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Relation Between Gamma and Chi-square Distributions

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿ يَرْفَعْ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ

أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ ﴾

صدق الله العلي العظيم

سورة المجادلة

الآية ١١

Dedication

We must and we move past our steps in the university life of the pause and go back to the years we spent in the university campus with the esteemed our professors who have given us so much effort great efforts in building tomorrow's generation to send the nation again ...

Before we offer our deepest gratitude and appreciation and love to those who carried the message holiest in life to those who paved the way for us science and knowledge ... All our professors Distinguished

To the fountain of patience and optimism and hope To each of the following in the presence of God and His Messenger, our dear parents….

To the taste of the most beautiful moments with our friends

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First and foremost, we would like to express all our undeniable and unforgettable thanks to the most graceful and most compassionate the almighty **ALLAH** that has provided us with lot of blessings that can never be counted.

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Contents

Introduction

Understanding and analyzing probabilistic data is fundamental in many fields, from the natural sciences to the social sciences. The kamma distribution and chi-square distribution play an important role in this context, as they allow researchers to effectively analyze data and understand the probability distributions of different phenomena.

This research aims to explore the concepts of the gamma distribution and chi-square distribution, and explain how they are used to analyze data and make predictions. In this context, the basic theories of both distributions will be reviewed and the focus will be on practical applications that can benefit a variety of research fields.

Objectives

- To learn a formal definition of the probability density function of a (continuous) exponential random variable.
- To learn key properties of an exponential random variable, such as the mean, variance, and moment generating function.
- To understand the steps involved in each of the proofs in the lesson.
- To be able to apply the methods learned in the lesson to new problems. To understand the motivation and derivation of the probability density function of a (continuous) gamma random variable.
- To understand the effect that the parameters α and θ have on the shape of the gamma probability density function.
- To learn a formal definition of the gamma function.
- To learn a formal definition of the probability density function of a gamma random variable.
- To learn key properties of a gamma random variable, such as the mean, variance, and moment generating function.
- To learn a formal definition of the probability density function of a chi-square random variable.
- To understand the relationship between a gamma random variable and a chi-square random variable.
- To learn key properties of a chi-square random variable, such as the mean, variance, and moment generating function.
- To learn how to read a chi-square value or a chi-square probability off of a typical chi-square cumulative probability table.

The importance of research and the relationship between them

The relationship between the gamma distribution and the chi-square distribution is considered an important relationship in the field of statistics and probability theory, as each of them is used in different contexts and for different purposes. However, there is a similarity and relationship between them.

Gamma Distribution:

The gamma distribution is generally used to model events that occur over a certain period of time, such as the expected life of products, or the waiting time for a service in service sectors, etc.
The gamma distribution represents many natural and economic phenomena that require a distribution of time variables.

Chi-Square Distribution:

The chi-square distribution is mainly used in analyzing statistical data, especially when testing hypotheses about the difference between two means or distributions.
The chi-square distribution is used in statistical hypothesis tests such as testing the hypothesis of non-correlation and testing the hypothesis of normal distribution.(Markovic, R.D. :1965)

The relationship between them:

The relationship between the gamma distribution and the chi-square distribution is that the chi-square distribution can be obtained as a result of collecting independent random variables distributed according to the gamma distribution, when it is raised to the square exponent. Thus, the chi-square distribution can be related to the gamma distribution when it is presented as a product of the gamma distribution. (Rameshwar, D. :2001)

Chapter One

(1.1) Gamma distribution

The gamma distribution is a type of probability distribution important in statistics and mathematics, and is a fundamental part of gamma distributions. The gamma distribution is the distribution of non-negative and heterogeneous random numbers, and it can have wide applications in fields such as natural sciences, engineering, biology, economics, etc. **(Glen-b (2014))**

The gamma distribution laws depend on two basic parameters: the shape parameter and the scale parameter. The main laws of kama distribution include:

