وزارة التعليم العالي والبحث العلمي

جـــــــــــــامعة بابل

كليه التربيه للعلوم الصرفه

Matrices in Graph Theory

بحث تقدمت به الطالبة

روواء كاظم محمد سلمان

الى / مجلس قسم الرياضيات

كلية التربية للعلوم الصرفة/جامعة بابل

وهو جزء من متطلبات نيل درجة البكالوريوس في الرياضيات

بإشراف

أ.د. احمد عبد علي عمران

٢٠٢٣م ١٤٤٤هـ

بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ

وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ

صدق الله العلي العظيم

المجادلة، ١١

الإهـــــــــــــــداء

ألحمد الله رب العالمين والصلاه والسلام على خاتم الانبياء والمرسلين إلهي لايطيب

الليل إلا بشكرك ولا يطيب النهار الا بطاعتك ولاتطيب اللحظات الابذكرك

ولاتطيب الآخرة إلا بعفوك ولا تطيب الجنة إلا برؤيتك

الى من بلغ الرسالة وأدى الأمانة ونصح الأمة إلى نبي الرحمه ونور العالمين سيدنا محمد ﷺ

الى من اسقتني الحب والحنان إلى رمز الحب وبلسم الشفاء إلى القلب الناصح

بالبياض إلى من أكبرت على يديها وعليها اعتمدت إلى شمعه تنير ظلمه حياتي الى من بوجودها اكتسب قوه ومحبه لا حدود لها إلى من عرفت معها معنى الحياة

امي الحبيبه

الى من كلله الله بالهيبة والوقار إلى من علمني العطاء بدون انتظار

غلي من احمل اسمه بكل افتخار

ابي الغالي

الى من هم اقرب إلي من روحي إلى من شاركوني حضن الام وبهم استمد عدتي واصراري

إخوتي الأعزاء

الى من وهبني الامل والتفاؤل في الحياة ونهلت من تجاربهم وكانوا ومازالوا عونا في حياتنا

الأساتذة الاجلاء

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رؤاء

شكر وعرفان

حروف نكتبها من نور ... صدقا وأمانة نطوقها بالعهد والوفاء نترجمها

شكرا وتبجيلا لفضائل وجلائل اعمالكم التي اشرابت لها هامة الزمان

وتظل اعمالكم شعلا تضئ عزة وشموخا فعندما يتوارث الناس روائع

الاشياء.. تكون منبعا للاصالة وفي الق التهذيب... هكذا عرفناكم

وانصهرت هممكم العالية بذلا وعطاء وامتزجت أرواحكم بالنبل والنقاء وكنتم قناويلا تحترق لتهب غيرها الضياء

وتختبئ الكلمات بعيدا عن عيون القلم لأنها طعم المستحيل

في التعبير عن الشكر والثناء ويبقى مانكتب وثيقة للصدق

والمحبة إعترافا لما قدمت لنا الأستاذ الفاضل وقائد السراب

أ.د. احمد عبد علي عمران

بكل فخر واعتزاز نتوجك اليوم ملك في محور العلم والمعرفة ونزف لك أسمى آيات الشكر المعبقة بعطر الفل والياسمين

Abstract

The rapid developments in overall life fields especially in discrete mathematics applications led to develop the technical used in daily life. In this search, we discuss two major subjects in discrete mathematics are Graphs and Matrices. At first, we are dealing the definitions of graph and matrix, in details, then, the history of them. After that, we view how to use the graph and matrices to solve the equations. Also, types of matrices discussed in details, in this work.

1 Introduction

In this paper we will discuss how problems like Page ranking and finding the shortest paths can be solved by using Graph Theory. At its core, graph theory is the study of graphs as mathematical structures. In our paper, we will first cover Graph Theory as a broad topic. Then we will move on to Linear Algebra.

2 Graph Theory

In this section we will explain what a graph is as well as the different properties of a graph such as degrees, trails, vertices, and edges.

2.1 Definition of a graph

A graph is a collection of vertices and edges. Vertices can be thought of as dots that are connected by edges.

