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Dirichlet: The Modern Architect of the Function Concept and Boundary Value Problems

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

﴿ وَمَا أُوتِيتُم مِّنَ الْعِلْمِ إِلاَّ قَلِيلاً ﴾

صدق الله العلي العظيم

سورة الاسراء (٨٥)

(الإهداء)

إلى حُجَّةِ اللهِ فِي أَرْضِهِ، وعَيْنَهُ فِي خَلْقِهِ، نُورَ اللهِ الَّذِي يَهْتَدِي بِهِ المهُتَدونَ مَوْلايَ صاحِبَ الزَّمان،

إلى إمام زَماني الحُجَّة ابن الحَسن (عَجَلَ الله فَرَجَهُ) .

إلى صاحب السيرة العطرة، والفكر المُستنير، فلقد كان لهُ الفَضل الأوَّل في بلوغي التعليم العالي

(والدي الحبيب) أطال الله في عمره .

إلى من وضعتني على طريق الحياة ، إلى القلب الحنون التي كانت دعواتها تحيطني

(امي الغالية) أطال الله في عمرها

إلى إخوتي ؛ من كان لهم بالغ الأثر في كثير من العقبات والصعاب .

إلى جميع أساتذتي الكرام ؛ ممن لم يتوانوا في مد يد العون لي

أُهدي لكم بحثي ....

والله ولي التوفيق

الشكر والتقدير :

الحمد لله أولاً وآخراً، وظاهراً وباطناً، حمد الشاكرين على توفيقه وتسديده، فما كان من توفيق في هذا العمل فمنه وحده، وما كان من تقصير فمن نفسي.

ثم أرفع آيات الشكر والعرفان، بقلبٍ يملؤه الرجاء، إلى مقام صاحب العصر والزمان، الحجة بن الحسن (عجل الله فرجه الشريف)، الذي ببركة وجوده تُنال الألطاف، وبأنواره تُستضاء دروب العلم والمعرفة، سائلاً المولى القدير أن يتقبل مني هذا القليل، ويجعله خطوةً في طريق مرضاته وخدمة دينه.

أتوجه بالشكر الجزيل للأستاذة أ.د. ازل جعفر موسى التي رافقتني في مسيرتي لإنجاز هذا البحث وكانت لها بصمات واضحة من خلال توجيهاتها وانتقاداتها البناءة والدعم الأكاديمي ،

كما اشكر عائلتي التي صبرت وتحملت معي ورفدتني بالكثير من الدعم ، واشكر الاصدقاء والأحباب وكل من قدم لي الدعم المادي أو المعنوي

وأخيراً اتوجه بشكر خاص لأساتذتي جميعاً لمساعدتهم لي في فهم الأمور التقنية المتعلقة بالبحث

(Abstract)

Peter Gustav Lejeune Dirichlet is regarded as the true architect who constructed the bridges of communication between mathematical abstraction and physical reality. His journey began with an early passion in Paris, enabling him to assimilate the genius of French analysis and develop it with unique German rigor, eventually becoming the legitimate heir to Gauss's mathematical throne at Göttingen and a beacon of knowledge for a generation of geniuses such as Riemann and Dedekind. The value of his fundamental contributions is manifested in his being the first to liberate the concept of a function from its narrow algebraic constraints, granting it a comprehensive logical definition based on the association between sets. He also laid the foundation for analytic number theory by using analytical tools to decode the mysteries of integers and proved his historic theorem on primes in arithmetic progressions. His brilliance did not stop there; he established the strict laws that governed Fourier series, rescuing them from scientific randomness by relying on the logic of Dirichlet boundary conditions, which today represent the core of mathematical physics. The significance of this problem lies in its incredible ability to predict what occurs deep within natural systems by

understanding the boundaries that surround them. Its practical applications emerge in the finest details of our contemporary lives, ranging from calculating stable heat distribution in objects and determining electrical potential in conductors to modeling complex fluid flows and studying Earth's gravitational fields, even extending to modern technologies like digital image processing and restoration. His legacy represents the fulcrum from which mathematics pivoted toward absolute abstraction, especially after proposing models for functions that challenged visual intuition, forcing scientists to move from reliance on geometric sketching to rigorous analytical proof. This shift paved the way for the emergence of advanced measurement and integration theories. On the physical side, his philosophy regarding boundary conditions was not merely a tool for solving equations but a profound vision linking system stability to minimal energy—a principle from which cosmologists drew their models for understanding gravity and spacetime. His genius is also evident in his employment of simple logic, such as the Pigeonhole Principle, to solve the most complex dilemmas, proving that his thought combined ultimate simplicity with unquestionable depth. Accordingly, researching Dirichlet’s work is an exploration of the primary roots from which the tree of contemporary applied sciences and engineering has been nourished. Every computer simulation of pressure, temperature distribution, or even data flow in modern networks owes its existence to the strict rules established by this extraordinary scientist, ensuring his contributions remain vibrant at the heart of every technological innovation based on the language of mathematics., confirming that Dirichlet's genius lies in transforming mathematics from an art of calculation into a science of rigorous logical structures."

