{X}) \quad \forall \, \mathcal{X} \in L \quad \forall \propto , \beta \in F$

(iv) $1 . \mathcal{X} = \mathcal{X} \quad \forall \, \mathcal{X} \in L \quad and \; 1 \; is \; the \; unity \; F$

Example 2.2:[2]

The set of real number R, with Ordinary addition and ordinary multi–application, is a linear space over (F , + , .) = (R, + , .). indeed,

(1) (IR , +) is an abelian an group

(2) $\propto . \mathcal{X} \in IR \quad \forall \mathcal{X} \in IR \quad , \propto \in IR$

(3) All other condition are sat is field (check)

This linear space (IR , + , .) is called real linear space .

Example 2.3 : [1]

The set of complex numbers C, with ordinary addition and ordinary multiplication, is a linear space over (F , + , .) = (C , + , .). indeed,

(1) (C , + , .) is an abelian an group

(2) $\propto . \mathcal{X} \in C \quad \forall \mathcal{X} \in C \quad , \propto \in C$

(3) All other conditions are sat is field

This linear space (C , + , .) is called complex linear space .

Definition 2.4 (Linear subspace): [2]

Let L be a linear space over a field F and Let $\emptyset \neq H \subset L$. Then H is called a linear subspace of L if H itself is a linear space over F.

Theorem 2.5 : [3]

Let H be a nonempty subset of a linear space L (F) . H is called a subspace of L if and only if $\propto \mathcal{X} + \beta y \in H \quad for\ all \quad \mathcal{X}, y \in H \quad and\ for\ all \quad \propto, \beta \in F.$

Definition 2.6 (Linear Transformation Mapping): [4]

Let L(F) and L' (F) be two Linear spaces over the same field F. A mapping

T : L → L' is called a Linear operator or Linear Transformation if

$$T(\propto \mathcal{X} + \beta y) = \propto T(\mathcal{X}) + \beta T(y) \quad \forall \mathcal{X}, y \in L, \forall \mathcal{X}, \beta \in F$$

Example 2.7 : [1]

Let T: $IR^3 \rightarrow R^2$ defined by $T(X_1, X_2, X_3) = (X_1, X_2)$

$\forall \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in IR$

(1) show that T is a linear transformation .

(2) If $X = (x_1, x_2, x_3) = (2, 1, -3)$, $Y = (y_1, y_2, y_3) = (0, -5, 1)$. compute T (2X) and T (X + Y).

Solution (1) :-

Let $X = (x_1, x_2, x_3) \in IR^3, Y = (y_1, y_2, y_3) \in IR^3$ *and* $\propto, \beta \in R$ *.Then*

$$T(\propto X + \beta Y) = T[\propto (x_1, x_2, x_3) + \beta(y_1, y_2, y_3)]$$

$$= T[\propto x_1 + \beta y_1, \propto x_2 + \beta y_2, \propto x_3 + \beta y_3]$$

$$= (\propto x_1, \beta y_1, \propto x_2 + \beta y_2)$$

$$= (\propto x_1, \propto x_2) + (\beta y_1 + \beta y_2)$$

$$= \propto (x_1, x_2) + \beta(y_1, y_2)$$

$$= \propto T(x_1, x_2, x_3) + \beta T(y_1, y_2, y_3) = \propto T(X) + \beta T(Y).$$

Solution (2) :- T (2X) = T (4 , 2 , -6) = (4 , 2).

T (X + Y) = T (2 , -4 , -2) = (2 , -4) .

Definition 2.8 : [2]

Let L be a linear space . A linear transformation

$T : L \rightarrow F$ is said to be Linear functional .

(Note: that F can be regarded as a linear space over F).

Example 2.9 : [1]

Let $L = F^n = \{(x_1, \dots\dots.., x_2) : x_1, \dots\dots.., x_2 \in F\}$ be a linear space over the field F . Let T : $F^n \to F$ defined by $T(x_1, \dots\dots., x_n)$

$$=\propto_1 x_1 + \dots\dots.. + \propto_n x_n \; \forall (x_1, \dots\dots., x_n) \in F^n \; and \propto_1, \dots.., x_n \in F.$$

Prove that T is a linear trans formation.

Solution : Let X = (x_1 , …. , x_n) , y = (y_1 , ……. , y_n) $\in F^n$ *and* $\propto, \beta, \in, F$.

Then $T(\propto x + \beta y) = T\,[\propto (x_1, \dots\dots.., x_n) + \beta(y_1, \dots\dots.., y_n)]$

$$= T\,(\propto x_1 + \beta\, y_1, \dots\dots\dots, \propto x_n + \beta y_n)$$

$$= \propto_1 (\propto x_1 + \beta\, y_1) + \cdots + \propto_n (\propto x_n + \beta y_n)$$

$$=\propto (\propto_1 x_1 + \cdots + \propto_n x_n) + \beta(x_1 y_1 + \propto_n y_n)$$

$$= \propto T(x_1, \dots, x_n) + \beta\, T(y_1, \dots., y_n)$$

Thus , T is a linear transformation (Linear functional) .

