جمهورية العراق
وزارة التعليم العالي و البحث العلمي
جامعة بـــابل
كلية التربيه للـــعلوم الصرفه
قسم الــرياضيات

The Brachistochrone

مشروع مقدم الـــى :
مجلس كلية التربية للعلـــوم الصرفة ـ جامعة بابل
وهو جزء من متطلبات نيل درجة البكالوريوس في الرياضيات

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

(يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ ۚ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ)

(١١) سورة المجادلة

صدق الله العلي العظيم

شكر وتقدير

لابد لنا ونحن نخطوا خطوتنا الأخيرة في الحياة الجامعية من وقفه نعود الى أعوام قضيناها في رحاب الجامعة مع اساتذتنا الكرام الذين قدموا لنا الكثير باذلين جهودا كبيره في بناء جيل الغد لتبعث الامة من جديد

ثم أتوجه بجزيل الشكر وعظيم الامتنان الى **(أ.د. ازل جعفر)** على ما بذلته جهد لغرض الاشراف على بحثي ومتابعتها لي بآرائها القيمة ومساعدتها لي بعمليتها فجزاها الله خير الجزاء

وقبل ان نمضي نقدم آيات الشكر والامتنان والتقدير والمحبة الى الذين حملوا أقدس رسالة في الحياة الى الذين مهدوا لنا طريق العلم والمعرفة ... الى جميع اساتذتها الافاضل

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The Brachistochrone

Abstract

The aim of this investigation is to determine a step-by-step solution to the Brachistochrone Problem, showing that Calculus of Variations is required to solve the problem, and that the quickest path between two points facilitated by gravity is the path of a cycloid. Ultimately, this investigation will show that differential equations and curricular knowledge in the field of Calculus (particularly the Euler-Lagrange concept) can be employed to solve problems with real world applications.

Chapter one: Introduction

Imagine coming home from work one cold winter evening and finding in your mailbox a peculiar challenge from one of your classmates. It states: Find the quickest path between two points in space. Ha! You sneer at the utter simplicity of the problem before you, insulted that your classmate,a respected intellectual, would waste your time with this no-brainer. "It's a straight line of course!" You shout with an expression of blind confidence, which quickly disappears as a small post-it note falls out from the envelope.

The Brachistochrone problem, the simplest of problems in Calculus of Variations, was first posed by mathematician Johann Bernoulli in his 1696 Acta Eruditorium as a direct challenge to all European mathematicians. Privately, Isaac Newton was the first to receive the challenge in his mailbox as he came home one day from working at the Royal Mint. Using Fermat's Theorem of Least Time, Newton solved the problem in a night, concluding using differential equations that the quickest path between two points in space was indeed a cycloid curve and not a straight line.

You stand speechless, baffled by this seemingly incoherent and strange concept which threatened to expose the inherent flaw in your reasoning, and the flaw in your inherent being. Where did you go wrong? Was your logic incoherent? How could there have been any other answer but the obvious, a straight line!? You dart back into the house in frustration, determined to search up the full solution online. You find the exact statement of Bernoulli's problem in *Acta Eruditorum*:

1.1 Fermat's Theorem of Least Time

Given two points A and B in the vertical plane, with A not lower than B, what is the curve traced out by a point ac ted on only by gravity, which starts at A and reaches B in the shortest time. [1]

Below it is a reference to Fermat's Theorem of Least Time:

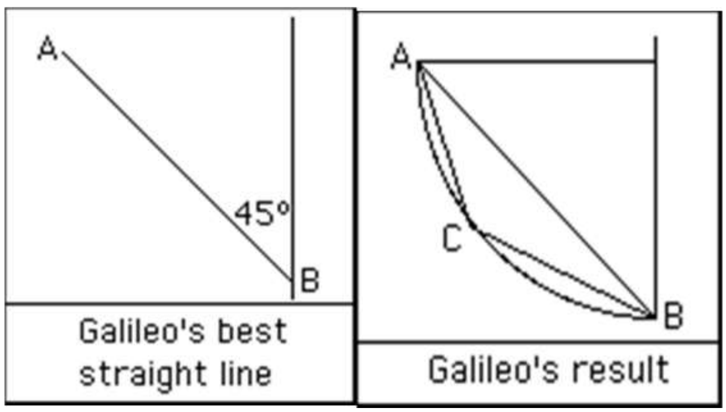
Fermat's principle or the principle of least time, named after French mathematician Pierre de Fermat, is the principle that the path taken between two points by a ray of light is the path that can be traversed in the least time [2]

