Republic of Iraq

Ministry of Higher Education and Scientific Research

University of Babylon

college of Education for Pure Sciences

Department of Mathematics

Elliptical Distributions

A graduation research submitted to the council of the Department of Mathematics in the college of Education for Pure Sciences / University of Babylon, as part of the requirements obtaining a bachelor's degree in mathematics.

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2026 A.D

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بسم الله الرحمن الرحيم

﴿ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ ﴾

صدق الله العلي العظيم

(سورة المجادلة، الآية 11)

الشكر والتقدير

أتقدم بخالص الشكر والتقدير إلى دكتورتي المُشرفة دكتوره رواسي عدنان حميد لما قدمته لي من توجيهات قيمة كان لها الأثر في إنجاز هذا البحث

كما أشكر السادة أعضاء اللجنة العلمية لتفضلهم بقراءة هذا العمل وتقييمه.

الاهداء

إلى من كان لهم الأثر الأسمى في مسيرتي العلمية، وإلى من غرسوا في نفسي حبّ المعرفة والسعي نحو التميز...

إلى والديّ الكريمين اللذين كانا ولا يزالان مصدر دعمي وقوتي وسببًا فيما وصلت إليه، فلهما مني خالص الامتنان وعظيم الوفاء.

إلى أساتذتي الأفاضل الذين أناروا دربي بالعلم، وأرشدوني إلى سُبل البحث الرصين والفكر المنهجي.

إلى كل من ساندني، ورافقني في هذه الرحلة العلمية، وأسهم بكلمة طيبة أو توجيه صادق...

أهدي هذا الجهد المتواضع، راجيًا أن يكون خطوة نافعة في طريق العلم والمعرفة

• Abstract

Elliptical distributions generalize multivariate normal models by defining density functions based on quadratic forms involving mean vectors and covariance matrices. This study presents their mathematical structure, key properties, and transformations, with emphasis on Mahalanobis distance and special cases. Analytical formulations and numerical examples are used to illustrate their behavior in multivariate statistical modeling.

Table of content

1. Introduction

Let $X \in R^n$ be a random vector defined on a probability space (Ω, F, P). In multivariate analysis, the primary objective is to study the joint behavior of components of X, including their dependence structure, dispersion, and geometric representation. Classical models such as the multivariate normal distribution assume specific structural properties, particularly light tails and linear dependence, which may not adequately describe real-world data.

Elliptical distributions arise as a natural generalization of multivariate normal distributions, preserving key mathematical properties while extending flexibility in modeling. A random vector X is said to follow an elliptical distribution if its probability density function depends on a quadratic form of the type:

$$(x - \mu)^T \Sigma^{-1} (x - \mu)$$

where $\mu \in R^n$ is the location vector and $\Sigma \in R^{n \times n}$ is a symmetric positive definite matrix.

The structure of this quadratic form induces level sets defined by:

$$(x-\mu)^T\Sigma^{-1}(x-\mu)=c$$

which represent ellipsoids in R^n. This geometric property is fundamental in understanding the distribution's behavior, as it links algebraic representation with spatial interpretation.

Elliptical distributions can be formally represented as:

$$X \sim EC_n(\mu, \Sigma, \psi)$$

where ψ is a characteristic generator function that determines the specific type of distribution. Different choices of ψ yield well-known distributions such as the multivariate normal, multivariate t, and Cauchy distributions.

The importance of elliptical distributions lies in their invariance under linear transformations. If:

$$Y = AX + b$$

then:

$$Y \sim EC_n(A\mu + b, A\Sigma A^T)$$

which simplifies analysis in higher dimensions.

This framework provides a unified mathematical model for representing multivariate data with symmetric structure while allowing for varying tail behavior, making it suitable for both theoretical analysis and applied statistical modeling.

2. Multivariate Framework and Statistical Structure

Let $X = (X_1, X_2, \ldots, X_n)^T$ be an n-dimensional random vector defined on a probability space. The fundamental structure of multivariate analysis is built upon two primary statistical quantities: the expectation vector and the covariance matrix. These are defined respectively as:

$$\mu = E[X]$$

$$\Sigma = E[(X - \mu)(X - \mu)^T]$$

where $\mu \in R^n$ and $\Sigma \in R^{n \times n}$, with Σ being symmetric and positive semi-definite.