Definition 2.10 (Normed Linear space): [2]

Let L(F) be a linear space over a field F . A mapping ‖ ‖ : L → IR is called norm if the following conditions hold

(1) $\|x\| \geq 0 \;\; \forall x \in L$. (positivity)

(2) $\|x\| = 0$ if and only if x = 0 .

(3) $\|x + y\| \leq \|x\| + \|y\| \; \forall\, x, y \in L$ (Triangle Inequality)

(4) $\|\propto x\| = |\propto|\, \|x\| \;\; \forall\, x \in L \;,\; \forall\, x \in F$.

(L , ‖ . ‖) is called normed Linear space .

Example 2.11 : [1]

Let L = IR be a linear space over IR with || . || : L → IR

Such that || x || = | x | . show that (R , || . ||) is a normed space .

Solution : we show that

(1) $\|x\| = |x| \geq 0 \ \forall x \in IR; \ hence \ \|x\| \geq 0$.

(2) Let $x \in IR$, $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0$.

(3) $\forall x \in IR , \forall x \in IR$.

$\|\propto x\| = |\propto x| = |\propto||x| = |\propto| \, \|x\|$.

(4) $\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\| \ \forall x , y \in IR$.

Example 2.12 : [2]

Let L = C be a complex Linear space over C with ||. || : C → IR

Such that || z || = | z | = $\sqrt{a^2 + b^2} \ \forall Z = a + ib$. show that (C , 11 11)

is a normed space.

Solution :- we show that

(1) $= |Z| = \sqrt{a^2 + b^2} \geq 0 \quad \forall Z = a + ib \in C \, ; hence \ \|Z\| \geq 0$.

(2) Let $z = a + ib \in C$

$$\|Z\| = |Z| = \sqrt{a^2 + b^2} = 0 \Leftrightarrow a = b = 0 \Leftrightarrow Z = o + io = o \, .$$

(3) Let z , w $\in C$

$\|z + w\|^2 \ (Z + W)(\overline{Z + W}) \ where \ \overline{Z + W} = conjugate \ of \ Z + W$

$$= (Z + W)(\overline{Z + W}) \ Z\bar{Z} + W\bar{W} + W\bar{Z} + \bar{W}Z$$

$$= Z\bar{Z} + W\bar{W} + W\bar{Z} + \overline{W\bar{Z}}$$

$$= Z\bar{Z} + W\bar{W} + 2Re\, W\bar{Z}$$

$\leq \|Z\|^2 + \|W\|^2 + 2\|W\|\|Z\| = (\|Z\| + \|W\|)^2$.

Thus , $\|Z + W\|^2 \leq (\|Z\| + \|W\|)^2, hence , \|Z + W\| \leq \|Z\| + \|W\|$.

(4) Let Z $\in C$, $\propto \in C$,

$\|\propto Z\| = |\propto Z| = |\propto (a + ib)|$

$= \sqrt{(\propto a)^2 + (\propto b)^2} = \sqrt{\propto^2 (a^2 + b^2)} = \sqrt{\propto^2} \sqrt{a^2 + b^2} = |\propto||Z| = |\propto|\|Z\|$.

Definition 2.13 : [3]

A sequence $< Un >$ in the normed space L is called convergent if $\exists\, 40 \in L$ **S.t** $\forall \in > 0 \; \exists\, K \in N : \|Un - 40\| < \in \; \forall\, n > K$.

Definition 2.14 : [4]

A sequence $< Un >$ in L is called a Cauchy Sequence if
$\forall \in > 0 \; \exists\, K (\in) \in N : \|Un - Um\| < \in \; \forall\, n, m > K$.

Definition 2.15 (Banach space): [4]

Let L be a normed space. Then, L is complete if every Cauchy sequence in L is convergent to a point in L. The complete normed space is called Banach space.

Example 2.16 : [1]

For $1 \leq P < \infty, We\ define\ the\ P-norm\ on\ IR^n\ (or\ C^n)$ by

$$\|(x_1, x_2, \dots\dots, x_n)\|_P = (|x_1|^P + |x_2|^P + \cdots |x_n|^P)^{1/P}$$

For P= ∞ We define the ∞, or maximum, norm by

$$\|(x_1, x_2, \dots, x_n)\|_\infty = \max \{|x_1|, |x_2|, \dots, |x_n|\}.$$

Then IR^n equipped with the P-norm is a finite – dimension al Banach space for $\exists \leq P \leq \infty$.

Example 2.17 : [2]

The space C ([a, b]) of continuous, real – valued functions on [a, b], with the sup – norm is a Banach space. More generally the space C (K) of continuous function on a compact metric space K equipped with the supnorm is a Banach space .

3. Inner product space

In mathematics, an inner product space (or, rarely a Hausdorff per-Hilbert space) is a real vector space or a complex vector space with an operation called an inner product. The inner product of two vectors in the space is a scatar, often denoted with angle brackets such as in $< a, b >$. Inner products allow formal definitions of intuitive geometric notions, such as lengths, angles, and orthogonally (Zero inner product) of vectors.