For the purpose of this paper, we will assume that the graph has at most one edge between any two vertices. Graphs can be very resourceful tools used in real life in order to help people see where they can go and the different routes they can take to get there. There are various examples of these graphs, but for now we will use Konigsberg Bridge Problem as an example. This problem consists of two islands in which seven bridges connect them and other various islands. The problem states that we can walk through the edges only once and we have to end up at the same place we started. However, we realized that it is impossible to go through every bridge exactly once and end up where we started. The Konigsberg Bridge problem can be represented in a graph where the edges can be trans-versed either way, making it an undirected graph. An undirected graph is a graph that does not contain any arrows on its edges, indicating which way to go. A directed graph, on the other hand, is a graph in which its edges contain arrows indicating which way to go.

3.1 Definition of a matrix

Graphs and matrices are closely related to each other. A matrix is a set of numbers arranged in rows and columns so as to form a rectangular array. Some matrices can provide valuable information about graphs like how many vertices are connected, how many walks there might be between 2 vertices, and more. We will cover how to find the number of vertices connected to each other, as well as how many walks there might be between 2 vertices, and more, further ahead.

Row one column two

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Row two column one

Figure 5: This is a Matrix

3.2 Matrix History

Historically, it was not the matrix but a certain number associated with a square array of numbers called the determinant that was first recognized. Only gradually did the idea of the matrix as an algebraic entity emerge. The term matrix was introduced by the 19th-century English mathematician James Sylvester, but it was his friend the mathematician Arthur Cayley who developed the algebraic aspect of matrices in two papers in the 1850s. Cayley first applied them to the study of systems of linear equations, where they are still very useful. They are also important because, as Cayley recognized, certain sets of matrices form algebraic systems in which many of the ordinary laws of arithmetic (e.g., the associative and distributive laws) are valid but in which other laws (e.g., the commutative law) are not valid.

3.1.1 Matrix operations

Now that we have defined a matrix, we will present basic operations with them. There are three types of Matrix operations that we will be covering in our paper. One of the operations that will be covered is Addition. While the order of the matrices may not matter while adding matrices, both matrices need to have the same number of rows and columns. Once it is confirmed that both matrices have the same number of rows and columns, the corresponding entries can be added. The matrices below show a demonstration of how 2 2 x 2 matrices would be added.

This is an example of matrix addition:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 10 & 6 \end{bmatrix}$$

Another type of matrix operation is matrix multiplication by a constant. This operation is analogous to the multiplication of a number in front of an expression in parentheses, using the distributive property. As we can see in this example, in constant matrix multiplication, each entry in the matrix is multiplied by 2 to find the final product.

This is an example of constant by matrix multiplication:

$$2 * \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 14 & 4 \end{bmatrix}$$

Finally, the remaining matrix operation is multiplication of two (or more) matrices with each other. When multiplying a matrix by another matrix the number of rows in the first matrix must be equal to the number of columns in the second matrix or else the two matrices can not be multiplied. Once it is clear that this requirement is met, the corresponding entry for column A of the first matrix has to be multiplied

by the corresponding entry of row A of the second matrix. Then the products of multiplying all the entries from column A of the first matrix and row A of the second matrix would be added to find the corresponding entry for the product. This process would have to be continued until each column of the first Matrix has been multiplied by each row of the second matrix. As this example demonstrates, this is how it would look.

This is an example of a matrix by matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 7 \\ 43 & 17 \end{bmatrix}$$

3.2 Determinants and Transpose

We will now define a property called Determinant for matrices. Determinant is a function that has a matrix as an input and a number as an output. For example, det([4]) = 4. When finding the determinant of a one by one matrix with a single entry, the determinant will always be the same as the input number. We will now explain how to find the determinant of larger matrices. To find the determinant of a 2 x 2 matrix we will need to do *ad-bc*, a being in row one column one, b being in row one column two, c being in row two column one, and d being in row 2 column 2. The next determinant we will find is of a 3 x 3 matrix. For this kind of problem a label must be placed on top of the first row as +, -, +. A + sign means the sign of the number stays the same, but a - sign means the sign of the number must be changed. Once the top row is labeled one can move on to finding the determinants of the numbers. To do this, start with the first number of the matrix which is the first number in column one and row one. To find the determinant of this number, the row and column that the number is in must be ignored. After that, a 2 x 2 matrix will be achieved which is needed to do $ad - bc$ which is the formula to find the determinant, as covered said before. This process mist be repeated two more times to find the determinants of the numbers in row one column two and row one column three. Finally, in order to find the determinant of a 4 x 4 matrix the same methods would be used, which first means labeling the first row with +, -, +, and -. After doing so, the rows and columns that the numbers are in must be cancelled out as one continues to solve. Doing this will result in a 3 x 3 matrix, which we have covered how to solve above. Although one can use the methods discussed above to solve for determinants, one can also use Theorem 3.1 which easily allows one to solve for any matrix of any size.

