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جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

Figen bam number

بحث مقدم الى جامعة بابل كلية التربية للعلوم الصرفة

وهو جزء من متطلبات الحصول على درجة البكالوريوس

اعداد الطالب

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اشراف

د. افتخار الشرع

2023 - A.D. **1443 -A.H**

بسم الله الرحمن الرحيم

فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ ۗ وَلَا تَعْجَلْ بِالْقُرْآنِ
مِن قَبْلِ أَن يُقْضَىٰ إِلَيْكَ وَحْيُهُ ۖ وَقُل رَّبِّ زِدْنِي
عِلْمًا

صدق الله العلي العظيم

طه ١١٤

أهداء

إلى من كلله الله بالهيبة والوقار.. إلى من علمني العطاء بدون انتظار.إلى من أحمل أسمه بكل افتخار..... والدي العزيز

إلى ملاكي في الحياة.. إلى معنى الحب وإلى معنى الحنان والتفاني.. إلى بسمتي في

الحياة وسر الوجود إلى من كان دعائها سر نجاحي وحنانها بلسم جراحي إلى أغلى

الحبايب... أمي الحبيبة

إلى من هم اقرب أليّ من روحي... إلى من شاركوني حضن الأم وبهم استمد

عزتي واصراري إخوتي وأخواتي

إلى الشموع التي احترقت من أجل أن تنير لنا الطريق، إلى من شجعني ووقف بجانبي

حتى نهاية الطريق... أساتذتي تقديرا ووفاء.

شكر وتقدير

لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية من وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذلك جهودا كبيرة في بناء جيل الغد لتبعث الأمة من جديد

إلى وقبل أن نمضي تقدم أسمى آيات الشكر والامتنان والتقدير والمحبة إلى الذين حملوا أقدس رسالة في الحياة الذين مهدوا لنا طريق العلم والمعرفة ......إلى جميع أساتذتنا الأفاضل

كما وأخص بالشكر الدكتورة افتخار لوقوفها معي طيلة فترة انجاز البحث ولما قدمت لي من نصائح قيمة آثرت ببحثي حتى خرج بهذا الشكل

الباحث

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Chapter one

introduction

Mitchell Feigenbaum:

Feigenbaum was born in Philadelphia, Pennsylvania, to Jewish emigrants from Poland and Ukraine. He attended Samuel J. Tilden High School, in Brooklyn, New York, and the City College of New York. In 1964 he began his graduate studies at the Massachusetts Institute of Technology (MIT). Enrolling for graduate study in electrical engineering, he changed his area to physics. He completed his doctorate in 1970 for a thesis on dispersion relations, under the supervision of Professor Francis E. Low. After short positions at Cornell University (1970–1972) and the Virginia Polytechnic Institute and State University (1972–1974), he was offered a longer-term post at the Los Alamos National Laboratory in New Mexico to study turbulence in fluids. Although a complete theory of turbulent fluids remains to be established, his research led him to study chaotic maps. In 1983, he was awarded a MacArthur Fellowship; and in 1986, alongside Rockefeller University colleague Albert Libchaber, he was awarded the Wolf Prize in Physics "for his pioneering theoretical studies demonstrating the universal character of non-linear systems, which has made possible the systematic study of chaos". He was a member of the Board of Scientific

Governors at The Scripps Research Institute. He was Toyota Professor at Rockefeller University from 1986 until his death.

his achievements:

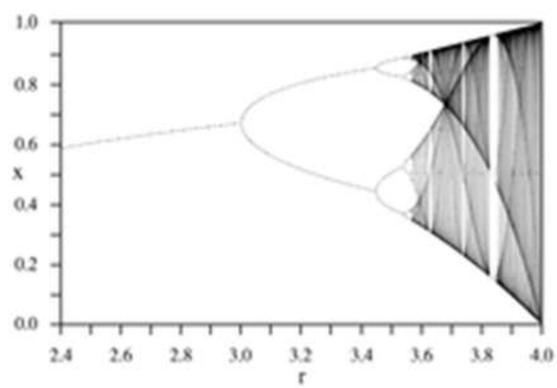
Some mathematical mappings involving a single linear parameter exhibit the apparently random behavior known as chaos when the parameter lies within certain ranges. As the parameter is increased towards this region, the mapping undergoes bifurcations at precise values of the parameter. At first there is one stable point, then bifurcating to an oscillation between two values, then bifurcating again to oscillate between four values and so on. In 1975, Dr. Feigenbaum, using the small HP-65 calculator he had been issued, discovered that the ratio of the difference between the values at which such successive period-doubling bifurcations occur tends to a constant of around 4.6692...[3] He was able to provide a mathematical argument of that fact, and he then showed that the same behavior, with the same mathematical constant, would occur within a wide class of mathematical functions, prior to the onset of chaos.[4]This universal result enabled mathematicians to take their first steps to unraveling the apparently intractable "random" behavior of chaotic systems. The "ratio of convergence" measured in this study is now known as the first Feigenbaum constant.