Inner product spaces generalize Euclidean vector spaces, in which the inner product is the dot product or scalar product of Cartesian coordinates. Inner product spaces of infinite dimension are widely used in functional analysis. Inner product spaces over the field of complex numbers are sometimes referred to as unitary spaces.

The first usage of the concept of a vector space with an inner product is due to Giuseppe peano in 1898.

Definition 3.1 : [1]

Let L is a linear space over F . A mapping $<., .>: L \times L \to F$ is called an inner production L if the following axioms hold

(1) $< x , x > \geq 0 \quad \forall x \in L$.

(2) $< x , x > = 0 \quad \Leftrightarrow x = 0$

(3) $\overline{< x,y >} = < y,x > \quad \forall x,y \in L \; where \; \overline{< x\,,y\; >} = conjugale \; of < y,x >$

(4) $< \propto x + \beta y > = \propto < x,Z > \; + \beta \; < y,Z > \quad \forall\, x\,,y\,,Z \; \in L\,.$

(L , < ., .>) is called inner product space or pre-Hilbert space.

Remark 3.2 :

(1) If F = IR then axiom (3) becomes $< x,y > = < y,x > \; \forall\, x\,,y\,, \in L$

(2) Every subspace of inner product space is an inner product space .

(3) $< u,\lambda V > = \; \bar{\lambda} < u,v > \; \forall \lambda \; \in F \; and \; u\,,v \; \in L$

Example 3.3 : [3]

Let L = IR^2 and Let $<.,.>: IR^2 \; X \; IR^2 \; \rightarrow F$ is defined as

$< X,Y > = x_1 y_1 + x_2\, y_2 \; \forall\, X,Y \; \in IR^2$ where $X = (x_1, x_2) \quad Y = (y_1, y_2)$.

Show that $<.,.>$ *is an inner product on* IR^2

Solution : (i) we check the I . P . S axioms

(1) $< X,X > = x_1^2 + x_2^2 \geq 0 \quad \forall\, X = \; (x_1\,,x_2) \in IR^2$

(2) $< X,X > = 0 \quad \Leftrightarrow \; x_1^2 + x_2^2 = 0 \;\; \Leftrightarrow \;\; x_1 = \; x_2 = 0 \; \Leftrightarrow \;\; X = (0\,,0)$

(3) $< X,Y > = x_1 y_1 + x_2 y_2 = < X\,,Y > \; (Since \; F = IR)$

(4) Let $\propto, \beta \; \in IR \; and \; Let \; X = \; (x_1, x_2)$, Y = $(y_1, y_2)\,, Z = (Z_1, Z_2)$

$< \propto X + \beta Y\,, Z > = \; < (\propto x_1 + \beta y_1\,, \propto x_2 \; + \beta y_2), (Z_1, Z_2) >$

$= (\propto x_1 + \beta y_1) Z_1\,, + (\propto x_2 \; + \beta y_2) Z_2$

$= (\propto x_1 Z_1 + \propto x_2 Z_2)\,, + (\; \beta y_1 Z_1 + \beta y_2 Z_2)$

$= \propto (x_1 Z_1 + x_2 Z_2)\,, + \beta(\, y_1 Z_1 + y_2 Z_2)$

$= \propto \langle X\,, Z \rangle\,, + \beta \langle Y, Z \rangle$

Thus , $\langle .,. \rangle$is an inner product on IR^2

Example 3.4 : [4]

Let L = F^n be a linear space and Let $\langle .,. \rangle : F^n \times F^n \to F$

Defined as $\langle X, Y \rangle = \sum_{i=1}^{n} x_i \, \bar{y} : \forall X, Y \in F^n$ *where*

$X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n)$.

Show that $\langle \, , \, \rangle$ is an inner product on F^n

Solution :-

(1) $\langle X, X \rangle = \sum_{i=1}^{n} x_i \, \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 \geq 0$

(2) $\langle X, X \rangle = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0 \iff x_i = 0 \;\forall\, i = 1, \dots, n$

$\iff X = (x_1, \dots, x_n) = (o, \dots, o) = O \, F^n$

(3) $\langle \overline{X, Y} \rangle = \sum_{i=1}^{n} x_i \, \bar{y}_i = \sum_{i=1}^{n} \bar{x}_i y_i = \sum_{i=1}^{n} y_i \bar{x}_i = \langle Y, X \rangle$

(4) Let $\propto, \beta \in F$ *and Let* $X, Y, Z \in F^n$

$\propto X + \beta Y, Z >= \sum_{i=1}^{n} (\propto x_i + \beta y_i) \, \bar{Z}_i = \propto \sum_{i=1}^{n} x_i z_i + \beta \sum_{i=1}^{n} y_i \bar{z}_i = \propto$
$\langle X, Z \rangle + \beta \langle Y, Z \rangle$.