Theorem 3.1. *For any square matrix A, switch two neighboring rows to get* A^I *.* $det(A) = -det\ A^I$

A Transpose of a matrix is when the columns can be reflected to become rows. When a transpose is applied twice, the result is the original matrix. We write this as, Transpose of A = A^T

$$\begin{aligned}&\det\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\\ =&a\det\begin{bmatrix}e & f\\ h & i\end{bmatrix} - b\det\begin{bmatrix}d & f\\ g & i\end{bmatrix} + c\det\begin{bmatrix}d & e\\ g & h\end{bmatrix}\\ =&a(e(i)-h(f)) - b(d(i)-g(f)) + c(d(h)-g(e))\\ =&x\end{aligned}$$

$$\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = 4(1)-2(3) = -2$$

$$\begin{aligned}&\det\begin{bmatrix}3 & 5 & 3\\ 2 & 7 & 1\\ 4 & 4 & 2\end{bmatrix}\\ =&3\det\begin{bmatrix}7 & 1\\ 4 & 2\end{bmatrix} - 5\det\begin{bmatrix}2 & 1\\ 4 & 2\end{bmatrix} + 3\det\begin{bmatrix}2 & 7\\ 4 & 4\end{bmatrix}\\ =&3(7(2)-4(1)) - 5(2(2)-4(1)) + 3(2(4)-7(4))\\ =&-30\end{aligned}$$

$$\begin{aligned}&\det\begin{bmatrix}1 & 2 & 5 & 2\\ 3 & 4 & 1 & 3\\ 2 & 3 & 5 & 2\\ 6 & 4 & 1 & 3\end{bmatrix}\\ =&1\det\begin{bmatrix}4 & 1 & 3\\ 3 & 5 & 2\\ 4 & 1 & 3\end{bmatrix} - 2\det\begin{bmatrix}3 & 1 & 3\\ 2 & 5 & 2\\ 6 & 1 & 3\end{bmatrix} + 5\det\begin{bmatrix}3 & 4 & 3\\ 2 & 3 & 2\\ 6 & 4 & 3\end{bmatrix} - 2\det\begin{bmatrix}3 & 4 & 1\\ 2 & 3 & 5\\ 6 & 4 & 1\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$$

Figure 6: The Transpose of a Matrix

4 Linear Algebra and Graph Theory

In this section we will talk about specific matrices and what these matrices can tell us about a graph.

4.1 Adjacency matrix

In this section we will cover the definition and uses of an adjacency matrix.

Definition 4.1. An adjacency matrix is a matrix of 0's and 1's based on whether or not two vertices have an edge between each other.

An adjacency matrix is a matrix of 0's and 1's based on whether or not two vertices have an edge between each other. If two vertices are connected to one another, the number 1 is inserted at the corresponding entry of the matrix. For example the corresponding entries for vertex 1 and 2 in the adjacency matrix is the first row, second column entry or the second row first column entry. If there is an edge between these two vertices, then a 1 can be placed in both of those entries. However, if two vertices are not connected, the number 0 can be used. For example, if someone were to reference row 1, column 4 and see the number 0, that means that there is no edge between vertex 4 and vertex 1.

Creating an adjacency matrix can also help determine the number of length k, k being any positive integer, walks from one vertex to another. For example, if the adjacency matrix is multiplied by itself once, which can be represented as $(A)^2$, and row 2 column 2 is referenced and the number 2 is there that means that there are 2 length two walks that start and end at vertex 2. In order to find the total number of length k walks in the entire graph, the adjacency matrix would have to multiplied by itself k times, which can be represented as $(A)^k$. Then that product would have to be multiplied by a vertical matrix of 1's. Finally, that product would have to be multiplied by a matrix of horizontal 1's in order to find the total number of length k walks in the graph. For example, in order to determine the number of length two walks in the graph, the adjacency matrix would be squared and then the rest of the steps would be followed normally The figures below show a labeled edge between two vertices in Graph A, the adjacency matrix of graph A, and how to find the number of length two walks in Graph A.

