Republic of Iraq
Ministry of Higher Education and Scientific Research
University of Babylon
college of Education for Pure Sciences
Department of Mathematics

Dirac delta function

A graduation research submitted to the council of the Department of Mathematics in the college of Education for Pure Sciences / University of Babylon, as part of the requirements obtaining a bachelor's degree in mathematics.

prepared by	Supervised by
Abbas Ghanem Jameel	Rawasy Adnan Hameed Al-Musarhad

2026 A.D **1447 A.H**

الشكر والتقدير

أتقدم بخالص الشكر والتقدير إلى أستاذي المشرف على دعمه المستمر وتوجيهاته القيمة وملاحظاته البنّاءة التي كان لها الأثر الكبير في إنجاز هذا البحث وتحسين مستواه العلمي.

كما أتوجه بجزيل الشكر إلى أعضاء هيئة التدريس في قسم الرياضيات لما قدموه من علمٍ ومعرفة، ولما وفرّوه من بيئة تعليمية مشجعة ساهمت في تطوير قدراتي وفهمي للمفاهيم الرياضية المتقدمة.

وأعبر عن امتناني العميق لعائلتي على صبرهم وتشجيعهم ودعمهم المتواصل طوال فترة إعداد هذا العمل، فقد كانوا مصدر قوة ودافعًا للاستمرار.

وأخيرًا، أشكر كل من ساهم بشكل مباشر أو غير مباشر في إنجاز هذا البحث، فلكم مني كل التقدير والامتنان.

الإهداء

أهدي هذا العمل إلى عائلتي، التي كانت دائمًا مصدر الدعم والمحبة والتشجيع طوال مسيرتي الدراسية. إن إيمانهم بي كان ولا يزال الدافع الأكبر للاستمرار والنجاح.

كما أهدي هذا البحث إلى أساتذتي ومعلميّ، الذين قدموا لي العلم بإخلاص وصبر، وساهموا في بناء فهمي للرياضيات والتفكير العلمي.

وأخيرًا، أهدي هذا العمل إلى كل طالب يسعى لفهم العلم ليس كمجرد معلومات، بل كوسيلة لبناء مستقبل أفضل.

Abstract

The Dirac Delta function is a fundamental concept in applied mathematics, defined as a distribution rather than a classical function. This work examines its mathematical definition, key properties, and role within distribution theory. It further explores its applications in physics, electrical engineering, Fourier analysis, and differential equations, supported by practical examples and analytical formulations.

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Introduction

The development of modern mathematics and physics has always been driven by the need to describe increasingly complex phenomena using precise and efficient tools. Among these tools, special mathematical constructs have emerged to bridge the gap between idealized theoretical models and real-world behavior. One of the most remarkable of these constructs is the Dirac delta function, a concept that, despite its seemingly simple definition, carries profound implications across multiple scientific disciplines. It was introduced by Paul Dirac in the early twentieth century as part of his work in quantum mechanics, where the need arose to represent quantities that are concentrated at a single point in space or time.

At first glance, the Dirac delta function appears to challenge the conventional understanding of what a function is. Unlike ordinary functions that assign a finite value to each point in their domain, the delta function is characterized by being zero everywhere except at a single point, where it is not merely large but conceptually infinite, while still maintaining a finite integral equal to one. This unusual behavior makes it an indispensable tool for modeling idealized impulses, such as instantaneous forces, point charges, or sudden energy transfers. As a result, it plays a central role in simplifying mathematical expressions that would otherwise be difficult or even impossible to handle using classical methods.

The importance of the Dirac delta function extends far beyond its original introduction in quantum physics. In mathematics, it is rigorously defined within the framework of distribution theory, where it serves as a fundamental example of a generalized function. This perspective allows mathematicians to manipulate the delta function in a consistent and logically sound manner, especially when dealing with differentiation and integration operations that would be undefined in the classical sense. In engineering, particularly in signal processing and systems analysis, the delta function is used to represent ideal input signals and to determine the behavior of systems through impulse responses. Its ability to “pick out” specific values from a function makes it an exceptionally powerful analytical tool.

This research aims to provide a comprehensive exploration of the Dirac delta function by examining its conceptual foundations, mathematical properties, and practical applications. The study begins with an overview of the fundamental definitions and interpretations associated with the delta function, followed by a detailed analysis of its key properties within the context of modern mathematical theory. Subsequently, the research investigates a range of applications in physics, electrical engineering, and differential equations, highlighting the versatility and effectiveness of this concept in solving real-world problems.