Thus , $\langle ., . \rangle$ is an inner product on F^n

Example 3.5 : [3]

Let L = C [0, 1] be a linear space over IR , and Let $\langle .,. \rangle : L \times L \to IR$ is defined by $\langle f, g \rangle : \int_0^1 f(x) g(x)$ *prove that* $\langle \, , \, \rangle$ *is an* inner product L .

Solution :-

(1) $\langle f, f \rangle = \int_0^1 f(x) \, f(x) \; dx = \int_0^1 [f(x)]^2 \, dx \geq 0$

(2) $\langle f, f \rangle = 0 \iff \int_0^1 [f(x)]^2 \, dx = 0 \iff [f(x)]^2 = 0 \;\forall\, x \in [0, 1]$

$\iff f(x) = 0 \;\forall\, x \in [0, 1] \iff f = \hat{o}$

(3) Let $x, \beta \in IR$ *and* $f, g, h \in L$

$\langle \propto f + \beta g, h \rangle = \int_0^1 (\propto f + \beta g)(x) \, L(x) \; dx$

$= \int_0^1 (\propto f(x) + \beta g(x)\, L(x)\ dx$

$= \propto \int_0^1 f(x)\, L(x) dx + \beta \int_0^1 g(x)\ L(x)\ dx$

$= \propto \langle f , h \rangle + \beta \langle g , h \rangle$

(4) $\langle f , g \rangle = \int_0^1 f(x)\, g(x) dx = \int_0^1 g(x)\ f(x)\ dx = \langle f , g \rangle$

Theorem 3.6 (General Cauchy Schwarz's Inequality): [3]

Let $(L, \langle .,. \rangle)$ is an Inner product space and Let $\| \ \| : L \to IR$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}\ \forall\, x \in L$. Then,

$|\langle x, y \rangle| \le \|x\|\, \|y\| \quad \forall\, x\, y \in L$.

Proof : If x = 0 or y = 0 then $\langle x, y \rangle = 0$, $and\ hence\ \langle x, y \rangle = 0$

$\le \|x\|\, \|y\|\ \ If\ y \ne 0 , put\ Z = \frac{y}{\|y\|}$ (I)

$\|Z\|^2 = \langle Z , Z \rangle = \langle \frac{y}{\|y\|} , \frac{y}{\|y\|} \rangle = \frac{1}{\|y^{\cdot}\|} \langle y , y \rangle$

$\frac{1}{\|y\|^2} \|y\|^2 = 7$ (II)

Next , it is enough to show that $|\langle x, Z \rangle| \le \|x\|$

because if $|\langle x, Z \rangle| \le \|x\|\ then\ from\ (I)$

$|\langle x, Z \rangle| = \left| \langle x, \frac{y}{\|y\|} \rangle \right| = \frac{1}{\|y\|} = |\langle x, Z \rangle| \le \|x\|$

$|\langle x, Z \rangle| \le \|x\|\, \|y\|$

Let $\propto \in F$ then $\langle x - \propto z , x - \propto z \rangle \ge 0$

$\langle x - \propto z , x - \propto z \rangle \ge 0$

$\langle x , x \rangle - \propto \langle z , x \rangle - \overline{\propto} \langle x , z \rangle + \propto \overline{\propto} \langle z , z \rangle \ge 0$

$$\|x\|^2 - \bar{\propto}\langle x , z\rangle - \propto \langle z , x\rangle + \propto\bar{\propto}\ \|z\|^2 \geq 0$$
= 1 fram (I)

$$\|x\|^2 - \langle x , z\rangle \overline{\langle x , z\rangle} + \langle x , z\rangle \overline{\langle x , z\rangle} - \bar{\propto}\langle x , z\rangle - \propto \langle z , x\rangle + \propto\bar{\propto} \geq 0$$

$$\|x\|^2 - |\langle x , z\rangle|^2 + \langle x , z\rangle (\overline{\langle x , z\rangle} - \bar{\propto}) - \propto ((\langle z , x\rangle - \propto ((\langle z , x\rangle - \bar{\propto})) \geq 0$$

$$\|x\|^2 - |\langle x , z\rangle|^2 + (\langle x , z\rangle - \propto)(\overline{\langle z , x\rangle - x}) \geq 0$$

$$\|x\|^2 - |\langle x , z\rangle|^2 + |\langle x , z\rangle - \propto|^2 \geq 0 \ \forall \propto \in F \qquad \text{(III)}$$

Put $\propto \langle x , z\rangle$, then (III) becomes

$$\|x\|^2 - |\langle x , z\rangle|^2 \geq 0 \implies |\langle x , z\rangle|^2 \leq \|x\|^2$$

$$|\langle x , z\rangle| \leq \|x\|$$

$$\left|\langle x , \frac{y}{\|y\|}\rangle\right| \leq \|x\| \quad \text{(using (I))}$$

$$|\langle x , y\rangle| \frac{1}{\|y\|} \leq \|x\|$$

$$|\langle x , y\rangle| \leq \|x\| \, \|y\| .$$

Theorem 3 . 7 : [4]

Every inner product space is a normed space and hence a metric space.