The logistic map is a prominent example of the mappings that Feigenbaum studied in his noted 1978 article: *Quantitative Universality for a Class of Nonlinear Transformations*.

Feigenbaum's other contributions include the development of important new fractal methods in cartography, starting when he was hired by Hammond to develop techniques to allow computers to assist in drawing maps. The introduction to the *Hammond Atlas* (1992) states Using fractal geometry to describe natural forms such as coastlines, mathematical physicist Mitchell Feigenbaum developed software capable of reconfiguring coastlines, borders, and mountain ranges to fit a multitude of map scales and projections. Dr. Feigenbaum also created a new computerized type placement program which places thousands of map labels in minutes, a task that previously required days of tedious labor In another practical application of his work, he founded Numerix with Michael Goodkin in 1996. The company's initial product was a software algorithm that dramatically reduced the time required for Monte Carlo pricing of exotic financial derivatives and structured products. Numerix remains one of the leading software providers to financial market participants.

The press release made on the occasion of his receiving the Wolf Prize summed up his works:

The impact of Feigenbaum's discoveries has been phenomenal. It has spanned new fields of theoretical and experimental mathematics ... It is hard to think of any other development in recent theoretical science that has had so broad an impact over so wide a range of fields, spanning both the very pure and the very applied



Bifurcation diagram of the logistic map. Feigenbaum noticed in 1975 that the quotient of successive distances between bifurcation events tends to 4.6692...

Behind the Feigenbaum Constant:

It’s called the Feigenbaum constant, and it’s about 4.6692016. And it shows up, quite universally, in certain kinds of mathematical—and physical—systems that can exhibit chaotic behavior.

It became a defining discovery in the history of chaos theory. But when it was first discovered, it was a surprising, almost bizarre result, that didn’t really connect with anything that had been studied before.

Somehow, though, it's fitting that it should have been Mitchell Feigenbaum—who I knew for nearly 40 years—who would discover it.

Trained in theoretical physics, and a connoisseur of its mathematical traditions, Mitchell always seemed to see himself as an outsider. He looked a bit like Beethoven—and projected a certain stylish sense of intellectual mystery. He would often make strong assertions, usually with a conspiratorial air, a twinkle in his eye, and a glass of wine or a cigarette in his hand.

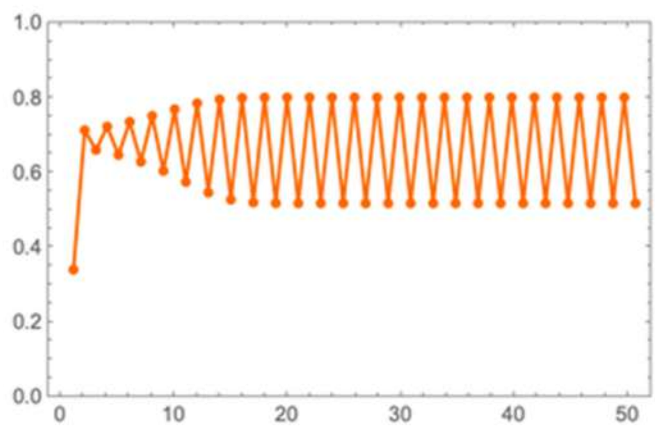
He would talk in long, flowing sentences which exuded a certain erudite intelligence. But ideas would jump around. Sometimes detailed and technical. Sometimes leaps of intuition that I, for one, could not follow. He was always calculating, staying up until 5 or 6 am, filling yellow pads with formulas and stressing Mathematica with elaborate algebraic computations that might run for hours.

He published very little, and what he did publish he was often disappointed wasn't widely understood. When he died, he had been working for years on the optics of perception, and on questions like why the Moon appears larger when it's close to the horizon. But he never got to the point of publishing anything on any of this.