The methodology adopted in this study is primarily analytical and descriptive, relying on established mathematical formulations and theoretical interpretations. By integrating insights from multiple disciplines, the research seeks to demonstrate not only how the Dirac delta function is defined, but also why it remains an essential component in both theoretical and applied sciences. Ultimately, this work aims to provide a clear and structured understanding of the Dirac delta function, making it accessible to students and researchers while preserving the depth and rigor required at the university level.

Chapter One:

Fundamental Concepts of the Dirac Delta Function

1-1-Introduction

The study of the Dirac delta function begins with a careful examination of its fundamental concepts, which form the foundation for understanding its broader mathematical and physical significance. Although the delta function is widely used in advanced applications, its true meaning cannot be fully appreciated without first exploring the ideas that define its structure and behavior. This chapter is therefore devoted to introducing the essential principles that distinguish the Dirac delta function from ordinary functions, while clarifying the context in which it operates.

A key aspect of this discussion is recognizing that the Dirac delta function does not fit within the traditional definition of a function. Instead, it is better understood as a mathematical idealization that captures the notion of an infinitely concentrated quantity at a single point. This perspective allows it to represent physical phenomena such as impulses and point sources in a simplified yet highly effective manner. In addition, the chapter will highlight how early interpretations, particularly those introduced by Paul Dirac, contributed to its widespread adoption in scientific analysis.

By the end of this chapter, the reader will have a clear understanding of the conceptual framework underlying the Dirac delta function, preparing them for a deeper exploration of its properties and applications in subsequent chapters.

1-2: Mathematical Definition of the Dirac Delta Function

The Dirac Delta function, denoted by $\delta(x)$, is introduced as a generalized mathematical object that cannot be described as an ordinary function. Instead, it is defined through limiting processes that capture its highly concentrated nature. One common representation is given by a sequence of functions that become increasingly narrow and tall around the origin:

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon}}e^{-\frac{x^2}{\epsilon}}$$

This expression provides an intuitive mathematical construction where the function approaches zero everywhere except near a single point.

To analyze its behavior, we distinguish between the cases $x \neq$ and $x = 0$. For all values away from the origin, the exponential term decays rapidly as $\epsilon \to 0$, leading to:

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0$$

At the origin, the situation is different. Substituting $x = 0$ into the expression gives:

$$\delta(0) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon}} \to \infty$$

This indicates that the function becomes unbounded at a single point. However, this divergence is controlled in such a way that the total integral remains finite:

$$\int_{-\infty}^{\infty} \delta\,(x)dx = 1$$

Thus, the Dirac Delta function is zero everywhere except at

$$X = 0$$

where it becomes infinitely large, while still preserving a finite total contribution. This unusual combination of properties distinguishes it from all classical functions.

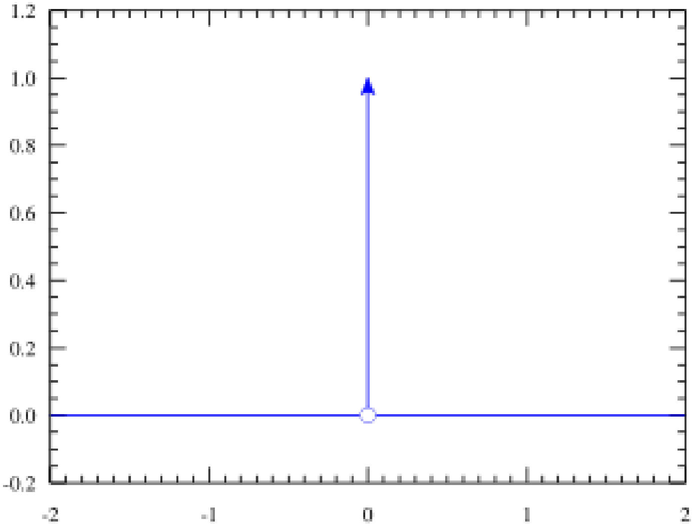
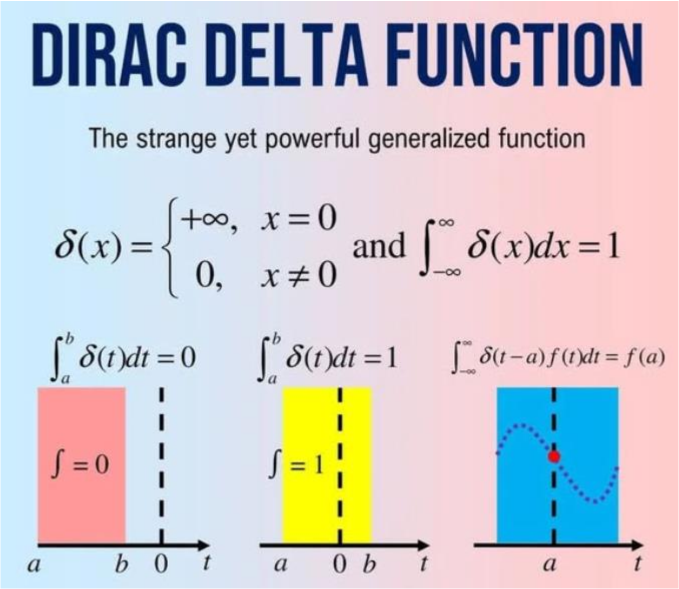