In other words, you discover that the first step to answering the Brachistochrone problem is to define the "quickest" path between two points A and B as the path which takes the shortest time, and not the path with the least distance because, under the influence of gravitational forces, objects in space must change velocity. Sounds reasonable enough! But you still refuse to admit defeat because you still don't have a clear picture of how a curve can best a straight line. Well as an analogy, in the 21st century most people use a GPS to help them find the quickest route to their destination, but often the GPS doesn't take into account that shorter routes may have a slower speed limit and therefore takes longer time to travel. The fastest route practically is often a highway, which is a much longer road, but takes shorter time to travel due to the faster speed limit. Now, applying this to travel on a two-dimensional plane by gravitational force only, the concept is the same. For over 300 years, the common misconception surrounding this classic problem in mathematics was that the shortest path was a straight line. However, as mathematics must ultimately shift from the theoretical to the practical, the straight-line solution does not work whengravity is taken into consideration.

1.2 The Five Mathematicians

As early as Galileo Galilei, the Brachistochrone problem has been studied and debated. Galileo'sversion of the problem was finding the straight line from a point A to a point B in the vertical linein which it would reach the quickest. To do this he calculated the time taken for the point to move from A to B in a straight line. He then proved that the point would reach B quicker if it firsttraveled to C then to B, where C was a point on a circle. [3]

Figure 1: Galileo's solution [5]



Although Galileo was correct in assuming that a circular arc corresponds to a faster travel, he had not quite arrived at the correct solution, the Brachistochrone curve. It wasn't until 58 years later when Johann Bernoulli formally posed the problem that five correct solutions were determined, each using a different method. The five solutions came from Johann Bernoulli himself, his brother and rival Jacob Bernoulli, his companion L'Hopital, his teacher Gottfried Leibniz, and finally, Isaac Newton. Upon publishing the problem, Johann and Leibniz tempted Newton by saying:

"There are fewer who are likely to solve our excellent problems, aye, fewer even among the very mathematicians who boast that [they]... have wonderfully extended its bounds by means of the golden theorems which (they thought) were known to no one, but which in fact had long previously been published by others." [1]

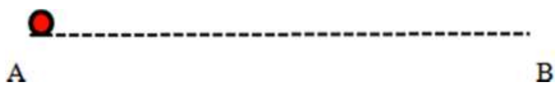
When Newton solved the question in one night and sent his solution to Johann Bernoulli without a sender name, Bernoulli said "Ahh... I recognize the lion by his paw." [4].

Eventually, the development of *Calculus of Variations*, a version of Calculus created by Leonard Euler specifically to define the terms of the Brachistochrone problem, combined elements in each of the five solutions to formulate the more modern solution we now recognize as the *Euler- Lagrange* method. This investigation will focus only on the method *Euler-Lagrange* as it is arguably the most powerful solution we have today.

1.3 Brainstorming

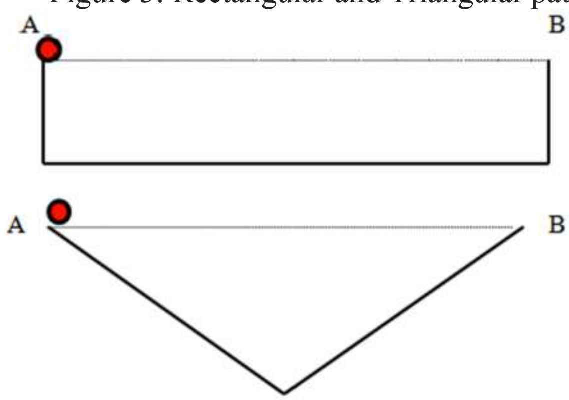
Before we delve into the complexities of Calculus of Variations, it is important to view the Brachistochrone problem as an intuitive concept. Suppose we want to find the quickest path fromA to B when they are both at ground level. Clearly, a rolling object such as a bead will take an infinitely large amount of time to get from A to B. In other words, it will never make the trip because according to the law of conservation of energy, the bead needs to change heights in orderto gain energy. We can quickly see that a straight line cannot be the solution.