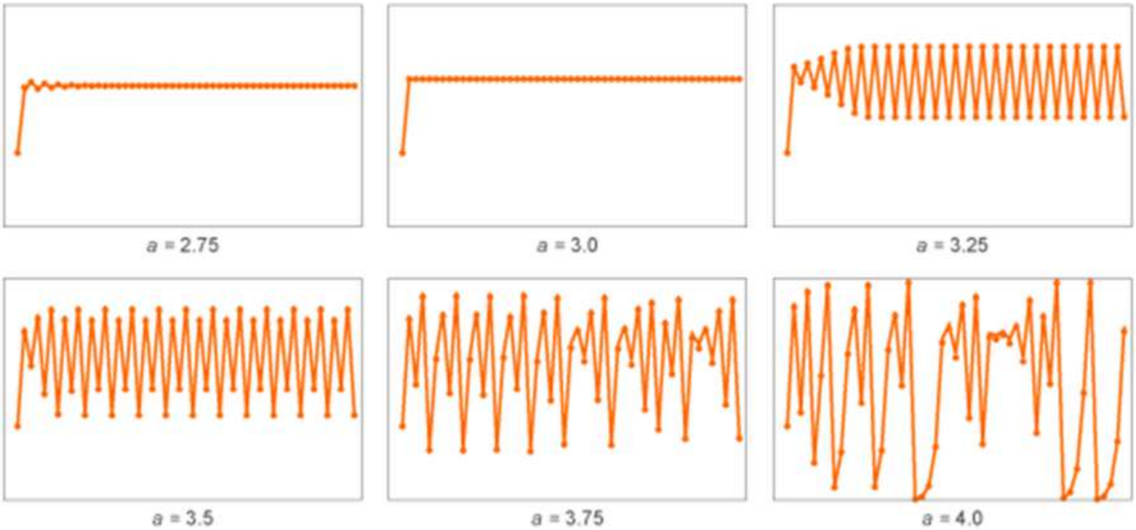
For more than 30 years, Mitchell's official position (obtained essentially on the basis of his Feigenbaum constant result) was as a professor at the Rockefeller University in New York City. (To fit with Rockefeller's biological research mission, he was themed as the Head of the "Laboratory of Mathematical Physics".) But he dabbled elsewhere, lending his name to a financial computation startup, and becoming deeply involved in inventing new cartographic methods for the *Hammond World Atlas*.

Mitchell's discovery:

The basic idea is quite simple. Take a value x between 0 and 1. Then iteratively replace x by $a\ x\,(1-x)$. Let's say one starts from $x = \frac{1}{3}$, and takes $a = 3.2$. Then here's what one gets for the successive values of x:

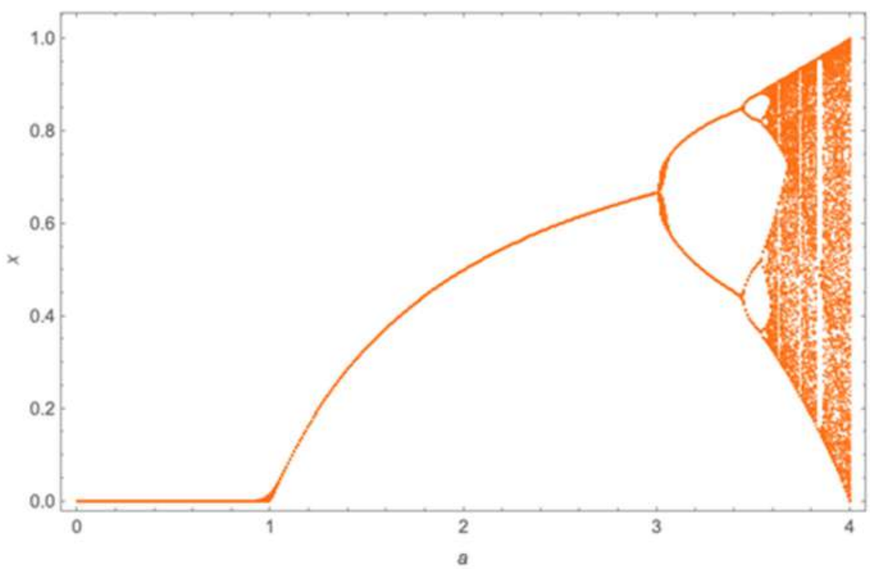


After a little transient, the values of x are periodic, with period 2. But what happens with other values of a? Here are a few results for this so-called "logistic map":



For small a, the values of x quickly go to a fixed point. For larger a they become periodic, first with period 2, then 4. And finally, for larger a, the values start bouncing around seemingly randomly.