Differential equations are the fundamental tool and the primary language used by scientists and engineers to describe natural and physical phenomena, ranging from planetary motion to heat distribution and fluid dynamics. However, solving these equations remains incomplete and lacks practical value without defining "Boundary Conditions," which govern the problem and determine the correct path for the solution.

Among the most significant and established of these are the Dirichlet and Neumann conditions. While Dirichlet conditions focus on specifying the value of the function itself at the boundaries, Neumann conditions concern the rate of change of the function (the derivative) along those boundaries. With the massive advancements in applied mathematics and the emergence of complex problems in modern physics and engineering, classical formulas alone are no longer sufficient. This has necessitated the development of "modern" perspectives and modified conditions tailored to the nature of partial differential equations, fractional calculus, and beyond.

This research aims to clarify the fundamental distinction between these two types of conditions, highlighting modern approaches in their application to solving differential equations. Throughout the following chapters, we will examine how the choice of boundary conditions affects the stability and accuracy of the solution, and identify the scenarios where modern Neumann conditions are preferred over traditional Dirichlet conditions—and vice versa—to ensure a mathematical model that best simulates realit

(Contents)

Chapter One

Biography

Chapter One: Biography

1.1 Family Background and school Education

Figure (1.1) Peter Gustav Lejeune Dirichlet

Johann Peter Gustav Lejeune Dirichlet, to give him his full name, was born in D¨uren (approximately halfway between Cologne and Aachen (= Aix-la-Chapelle)) on February 13, 1805

Lejeune Dirichlet Dirichlet first attended an elementary school and when this became insufficient, a private school. There he also got instruction in Latin as a preparation for the secondary school (Gymnasium), where the study of the ancient languages constituted an essential part of the training. Dirichlet's inclination for mathematics became apparent very early. He was not yet 12 years of age when he used his pocket money to buy books on mathematics, and when he was told that he could not understand them, he responded, anyhow that he would read them until he understood them

Dirichlet showed a special interest in mathematics and history, in particular in the then recent history following the French Revolution. It may be assumed that Dirichlet's later free and liberal political views can be traced back to these early studies and to his later stay in the house of General Foy in Paris

Dirichlet had mathematics lessons with Georg Simon Ohm (1789–1854), well known for his discovery of Ohm’s Law (1826); after him the unit of electric resistance got its name.[5]

1.2 Study in Paris

Dirichlet’s parents still had friendly relations with some families in Paris since the time of the French rule, and they let their son go to Paris in May 1822 to study mathematics. Dirichlet studied at the Coll`ege de France and at the Facult´e des Sciences, where he attended lectures of noted professors such as S.F. Lacroix (1765–1843), J.-B. Biot (1774–1862), J.N.P. Hachette (1769–1834), and L.B. Francœur (1773–1849). He also asked for permission to attend lectures as a guest student at the famous.

There is no doubt that the study of Gauß’ magnum opus left a lasting impression on Dirichlet which was of no less importance than the impression left by his courses. We know that Dirichlet studied the Disquisitiones arithmeticae several times during his lifetime, and we may safely assume that he was the first German mathematician who fully mastered this unique .[17]

1.3 Transfer to Berlin and Marriage

From the very beginning, Dirichlet also had applied for permission to give lectures at the University of Berlin, and in 1831 he was formally transferred to the philosophical faculty of the University of Berlin with the further duty to teach at the Military School. There were, however, strange formal oddities about his legal status.[10]

Figure (1.2) Rebecka Mendelssohn

Dirichlet took an interest in Rebecka, Rebecka Henriette Lejeune Dirichlet (née Rebecka Mendelssohn; 11 April 1811 – 1 December 1858) was a granddaughter of Moses Mendelssohn and the youngest sister of Felix Mendelssohn and Fanny Mendelssohn and although she had many suitors, she decided for Dirichlet. Lackmann ([Lac]) characterizes