Figure 1



DIRAC DELTA FUNCTION

The strange yet powerful generalized function

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_a^b \delta(t)dt = 0 \qquad \int_a^b \delta(t)dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$$

Figure 2

1-3: Physical and Intuitive Interpretation of the Dirac Delta Function

The Dirac Delta function can be understood more intuitively by interpreting it as an idealized impulse—a quantity that is concentrated at a single point in space or time, yet produces a finite effect. In physical systems, many phenomena occur over extremely short intervals or within negligibly small regions, and the delta function provides a convenient way to model such behavior mathematically.

In time-dependent systems, an impulse applied at $t = t_0$ is represented as:

$$\delta(t - t_0)$$

This function is zero everywhere except at $t = t_0$, where it is infinitely large in such a way that its total effect remains finite:

$$\int_{-\infty}^{\infty} \delta\,(t - t_0)dt = 1$$

If the impulse has a magnitude A, it is written as:

$$A\delta(t - t_0)$$

which Satisfies:

$$\int_{-\infty}^{\infty} A\,\delta(t - t_0)dt = A$$

In spatial contexts, the Dirac Delta function is used to represent point sources such as charges or masses. For example, a point mass located at $x = a$ can be modeled as:

$$\rho(x) = m\delta(x - a)$$

ensuring that the total mass is preserved:

$$\int_{-\infty}^{\infty} \rho\,(x)dx = m$$

Thus, the delta function provides a powerful bridge between mathematical abstraction and physical reality by allowing localized phenomena to be expressed in continuous models.

1-4: Comparison Between the Dirac Delta Function and Ordinary Functions

The Dirac Delta function differs fundamentally from ordinary functions, particularly in how it is defined and how it behaves under mathematical operations. A classical function $f(x)$ assigns a finite value to every point $x \in R$ and it is typically continuous or piecewise continuous. In contrast, the Dirac Delta function is not defined by pointwise values but by its effect within an integral.

For ordinary functions, integration accumulates area in the usual sense:

$$\int_a^b f(x)dx$$

represents the area under the curve. However, for the delta function:

$$\delta(x) = 0 \text{ For } x \neq 0$$

yet it satisfies:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

which cannot be achieved by any standard function.

Another key distinction appears in multiplication. For a regular function:

$$f(x) \,.\, g(x)$$

is defined pointwise, whereas with the delta function:

$$F(x)\delta(x-a) = f(a)\delta(x-a)$$

which reflects its distributional nature rather than classical multiplication.

Finally, limits of ordinary functions do not typically produce objects like $\delta(x)$, except in a generalized sense:

$$\delta(x) = \lim_{\epsilon\to 0} f_{\epsilon}(x)$$

where $f_{\epsilon}(x)$ is a sequence of functions with shrinking support. This highlights that the Dirac Delta function exists beyond the scope of traditional function theory.

Chapter Two :

Mathematical properties and Analytical Framework of the Dirac

2-1-Introduction

Following the foundational concepts presented in the previous chapter, the study now moves toward a deeper and more structured examination of the Dirac delta function from a mathematical perspective. While Chapter One focused on the intuitive meaning and basic definition of the delta function, this chapter is concerned with analyzing its formal properties and understanding how it behaves within a rigorous analytical framework. This transition is essential, as the true power of the Dirac delta function lies not only in its conceptual simplicity but also in its precise mathematical behavior when subjected to various operations.