In the context of elliptical distributions, these two parameters fully determine the location and dispersion structure, while the shape of the distribution is governed by a generator function. A key representation of elliptical random vectors is given by the stochastic decomposition:

$$X = \mu + RAU$$

where $R \geq 0$ is a scalar random variable controlling radial magnitude, A is a matrix such that $\Sigma = AA^T$, and U is a random vector uniformly distributed on the unit sphere in R^n, i.e.:

$$U^T U = 1$$

This representation highlights the decomposition of a multivariate distribution into radial and angular components. The vector U determines direction, while R determines distance from the center μ. This separation is fundamental in understanding elliptical symmetry.

The covariance structure plays a central role in shaping the geometry of the distribution. Using eigen decomposition, the covariance matrix can be written as:

$$\Sigma = Q\Lambda Q^T$$

where Q is an orthogonal matrix of eigenvectors and Λ is a diagonal matrix of eigenvalues:

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

This decomposition implies that the elliptical contours are aligned along eigenvectors, while the eigenvalues determine the scaling along each principal axis.

A transformation of variables can be defined using whitening transformation:

$$Z = \Sigma^{-1/2}(X - \mu)$$

which yields:

$$E[Z] = 0, \;\; \mathrm{Cov}(Z) = I_n$$

where I_n is the identity matrix. This transformation standardizes the elliptical distribution into a spherically symmetric form.

The quadratic form that defines elliptical distributions is given by:

$$Q(X) = (X - \mu)^T \Sigma^{-1} (X - \mu)$$

This scalar quantity plays a central role in both density functions and geometric interpretation. Specifically, level sets of this function define ellipsoids:

$$Q(X) = c$$

where c>0 is a constant. These level sets describe surfaces of equal probability density.

In addition, the spectral properties of Σ allow interpretation of variability in different directions. The variance along eigenvector q_i is given by:

$$\mathrm{Var}(q_i^T X) = \lambda_i$$

which shows that dispersion is direction-dependent.

Another important property is invariance under affine transformations. For any matrix $A \in R^{m\times n}$ and vector $b \in R^m$, define:

$$Y = AX + b$$

Then:

$$E[Y] = A\mu + b$$

$$\mathrm{Cov}(Y) = A\Sigma A^T$$

This property ensures that elliptical distributions remain within the same family under linear transformations, preserving structural consistency.

The multivariate framework also introduces the concept of standardized distance, known as Mahalanobis distance:

$$D^2 = (X - \mu)^T \Sigma^{-1} (X - \mu)$$

which measures deviation accounting for covariance structure. Unlike Euclidean distance, this metric adapts to correlation between variables.

Overall, the multivariate structure of elliptical distributions combines linear algebra, probability theory, and geometry into a unified framework. This allows for a consistent mathematical treatment of high-dimensional data, forming the basis for further analytical derivations and computational applications.

3. Elliptical Distribution Definition and Generator function

An n-dimensional random vector $X \in R^n$ is said to follow an elliptical distribution if its probability density function can be expressed in the form:

$$f_X(x) = k_n|\Sigma|^{-\frac{1}{2}}, g!\,((x-\mu)^T\Sigma^{-1}(x-\mu))$$

where $\mu \in R^n$ is the location parameter, $\Sigma \in R^{n\times n}$is a symmetric positive definite scale matrix, $g(\cdot)$is a non-negative generator function, and k_n is a normalizing constant ensuring:

$$\int_{R^n} f_X(x), dx = 1$$

The scalar quantity:

$$Q(x) = (x-\mu)^T\Sigma^{-1}(x-\mu)$$

is the fundamental quadratic form that governs the structure of the distribution. All elliptical distributions are functions of this single scalar argument, which implies that their level sets satisfy:

$$Q(x) = c$$

for constant c>0. These level sets define ellipsoids in R^n, confirming the geometric interpretation of the distribution.

Different choices of the generator function g(·)produce different members of the elliptical family.

For example:

- Multivariate Normal:

$$g(t) = \exp\left(-\frac{1}{2}t\right)$$

- Multivariate t-distribution:

$$g(t) = \left(1 + \frac{t}{\nu}\right)^{-\frac{\nu+n}{2}}$$

- Multivariate Cauchy:

$$g(t) = (1 + t)^{-\frac{n+1}{2}}$$

This unified formulation allows all elliptical distributions to share common properties such as symmetry about, dependence on, and invariance under affine transformations. The structure is entirely determined by (x), making it the central component of the theory.