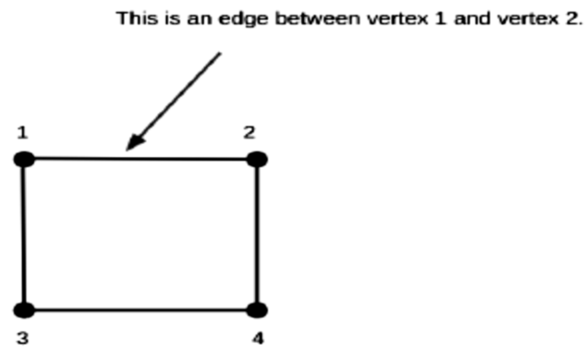

Figure 7: an image of graph A with a labeled edge between vertex 1 and vertex 2.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

This is the adjacency matrix for Graph A which is important in finding the number of length k walks in Graph A.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

This is the first step that must be taken to find the number of length two walks. The adjacency matrix is being squared.

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

This is the second step of finding the number of length 2 walks in Graph A. The squared adjacency matrix or $(A)^2$ is being multiplied by a vertical matrix of 1’s.

$$\begin{bmatrix}4\\4\\4\\4\end{bmatrix} * \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}16\end{bmatrix}$$

This is the last step of finding the number of length 2 walks in Graph A. The vertical matrix of 4's is being multiplied by the horizontal matrix of 1's to get a matrix of 16.

These matrices show the process used to find the number of length k walks in a graph. This example shows the process used to find the number of length two walks in graph A. First the adjacency matrix is multiplied by itself to get a matrix of 2's and 0's. Then that product is multiplied by a vertical matrix of 1's to get a matrix product of vertical 4's. Then the vertical matrix of 4's is multiplied by a horizontal matrix of 1's to get a matrix of 16. This means that there are 16 length 2 walks in graph A.

4.2 Incidence matrix

In this section we will cover the definition of an incidence matrix.

Definition 4.2. An incidence matrix is the matrix of a directed graph, or a graph with directional edges. An Incidence matrix is the matrix of a directed graph, or a graph with directional edges. Each column in an incidence matrix represents an edge between two vertices. The incidence matrix is made up of 1's, -1's, and 0's. The number 1 represents leaving a vertex. The number -1 represents arriving at a vertex. The number 0 means that the vertex is not involved. The figure below is an example of an incidence matrix.

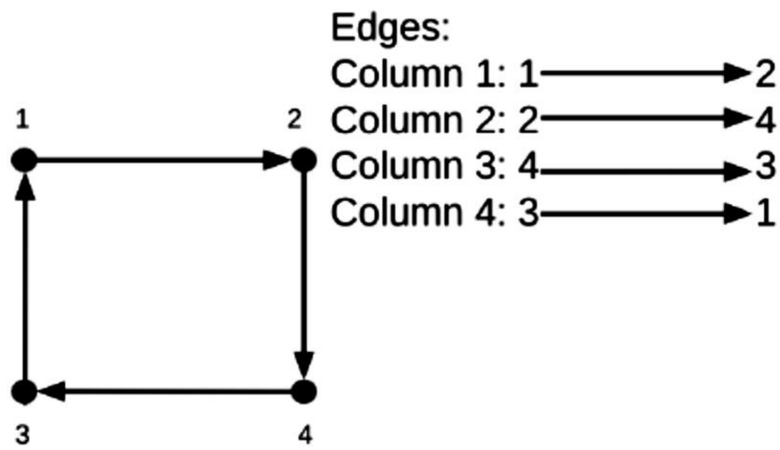

Figure 8: An image of the directed version of Graph A where each edge represents a particular column in the incidence matrix.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

This is the Incidence matrix for the direction version of Graph A. Again, 1 represents leaving a vertex, -1 represents arriving at a vertex, and 0 means that the vertex is not involved.

4.3 Laplacian matrix

In this section we will cover the two formulas used to find the Laplacian Matrix.

Definition 4.3. Laplacian Matrix can be defined as such: $L_0(L_0)^T$ where L0 represents the Incidence matrix and $L^T{}_0$ represents the transpose of an Incidence matrix.

If one wishes to find the Laplacian matrix of an undirected graph, they can assign directions to the edges in the graph. The Laplacian Matrix can also be found by using a much simpler formula known as $D - adj(G)$ In this formula, D is a matrix where diagonal entries are degrees of vertices and all other entries are 0 and $adj(G)$ represents the Adjacency matrix.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

The Incidence matrix or L_0 is being multiplied by the transpose of the Incidence matrix or $(L_0)^T$.