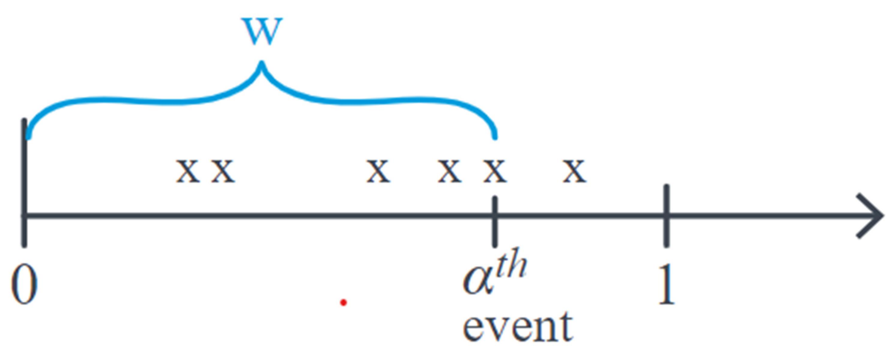
1. **Gamma Distribution**: Gamma distribution is used to model events that are related to time, such as the duration of production operations or the waiting period before a specific event occurs. The shape and appearance of the gamma distribution is controlled by the shape rate and the restriction rate. (Grimmett and stirzaker, 2001).

2. **Beta Distribution** : The beta distribution represents a continuous probability distribution that is often used to model probability success/failure ratios. The shape and appearance of the beta distribution is controlled using the shape rate and the bound rate. (Gauss et al., 2012)

3. **Incomplete Gamma Distribution** : It is a type of gamma distribution that appears in calculating the numerical integration of beta functions.Gamma distribution laws contain several important properties, such as the mean, standard deviation, and shape of the distribution. These laws have the ability to represent a variety of real-world phenomena, making them a valuable tool in analyzing data and predicting future events. (Saralees, 2008)

(1.2) Gamma Distributions :

In the previous lesson, we learned that in an approximate Poisson process with mean **λ**, the waiting time **X** until the first event occurs follows an exponential distribution with mean $\theta = \frac{1}{\lambda}$. We now let **W** denote the waiting time until the **a**th event occurs and find the distribution of **W**. we could represent the situation as (Montenegro :2012)



(1.3) The Gamma Function :

The gamma function, denoted $\boldsymbol{\Gamma(t)}$, is defined, for t > 0, by:

$$\Gamma(t) = \int_0^{\infty} y^{t-1} e^{-y} dy$$

We'll primarily use the definition in order to help us prove the two theorems that follow.

(1.3.1) Theorem :

Provided t > 1 :

$$\boldsymbol{\Gamma(t) = (t-1) \times \Gamma(t-1)}$$

Proof

We'll use integration by parts with**:**

u=y $^{t-1}$ ***and dv =e*** $^{-y}$***dy***

to get:

$$du = (t-1)y^{t-2} \text{ and } v = -e^{-y}$$

Then, the integration by parts gives us:

$$\Gamma(t) = \lim_{b\to\infty}[-y^{+t}e - y]_{y=0}^{x_{l=b}} \quad (t-1) \quad = \int_0^\infty y^{t-1}e^{-y}dy$$

Evaluating at y =b and y=0 for the first term, and using the definition of

The gamma function (provided t-1>0) for the second term we have :

$$\Gamma(t) = \lim_{b\to\infty}[\frac{b^{t-1}}{e^b}] + (t-1)\Gamma(t-1)$$

Now, if we were to be lazy, we would just wave our hands, and say that the first term goes to 0, and therefore:

$$\Gamma(t) = (t-1) \times \Gamma(t-1)$$

Provided t > 1 , as was to be proved.
Let's not be too lazy though! Taking the limit as b goes to infinity for that first term, we get infinity over infinity . Ugh! Maybe we should have left well enough alone! We can take the exponent and the natural log of the numerator without changing the limit. Doing so, we get:

$$-\lim_{b\to\infty}[\frac{b^{t-1}}{e^b}] = -\lim_{b\to\infty}\{\frac{exp\,[(t-1)ln\,b]}{exp\,(b)}\}$$

Then, because both the numerator and denominator are exponents, we can write the limit as:

$$-\lim_{b\to\infty}[\frac{b^{t-1}}{e^b}] = -\lim_{b\to\infty}\{exp\,[(t-1)ln\,b - b]\} \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Manipulating the limit a bit more, so that we can easily apply Hospital's Rule, we get:

$$-\lim_{b\to\infty}[\frac{b^{t-1}}{e^b}] = -\lim_{b\to\infty}\{exp\,[(t-1)\,b\left(\frac{ln\,b}{b}-1\right)]\}$$