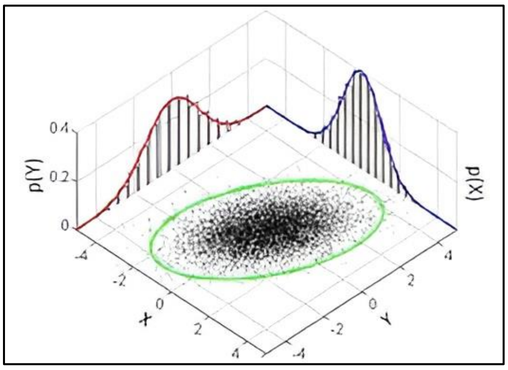

Figure 1

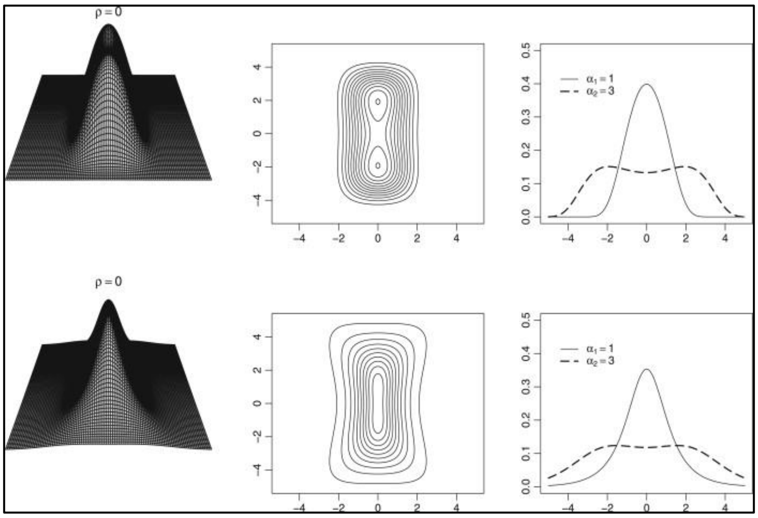

Figure 2

4. Mahalanobis Distance and Geometric Interpretation

The Mahalanobis distance is a fundamental measure associated with elliptical distributions and is defined as:

$$D^2(x) = (x - \mu)^T \Sigma^{-1} (x - \mu)$$

This quantity represents a normalized squared distance between a point x and the center μ, taking into account the covariance structure Σ. Unlike the Euclidean distance, it incorporates correlations between variables and scale differences, making it invariant under linear transformations.

Geometrically, the condition:

$$D^2(x) = c$$

defines a family of concentric ellipsoids centered at μ. Each ellipsoid corresponds to points with equal probability density under an elliptical distribution. The shape and orientation of these ellipsoids are determined by the eigenvalues and eigenvectors of Σ, where larger eigenvalues correspond to elongated axes.

By performing the transformation:

$$Z = \Sigma^{-\frac{1}{2}} (X - \mu)$$

the Mahalanobis distance reduces to:

$$D^2(x) = Z^T Z$$

which represents the squared Euclidean norm in the transformed space. This shows that elliptical distributions can be interpreted as spherical distributions under an appropriate linear transformation.

In statistical applications, large values of $D^2(x)$ indicate potential outliers, often evaluated using chi-square thresholds:

$$D^2(x) \sim \chi^2_n$$

This property is widely used in multivariate hypothesis testing and anomaly detection.

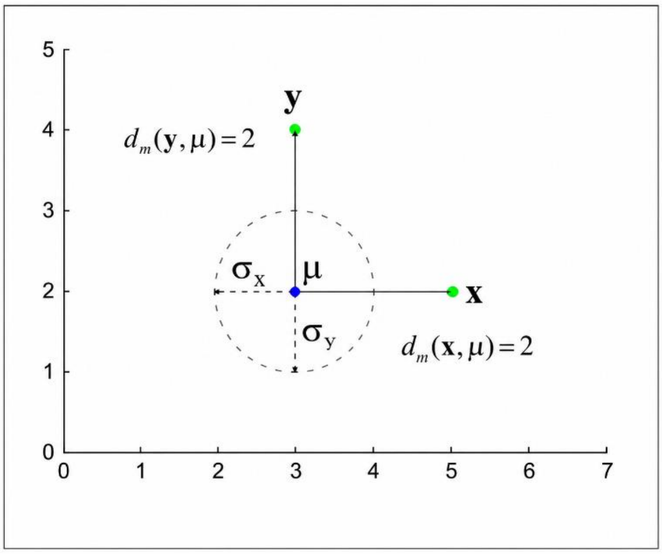

Figure 3

5. Linear Transformations and Stability of Elliptical Family

A key property of elliptical distributions is their invariance under affine transformations. Let $X \sim EC_n(\mu, \Sigma, g)$, and consider a linear transformation:

$$Y = AX + b$$

where $A \in R^{m \times n}$ and $b \in R^m$. Then the transformed vector Y also follows an elliptical distribution:

$$Y \sim EC_m(A\mu + b, A\Sigma A^T, g)$$

The expectation and covariance transform as:

$$E[Y] = A\mu + b$$

$$Cov(Y) = A\Sigma A^T$$

This property ensures structural stability of the elliptical family under scaling, rotation, and translation.

Moreover, the quadratic form transforms consistently:

$$(x - \mu)^T \Sigma^{-1} (x - \mu) = (y - A\mu - b)^T (A\Sigma A^T)^{-1} (y - A\mu - b)$$

This invariance confirms that the elliptical geometry is preserved in transformed spaces, maintaining ellipsoidal level sets and consistent probabilistic interpretation across dimensions.

6. Special Cases and Limiting Distributions

Elliptical distributions include several important special cases that are widely used in statistical modeling. The most fundamental case is the multivariate normal distribution, defined by:

$$f(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \mathrm{ex}\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

which corresponds to the generator function $g(t) = \exp(\frac{-t}{2})$. This model is widely used due to its mathematical simplicity and strong theoretical properties.

Another important case is the multivariate t-distribution:

$$f(x) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{n/2}|\Sigma|^{\frac{1}{2}}}\left(1+\frac{1}{\nu}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)^{-\frac{\nu+n}{2}}$$

As $\nu \to \infty$, it converges to the multivariate normal distribution. For small ν, it exhibits heavier tails.

The multivariate Cauchy distribution is obtained when $\nu = 1$:

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}|\Sigma|^{\frac{1}{2}}}(1+(x-\mu)^T\Sigma^{-1}(x-\mu))^{-\frac{n+1}{2}}$$

These limiting cases highlight the flexibility of elliptical distributions in modeling both light-tailed and heavy-tailed data within a unified mathematical framework.

7. Covariance Structure and Eigen Decomposition

The covariance matrix $\Sigma \in R^{n\times n}$ plays a central role in defining the geometry of elliptical distributions. Since Σ is symmetric and positive definite, it admits an eigen decomposition of the form:

$$\Sigma = Q\Lambda Q^T$$

where Q is an orthogonal matrix whose columns are eigenvectors q_i, and Λ is a diagonal matrix containing eigenvalues $\lambda_i > 0$:

$$\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

Using this decomposition, the quadratic form becomes:

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = (x-\mu)^T Q\Lambda^{-1}Q^T(x-\mu)$$

By defining a rotated coordinate system:

$$z = Q^T(x-\mu)$$

the expression simplifies to:

$$D^2 = \sum_{i=1}^{n} \frac{z_i^2}{\lambda_i}$$

This shows that each eigenvalue λ_i controls the scaling along the corresponding eigenvector direction. Larger eigenvalues produce wider spread, while smaller eigenvalues produce tighter concentration.

Thus, the ellipsoidal geometry is fully determined by the spectral properties of Σ, where eigenvectors define orientation and eigenvalues define axis lengths.

8. Semi Elliptical (Spherical cases) and Relation to Elliptical Distributions

A special case of elliptical distributions arises when the covariance matrix takes the form:

$$\Sigma = I_n$$

where I_n is the identity matrix. In this case, the quadratic form simplifies to:

$$(x - \mu)^T(x - \mu) = |x - \mu|^2$$

and the density function becomes:

$$f(x) = k \cdot g(|x - \mu|^2)$$

Such distributions are referred to as spherical distributions, as their iso-density contours are spheres centered at μ.