This equation shows the first equation that could have been used to find the Laplacian matrix. In this equation, the first matrix or the Incidence matrix which is represented by L_0, is multiplied by thetranspose of the Incidence matrix, or $L^T{}_0$. The product of these two factors is the Laplacian matrix or $L_0(L_0)^T$.

We also could have done...

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

This shows the adjacency matrix or $adj(G)$ is being subtracted from the diagonal matrix or D.

This equation represents the other equation that could have been used to find the Laplacian matrix. In this equation, the Adjacency matrix, or $adj(G)$ is subtracted from D or the Diagonal matrix. The difference between these two matrices is equal to the Laplacian matrix as well.

Remark: The directions assigned to a graph do not make a difference once the final Laplacian matrix is found. While the Incidence matrix will be different based on how the edges are directed, once the Incidence matrix is multiplied by its transpose, the Laplacian matrix will be the same no matter what the Incidence matrix is.

4.4 Diagonal Matrix

A square matrix in which every element except the principal diagonal elements is zero is called a Diagonal Matrix. A square matrix D = $[d_{ij}]$n x n will be called a diagonal matrix if $d_{ij} = 0$, whenever i is not equal to j.

4.4.1 Properties of Diagonal Matrix

Property 1: Same order diagonal matrices gives a diagonal matrix only after addition or multiplication.

Property 2: Transpose of the diagonal matrix D is as the same matrix. $D = D^T$

Property 3: Under Multiplication, Diagonal Matrices are commutative, i. e. PQ = QP

What is Block Diagonal Matrix?

A matrix which is split into blocks is called a block matrix. In such type of square matrix, off-diagonal blocks are zero matrices and main diagonal blocks square matrices. Here, the non-diagonal blocks are zero. D = 0 when i is not equal to j, then D is called a block diagonal matrix.

n x n order block diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & . & . & 0 \\ 0 & a_{22} & 0 & . & . & 0 \\ 0 & 0 & a_{33} & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Inverse of a Diagonal Matrix

If the elements on the main diagonal are the inverse of the corresponding element on the main diagonal of the D, then D is a diagonal matrix.

$$Let\ D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Determinants of the above matrix:

$$|D| = a_{11} \begin{bmatrix} a_{22} & 0 \\ 0 & a_{33} \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & a_{33} \end{bmatrix} + 0 \begin{bmatrix} 0 & a_{22} \\ 0 & 0 \end{bmatrix}$$

$= a_{11}a_{22}a_{33}$

$$AdjD = \begin{bmatrix} a_{22}a_{33} & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$$

$$D^{-1} = \frac{1}{|D|} adjD$$

=

$$\frac{1}{a_{11}a_{22}a_{33}} \begin{bmatrix} a_{22}a_{33} & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

Anti - Diagonal Matrix

If the entries in the matrix are all zero except the ones on the diagonals from lower left corner to the other upper side(right) corner are not zero, it is anti - diagonal matrix.

$$\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

The importance of matrices in our live

that matrices have a very great importance within society and the various fields of life, and the pen of their innovation is Al-Kharizmi, and that importance lies in several simple mathematical abbreviations, among the importance of matrices, the following:

- Matrices are used in many life and scientific applications such as mathematical applications or in some fields of science such as physics and chemistry. Matrices can also be used extensively in the representation of compact discs, while it consists of a huge number of numbers, by relying on alternatives, instead of making Many complex accounts

- Matrices are used in the process of statistics and probabilities, and it is a theory in which matrices are applied in the form of many random squares, through the so-called probability transfers, and that method is carried out through the so-called non-transferable process of negative results.
- Matrices are used in theories of great importance such as symmetry and transformations, and these theories are of great importance in the field of physics, and they are fundamental in modern physics, especially in the field of particles.
- Matrices are used in an important and fundamental way in many sciences and branches such as mechanics, physics, engineering optics and electromagnetism, quantum mechanics, analytical geometry, computer graphics, probability and statistics theories, 3D graphics processing , and in economics.

- Matrices are very important in a lot of scientific theories such as graph theory, theory of analysis and geometry, theory of linear combinations, theory of geometric optics , and theory of electronics

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