Rebecka as the linguistically most gifted, wittiest, and politically most receptive of the four children. She experienced the radical changes

during the first half of the nineteeth century more consciously and critically than her siblings.[9] These traits are clearly discernible also from her letters quoted by her nephew Sebastian Hensel ([H.1], [H.2]). The engagement to Dirichlet took place in November 1831. After the wedding in May 1832, the young married couple moved into a flat in the parental house, Leipziger Str, began the golden age of mathematics in Berlin.[15] [11]

1.4 Dirichlet in Göttingen

When Gauß died on February 23,1855, the University of Göttingeunanimously wanted to win Dirichlet as his successor. Given the difficulties faced in Berlin, he decided to accept the offer and immediately moved to Göttingen with his family. Kummer was called to assume his position as a professor of mathematics in Berlin,[3]

Dirichlet died on May 5, 1859, one day earlier than his faithful friend Alexander von Humboldt, who died on May 6, 1859, in his

90th year of life. The tomb of Rebecka and Gustav Lejeune Dirichlet in Göttingen still exists and will soon be in good condition again, when the 2006 restorative work is[6]

Figure (1.3) Grave of Dirichlet in Göttingen

Chapter Two

Significant Contributions

2.1 The modern concept of function

The Dirichlet function (English: Dirichlet function) is the characteristic function of the set of rational number Q defined on the set of real numbers R, usually denoted as D(x). This function is named after the German mathematician Dirichlet While trying to gauge the range of functions for which convergence of the Fourier series can be shown, Dirichlet defines a function by the property that "to any x there corresponds a single finite y", but then restricts his attention to piecewise continuous functions. Based on this, he is credited with introducing the modern concept of a function, as opposed to the older vague understanding of a function as an analytic formula.[1]

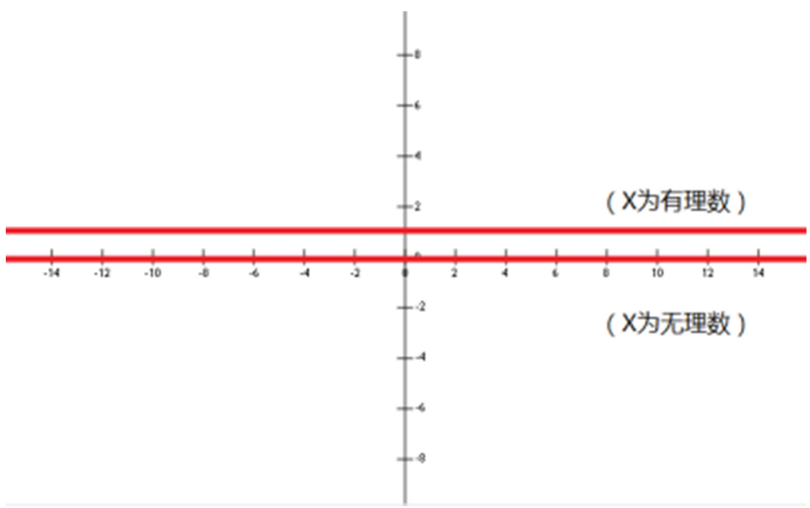

Figure (2.1) Dirichlet function

(The Dirichlet Function Table) :

Input value (x)	Number Type	The Output The Function	Logical Justification
X=0.5	Rational Number	1	Because 0.5=$\frac{1}{2}$
$x = \pi$	Irrational Number	0	Non-periodic mathematical constant
$x = \sqrt{2}$	Irrational Number	0	Roots of deaf Numbers
$x = 4$	Rational Number	1	An Intger

2.2 Pigeonhole Principle

The pigeonhole principle is also called the Dirichlet drawer principle, after the nineteenthcentury German mathematician Dirichlet, who often used this principle in his work. The pigeonhole principle can be used to prove a useful corollary about functions.

The pigeonhole principle If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects

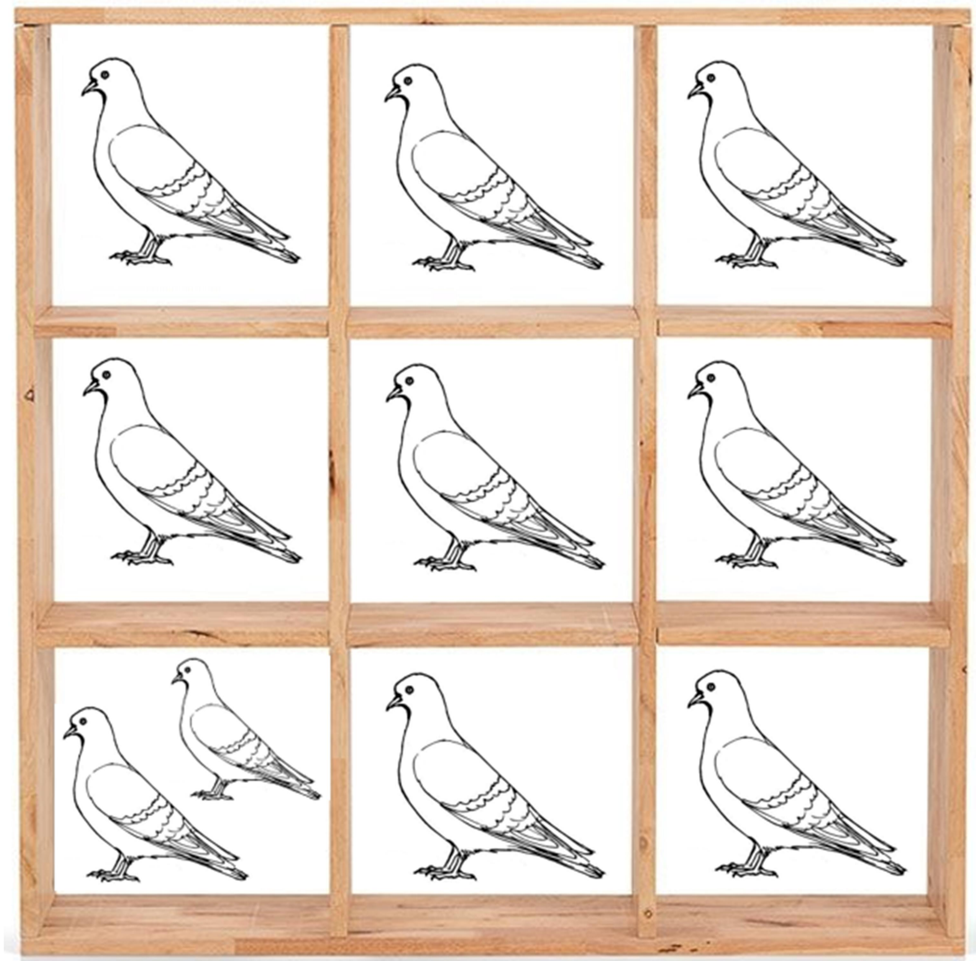

Figure (2.2) There Are More Pigeons Than Pigeonholes

2.3 We quote a few highlights of Dirichlet's œuvre showing him at the peak of his creative power

2.3.1 A. Fourier Series :

Dirichlet was the first mathematician to prove rigorously for a fairly wide class of functions that such an expansion is possible. His justly famous memoir on this topic is entitled Sur la convergence des s´eries trigonom´etriques qui servent `a repr´esenter une fonction arbitraire entre des limites donn´ees , He points out in this work that some restriction on the behaviour of the function in question is necessary for a positive solution to the problem, since, e.g. the notion of integral "ne signifie quelque chose" for the (Dirichlet) function,[8]

$$f(x) = \begin{cases} 1 & for \quad x \in Q, \\ 0 & for \quad x \in R\backslash Q, \end{cases}$$

2.3.2 Dirichlet's Theorem on Primes in Arithmetical Progressions :

richlet's mastery in the application of analysis to number theory manifests itself most impressively in his proof of the theorem on an infinitude of primes in any arithmetic progression of the form (a+km)k≥1, where **a** and **m** are coprime natural numbers.[8]

2.3.3 Dirichlet's Class Number Formula :

Dirichlet first sketched his results on this topic and on the mean value of certain arithmetic functions in 1838 in an article in Crelle's Journal and elaborated on the matter in full detail in a very long memoir of 1839–1840, likewise in Crelle's Journal.

Following Gauß, Dirichlet considered quadratic forms

$$ax^2 + 2bxy + cy^2$$

2.3.4 Dirichlet's Unit Theorem

An algebraic integer is, by definition, a zero of a monic polynomial with integral coefficients. This concept was introduced by Dirichlet in a letter to Liouville , but his notion of what Hilbert later called the ring of algebraic integers in a number field remained somewhat imperfect, since for an algebraic integer ϑ he considered only the set Z[ϑ] as the ring of integers of Q(ϑ). Notwithstanding this minor imperfection, he succeeded in determining the structure of the unit group of this ring in his poineering memoir Zur Theorie der complexen Einheiten (On the theory of complex units.[8]