More generally, any elliptical random vector:

$$X \sim EC_n(\mu, \Sigma)$$

can be transformed into a spherical form through the linear transformation:

$$Z = \Sigma^{\frac{-1}{2}}(X - \mu)$$

which satisfies:

$$E[Z] = 0, \; Cov(Z) = I_n$$

Thus, spherical distributions represent a standardized form of elliptical distributions, where the covariance structure has been removed. This transformation shows that elliptical distributions can be viewed as scaled and rotated versions of spherical distributions, preserving their fundamental geometric structure.

9. Properties of Elliptical Distributions

Elliptical distributions possess several fundamental mathematical properties that arise directly from their dependence on the quadratic form:

$$Q(x) = (x - \mu)^T \Sigma^{-1} (x - \mu)$$

In the two-dimensional case ($n = 2$), the iso-density contours defined by:

$$f(x_1, x_2) = c$$

are ellipses or unions of ellipses. More generally, in R^n, these contours satisfy:

$$(x - \mu)^T \Sigma^{-1} (x - \mu) = c$$

which represent ellipsoids centered at μ. All such ellipsoids are scaled versions of one another.

A fundamental special case is obtained when the generator function is:

$$g(z) = e^{\frac{-z}{2}}$$

which yields the multivariate normal distribution. In this case, the support is unbounded since:

$$g(z) > 0 \ \ \forall z \geq 0$$

However, in general, elliptical distributions may be bounded or unbounded. A distribution is bounded if:

$$g(z) = 0 \ \ \text{for } z > c$$

for some constant c.

Due to the quadratic dependence on x, all elliptical distributions are symmetric about μ:

$$X - \mu \overset{d}{=} -(X - \mu)$$

Certain elliptical distributions, such as the Cauchy distribution, do not possess finite moments. In such cases:

$$E[X] \text{ does not exist}$$

Let X be partitioned into sub-vectors:

$$X = (X_1, X_2)$$

If $\mathrm{Cov}(X_1, X_2) = 0$, then (provided expectations exist):

$$E[X_1 \mid X_2] = E[X_1]$$

which implies mean independence.

Elliptical distributions are also closed under linear transformations. For any matrix D with full row rank:

$$Y = DX$$

then: $Y \sim EC$

This implies that any linear combination:

$$a^T X$$

and any sub-vector of X is also elliptically distributed.

These properties collectively define the structural behavior of elliptical distributions and distinguish them from general multivariate models.

10. Applications of Elliptical Distributions

Elliptical distributions are widely used in multivariate statistical modeling due to their dependence on the quadratic form:

$$D^2 = (x - \mu)^T \Sigma^{-1} (x - \mu)$$

In **statistical analysis**, this measure is applied for outlier detection by comparing:

$$D^2 \sim \chi_n^2$$

In **financial modeling,** asset return vectors are often represented as:

$$R \sim EC_n(\mu, \Sigma)$$

and portfolio returns are expressed as:

$$P = w^T R$$

where w is a weight vector.

Elliptical distributions are also used in **robust statistics**, particularly through heavy-tailed models such as the multivariate t-distribution:

$$g(t) = \left(1 + \frac{t}{\nu}\right)^{-\frac{\nu+n}{2}}$$

which improves resistance to extreme observations.

Additionally, their invariance under linear transformations:

$$Y = AX + b$$

makes them suitable for dimensionality reduction and multivariate data analysis. These applications demonstrate the practical importance of elliptical distributions in modeling complex correlated data.

11.Examples

Example 1: Computation of Mahalanobis Distance

Let $X \in R^2$ be a random vector with parameters:

$$\mu = [1 \quad 2], \ \Sigma = [2 \quad 1 \quad 1 \quad 3]$$

Consider the observation:

$$x = [4 \quad 6]$$

The objective is to compute the Mahalanobis distance:

$$D^2 = (x - \mu)^T \Sigma^{-1} (x - \mu)$$

solution

Step 1: Compute the deviation vector

$$x - \mu = [4 - 1\ 6 - 2][3 \quad 4]$$

Step 2: Compute the determinant of Σ

$$|\Sigma| = (2)(3) - (1)(1) = 6 - 1 = 5$$

Step 3: Compute the inverse of Σ

$$\Sigma^{-1} = \frac{1}{5}[3 \quad -1\ -1 \quad 2]$$

Step 4: Multiply

$$\Sigma^{-1}(x - \mu) = \frac{1}{5}[3 - 1 - 12][3\ 4]\frac{1}{5}[(3)(3) + (-1)(4)(-1)(3) + (2)(4)]\frac{1}{5}[9 - 4\ -3 + 8]\frac{1}{5}[5 \quad 5][1 \quad 1]$$

Step 5: Compute the final distance

$$D^2 = [3 \quad 4][1 \quad 1] = 3 + 4 = 7$$

Result:

$$D^2 = 7$$

Interpretation

For n = 2, compare with the chi-square threshold:

$$\chi^2_{0.95,2} \approx 5.99$$

Since:

$$7>5.99$$

The observation lies outside the typical elliptical region and can be considered a potential outlier.