Now, let's take the limit as b goes to infinity:

(1.3.2) Theorem :

If t = n , a positive integer, then

$$\boldsymbol{\Gamma(n) = (n-1)!}$$

Proof

Using the previous theorem :

$$\Gamma(n) = (n-1)\Gamma(n-1)$$
$$= (n-1)(n-2)\,\Gamma(n-2)$$
$$= (n-1)(n-2)(n-3)\ldots(2)(1)\,\Gamma(1)$$

And, since by the definition of the gamma function

$$\Gamma(1) = \int_0^\infty y^{t-1}e^{-y}dy = \int_0^\infty e^{-y}dy = 1$$

We have

$$\boldsymbol{\Gamma(n) = (n-1)!}$$

As was to be proved

(1.4) Gamma Properties

A continuous random variable ***X*** follows a gamma distribution with parameters $\Theta > 0$ and $\alpha > 0$ if its probability density function is:

$$\int(X) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\; x^{\alpha-1}\,e^{-x/\theta}$$

For $\boldsymbol{x} > \theta$

Before we get to the three theorems and proofs, two notes:
1. We consider $\boldsymbol{\alpha > 0}$ positive integer if the derivation of the p.d.f. is motivated by waiting times until α events. But the p.d.f. is actually a valid p.d.f. for any $\boldsymbol{\alpha > 0}$ (since $\boldsymbol{\Gamma(\alpha)}$ is defined for all positive α).
2. The gamma p.d.f. reaffirms that the exponential distribution is just a special case of the gamma distribution. That is, when you put α = 1 into the gamma p.d.f., you get the exponential p.d.f.

Gamma Example :

Engineers designing the next generation of space shuttles plan to include two fuel pumps —one active, the other in reserve. If the primary pump malfunctions, the second is automatically brought on line. Suppose a typical mission is expected to require that fuel be pumped for at most 50 hours. According to the manufacturer's specifications, pumps are expected to fail once every 100 hours. What are the chances that such a fuel pump system would not remain functioning for the full 50 hours? **(** Balakrishnan 2009)

Example:

We are given that λ , the average number of failures in a 100-hour interval is 1. There fore,Θ , the mean waiting time until the first failure is $\frac{1}{\lambda}$, or 100 hours. Knowing that, let's now let Y denote the time elapsed until the α = 2nd pump breaks down. Assuming the failures follow a Poisson process, the probability density function of is Y:

$$\int y(\mathbf{y}) = \frac{1}{100^2\Gamma\,(2)}\; e^{-y/100}\, y^{2-1} = \frac{1}{10000}\; ye^{-y/100}$$

for y > 0. Therefore, the probability that the system fails to last for 50 hours is:

$$\text{p(y<50)} = \int_0^{50} \frac{1}{10000}\; ye^{-y/100}\, dy$$

Integrating that baby is going to require integration by parts. Let's let:

$$\text{u = y and du} = e^{-y/100}$$

So that :

$$\text{du = dy and u} = 100e^{-y/100}$$

(1.5) Derivation of the Probability Density Function

Just as we did in our work with deriving the exponential distribution, our strategy here is going to be to first find the cumulative distribution function F(w) and then differentiate it to get the probability density function f (w) . Now, for $w > 0$ and $\lambda > 0$, the definition of the cumulative distribution function gives us: (Saralees Nadarajah :2008)

$$F(W) = P(W \leq W)$$

The rule of complementary events tells us then that:

$$F(W) = 1 - P(W \geq W)$$

Now, the waiting time W is greater than some value ***W*** only if there are fewer than α events in the interval [0,**W**] . That is:

F(w) =1-p (fewer than α events in [0,w])

A more specific way of writing that is:

F(w) =1-p (0 events or 1 event or ….or (α – 1) events in [0,w])

Those mutually exclusive "ors" mean that we need to add up the probabilities of having 0 events occurring in the interval [0 ,w] , 1 event occurring in the interval [0 ,w] , ..., up to **(α – 1)** events in[0 ,w] . Well, that just involves using the probability mass function of a Poisson random variable with mean λw . That is:

$$f(w) = \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{(-\lambda w)}}{k!}$$

Now, we could leave F (w) well enough alone and begin the process of differentiating it, but it turns out that the differentiation goes much smoother if we rewrite F(w) as follows

$$\mathrm{f}(w) = 1 - e^{-\lambda w} \sum_{k=1}^{\alpha-1} \frac{1}{k!} \left[(\lambda w)^{-k} e^{-\lambda w} \right]$$

As you can see, we merely pulled the k = 0 out of the summation and rewrote the probability mass function so that it would be easier to administer the product rule for differentiation.