2.3.5 Dirichlet's Principle

We pass over Dirichlet's valuable work on definite integrals and on mathematical physics in silence , but cannot neglect mentioning the so-called Dirichlet Principle, since it played a very important role in the history of analysis . Dirichlet's Problem concerns the following problem: Given a (say, bounded) domain $G \subset R$ 3 and a continuous real-valued function f on the (say, smooth) boundary ∂G of G

find a real-valued continuous function u, defined on the closure G of G, such that u is twice continuously differentiable on G and satisfies Laplace's equation[8]

$$\Delta u = 0 \quad on\ G$$

Chapter Three

Applied Applications

3.1 The Dirichlet Problem in Two Dimensions

In this section we briefly discuss the solution of elliptic boundary value problems in two dimensional bounded domains, The most common boundary condition is to specify the value of the function on the boundary; this type of constraint is called a Dirichlet1 boundary condition. For example, if we specify Dirichlet boundary conditions for the interval domain [a, b], then we must give the unknown at the endpoints a and b; this problem is then called a Dirichlet BVP. In two dimensions we have to specify the boundary values along the entire boundary curve and in on the entire boundary surface.[14]

3.1.1 Dirichlet Boundary Value Problem

for many applications is makes sense to study to corresponding steady state (time independent) problem. The solution be- ing time independent means its time derivative is zero. This leads to the so-called Laplace equation and the Dirichlet problem.[6]

$$\Delta u(x) = 0 \quad , x \in \Omega$$

$$\Delta u(x) = F(x) \quad , x \in \partial\Omega$$

This is a very important problem in mathematics and engineering and is one of the most studied problems in theory and practice. Often in R 2 and R 3 we use the variables x, y and z instead of x1, x2 and x3 and we write x = (x, y) or x = (x, y, z). So with this notation the Dirichlet problem for Laplace's equation becomes:

$$\Delta u(x) = \frac{\partial^2 u}{\partial \mathrm{x}^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad in \;\; \Omega$$

$$u(x) = F(x) \quad x \in \partial\Omega$$

When the region is a rectangle R = [0, a] × [0, b], the boundary conditions will be given on each edge separately as:[14]

$$\boldsymbol{u(x,0) = f_1(x)} \qquad \boldsymbol{u(x,b) = f_2(x)\,,} \qquad \boldsymbol{0 < x < a}$$

$$\mathbf{u(a,y) = g_2(y)} \qquad \mathbf{u(0,y) = g_1(y)} \qquad \boldsymbol{0 < y < b}$$

3.2 Applications in Machine Learning

In the realm of machine learning and statistical modeling, the Dirichlet distribution plays a crucial role in probabilistic modeling and Bayesian inference. It serves as a distribution over distributions, making it particularly useful in scenarios where uncertainty exists in categorical data. From topic modeling to clustering applications, the Dirichlet distribution is a foundational concept that every data scientist should understand. In this blog, we will explore the Dirichlet distribution, its mathematical formulation, key properties, and real-world applications in machine learning.[11]

3.2.1 Key Properties of the Dirichlet Distribution:

1. **Simplex Constraint**: The Dirichlet distribution ensures that the sum of probabilities always equals 1, making it suitable for modeling categorical distributions.
2. **Concentration Parameters**: The values of α\alpha control the shape of the distribution. Higher values of α\alpha lead to more uniform distributions, while lower values result in more sparsity.
3. **Conjugate Prior**: The Dirichlet distribution is the conjugate prior for the multinomial distribution, making it useful in Bayesian inference.

3.2.2 Applications in Machine Learning

1. Topic Modeling (Latent Dirichlet Allocation — LDA)

One of the most well-known applications of the Dirichlet distribution is in **Latent Dirichlet Allocation (LDA)**, a popular topic modeling technique. In LDA, documents are represented as mixtures of topics, and topics are distributions over words. The Dirichlet distribution acts as a prior for topic distributions, ensuring sparsity and diversity in topic assignments.[7]

2. Clustering and Mixture Models

Dirichlet distributions are widely used in **Dirichlet Process Mixture Models (DPMM)**, which are non-parametric Bayesian models for clustering. Unlike traditional clustering methods, DPMMs allow for an infinite number of clusters, making them highly flexible for unsupervised learning tasks.[7]

3. Bayesian Optimization and Hyperparameter Tuning

In Bayesian optimization, the Dirichlet distribution can be used to model uncertain categorical parameters, helping in efficient hyperparameter tuning of machine learning models.[7]

4. Genetic and Biological Data Analysis

The Dirichlet distribution is also used in bioinformatics for modeling DNA sequence variation, population genetics, and evolutionary data analysis.[7]

Conclusion

The Dirichlet distribution is a fundamental concept in Bayesian inference and probabilistic modeling. Its applications range from topic modeling and clustering to hyperparameter tuning and genetic data analysis. By mastering this concept, data scientists can enhance their ability to work with probabilistic models efficiently.

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