Proof :-

Let $(L , \langle . \ , \ . \rangle)$ is an Inner product space and Let the function $\| \ \| : L \to IR$ is defined by $\| x \| = \sqrt{\langle x, x\rangle} \ \forall x \in L$ To prove $\| . \|$ is a norm on L

(1) Since $\langle x , x\rangle \geq 0 \ \forall x \in L \implies \| x \| = \sqrt{\langle x, x\rangle} \geq 0 \ \forall x \in L$

(2) $\| x \| = 0 \iff \sqrt{\langle x, x\rangle} = 0 \iff \langle x, x\rangle = 0 \iff x = 0 \ X$

(3) Let $\forall x \in F \ and \ x \in L$

$$\|\propto x\|^2 = \langle \propto x , \propto x\rangle = \propto\bar{\propto} \langle x , x\rangle |\propto|^2 \, \|x\|^2$$

Thus, $\|\propto x\| = |x| \ \|x\|$

(4) T. P. $\|x+y\| \le \|x\| + \|y\| \ \forall\, x, y \in L$

$\|x+y\|^2 = \langle x+y\,,\ x+y\rangle$

$= \langle x\,,\ x\rangle + \langle y\,, x\rangle + \langle x\,,\ y\rangle + \langle y\,, y\rangle$

$\|x\|^2 + \langle\overline{x\,,\ y}\rangle + \langle\, x\,, y\rangle + \|y\|^2$

$\|x\|^2 + 2Re\,\langle\, x\,, y\rangle + \|y\|^2$

$\le \|x\|^2 + 2\,|\,\langle x\,, y\rangle| + \|y\|^2$

$\le \|x\|^2 + 2\,\|\,x\|\ \|y\| + \|y\|^2$ (by Cauchy Schwarz)

$= (\|\,x\| + \|y\|)^2$

$Thus,\ \|x+y\| \ \le\ \|x\| + \|y\|$

Theorem 3 . 8 : [1]

Let $(L, \langle\ .,.\ \rangle)$ is an I. P. S. and $x, y \in L$. Then

(1) $\|x+y\|^2 = \|x\|^2 + 2\,Re\,\langle x\,, y\rangle + \|y\|^2$ (Polarization Identity)

(2) $\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$ (Law of parallelogram)

Proof :-

(1) $\|x+y\|^2 = \langle x+y\ ,\ x+y\rangle$

$\langle x, x\,\rangle + \langle y, x\,\rangle + \langle x, y\,\rangle + \langle y, y\,\rangle$

$= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2$

$= \|x\|^2 + 2\,Re\,\langle x, y\rangle + \|y\|^2$

(2) T. P. $\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$

By part (1), $\|x+y\|^2 = \|x\|^2 + 2\,Re\,\langle x, y\rangle + \|y\|^2$

$\|x-y\|^2 = \langle x-y\,,\ g-y\rangle$

$= \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle$

$= \|x\|^2 + \overline{\langle x, y\rangle} - \langle x, y\rangle + \|y\|^2$

$= \|x\|^2 - 2\, Re\, \langle x, y\rangle + \|y\|^2$ (II)

By Summing up (I) and (II) We get $\|x + y\|^2 + \|x - y\|^2$

$= 2\, \|x\|^2 + 2\, \|y\|^2$

Remark 3 . 9 :

Any normed Linear space generated from inner product space must satisfies the two Laws of Theorem 3.8

Definition 3. 10 Orthogonal Elements[3]

Let (L, ⟨. , .⟩) be an I. P. S. and x, y $\in L$. Then x is said to be orthogonal on y (denoted by x y) if and only if $< x , y \geq o$.

Example 3. 11[4]

Let $L = IR^2$ is I. P. S. such that $\langle X , Y\rangle = x_1\, y_1 + x_2\, y_2$ is usual inner product $\forall\, X = (x_1 , x_1) , Y = (y_2 , y_2) \in IR^2$

Let $X = (-6 , 3) , Y = (2 , -1) , Z = (1 , 2)$.

Show that $X \perp Z , Y \perp Z$ *and* $Y \not\perp X$.

Solution: $\langle X , Z\rangle = \langle(-6 , 3), (1 , 2)\rangle$

= -6 + 6 = 0 . Hence, $X \perp Z$.

4. Hilbert Space

Definition 4. 1 [5]

Hilbert space is an Inner product space $(L,\langle .\ ,\ .\rangle)$ which is a Banach space with respect to $\|x\| = \sqrt{\langle x\, ,x\rangle}$.