Now, let's do that differentiation! We need to differentiate F(w) with respect to w to get the probability density function f(w) . Using the product rule, and what we know about the derivative of $e^{\lambda w}$ and $(\lambda w)^{k}$, we get :

$$\boldsymbol{f}(\boldsymbol{w}) = \boldsymbol{F}'^{(\boldsymbol{w})} = \boldsymbol{\lambda} \boldsymbol{e}^{-\boldsymbol{\lambda w}} - \sum_{k=1}^{\alpha-1} \frac{1}{k!} \left[(\boldsymbol{\lambda w})^{\boldsymbol{k}} . \left(-\boldsymbol{\lambda w}^{-\boldsymbol{\lambda w}}\right) \boldsymbol{e}^{-\boldsymbol{\lambda w}} . \boldsymbol{k} (\boldsymbol{\lambda w})^{-\boldsymbol{k}} . \boldsymbol{\lambda} \right]$$

Pulling $\boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}}$ out of the summation, and dividing k by k! (to get $\frac{\mathbf{1}}{(\boldsymbol{k}-\mathbf{1})!}$) in the second term in the summation, we get that $\boldsymbol{f}(\boldsymbol{w})$ equals:

$$\boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} + \boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} \left[\sum_{k=0}^{\alpha-1} \left\{ \frac{(\lambda w)^{k}}{k!} - \frac{(\lambda w)^{k-1}}{(k-1)!} \right\} \right]$$

Evaluating the terms in the summation at k =1 , k=2 ,up to k =α-1 , we get that $\boldsymbol{f}(\boldsymbol{w})$ equals :

$$\boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} + \boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} \left[(\boldsymbol{\lambda w} - 1) + \left(\frac{(\lambda w)^{2}}{2!} \boldsymbol{\lambda w} \right) + .. + \left(\frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} - \frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} \right) \right]$$

Do some (lots of!) crossing out ($\boldsymbol{\lambda w} - \boldsymbol{\lambda w} = \mathbf{0}$, for example), and a bit more simplifying to get that $\boldsymbol{f}(\boldsymbol{w})$ equals:

$$\boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} + \boldsymbol{\lambda e}^{-\boldsymbol{\lambda}} \left[-1 + \frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} \right] = \boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} - \boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} + \frac{\boldsymbol{\lambda e}^{-\boldsymbol{\lambda}} \ (\boldsymbol{\lambda e})^{\boldsymbol{\alpha}-\boldsymbol{\lambda w}}}{(\alpha-1)!}$$

And since $\boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} = \boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} = \mathbf{0}$, we get that $\boldsymbol{f}(\boldsymbol{w})$ equals:

$$= \frac{\boldsymbol{\lambda e}^{-\boldsymbol{\lambda w}} (\boldsymbol{\lambda e})^{\boldsymbol{\alpha}-\boldsymbol{\lambda w}}}{(\alpha-1)!}$$

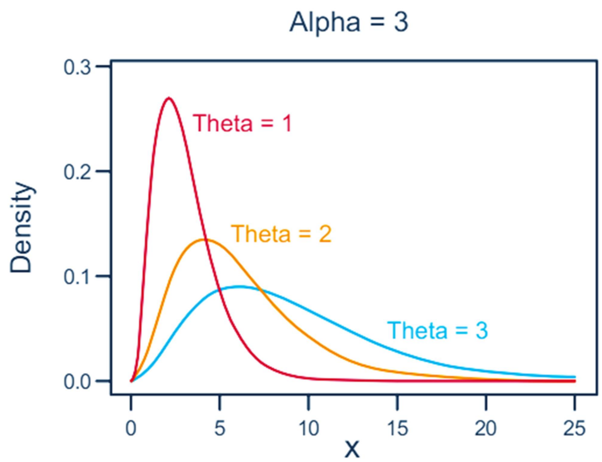
Are we there yet? Almost! We just need to reparametrize (if $\Theta = \frac{\mathbf{1}}{\boldsymbol{\lambda}}$, then $\lambda = \frac{\mathbf{1}}{\Theta}$). Doing so, we get that the probability density function of $\boldsymbol{W}$, the waiting time until the α^{th} event occurs, is:

$$f(w) = \frac{1}{(\alpha-1)!\,\theta^{\alpha}}\; e^{-\lambda/\theta}\, w^{\alpha-1}$$