A central theme of this chapter is the exploration of the fundamental properties that define the Dirac delta function, such as its behavior under integration, translation, and scaling. These properties form the backbone of its applications and are crucial for working with the delta function in both theoretical and applied contexts. Particular attention will be given to the sifting property, which allows the delta function to extract the value of another function at a specific point, as well as its role in simplifying complex integrals.

In addition to these basic properties, the chapter will examine how the Dirac delta function interacts with common mathematical operations, including differentiation and convolution. Since the delta function does not belong to the class of ordinary functions, these operations must be interpreted within the framework of distribution theory, as established by Laurent Schwartz. This approach provides the necessary tools to handle the delta function in a consistent and mathematically sound manner.

2-2: Fundamental Properties of the Dirac Delta Function

The Dirac Delta function is defined through a set of fundamental properties that describe how it behaves under integration and transformation. These properties form the operational backbone of the delta function and are essential for solving applied mathematical problems. Each property is presented below with its mathematical formulation and a brief interpretation.

2-2-1 Normalization Property

The normalization condition ensures that the total effect of the delta function is equal to one:

$$\int_{-\infty}^{\infty} \delta\,(x)dx = 1$$

This property guarantees that although the function is infinitely concentrated at a single point, its total contribution remains finite and well-defined.

2-2-2 Sifting (Sampling) Property

The most important property is the ability of the delta function to extract the value of another function at a specific point:

$$\int_{-\infty}^{\infty} f\,(x)f(x-a)dx = f(a)$$

more generally:

$$\int_{-\infty}^{\infty} f\,(x)f(x-a)dx = \begin{cases} f(a), a \in R \\ 0, \text{otherwise} \end{cases}$$

This property defines the delta function completely in distribution theory.

2-2-3 Shifting Property

The delta function can be translated along the real axis without changing its structure:

$$\delta(x-a)$$

and within integrals:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

This shows that the location of the impulse determines the evaluation point.

2-2-4 Scaling Property

When the argument of the delta function is scaled, its amplitude adjusts accordingly:

$$\delta(ax) = \frac{1}{|a|}\delta(x), \qquad a \neq 0$$

More generally:

$$\delta(ax-b) = \frac{1}{|a|}\delta(x-\frac{b}{a})$$

This property is crucial when performing change of variables in integrals.

2-2-5 Symmetry Property

The delta function is an even distribution:

$$\delta(-x) = \delta(x)$$

and for shifted arguments:

$$\delta(a - x) = \delta(x - a)$$

This reflects that the delta function has no directional dependence.

2-2-6 Derivative Property

The derivative of the delta function is defined in the distributional sense:

$$\int_{-\infty}^{\infty} f(x)\delta'(x - a)dx = -f'(a)$$

This property shows that differentiation transfers from the delta function to the test function with a sign change.

Together, these properties define the Dirac Delta function as a powerful analytical tool, enabling precise manipulation of localized effects in mathematical physics and engineering systems.

2-3 Dirac Delta Function and Distribution Theory

The Dirac Delta function is most rigorously defined within the framework of distribution theory, where it is treated as a linear functional acting on a space of smooth test functions. Instead of assigning pointwise values, the delta function is characterized entirely by how it behaves under integration. For any test function $\varphi(x)$, the defining relation is:

$$\langle \delta(x), \varphi(x) \rangle = \int_{-\infty}^{\infty} \delta(x)\rho(x)dx = \varphi(x)$$

More generally, for a shifted delta:

$$\langle \delta(x-a), \varphi(x) \rangle = \varphi(a)$$

This formulation shows that the Dirac Delta is a linear operator, satisfying:

$$(\delta, \alpha\varphi_1 + \beta\varphi_2) = \alpha\varphi_1(0) + \beta\rho_2(0)$$

which confirms its linearity within the space of test functions.

In distribution theory, differentiation is extended beyond classical limits. The derivative of the delta function is defined through its action:

$$\langle \delta'(x), \varphi(x) \rangle = -\varphi'(0)$$

and more generally:

$$\int_{-\infty}^{\infty} \delta'(x-a)\varphi(x)dx = -\varphi'(a)$$

This allows differentiation of discontinuous or non-smooth functions in a consistent way.