Example 4. 2[5]

Consider the I. P. S. $(\ IR^n\ ,\langle .\ ,\ .\rangle) or\ (\ C^n\ ,\langle .\ ,\ .\rangle)$ such that

$$\langle\ X,Y\rangle\ = \sum_{i=1}^{n} x_i\ \overline{y_i}\ \ where\ X = (x_1\ ,\dots,x_4), Y = (y_1\ ,\dots,y_4)$$

$$\in IR^n\ \ (or\ C^n).$$

Show that $(\ IR^n\ ,\langle .\ ,\ .\rangle) or\ (C^n\ ,\langle .\ ,\ .\rangle)$ is Hilbert space

Solution:

$$Since\ \sqrt{\langle X\ ,x\rangle}\ = \left[\sum_{i=1}^{n} x_i\ \overline{x_i}\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} |x_i|^2\ \right]^{\frac{1}{2}} = \|X\|$$

From Example , $IR^n(\ or\ C^n\ )$ is a Banach space w.r.t.

$\|X\| = \sqrt{\langle x\ ,x\rangle}$, and thus , $(\ IR^n\ ,\langle .\ ,\ .\rangle)(\ or\ C^n\ ,\langle .\ ,\ .\rangle)$ is a Hilbert space.

Example 4. 3[5]

The space $C[-1\ ,1]$ with the inner product defined by $\langle f,g\rangle$

$$= \int_{-1}^{1} f(x)\ g(x)dx\ \ \ is\ not\ a\ Hilbert\ space.$$

Solution: Let

$$f_n\ (\text{x}) = \begin{cases} 0 & \text{if} \quad -1 \le x \le 0 \\ \text{nx} & \text{if} \quad 0 < x < \frac{1}{n} \\ & \end{cases}$$

$1 \quad \text{if} \quad \frac{1}{4} \leq x \leq 1$

$\|f_n - f_m\|^2 = \langle f_n - f_m , f_n - f_m \rangle$

Suppose $n > m$, $then \frac{1}{n} < \frac{1}{m}$. We must find $f_n(x) - f_m(x)$

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

and

$$f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ mx & \text{if} \quad 0 < x < \frac{1}{m} \\ 1 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ 1 & \text{if} \quad 0 < x \leq 1 \end{cases}$$

Thus, f $\notin C\,[-1\,,1]$. $Then, \langle f_n \rangle$ is not convergent in $C\,[-1\,,1]$. $i.e.,$ The space is not Hilbert space.

Remark 4. 4 [5]

Every Hilbert space is a Banach space but the converse is not true. For example, the space $C\,[a\,,b]$ $with\ \|f\| = max\,\{|f(x)|: x \in [a,b]\}$

is Banach space. However, $C\,[a\,,b]$ is not a Hilbert space since it does not Satisfy parallel gram Law; that is $\|.\|$ cannot be obtained from inner product.

The Gram 4. 5 [5]

We define the projection operator by

$$proj_u\,(v) = \frac{\langle V,u\rangle}{\langle u,u\rangle}\,u\,,$$

Where $\langle V,\ u\rangle$ denotes the inner product of the vectors v and u . This operator projects the vector v orthogonally onto the line spanned by vector u. If u = o, we define $proj_o\,(v) := 0$ i. e. the projection map $proj_o$ is the zero map, sending every vector to the zero vector.

The Gram – Schmidt process then works as follows

$$u_1 = v_1\,, \qquad e_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = v_2 - proj_{u_1}\,(v_2)\,, \qquad e_2 = \frac{u_2}{\|u_2\|}$$

$$e_3 = \frac{u_3}{\|u_3\|}$$

$$u_3 = v_3 - proj_{u_1}\,(v_3) - proj_{u_2}\,(v_3)$$

$$u_4 = v_4 - proj_{u_1}\,(v_4) - proj_{u_2}\,(v_4) - proj_{u_3}\,(v_4),$$

$$e_4 = \frac{u_4}{\|u_4\|}$$

$$u_k = v_k - \sum_{0=1}^{k-1} proj_{u_1}\,(v_k), \qquad e_k = \frac{u_k}{\|u_k\|}\,.$$

The sequence $u_1 , \dots , u_k$ is the required system of orthogonal vectors, and the normalized vectors $e_1 , \dots , e_k$ from an orthonormal set. The calculation of the sequence $u_1 , \dots , u_k$ is know as, Gram- Schmidt orthogonalization.

While the calculation of the sequence $e_1, \ldots, e_k$ is know as Gram- Schmidt orthonormalization as the vectors are normalized.

Example 4. 6 [5]

Euclidean space

Consider the following set of vectors in R^2 (with the conventional inner product).

$$S = \left\{v_1 = \begin{bmatrix}3\\1\end{bmatrix}, v_2 = \begin{bmatrix}2\\2\end{bmatrix}\right\}.$$

Now, perform Gram- Schmidt, to obtain an orthogonal set of vectors.

$$u_1 = v_1 = \begin{bmatrix}3\\1\end{bmatrix}$$

$$u_1 = v_1 - proj_{u_1}(v_2) = \begin{bmatrix}2\\2\end{bmatrix} - proj_{\begin{bmatrix}3\\1\end{bmatrix}} \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} - \frac{8}{10} \begin{bmatrix}3\\1\end{bmatrix}$$

$$= \begin{bmatrix}-215\\615\end{bmatrix}.$$

We check the vectors u_1 and u_2 are indeed orthogonal

$$\langle u_1, u_2\rangle = \langle \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-215\\615\end{bmatrix}\rangle = -\frac{6}{5} + \frac{6}{5} = 0$$

Noting that if the dot product of two vectors is o then they are orthogonal.