For w > 0 , Θ >0 .and α > 0

(1.6) Effect of Θ and α on the Distribution :

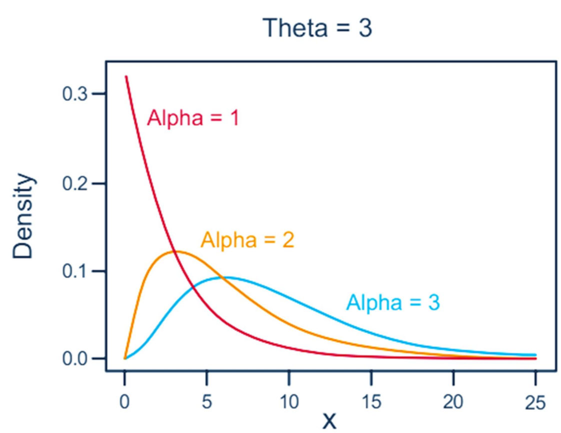
Recall that Θ is the mean waiting time until the first event, and α is the number of events for which you are waiting to occur. It makes sense then that for fixed α , as Θ increases, the probability "moves to the right," as illustrated here with α fixed at 3, and Θ increasing from 1 to 2 to 3:



The plots illustrate, for example, that if we are waiting for α =3 events to

occur, we have a greater probability of our waiting time X being large if our mean waiting time until the first event is large (Θ =3 say) than if it is small(Θ =1 say)

It also makes sense that for fixed Θ, as α increases, the probability "moves to the right," as illustrated here with Θ fixed at 3, and α increasing from 1 to 2 to 3:



The plots illustrate, for example, that if the mean waiting time until the first event is $\theta = 3$, then we have a greater probability of our waiting time X being large if we are waiting for more events to occur ($\alpha = 3$, say) than fewer ($\alpha = 1$, say).

(1.7) The distributive function, gamma distribution (Rameshwar, :2001)

The cumulative distribution function **(CDF)** of the gamma distribution is a function that calculates the cumulative probability for a given random variable $\boldsymbol{x}$ in the gamma distribution. It is represented by $\boldsymbol{F(x;k,\theta)}$, where k is the shape parameter and $\boldsymbol{\theta}$ is the scale parameter. The cumulative distribution function **(CDF)** of the gamma distribution is defined by the following equation: (Montenegro : 2012)

$$F(x;k,\theta)=\Gamma(k)1\gamma(k,\theta x)$$

Here, $\gamma(k,\theta x)$ represents the lower incomplete gamma function evaluated at θx, and (k) is the gamma function.

The gamma distribution is characterized by its shape parameter $\boldsymbol{k}$ and scale parameter $\boldsymbol{\theta}$. It is a continuous probability distribution that is often used to model the waiting time until a given number of events occur, or the time until the nth event occurs in a Poisson process.

Example:

Let's consider a gamma distribution with shape parameter $k=2$ and scale parameter $\theta=3$. We want to calculate the cumulative probability for the gamma distribution at $x=4$.

Using the cumulative distribution function, we can calculate the value as follows:

$$F(4;2,3)= \frac{\mathbf{1}}{\mathbf{\Gamma(2)}}\gamma(2,\frac{4}{3})$$

$$F(4;2,3)=\frac{1}{1}\gamma(2,\frac{4}{3})$$

$$F(4;2,3) = \gamma(2,\frac{4}{3})$$

(1.٧) The arithmetic mean of the gamma distribution

The arithmetic mean, also known as the expected value or the mean, of a gamma distribution is a measure of the central tendency of the distribution. It represents the average value that is expected to occur over many repetitions of the random experiment described by the gamma distribution.