One can summarize this by plotting the values of x (here, 300, after dropping the first 50 to avoid transients) reached as a function of the value of a:



As a increases, one sees a cascade of "period doublings". In this case, they're at $a = 3$, $a \simeq 3.449$, $a \simeq 3.544090$, $a \simeq 3.5644072$. What Mitchell noticed is that these successive values approach a limit (here $a_\infty \simeq 3.569946$) in a geometric sequence, with $a_\infty - a_n \sim \delta^{-n}$ and $\delta \simeq 4.669$.

That's a nice little result. But here's what makes it much more significant: it isn't just true about the specific iterated map $x \longrightarrow a\,x\,(1 - x)$; it's true about any map like that. Here, for example, is the "bifurcation diagram" for $x \longrightarrow a \sin(\pi \sqrt{x})$:

The details are different. But what Mitchell noticed is that the positions of the period doublings again form a geometric sequence, with the exact same base: $\delta \simeq 4.669$. It's not just that different iterated maps give qualitatively similar results; when one measures the convergence rate this turns out be exactly and quantitatively the same—always $\delta \simeq 4.669$. And this was Mitchell's big discovery: a quantitatively universal feature of the approach to chaos in a class of systems.

The Big Discovery:

The Navier–Stokes equations are very hard to work with. In fact, to this day it's still not clear how even the most obvious feature of turbulence—its apparent randomness—arises from these equations. (It could be that the equations aren't a full or consistent mathematical description, and one's actually seeing amplified microscopic molecular motions. It could be that—as in chaos theory and the Lorenz equations—it's due to amplification of randomness in the initial conditions. But my own belief, based on work I did in the 1980s, is that it's actually an intrinsic computational phenomenon—analogous to the randomness one sees in my rule 30 cellular automaton.)

So how did Mitchell approach the problem? He tried simplifying it—first by going from equations depending on both space and time to ones depending only on time, and then by effectively making time discrete, and looking at iterated maps. Through Paul Stein, Mitchell knew about the (not widely known) previous work at Los Alamos on iterated maps. But Mitchell didn't quite know where to go with it, though having just got a swank new HP-65 programmable calculator, he decided to program iterated maps on it.

Then in July 1975, Mitchell went (as I also did a few times in the early 1980s) to the summer physics hang-out-together event in Aspen, CO. There he ran into Steve Smale—a well-known mathematician who'd been studying dynamical systems—and was surprised to find Smale talking about iterated maps. Smale mentioned that someone had asked him if the limit of the period-doubling cascade $a_\infty \simeq 3.56995$ could be expressed in terms of standard constants like π and $\sqrt{2}$. Smale related that he'd said he didn't know. But Mitchell's interest was piqued, and he set about trying to figure it out.

He didn't have his HP-65 with him, but he dove into the problem using the standard tools of a well-educated mathematical physicist, and had soon turned it into something about poles of functions in the complex plane—about which he couldn't really say anything. Back at Los Alamos in August, though, he had his HP-65, and he set about programming it to find the bifurcation points a_n.

The iterative procedure ran pretty fast for small n. But by $n = 5$ it was taking 30 seconds. And for $n = 6$ it took minutes. While it was computing, however, Mitchell decided to look at the a_n values he had so far—and noticed something: they seemed to be converging geometrically to a final value.

At first, he just used this fact to estimate a_∞, which he tried—unsuccessfully—to express in terms of standard constants. But soon he began to think that actually the convergence exponent δ was more significant than a_∞—since its value stayed the same under simple changes of variables in the map. For perhaps a month Mitchell tried to express δ in terms of standard constants.

But then, in early October 1975, he remembered that Paul Stein had said period doubling seemed to look the same not just for logistic maps but for any iterated map with a single hump. Reunited with his HP-65 after a trip to Caltech, Mitchell immediately tried the map $x \longrightarrow \sin(x)$—and discovered that, at least to 3-digit precision, the exponent δ was exactly the same.