For non- zero vectors, we can then normalize the vectors by dividing out their sizes as shown above.

$$e_1 = \frac{1}{\sqrt{10}} \begin{bmatrix}3\\1\end{bmatrix}$$

$$e_2 = \frac{1}{\sqrt{\frac{40}{25}}} \begin{bmatrix}-215\\615\end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix}-1\\3\end{bmatrix}.$$

Theorem 4. 7 [5]

Let V be an inner product space and V_0 be a finite- dimensional subspace of V. Then any vector $x \in V$ is uniquely represented as $x = p + o$, where $P \in V_o \; and \; o \perp V_o$

The component P is the orthogonal projection of the vector x onto the subspace V_o . The distance from x to the subspace V_o is $\|o\|$.

If $v_1 , v_2 \ldots , v_n$ is an orthogonal basis for V_o then.

$$P = \frac{\langle x , v_1 \rangle}{\langle v_1 , v_2 \rangle} v_1 + \frac{\langle x , v_2 \rangle}{\langle v_2 , v_2 \rangle} v_2 + \cdots + \frac{\langle x , v_n \rangle}{\langle v_n , v_n \rangle} v_n .$$

Theorem 4. 8 [4] [The projection theorem]

Let δ CH be a Hilbert subspace and Let $x \in H$. Then

1. There exists a unique element $x^2 \in \delta$ (called the projection of x onto δ) such that.

   $$\|x - x^2\| = \inf_{y \in \delta} \|x - y\|$$

   Where $\| . \|$ is the norm generated by the inner product associated with H.
2. x^2 is (uniquely) characterized by

   $(x - x^2) \in \delta^{\perp}$

Proof:

In order to prove part 1 we being by noting that δ .

Since it is a Hilbert subspace, is both complete and convex.

Now fix $x \in H$ and define

$$d = \inf_{y \in \delta} \|x - y\|^2$$

Clearly d exists since the set of squared norms $\|x - y\|^2$ is a set of real numbers bounded below by o. Now since d is the greatest lower bound of $\|x - y\|^2$ there exists a sequence $(y_k)_{k=1}^{\infty}$ from δ such that, for each $\in > 0$, there exists an N_ε such that

$$\|x - y_k\|^2 \le d + \varepsilon$$

For all $K \ge N_\varepsilon$. we now want to show that any such sequence (y_k) is a Cauchy sequence. For that purpose, define

$$u = x - y_m$$

$$v = x - y_n$$

Now applying the parallelogram identity to u and v , we get.

$$\|2x - y_m - y_n\|^2 + \|y_n - y_m\|^2 \; 2(\|x - y_m\|^2 + \|x - y_n\|^2)$$

Which may be manipulated to become.

$$4\left\|x - \frac{1}{2}(y_m - y_n)\right\|^2 + \|y_n - y_m\|^2$$

$$= 2(\|x - y_m\|^2 + \|x - y_n\|^2)$$

Now since δ is convex, $\frac{1}{2}(y_m + y_n) \in \delta$ and consequently

$\left\|x - \frac{1}{2}(y_m + y_n)\right\|^2 \ge d$. It follows that

$$\|y_m - y_n\|^2 \le 2(\|x - y_m\|^2 + \|x - y_n\|^2) - 4d$$

Now consider any $\varepsilon > o$, choose a corresponding N_ε such that $\|x - y_k\|^2 \le d + \varepsilon\, 14\, for\ all\ K \ge N_\varepsilon$ (such an N_ε exists we have seen).

Then, for all n , m $\ge N_\varepsilon$, *we have*

$$\|y_m - y_n\|^2 \leq 2(\|x - y_m\|^2 + \|x - y_n\|^2) - 4d \leq \varepsilon$$

Hence (y_k) is a Cauchy sequence. By the completeness of δ, It converges to some element $\hat{x} \in \delta$. By the continuity of the inner product,

$\|x - \hat{x}\|^2 = d$. Hence $\hat{x}$ is the projection we seek. To show that $\hat{x}$ is unique, consider another projection $y \in \delta$ and the sequence $(\hat{x}, y, \hat{x}, y, \hat{x}, y, \ldots)$. By the argument above, this is a Cauchy sequence. But then $\hat{x} = y$. Hence (1) is proved. The proof of part (2) comes in two parts.

First we show that any $\hat{x}$ that satisfies also satisfies.