Figure 2: Straight Line from A to B [5]



Considering a rectangular or triangular path,

Figure 3: Rectangular and Triangular paths



These wouldn't work either despite satisfying the law of conservation of energy because they bothrequire that the bead change direction instantaneously. In the rectangular path, the object must make a 90 degrees' direction change as it descends and ascends again, while in the triangular path,the object must make a sharp turn at the bottom of the triangle. The only way for the object to travel the full path and change directions instantaneously in both paths is if there is an infinite amount of energy applied to the object. Therefore, it can be concluded that the quickest path cannot contain angled points, or kinks and corners in the A-B interval. In terms of Calculus, any paths with the potential to be a Brachistochrone must be continuous and differentiable between A and B. But there are infinitely many different differentiable paths between A and B so how do we choose the best one? Well first we have to define any path from A to B (A is not lower than B keep in mind) in terms of an equation.

If the distance of any type of path between points A and B is *d*, then the velocity of any object along that path would be:

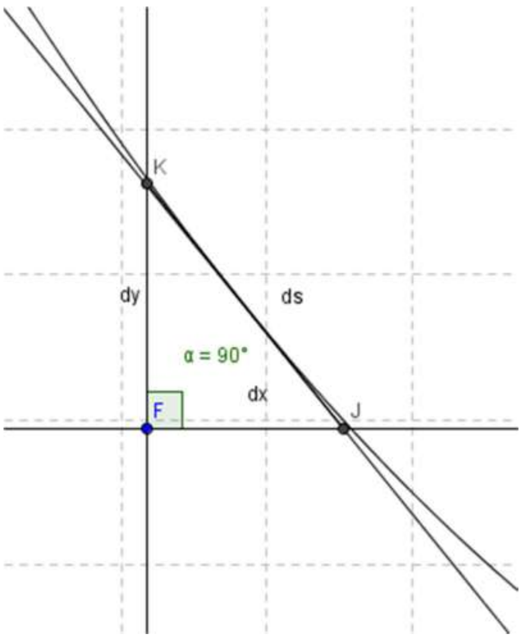
$$v = \frac{ds}{dt}\text{, } where\ t = \int_a^b \frac{ds}{v}$$

According to the Law of Conservation of Energy, the total mechanical energy of any objecttravelling from A to B must be constant:

$$E_p = E_k$$
$$\frac{1}{2}mv^2 = mgy$$
$$v = \sqrt{2gy}$$

According to the Pythagorean Theorem,

Figure 4: Sample Curve of Any path from A to B, graphed on *MacBook Pro* Grapher



$$ds^2 = dy^2 + dx^2$$
$$ds = \sqrt{dy^2 + dx^2}$$

Dividing all terms by dx,

$$\frac{ds}{dx} = \frac{\sqrt{dy^2 + dx^2}}{\sqrt{dx^2}}$$
$$\frac{ds}{dx} = \frac{\sqrt{dy^2 + dx^2}}{\sqrt{dx^2}}$$
$$ds \quad \sqrt{\acute{y}^2 + 1dx}$$

If we look at this equation closely, we see that it resembles the formula for the arc length of a graph, which is what we want because essentially, the arc length formula can be used to determinethe length of any path from A to B in a 2-D plane:

$$s = \int \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

Hence, substituting $ds = \sqrt{\ y'^2 + 1} dx$ and $v = \sqrt{2gy}$ into the equation,

$$t = \int_a^b \frac{\sqrt{y'^2 + 1}}{\sqrt{2gy}} dx = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{y'^2 + 1}{y}} dx$$

Already, this equation shows that for any path between A and B, defined by the graph of y, the

time taken to travel the path varies by $\sqrt{\frac{y'^2+1}{2gy}}$. Unlike a normal function where the dependent variables x and y vary, this time the whole function varies. In short, this is known as a "functional" group and is an integral part to the method using Calculus of Variations.

Chapter two :Calculus of Variations

2.1 The Euler-Lagrange Equation

Calculus of Variations was first used by Euler, who collaborated with J. L. Lagrange to generalizeNewton's solution to the Brachistochrone problem. It is defined as using "calculus to find the maxima and minima of a function which depends for its values on another function or a curve." [3]. Generally, we can solve for the minimum time taken to travel from point A to B t= $\int_a^b \sqrt{\frac{y'^2+1}{2gy}}$ by minimizing the function $f(x) = \sqrt{\frac{y\prime^2+1}{y}}$

To minimize $f(x)$, we will use the fundamental *Euler-Lagrange Equation* [5]:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$$

In this case, F is the function we are trying to minimize.

Note: In *Calculus of Variations*, ∂ denotes the partial derivative of a function (basically, you areonly taking the derivative of the desired variable, or *part* of the function).