The arithmetic mean μ of a gamma distribution with shape parameter k and scale parameter θ is given by the formula: (Montenegro :2012)

$$\mu = k\theta$$

This formula indicates that the arithmetic mean of a gamma distribution is equal to the product of the shape parameter k and the scale parameter θ.

Example:

Let's consider a gamma distribution with shape parameter $k=2$ and scale parameter $\theta=3$. We can calculate the arithmetic mean as follows:

$\mu = k\theta = 2 \times 3 = 6$

So, the arithmetic mean of this gamma distribution is $\mu=6$. This means that, on average, we expect the value of the random variable to be approximately 6.

(1.٨) Variance in gamma distribution

The variance of a gamma distribution represents the spread or dispersion of the distribution around its mean. It is denoted by $\mathrm{Var}[X]$ and can be calculated using the following formula: (Stirzaker, D. R. :2001)

$$\mathrm{Var}[X]=k\theta^2$$

- k is the shape parameter (also known as the scale parameter).
- θ is the scale parameter.

This formula indicates that the variance of a gamma distribution is equal to the product of the shape parameter k and the square of the scale parameter θ.

Example:

Let's consider a gamma distribution with a shape parameter $k=3$ and a scale parameter $\theta=2$. We can calculate the variance as follows:

$$\mathrm{Var}[X]=k\theta^2=3\times2^2=12$$

So, the variance of this gamma distribution is $\mathrm{Var}[X]=12$. This means that the spread or dispersion of the distribution around its mean is approximately 12.

(1.٩) The torque-generating function in the gamma distribution

I believe there might be a misunderstanding or confusion regarding the term "torque-generating function" in the context of the gamma distribution. The term "moment-generating function" (MGF) is commonly associated with probability distributions, including the gamma distribution. (Saralees Nadarajah :2008)

The moment-generating function $MX^{(t)}$ of a random variable X is defined as the expected value of etX, where t is a parameter. For the gamma distribution, the moment-generating function can be expressed as:

$$Mx(t)=(1-\theta t)^{-k}$$

- k is the shape parameter.
- θ is the scale parameter.

The moment-generating function provides a convenient way to calculate moments of a distribution. Specifically, the n-th moment of a random variable X can be obtained by taking the n-th derivative of the moment-generating function and evaluating it at $t=0$.

Example:

Let's consider a gamma distribution with parameters $k=2$ and $\theta=3$. We can calculate the moment-generating function for this distribution as follows:

$$Mx(t)=(1-3t)^{-2}$$

This is the moment-generating function for the gamma distribution with parameters $k=2$ and $\theta=3$. We can use this function to calculate moments of the distribution, such as the mean and variance, by taking derivatives and evaluating them at $t=0$.

Chapter Two
The chi-square Distribution

Chi-square distributions are one of the important distributions in mathematical statistics, and they are a probability distribution that is used in analyzing statistical data. The chi-square distribution is used in several applications, such as statistical tests of variances and structural tests of goodness of fit to data.(Kennard, R.W.; 1970),

(2-1) The chi-square distribution :

The chi-square distribution is a probability distribution that is widely used in statistics, particularly in hypothesis testing and confidence interval construction for the variance of a normally distributed population.

The probability density function (PDF) of the chi-square distribution with ***k*** degrees of freedom is given by:

$$f(x;k)=\frac{1}{2^{k/2}\,\Gamma(k/2)}\,x^{(k/2)-1}\,e^{-x/2}$$

- x : is the random variable.
- k : is the degrees of freedom parameter.
- Γ : is the gamma function.

Example:

Suppose we have a random variable X that follows a chi-square distribution with 5 degrees of freedom (k=5). We want to find the probability $P(X\leq 10)$. Using the formula for the chi-square distribution with k=5, we have:

$$f(x;5)=\frac{1}{2^{5/2}\,\Gamma(5/2)}\,x^{(5/2)-1}e^{-x/2}$$

$$f(x;5)=\frac{1}{2^{2.5}\,\Gamma(2.5)}\,x^{(5/2)-1}e^{-x/2}$$

1

$f(x;5)=2^{2.5}\ \Gamma(2.5)\quad x^{(5/2)-1}e^{-x/2}$

Now, plug in $=5$ and $x=10$ into the PDF formula to find $P(X\leq 10)$.