He was immediately convinced that he'd discovered something great. But Stein told him he needed more digits to really conclude much. Los Alamos had plenty of powerful computers—so the next day Mitchell got someone to show him how to write a program in FORTRAN on one of them to go further—and by the end of the day he had managed to compute that in both cases δ was about 4.6692.

Chapter tow

THE QUADRATIC FAMILY:

A group of quadratic functions which all share a common characteristic is called family of quadratic functions.

To know more about different families of quadratic functions, we have to know the different forms in which quadratic functions can be expressed.

Let us come to know the different forms of quadratic functions.

Different Forms of Quadratic Functions

Quadratic functions can be expressed in the following three different algebraic forms.

Standard form : $f(x) = ax^2 + bx + c$

Factored form : $f(x) = a(x - r)(x - s)$

Vertex form : $f(x) = a(x - h)^2 + k$

In vertex form, vertex of the parabola is (h, k) and the axis of symmetry is $x = h$.

Family of parabolas :

A group of parabolas which all share a common characteristic.

Families of Quadratic Functions:

Family 1 :

If the values of a and b are varied in a quadratic function expressed in standard form, $f(x) = ax^2 + bx + c$, a family of parabolas with the same y–intercept is created.

Common characteristic :

Same y - intercept

Family 2 :

If the value of a is varied in a quadratic function expressed in factored form, $f(x) = a(x - r)(x - s)$, a family of parabolas with the same x–intercepts and axis of symmetry is created.

Common characteristic :

ame x - intercepts and Axis of symmetry

Family 3 :

If the value of a is varied in a quadratic function expressed in vertex form, $f(x) = a(x - h)^2 + k$, a family of parabolas with the same vertex and axis of symmetry is created.

Common characteristic :

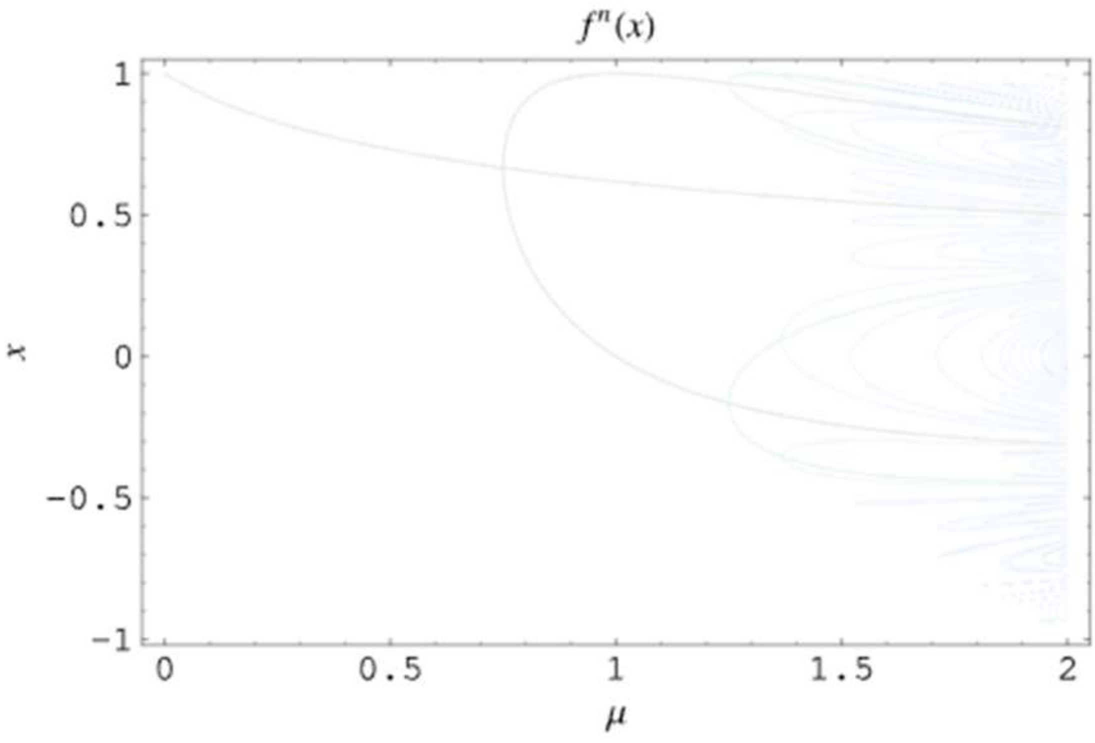
Same vertex and Axis of symmetry

What is astonishing is that this constant d_∞ seems to be universal. "That is, for many families of one-humped functions like the family of quadratic functions, bifurcacions occur in such a regular fashion that the distances between successive pairs of Sifurcation points approach the very same value d! It is for this reason that d is called a universal constant. Morc particularly, it is referred to as the Feigenbaum constant, because Feigenbaum was the first to discover it and its universality,