Applying the equation F=f$(x) = \sqrt{\frac{\acute{y}^2+1}{y}}$ to the Euler-Lagrange equation and using conventional calculus rules of derivation

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

LHS:

$$\boldsymbol{\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{\frac{y'^2+1}{y}}\right)}$$

$$\boldsymbol{= \sqrt{y'^2+1}\,\frac{\partial}{\partial y}\left(\frac{-\frac{1}{2}}{y}\right)}$$

$$\boldsymbol{= \sqrt{y'^2+1}\left(-\frac{1}{2}y^{-\frac{3}{2}}\right)}$$

$$\boldsymbol{\therefore \frac{\partial F}{\partial y} = -\frac{1}{2}\left(\sqrt{\frac{y'^2+1}{y^{\frac{2}{3}}}}\right)}$$

RHS:

To evaluate $\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)$, we will first evaluate the function F in terms of y', and then derive it in terms of x.

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y'}\left(\sqrt{\frac{y'^2+1}{y}}\right)$$

$$= \frac{1}{\sqrt{y}}\frac{\partial}{\partial y'}\left(1+y'^2\right)^{1/2}$$

$$= \frac{1}{\sqrt{y}}\frac{1}{2}\left(1+y'^2\right)^{-\frac{1}{2}}2y'$$

$$\therefore \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{y\left(1+y'^2\right)}}$$

Taking the derivative of $\frac{\partial F}{\partial y}$ with respect to x,

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\frac{y\prime}{\sqrt{y(1+y\prime^2)}}$

Using substitution to simplify the process, we let a=y, b=y' and c=y''. Hence,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\frac{b}{\sqrt{a(1+b^2)}}$$

Now, we must use the quotient rule to differentiate. $let\ s$(x)=b , and let v (x)=$\sqrt{a(1+b^2)}$. Hence, s'(x)=c and using the product rule,

$$v'(x) = \left(\frac{1}{2}a - \frac{1}{2}b\right)\left(\sqrt{1+b^2}\right) + \left(\sqrt{a}\right)\left(\frac{1}{2}(1+b^2)^{-\frac{1}{2}}2bc\right)$$

$$= \frac{bc\ \sqrt{a}\left(2\sqrt{a}\right) + (b)\left(\sqrt{1+b^2}\right)(\sqrt{1+b^2})}{2\sqrt{a(1+b^2)}}$$

$$v'(x) = \frac{2abc + b(1+b^2)}{2\sqrt{a(1+b^2)}}$$

Applying the quotient rule,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{c\sqrt{a(1+b^2)} - (b\frac{2abc + b(1+b^2)}{2\sqrt{a(1+b^2)}})}{a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}}\frac{2a(1+b^2)c - 2ab^2c - b^2(1+b^2)}{2a(1+b^2)}$$

$$= \frac{1}{\sqrt{a(1+b^2)}}\left(\frac{c}{1+b^2} - \frac{b^2}{2a}\right)$$

$$\therefore \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{1}{\sqrt{y\left(1+y'^2\right)}}(\frac{y''}{1+y'^2} - \frac{y'^2}{2y})$$

LHS=RHS:

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$-\frac{1}{2}\left(\sqrt{\frac{y'^2+1}{y'^{\frac{3}{2}}}}\right) = \frac{1}{\sqrt{y(1+y'^2)}}\left(\frac{y''}{1+y'^2} - \frac{y'^2}{2y}\right)$$

Using the same substitution, a=y, b=y' and c=y'',

$$-\frac{1}{2}\left(\sqrt{\frac{b^2+1}{a^{\frac{3}{2}}}}\right) = \frac{1}{\sqrt{a(1+b^2)}}\left(\frac{c}{1+b^2} - \frac{b^2}{2a}\right)$$

$$\left(\sqrt{a(1+b^2)}\right)\left(-\frac{1}{2}\sqrt{\frac{b^2+1}{a\frac{3}{2}}}\right) = \frac{c}{1+b^2} - \frac{b^2}{2a}$$

$$\left(\sqrt{a(1+b^2)}\right)\left(-\frac{1}{2}\sqrt{\frac{b^2+1}{a\frac{3}{2}}}\right) + \frac{b^2}{2a} = \frac{c}{1+b^2}$$

$$\frac{c}{1+b^2} = \frac{-(1+b^2)}{2a} + \frac{b^2}{2a}$$

$$\frac{c}{1+b^2} = \frac{-1}{2a}$$

$$2ac = -1 - b^2$$

$$\therefore 2y * y'' + 1 + y'^2 = 0$$

The proceeding step after we have reached this equation is where the difficulty arises. As with all higher order differential equations, they are extremely difficult to solve, however, we can still simplify this equation to a second-order differential equation. Here was Euler's method:

If we multiply the equation by y', then:

$$2y * y' * y'' + y' + y'^3 = 0$$

Essentially, if we work backwards from the product rule, we find that:

$$\frac{d}{dx}(y + y * y'^2) = 2y * y' * y'' + y' + y'^3$$

Therefore, if we integrate the original equation,

$$\int 2y * y' * y'' + y' + y'^3 \, dx = \int 0 \, dx$$

$$\int \frac{d}{dx}(y + y * y'^2) = c$$

$$y + y * y'^2 = c$$

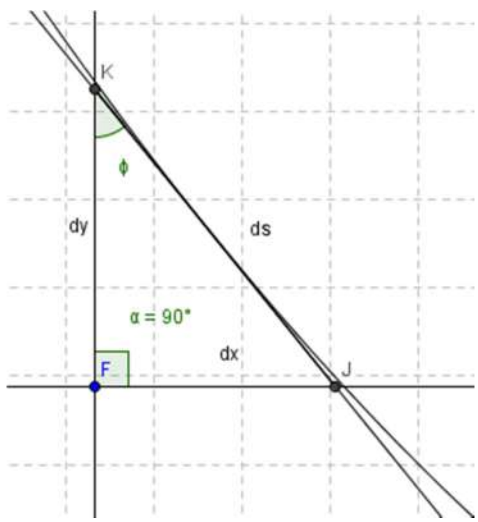
Solving for y’,

$$y' = \sqrt{\frac{c-y}{y}}$$

$$\frac{dy}{dx} = \sqrt{\frac{c-y}{y}}$$

Now that we have expressed y’ in terms of y, which must remember is the equation of the path between points A and B, we have to somehow substitute y’ out of the equation. This can be done using parametric equations, since we can’t express y in terms of y’, we have to express it in terms of another variable. In this case, that variable is the angle of descent of an object.

Figure 5: Adding a parameter, ∅ graphed on Macbook Pro Grapher



Since this graph is the generalization for any curve that establishes a path between points A andB, we can define our answer in terms of these parameters, specifically the angle ∅.

Therefore,

$$\frac{dy}{dx} = \sqrt{\frac{c-y}{y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{c-y}}$$

$$\tan\emptyset == \sqrt{\frac{y}{c-y}}$$

Converting this to parametric form,

$$(\frac{\sin\emptyset}{\cos\emptyset})^2 = \frac{y}{c-y}$$

$$\tan\emptyset == \sqrt{\frac{y}{c-y}}$$

Converting this to parametric form,

$$(\frac{\sin\emptyset}{\cos\emptyset})^2 = \frac{y}{c-y}$$

$$(c-y)(sin\emptyset)^2 = y(cos\emptyset)^2$$

$$(c)(sin\emptyset)^2 = y[(cos\emptyset)^2 + (sin\emptyset)^2]$$

$$(c)(sin\emptyset)^2 = y$$

Simplifying $(sin\emptyset)^2 = \frac{1-\cos 2\emptyset}{0}$,

$$y = \frac{c(1-\cos 2\emptyset)}{2}$$

Solving for the x-coordinate using the Chain Rule,

$$\frac{dx}{d\emptyset} = \frac{dx}{dy} * \frac{dy}{d\emptyset}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{c-\acute{y}}} \quad \frac{dy}{d\emptyset} = c * \sin 2\emptyset = c * 2\sin\emptyset\cos\emptyset$$

$$\frac{dx}{d\emptyset} = \sqrt{\frac{y}{c-y}}(c * 2\sin\emptyset\cos\emptyset)$$

Substituting in $y = (c)(sin\emptyset)^2$,

$$\frac{dx}{\emptyset} = \sqrt{\frac{(c)(\sin\emptyset)^2}{c-(c)(\sin\emptyset)^2}}(c * 2\sin\emptyset\cos\emptyset)$$

$$\frac{dx}{d\emptyset} = \sqrt{\frac{(\sin\emptyset)^2}{1 - (\sin\emptyset)^2}}(c * 2\sin\emptyset\cos\emptyset)$$

$$\frac{dx}{d\emptyset} = (2c)(\sin\emptyset)^2$$

$$\frac{dx}{d\emptyset} = c(1 - \cos 2\emptyset)$$

$$\int dx = \int c(1 - \cos 2\emptyset)d\theta$$

$$x = c(\emptyset - \frac{1}{2} sin2\emptyset)$$

$$x = \frac{c}{2}(2\emptyset - \sin 2\emptyset)$$

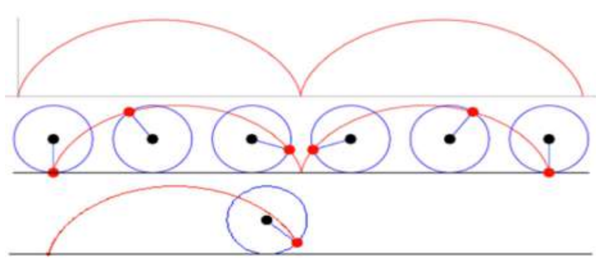
In summary, the two parametric equations that define the path of least time are:

1. $x = \frac{2}{c}(2\emptyset - \sin 2\emptyset)$
2. $y = \frac{c(1-\cos 2\emptyset)}{2}$

The exciting discovery of these two parametric equations surprised the five mathematicians because they resemble the very well-known equations of a cycloid, which is a curve that is created by drawing out the path of a point on the circumference of a circle rolling on a flat surface. What was amazing about the cycloid was that it was the solution to the Brachistochrone problem, the solution that Galileo came close to when he proposed that a circular arc path would yield the fastest travel time. A cycloid is not a circle, but it is definitely a Brachistochrone curve, and the following calculations will prove why.

2.2 The equation of a cycloid

Figure 6: The cycloid [6]



So far, using Calculus of Variations, we have shown that in the process of minimizing the time of travel between two points, we have come across the two parametric equations of a cycloid:

1. $x = \frac{2}{c}(2\emptyset - \sin 2\emptyset)$

2. $y = \frac{c(1-\cos\ \emptyset)}{2}$

But how do we know these equations define a cycloid? Well, if we consider that a cycloid is created by rotating a circle along a flat surface, then we know that the point on the circumference of the circle we are tracing will be moving clockwise. Knowing that the co-ordinates of a circle are defined by x = $\mathrm{r}\cos\emptyset$, and$y = \mathrm{r}\sin\emptyset$, then moving clockwise means that the co-ordinates change to $x = -r\sin\emptyset$ and$y = r\cos\emptyset$. Furthermore, because the circle that draws out the cycloid moves in the positive direction along the x-axis, this yields the equation that allows t to represent the angle the circle has moved:

$$\Delta x = 2\pi r \frac{\emptyset}{2\pi}$$

$$x = -r\sin\emptyset + r\emptyset$$

$$x = r(\emptyset - \sin\emptyset)$$

As for the y-coordinate, we must assume that the center of the circle is at (r , r) so that the bottom of the cycloid is on the x-axis. Hence:

$$y = r - r\cos\emptyset$$

$$y = r(1 - \cos\emptyset)$$

Now, we have proven that the equations we derived earlier are indeed equations for the cycloid. As we can see, the equations have the exact same form, with$r = \frac{c}{2}$. In other words, C is equal to the diameter of the circle which drew the cycloid.

Chapter three: Application

3.1 Solving for time of travel

So, much like how Bernoulli, Newton, Lagrange, and Euler discovered each in their own way that the solution to the Brachistochrone problem is indeed a cycloid, we have through this investigation shown that minimizing the time taken to travel from point A to B facilitated only by gravity will result in the cycloid equations. Intuitively, we also understand that in the real world, if an object were to roll down a straight ramp, it would require more time than if it rolled down a cycloid ramp. The only logical way to proceed would be to solve for the time that it actually takes for an object to travel down such a ramp in order to gain some quantifiable data.

So, if we recall the initial investigation of time of travel, we have the equation:

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{y'^2 + 1}{y}} dx$$

Since we now know that the path must be a cycloid, we can use the differential equation for a cycloid we determined earlier to solve for time with respect to the parameters of the cycloid equation. We know:

$$y' = \frac{dy}{dx} = \sqrt{\frac{c - y}{y}}$$

And since C=2r from our exploration on cycloids,

$$\frac{dy}{dx} = \sqrt{\frac{2r - y}{y}}$$

$$\frac{dy}{\sqrt{\frac{2r - y}{y}}} = dx$$

Hence, substituting the cycloid differential equation into our initial equation for time,

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{y'^2 + 1}{y}} dx$$

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{\frac{2r-y}{y}+1}{y}} * \frac{dy}{\sqrt{\frac{2r-y}{y}}}$$

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{\frac{2r-y+y}{y}}{y\left(\frac{2r-y}{y}\right)}} dy$$

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{2g}{y(2r-y)}} dy$$

Rearranging the equation,

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{-2r}{[\,(r-y)^2 - r^2]}} dy$$