This example demonstrates how Mahalanobis distance incorporates covariance structure to measure deviation more accurately than Euclidean distance.

Example 2: Covariance Matrix Computation

Let a dataset consist of the following observations in R^2:

$$x_1 = [1 \quad 2],\ x_2 = [2 \quad 3],\ x_3 = [3 \quad 5]$$

Step 1: Compute the mean vector

$$\mu = \frac{1}{3}(x_1 + x_2 + x_3)\frac{1}{3}[1 + 2 + 3\ 2 + 3 + 5]\left[2 \quad \frac{10}{3}\right]$$

Step 2: Compute deviations

$$x_1 - \mu = \begin{bmatrix} -1 & -\frac{4}{3} \end{bmatrix}, \qquad x_2 - \mu = \begin{bmatrix} 0 & -\frac{1}{3} \end{bmatrix}, \qquad x_3 - \mu = \begin{bmatrix} 1 & \frac{5}{3} \end{bmatrix}$$

Step 3: Compute covariance matrix

$$\Sigma = \frac{1}{3}\sum_{i=1}^{3}(x_i - \mu)\,(x_i - \mu)^T$$

$$\Sigma = \frac{1}{3}\left(\begin{bmatrix} 1 & \frac{4}{3} \\ \frac{4}{3} & \frac{16}{9} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{9} \end{bmatrix} + \begin{bmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & \frac{25}{9} \end{bmatrix}\right) \quad \begin{bmatrix} \frac{2}{3} & 1 \\ 1 & \frac{14}{9} \end{bmatrix}$$

Result:

$$\Sigma = \begin{bmatrix} \frac{2}{3} & 1 & 1 & \frac{14}{9} \end{bmatrix}$$

This matrix describes both variance and correlation between variables, forming the basis of elliptical geometry.

Example 3: Inverse of Covariance Matrix

Let the covariance matrix be:

$$\Sigma = [4 \quad 2 \qquad 2 \quad 3]$$

Step 1: Compute determinant

$$|\,\Sigma\,| = (4)(3) - (2)(2) = 12 - 4 = 8$$

Step 2: Compute inverse matrix

$$\Sigma^{-1} = \frac{1}{|\Sigma|}[3 \quad -2 \; -2 \quad 4]$$

$$\Sigma^{-1} = \frac{1}{8}[3 \quad -2 \; -2 \quad 4]$$

Result:

$$\Sigma^{-1} = \left[\frac{3}{8} \quad -\frac{1}{4} \; -\frac{1}{4} \quad \frac{1}{2}\right]$$

This inverse matrix is essential in elliptical distributions since it directly appears in the quadratic form:

$$(x-\mu)^{T}\Sigma^{-1}(x-\mu)$$

It determines the scaling and orientation of the ellipsoidal contours.

Example 4: Outlier Detection Using Chi-Square Rule

Let a bivariate observation be:

$$x = [5 \quad 8]$$

with parameters:

$$\mu = [2 \quad 3], \;\; \Sigma = [2 \quad 0 \quad\quad 0 \quad 2]$$

Step 1: Compute deviation

$$x - \mu = [3 \quad\quad 5]$$

Step 2: Inverse covariance matrix

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Step 3: Mahalanobis distance

$$D^2 = (x-\mu)^T \Sigma^{-1} (x-\mu)$$

$$D^2 = [3 \quad 5] \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} [3\ 5]$$

$$D^2 = \frac{1}{2}(9+25) = \frac{34}{2} = 17$$

Step 4: Decision rule

For n = 2:

$$\chi^2_{0.95,2} \approx 5.99$$

Since:

$$17 > 5.99$$

Result:

The observation is classified as an outlier.

This demonstrates how elliptical distributions are used in statistical decision-making through distance-based thresholds.