We conclude hy noting that the quadratic family (Q_U) is one of the most illustrious parametrized families. Its functions are easy to describe, and have properties many more complicated functions have. Moreover, there is an enormous wealth of information concerning the family, spurred in part by the captivating article in the magazine Nature by Robert May (1975). Books by Pierre Collet and JeanPierre Eckmann (1980) and by Chris Preston (1983) give detailed analysis of functions like quadratic functions. We will once again study properties of the family

Feigenbaum Constant:

characterizes the geometric approach of the bifurcation parameter to its limiting value as the parameter is increased for fixed . The plot above is made by iterating equation *($f(x)=1-\mu|x|^r$)* with several hundred times for a series of discrete but closely spaced values of , discarding the first hundred or so points before the iteration has settled down to its fixed points, and then plotting the points remaining



Asimilar plot that more directly shows the cycle may be constructed by plotting $f^n(x)-x$ as a function of μ. The plot above (Trott, pers. comm.) shows the resulting curves for n=1, 2, and 4.

Let μ=1 be the point at which a period 2^n-cycle appears, and denote the converged value by μ_∞. Assuming geometric convergence, the difference between this value and μ_n is denoted

$$\lim_{n\to\infty} \mu\infty - \mu n = \frac{r}{\delta}$$

where Γ is a constant and $\delta > 1$ is a constant now known as the Feigenbaum constant. Solving for δ gives

$$\delta = \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}$$

(Rasband 1990, p. 23; Briggs 1991). An additional constant , defined as the separation of adjacent elements of period doubled attractors from one double to the next, has a value

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}}$$

where d_n is the value of the nearest cycle element to 0 in the 2^n cycle (Rasband 1990, p. 37; Briggs 1991)

For equation (1) with r=2, the onsets of bifurcations occur at $\mu = 0.75$, 1.25, 1.368099, 1.39405, 1.399631, ..., giving convergents to δ for n=1, 2, 3, ... of 4.23374, 4.5515, 4.64617, ...

For the logistic map,

δ =4.669201609102990

Γ =2.637

μ_∞ =3.569945672

α 2.502907875

(OEIS A006890, A098587, and A006891; Broadhurst 1999; Wolfram 2002, p. 920), where is known as the Feigenbaum constant and is the associated "reduction parameter

Briggs (1991) calculated to 84 digits, Briggs (1997) to 576 decimal places (of which 344 were correct), and Broadhurst (1999) to 1018 decimal places. It is not known if the Feigenbaum constant is algebraic, or if it can be expressed in terms of other mathematical constants (Borwein and Bailey 2003, p. 53).

Briggs (1991) calculated to 107 digits, Briggs (1997) to 576 decimal places (of which 346 were correct), and Broadhurst (1999) to 1018 decimal places.

Amazingly, the Feigenbaum constant and associated reduction parameter are "universal" for all one-dimensional maps f(x) if *f(x)* has a single locally quadratic maximum. This was conjecture by Feigenbaum, and demonstrated rigorously by Lanford (1982) for the case r=2, and by Epstein (1985) for all r< 14.

More specifically, the Feigenbaum constant is universal for one-dimensional maps if the Schwarzian derivative

$$D_{\text{Schwarzian}} \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

is negative in the bounded interval (Tabor 1989, p. 220). Examples of maps which are universal include the Hénon map, logistic map, Lorenz attractor, Navier-Stokes truncations, and sine map x_{n+1}=a sin Πx_n The value of the Feigenbaum constant can be computed explicitly using functional group renormalization theory. The universal constant also occurs in phase transitions in physics.

The value of for a universal map may be approximated from functional group renormalization theory to the zeroth order by solving

$$1-\alpha^{-1} = \frac{1-\alpha^{-2}}{\left[1-\alpha^{-2}\left(1-\alpha^{-1}\right)\right]^2},$$

which can be rewritten as the quintic equation

$$\alpha^5 + 2\,\alpha^4 - 2\,\alpha^3 - \alpha^2 + 2\,\alpha - 1 = 0.$$

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