Substituting in $y = r(1 - \cos \emptyset)$ to the denominator $[(r-y)^2 - r^2]$,

$$\begin{aligned}[(r-y)^2 - r^2] &= (r - r + r\sin\emptyset)^2 - r^2 \\ &= (rcos\emptyset)^2 - r^2 \\ &= r^2(cos^2\emptyset - 1) \\ &= -r^2 sin^2\emptyset\end{aligned}$$

Simplifying the integral,

$$t = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{2r}{r^2 sin^2\emptyset}} dy$$

Now, in order to integrate in terms of ∅, we must substitute dy with an equation in terms of d∅:

Since $y = r(1 - \cos \emptyset)$,

$$\frac{dy}{d\emptyset} = r\sin\emptyset$$

$$dy = rsin\emptyset\ d\emptyset$$

Substituting $dy = rsin\emptyset\ d\emptyset$,

$$t = \frac{1}{2g} \int_a^b \sqrt{\frac{2r}{r^2 sin^2\emptyset}} r\sin\emptyset\ d\emptyset$$

$$= \frac{1}{\sqrt{2g}} \int_a^b \sqrt{2r}\ d\emptyset$$

$$t = \sqrt{\frac{r}{g}} * \emptyset$$

Since r and ∅ are both variables for the parametric equations of the cycloid, we can now calculate the shortest time of travel between points A and B. We have officially solved the Brachistochrone problem as we now have an equation to represent the shortest time.

The Brachistochrone curve equation can be used to demonstrate a variety of properties. First and foremost, we can show that the time required to travel down a Brachistochrone curve is indeed faster than the time required to travel down a straight ramp. If we set points A (0,0) and B (1, -1) in 2-D space as the start and end points of travel (note the y coordinate of B must be lower or equal to the y-coordinate of A in order to make a ramp), then we can draw a cycloid connecting the points with the following equation:

Using systems of equations and the parametric equations of a cycloid,

Since point B (1 , -1)

Y: 1= $r(1-\cos\emptyset)$
X: -1= $r(\emptyset-\sin\emptyset)$

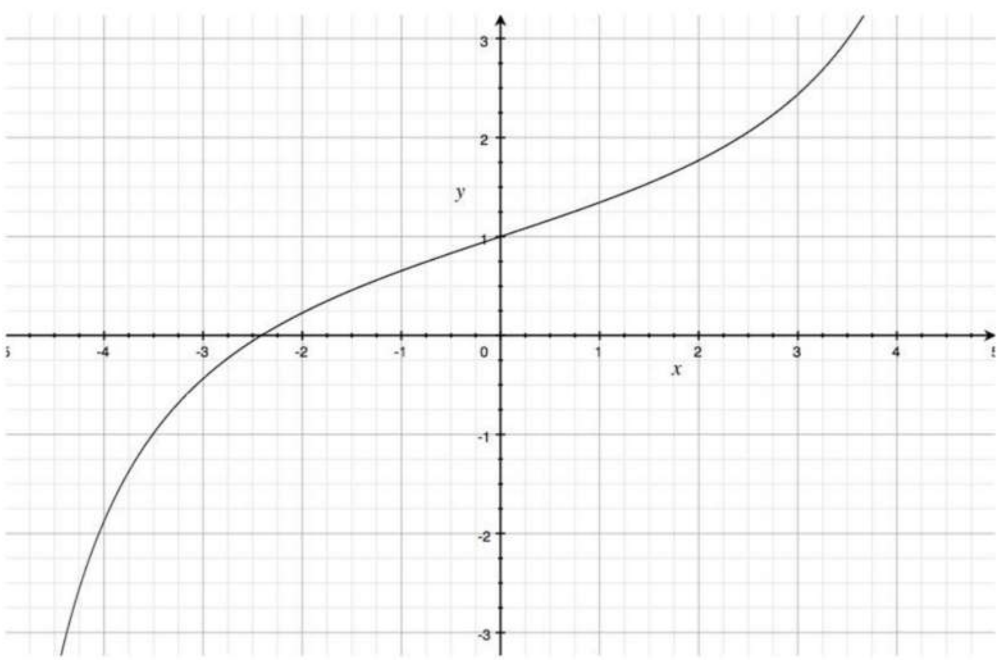
Hence, Y: $r=\frac{1}{(1-\cos\emptyset)}$