Another key concept is the representation of distributions as limits of sequences of regular functions. For example, the Dirac Delta can be expressed as:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{x^2}{\epsilon}}$$

This shows how classical functions can approximate the delta function in the limit, preserving its integral properties.

Thus, within distribution theory, the Dirac Delta function becomes a well-defined mathematical object, enabling rigorous treatment of impulses, discontinuities, and localized phenomena in advanced analysis.

Chapter Three:

Practical Applications of the Dirac Delta Function

3-1 Introduction

This chapter explores the practical applications of the Dirac Delta function in physics, engineering, and mathematical analysis. It focuses on how this distribution is used to model point sources, impulse signals, and localized effects. Through structured formulations, the Dirac Delta function simplifies complex systems, enabling precise representation of instantaneous phenomena and providing effective solutions in Fourier analysis and differential equations across various scientific and engineering contexts.

3-2: Physics Applications

The Dirac Delta function is widely used in physics to model highly localized physical quantities such as point charges, point masses, and instantaneous forces. Its main advantage is the ability to represent continuous systems that contain discrete or singular sources in a mathematically consistent way.

In electrostatics, a point charge located at $x = a$ is represented using the delta function as:

$$\rho(x) = q\delta(x - a)$$

This ensures that the total charge is preserved:

$$\int_{-\infty}^{\infty} \rho\,(x)dx = \int_{-\infty}^{\infty} g\,\delta(x - a)dx = q$$

Similarly, in mechanics, a point mass concentrated at a single position can be modeled as:

$$m(x) = m_0\delta(x - a)$$

with total mass:

$$\int_{-\infty}^{\infty} m\,(x)dx = m_0$$

In dynamics, instantaneous forces (impulses) are expressed as:

$$F(t) = f_0\delta(t - t_0)$$

which leads to the impulse-momentum relation:

$$\int_{-\infty}^{\infty} f\,(t)dt = F_0$$

Example

Consider a force applied at $t = 2$ seconds :

$$f(t) = 5\delta(t-2)$$

The total impulse delivered is:

$$I = \int_{-\infty}^{\infty} 5\, f(t-2)dt = 5$$

This shows that the entire effect of the force is concentrated at a single instant while still producing a finite measurable result.

Thus, the Dirac Delta function provides a precise mathematical tool for modeling idealized physical systems where localization and instantaneous effects are essential.

3-3: Electrical Engineering Applications

In electrical engineering, the Dirac Delta function is used to model impulse signals and analyze linear time-invariant (LTI) systems. The impulse is defined as: $\delta(t)$

and satisfies:

$$\int_{-\infty}^{\infty} \delta\,(t)dt = 1$$

An input impulse applied at time t_0 is written as:

$$x(t) = f(t - t_0)$$

For an LTI system with impulse response $h(t)$, the output is given by convolution:

$$y(t) = \int_{-\infty}^{\infty} x\,(\tau)h(t - \tau)d\tau$$

Substituting the impulse input:

$$y(t) = \int_{-\infty}^{\infty} \delta\,(T - t_0)h(t - \tau)d\tau$$

we obtain:

$$y(t) = h(t - t_0)$$

Example

Given:

$$x(t) = 3\delta(t - 2),\;\; h(t) = e^{-t}u(t)$$

solution

Step 1: Write convolution

$$y(t) = \int_{-\infty}^{\infty} 3\,\delta(\tau - 2)h(t - \tau)d\tau$$

Step 2: Factor constant

$$y(t) = 3\int_{-\infty}^{\infty} \delta\,(\tau - 2)h(t - \tau)d\tau$$

Step 3: Apply sifting property

$$y(t) = 3h(t - 2)$$

Step 4: Substitute

$$y(t) = 3e^{-(t-2)}u(t - 2)$$

Thus, the impulse shifts and scales the system response:

$$y(t) = 3e^{-(t-2)}u(t - 2)$$

This demonstrates how the delta function simplifies system analysis into direct evaluation rather than full integration.