$P(X\leq 10)=\int_0^{10} f(x;5)dx$

After integrating the PDF function over the interval $[0,10]$, you can find the desired probability.

(2-2) A terrible chi-distribution table :

Apologies for the inconvenience. Here's a chi-square distribution table with degrees of freedom ranging from 1 to 10 and significance levels of 0.10, 0.05, 0.025, 0.01, and 0.001: (Giovana, :2011).

Degrees of Freedom	0.10	0.05	0.025	0.01	0.001
1	2.71	3.84	5.02	6.63	10.83
2	4.61	5.99	7.38	9.21	13.82
3	6.25	7.82	9.35	11.34	16.27
4	7.78	9.49	11.14	13.28	18.47
5	9.24	11.07	12.83	15.09	20.52
6	10.64	12.59	14.45	16.81	22.46
7	12.02	14.07	16.01	18.48	24.32
8	13.36	15.51	17.53	20.09	26.12
9	14.68	16.92	19.02	21.67	27.88
10	15.99	18.31	20.48	23.21	29.59

This table provides critical values for various significance levels and degrees of freedom

(2-3) The arithmetic median function is a chi-square distribution:

The arithmetic median function for a chi-square distribution depends on the degrees of freedom If k is even, the median is equal to k, and if k is odd, the median is less than k by $\frac{2}{3}$.

Mathematically, the arithmetic median function M can be defined as follows: (Stirzaker, D. R. :2001)

$$M = \begin{cases} k & \text{if } k \text{ is even} \\ k - \frac{2}{3} & \text{if } k \text{ is odd} \end{cases}$$

- M is the arithmetic median function.
- k is the degrees of freedom.

Example:

if we have a chi-square distribution with 6 degrees of freedom, the median will be equal to 6 because 6 is even. If we have a chi-square distribution with 5 degrees of freedom,

the median will be less than 5 by $\frac{2}{3}$, so it will 5 - $\frac{2}{3}$ = $4\frac{1}{3}$.

Definition of the arithmetic mean and variance of the chi-square distribution :

The arithmetic mean and variance of the chi-square distribution can be defined as follows:

1. Arithmetic Mean (μ): For a chi-square distribution with k degrees of freedom, the arithmetic mean is equal to $\mu=k$
2. Variance (σ^2): The variance of a chi-square distribution is twice the degrees of freedom. $\sigma^2=2k$

Example: Suppose we have a chi-square distribution with 8 degrees of freedom. Using the formulas above, we can calculate the arithmetic mean and variance:

1. Arithmetic Mean: $\mu=8$
2. Variance: $\sigma^2=2\times8=16$

So, for this chi-square distribution, the arithmetic mean is 8 and the variance is 16.

(2-3) The moment-generating function in the chi-square distribution

The moment generating function (MGF) in the chi-square distribution is the function that helps in calculating the initial moments (moments) of this distribution. The moment-generating function of the chi-square distribution is given by the following expression: (Rameshwar, D:2001)

$$M(t)=(1-2t)^{-\frac{k}{2}}$$

- t is the independent variable.
- k is the degrees of freedom.

Let's illustrate this with an example:

Suppose we have a chi-square distribution with 4 degrees of freedom. Using the MGF formula, we can calculate the MGF as follows:

$$M(t)=(1-2t)^{-\frac{4}{2}}=(1-2t)-2$$

Now, let's find the first moment ($E[X]$) of the chi-square distribution using the MGF. We can do this by differentiating the MGF with respect to t and evaluating it at $t=0$. The first moment is given by:

$$E[X]=M'(0)=-\frac{d}{dt}\ ((1-2t)^{-2})\ |t=0$$

$$=(-2)(-2)(1-2t)^{-3}|t=0\ =4$$

So, the first moment ($E[X]$) of the chi-square distribution with 4 degrees of freedom is 4.

Chapter Three

(3-1) The relationship between gamma and chi-square

The relationship between the gamma distribution and the chi-square distribution lies in their mathematical formulation and derivation.

1. **Gamma Distribution:** The gamma distribution is a continuous probability distribution that is often used to model the time until an event occurs. It is characterized by two parameters, shape (k) and scale (θ).
2. **Chi-square Distribution:** The chi-square distribution is a special case of the gamma distribution, where the shape parameter (k) equals the degrees of freedom. It is obtained by summing the squares of independent standard normal random variables.