Substituting this into X: -1= $r(\emptyset-\sin\emptyset)$,

-1= $\frac{1}{(1-\cos\emptyset)}(\emptyset-\sin\emptyset)$

$$Graphing: \frac{1}{(1-\cos\emptyset)}(\emptyset-\sin\emptyset)+1=0$$

Figure 7: Graphed using *Macbook Pro Grpher*



From analyzing this graph, it is clear that the x-intercept occurs at $\emptyset\approx-2.4$ in radians. Hence, solving for r,

$$r = \frac{1}{(1 - \cos(-2.4))} \approx 0.58\, units$$

Using r=0.58 units and $\emptyset \approx -2.4$ rad to solve the time of travel (considering that the gravitational acceleration of any object on the earth is 9.81m*s^{-2}),

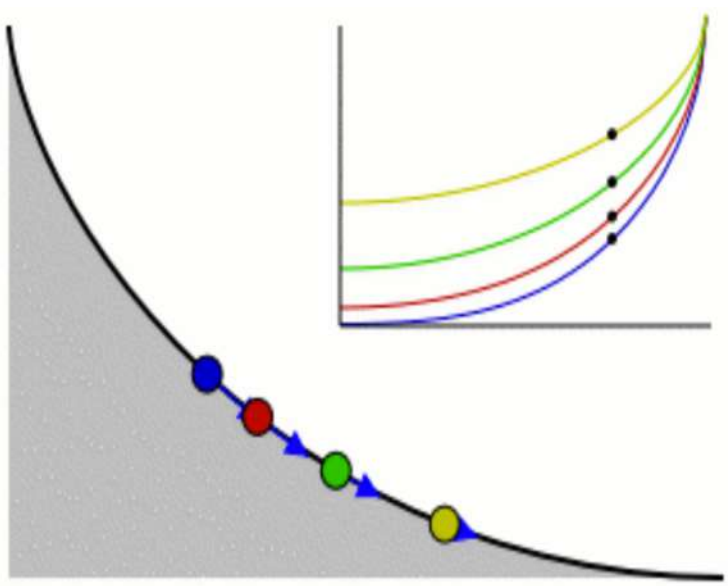
$$t = \sqrt{\frac{r}{g}} * \emptyset = \sqrt{\frac{0.58}{9.81}} * -2.4 = 0.5836\, s \approx 0.5.8\, s$$

Generally, this is a much shorter time than if an object were to slide down a ramp from point A (0,0) to B (1, -1).

3.2 Real World Applications

There are many real-world applications to the Brachistochrone curve, for example, an extension of the curve is the concept of the Tautochrone curve or Isochrone curve which is basically a Brachistochrone curve that demonstrates the Tautochrone property. The Tautochrone property is satisfied when objects rolling down a cycloidal ramp regardless of where they start, take the same amount of time to reach the bottom of the ramp. This makes every point on the cycloid ramp isochronous, hence the name Isochrone.

Figure 8: Tautochrone Property [6]



As you can see, the balls on this ramp, regardless of where they started rolling from, will reach the bottom at exactly the same time. With the Tautochrone property, the real world applications could include: building creative roller coasters, industrial applications where mechanical operations must be timed perfectly, or in the field of physics and motion. The one largest limitation to the investigation of Brachistochrone curves is that they only work when gravity is the only force pushing an object down a cycloid ramp, which rarely is ever the case unless you are conducting the experiment in an isolated setting such as a laboratory. On a large scale, where air resistance, friction, and other parasitic forces hinder the experiment, the accuracy of the ideal model may be subject to error.

Conclusion

Conclusion

The marvel of mathematics is how a simple problem can lead to complex and often unfathomable concepts and solutions. The Brachistochrone curve, what started out as a simple challenge to deduce the shortest path between two objects in space facilitated by gravity, turned into a whole field of mathematics called Calculus of Variations. Overall, we can conclude that it is the inquisitive curiosity and insatiable hunger that scientists and mathematicians have for defining and discerning the world that allows for such elegant and sophisticated evolutions.

Through this investigation in which I have employed the very fundamental, yet crucial skills of calculus, I have by extension and not by purpose, inadvertently gained the knowledge on how to solve partial differentiation equations and how to represent equations in parametric form. And that is the desire to explore independently. All brilliant mathematicians have done it their own way, and this is just my first step into that same realm of discovery.

The journey to discovery does not end here, however, because throughout this investigation, I have jarringly come to realize my own shortcomings and mathematical flaws such as forgetting to square root or changing a positive sign to a negative sign. These all cost me days of frustration and irritation.

In the future, I will stand by the principle that the best math's is explored independently, but the occasional challenge wouldn't hurt.

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