Example 5: Linear Transformation of Elliptical Vector

Let:

$$X \sim EC_2(\mu, \Sigma)$$

where:

$$\mu = [1\ 2],\ \ \Sigma = [2 \quad 1 \qquad 1 \quad 2]$$

Consider the linear transformation:

$$Y = AX + b$$

where:

$$A = [1 \quad 2 \qquad 0 \quad 1],\ \ b = [1\ \ 0]$$

Step 1: Mean transformation

$$E[Y] = A\mu + b$$

$$A\mu = [1 \quad 2 \qquad 0 \quad 1]\,[1 \qquad 2]\,[5 \qquad 2]$$

$$E[Y] = [6 \qquad 2]$$

Step 2: Covariance transformation

$$\mathrm{Cov}(Y) = A\Sigma A^T$$

$$A\Sigma = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\mathrm{Cov}(Y) = \begin{bmatrix} 14 & 9 \\ 9 & 4 \end{bmatrix}$$

Result:

$$Y \sim EC_2(A\mu + b, A\Sigma A^T)$$

This example shows that elliptical distributions remain closed under linear transformations, preserving their structural form.

Example 6: Spherical Transformation

Let:

$$X \sim EC_2(\mu, \Sigma)$$

where:

$$\mu = [1 \quad 2], \ \Sigma = [4 \quad 0 \quad 0 \quad 9]$$

Step 1: Compute $\Sigma^{-\frac{1}{2}}$

$$\Sigma^{-\frac{1}{2}} = \left[\frac{1}{2} \quad 0 \quad 0 \quad \frac{1}{3}\right]$$

Step 2: Apply transformation

$$Z = \Sigma^{-\frac{1}{2}}(X - \mu)$$

Step 3: Properties of Z

$$E[Z] = 0$$

$$Cov(Z) = I_2$$

Result:

$$Z \sim EC_2(0, I_2)$$

This transformation converts an elliptical distribution into a spherical one, where:

$f(z) = g(z^T z)$

Thus, the covariance structure is removed, and the distribution depends only on the Euclidean norm:

$$|z|^2$$

This example demonstrates how elliptical distributions can be standardized into spherical form, simplifying analysis and interpretation.

Example 7: Comparison Between Normal and t Elliptical Distributions

Consider two elliptical distributions with the same parameters:

$$\mu = [0\ 0],\ \ \Sigma = [1 \quad 0 \qquad 0 \quad 1]$$

Case 1: Multivariate Normal

$$f_N(x) = \frac{1}{2\pi} \exp\left(-\frac{1}{2} x^T x\right)$$

Case 2: Multivariate t (with $\nu = 2$)

$$f_t(x) = C\left(1 + \frac{x^T x}{2}\right)^{-2}$$

Comparison

For large values of $|x|$:

$$f_N(x) \to 0 \ \text{ exponentially}$$

$$f_t(x) \to 0 \ \text{ polynomially}$$

Result

Normal distribution:

- Light tails
- Rapid decay

T-distribution:

- Heavy tails
- Higher probability for extreme values

This shows that different generator functions $g(t)$ lead to different tail behaviors, even when $\mu$and Σ are identical.

12.Conclusion

Elliptical distributions provide a unified mathematical framework for modeling multivariate data through the quadratic form $\left((x-\mu)^T\Sigma^{-1}(x-\mu)\right)$. This structure yields ellipsoidal iso-density contours centered at (μ), linking algebraic representation with geometric interpretation. Core properties central symmetry, affine invariance, and closure under linear transformations—ensure analytical tractability, while the covariance matrix (Σ) governs orientation and scale via its spectral decomposition.

Special cases such as the multivariate normal, t, and Cauchy distributions demonstrate how different generator functions control tail behavior, enabling both light- and heavy-tailed modeling within the same class. The spherical transformation ($Z=\Sigma^{\frac{-1}{2}}(X-\mu)$) further reveals that all elliptical models are standardized versions of a spherically symmetric form.

Practically, the Mahalanobis distance (D^2) provides a covariance-aware metric for inference and outlier detection, often assessed via chi-square thresholds. Applications in multivariate statistics, robust modeling, and finance (e.g., portfolio returns ($w^T R$)) highlight the versatility of this family. Overall, elliptical distributions balance mathematical rigor with modeling flexibility, making them a powerful tool for analyzing correlated, high-dimensional data.

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