Suppose, then, that $\hat{x}$ satisfies. Define $w = x - \hat{x}$ and consider an element $y = \hat{x} + \propto z$ where $Z \in \delta$ and $\propto \in IR$. Since δ is a vector space, it follows that $y \in \delta$. Now since $\hat{x}$ satisfies, y is no closer to x then $\hat{x}$ is. Hence

$$\|w\|^2 \leq \|w - \propto z\|^2 = (w - \propto z, w - \propto z) =$$
$$= \|w\|^2 + \propto^2 \|z\|^2 - 2 \propto (\varepsilon, z)$$

Simplifying, we get

$$0 \leq \propto^2 = \|z\|^2 - 2 \propto (w, z)$$

This is true for a u scalars $\propto$. In particular, set

$\propto = (\varepsilon, z)$. We get $\quad 0 \leq (w, z)^2 (\|z\|^2 - 2)$

For this to be true for all $z \in \delta$ we must have (w , z) = o

For all $z \in \delta$ such that $\|z\|^2 < 2$. But then (why 2) we must have (w, z) = o for all $z \in \delta$. Hence $w \in \delta^{\perp}$. Now we went to prove the converse , i. e. that if $\hat{x}$ satisfies, then it also satisfies. Thus consider an element $\hat{x} \in \delta$ which and Let $y \in \delta$. Mechanical calculation reveal that.

$$\|x - y\|^2 = (x - \hat{x} + \hat{x} - y, x - \hat{x} + \hat{x} - y) =$$

$$= \|x - \hat{x}\|^2 + \|\hat{x} - y\|^2 + (x - \hat{x}\,, \hat{x} - y) =$$

Now since $(x - \hat{x}\,) \in \delta^{\perp}$ and $(\hat{x} - y) \in \delta$ (recall that δ is recta space), the last term disappears , and our minimization problem becomes (disregarding the con-stant term $\|x - \hat{x}\|^2$)

$$\min_{y \in \delta} \|\hat{x} - y\|$$

Clearly $\hat{x}$ solves this problem (Nate that it doesn't matter for the solution whether we minimize a norm or its square) Indeed. Since $\|\hat{x} - y\| = 0$ implies $\hat{x} = y$ we may conclude that: f Same $\hat{x}$, then it is the unique solution.

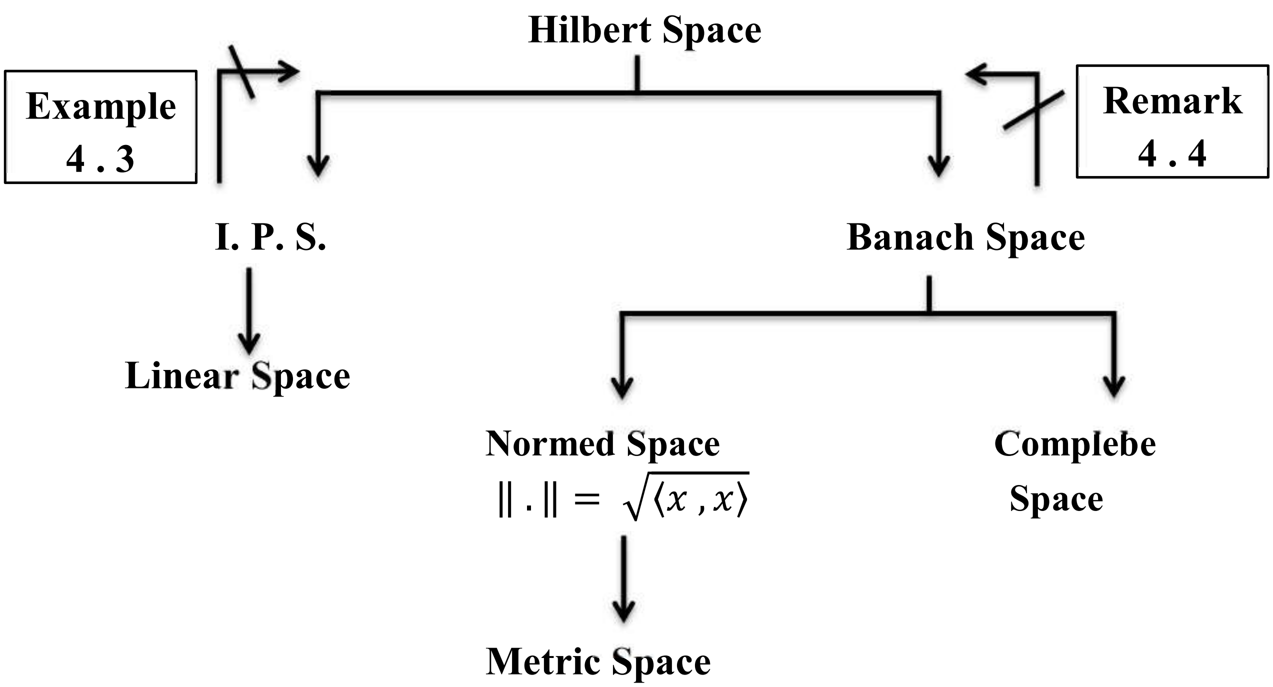

Figure 1: The relationship between Hilbert's space and other spaces.

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