Specifically, if X follows a gamma distribution with shape parameter $k=n/2$ and scale parameter $\theta=2$ (where n is the degrees of freedom), then X follows a chi-square distribution with n degrees of freedom.

In summary, the chi-square distribution is a special case of the gamma distribution, where the shape parameter equals the degrees of freedom. This relationship allows for easy conversion between the two distributions. (Saralees Nadarajah :2008).

(3-2) The relationship between gamma distribution and chi-square?

The relationship between the gamma distribution and the chi-square distribution arises from the definition of the chi-square distribution as the sum of the squares of independent standard normal random variables.

Specifically, if $X_1,X_2,\ldots,X_n$ are independent and identically distributed standard normal random variables ($N(0,1)$), then the sum of their squares:

$$\chi^2=X_1^2+X_2^2+\ldots+X_n^2$$

follows a chi-square distribution with n degrees of freedom.

The gamma distribution, on the other hand, is a continuous probability distribution that generalizes the concept of waiting times between Poisson distributed events. It is often used to model continuous positive random variables, such as time, size, or count data.

The relationship between the gamma distribution and the chi-square distribution comes from the fact that the chi-square distribution can be obtained as a special case of the gamma distribution when the shape parameter of the gamma distribution is $k=n/2$ and the rate parameter is $\theta=2$, where n is the degrees of freedom of the chi-square distribution.

In summary, the chi-square distribution is a special case of the gamma distribution, and they are related through the sum of squares of independent standard normal random variables. (Markovic, R.D: 1965).

Example

Suppose X follows a gamma distribution with shape parameter $k=3$ and scale parameter $\theta=2$. Calculate the probability that the square of $(2X^2)$ exceeds 8. (Glen-b :2014)

Solution:

Given that X follows a gamma distribution with $k=3$ and $\theta=2$, we know that X^2 follows a chi-square distribution with $2k$ degrees of freedom.

1. First, find the degrees of freedom for the chi-square distribution: $\text{Degrees of Freedom}=2k=2\times3=6$
2. Using the chi-square distribution table or calculator, find the probability that $\chi^2>8$ with 6 degrees of freedom.
   From the table, $P(\chi^2>8)$ for $\nu=6$ is approximately 0.145.

So, the probability that X^2 exceeds 8 is approximately 0.145.

Example

Let X be a random variable following a gamma distribution with shape parameter $k=4$ and scale parameter $\theta=2$. Determine the probability that the square of (X^2) exceeds 10. (Glen-b :2014)

Solution:

Given that X follows a gamma distribution with $k=4$ and $\theta=2$, we know that X^2 follows a chi-square distribution with $2k$ degrees of freedom.

1. First, find the degrees of freedom for the chi-square distribution: Degrees of Freedom$=2k=2\times4=8$
2. Using the chi-square distribution table or calculator, find the probability that $\chi^2>10$ with 8 degrees of freedom.

   From the table, $P(\chi^2>10)$ for $v=8$ is approximately 0.194.

So, the probability that X^2 exceeds 10 is approximately 0.194.

Conclusion

The relationship between the gamma distribution and the chi-square distribution shows its importance in many fields, including statistics, probability, medical science, engineering, and others. Here are some benefits of this relationship:

1. Applications of statistics and probability: The gamma distribution is used to model events that occur with a specific frequency, such as the time it takes for a specific event to occur, while the chi-square distribution is used to test hypotheses of differences between group variables.

2. Data analysis: The gamma distribution is used in data analysis to distribute time or time size between specific incidents, while the chi-square distribution is used to estimate the standard deviation and hypothesis tests.

3. Engineering and Medical Sciences: The gamma distribution is used to model the time required for a specific event to occur, such as the time it takes for traffic to flow across a road, or the time it takes for a medical system to treat a patient, while the chi-square distribution is used to estimate the standard deviation of sizes and measures. in engineering and medical sciences.

4. Estimating Parameters: The gamma distribution can be used to estimate different parameters, such as the event rate or mean, while the chi-square distribution is used to estimate the standard deviation and variance.

In short, understanding and using the relationship between the gamma distribution and the chi-square distribution can help analyze data and use it in multiple fields to understand natural phenomena and make correct decisions.

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