3-4: Fourier Transform Applications

The Dirac Delta function plays a central role in Fourier analysis due to its perfect localization in time and complete spread in frequency. It is defined by: $\delta(t)$

with normalization:

$$\int_{-\infty}^{\infty} \delta\,(t)dt = 1$$

Fundamental Fourier Properties

Fourier transform of delta:

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta\,(t)e^{-j\omega t}dt = 1$$

Shifted delta:

$$F\{\delta(t - t_0)\} = e^{-j\omega t_0}$$

Inverse relation:

$$F^{-1}\{1\} = \delta(t)$$

Orthogonality identity:

$$\int_{-\infty}^{\infty} e^{j\omega t} e^{-j\omega' t}dt = 2\pi\delta(\omega - \omega')$$

Example

Given:

$$x(t) = 4\delta(t - 3)$$

Find Fourier transform .

solution

Step 1: Apply definition

$$X(\omega) = \int_{-\infty}^{\infty} 4\,\delta(t-3)e^{-j\omega t}dt$$

Step 2: Factor constant

$$X(\omega) = 4\int_{-\infty}^{\infty} \delta\,(t-3)e^{-j\omega t}dt$$

Step 3: Apply sifting property

$$\int_{-\infty}^{\infty} f\,(t)\delta(t-a)dt = f(a)$$

$$X(\omega) = 4e^{-j\omega\cdot 3}$$

Step 4: Final result

$$X(\omega) = 4e^{-j3\omega}$$

This shows that a time-shifted impulse becomes a pure phase term in frequency domain, confirming:

$$\delta(t-t_0) \rightarrow e^{-j\omega t_0}$$

Thus, the Dirac Delta function serves as a fundamental building block in spectral analysis and frequency-domain representation.

3-5: Differential Equations and Green’ s Functions

The Dirac Delta function is used to solve linear differential equations with localized sources by introducing Green’s functions. Consider a linear operator :

$$L[y(x)] = f(x)$$

Define the Green’s function by:

$$L[G(x,\xi)] = \delta(x-\xi)$$

Then the solution is:

$$y(x) = \int_{-\infty}^{\infty} G\,(x,\xi) f(\xi) d\xi$$

Example

Solve:

$$y''(x) = \delta(x-a)$$

solution

Step 1: Integrate once

$$y'(x) = \int \delta\,(x-a)dx = H(x-a) + C_1$$

where is the Heaviside function.

Step 2: Integrate again

$$y(x) = \int H\,(x-a)dx = (x-a)H(x-a) + C_1 x + C_2$$

Step 3: Piecewise form

$$H(x-a)=\begin{cases}0, & x< a\\ 1, & x\geq a\end{cases}$$

Thus:

$$y(x)=\begin{cases}C_1x+C_2, & x>a\\ (x-a)+C_1x+C_2, & x\geq a\end{cases}$$

Step 4: Discontinuity condition

Integrating around :

$$\int_{a-\epsilon}^{a+\epsilon} y''(x)dx=\int_{a-\epsilon}^{a+\epsilon} \delta(x-a)dx=1$$

$$y'(a^{+})-y'(a^{-})=1$$

This shows that the delta function introduces a jump in the derivative while keeping continuous, making it essential for modeling point sources in differential equations.

Conclusion

The Dirac Delta function emerges throughout this work as a unifying tool that transforms complex localized phenomena into tractable mathematical expressions. Its definition as a distribution rather than a classical function allows it to operate consistently within integrals, differential equations, and transform methods. The core behavior is captured by its normalization and sampling properties:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1, \quad \int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

These relations enable immediate evaluation of otherwise complicated expressions.

In applied contexts, the delta function models point sources and impulses with precision. In systems analysis, convolution simplifies dramatically when impulses are involved:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \quad x(t) = \delta(t-t_0) \Rightarrow y(t) = h(t-t_0)$$

In Fourier analysis, it provides a direct bridge between time and frequency domains:

$$F\{\delta(t-t_0)\} = e^{-j\omega t_0}, \quad F^{-1}\{1\} = \delta(t)$$

For differential equations, it enables Green’s function constructions:

$$L[G(x,\xi)] = \delta(x-\xi), \quad y(x) = \int G\,(x,\xi)f(\xi)d\xi$$

Across these domains, the same structure persists: localization in one domain corresponds to spread in another, and integration acts as the primary mechanism of interpretation. This consistency explains why the Dirac Delta function is indispensable in physics, engineering, and modern analysis. It is not merely a symbolic object, but a precise operational framework that converts singular behavior into solvable mathematical form.

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