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Selection properties in fuzzy metric spaces

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By

Sajad Hassan Jakhr

Supervised by

Mayada Ali Kareem

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بسم الله الرحمن الرحيم

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الى كل من كان له فضل في مسيرتي الدراسية ،

وساعدني ولو باليسير،

الأهل، والأصدقاء، والأساتذة المُبجَّلين . .

Abstract

We introduce and study some boundedness properties in fuzzy metric spaces. These properties are related to the classical covering properties of Menger, Hurewicz and Rothberger.

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1. Introduction

In the literature there are several different definitions of fuzzy metric spaces [11], [7], [3], [4]. In particular, Kramosil and Michalek [11] introduced fuzzy metric spaces based on the notion of continuous triangular norms that were the first time applied in [16] to modify the definition of probabilistic metric spaces introduced by K. Menger [2]. By a slight modification of the Kramosil-Michalek definition, George and Veeramani [3], [4] introduced and studied fuzzy metric spaces and topological spaces induced by fuzzy metric (see also [5], [6]). The notion of intuitionistic fuzzy metric spaces was introduced by Park in [12] .Both fuzzy metric and intuitionistic fuzzy metric spaces have many applications in different areas of mathematics as well as in engineering and in many branches of the quantum particle physics. In [6] it was shown that Park's definition of intuitionistic fuzzy metric spaces contains extra conditions and can be derived, in an equivalent manner, from the definition of fuzzy metric spaces.

We investigate fuzzy metric spaces in connection with several kinds of boundedness properties related to selection principles and studied already in other mathematical structures, such as uniform spaces [8] and topological groups [1].

The paper is organized in such a way that after this introduction in Section 2 we give basic definitions concerning fuzzy metric spaces. In Section 3 we define FM-, FH- and FR-boundedness properties which are the central objects of this article and give some examples. Section 4 is devoted to the main results describing the considered properties, in particular under basic operations: subspaces and products. Finally, in Section 5 we mention some possible directions of investigation related to game theory and L-fuzzy metric spaces.

2. Preliminaries

We begin with basic definitions about fuzzy metric spaces following [3]. First of all recall that if X is a nonempty set, then a fuzzy set A in X is a function from X into $[0, 1]$.

Definition 2.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if the following conditions are satisfied:

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0, 1]$;

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, $(a, b, c, d \in [0, 1])$.

Definition 2.2. ([3]) A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary nonempty set,
$*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying for all $x, y \in X$ and all $s, t > 0$ the following conditions:

$(M.1)$ $M(x, y, t) > 0$;
$(M.2)$ $M(x, y, t) = 1$ if and only if $x = y$;
$(M.3)$ $M(x, y, t) = M(y, x, t)$;
$(M.4)$ $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
$(M.5)$ $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is a continuous function.

The pair $(M, *)$ (or only M) is called a *fuzzy metric* on X.

The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. Note also that
$M(x, y, \cdot)$ is a non-decreasing function (with respect to t) for all $x, y \in X$.
Let $(X, M, *)$ be a fuzzy metric space. Given $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$, the set $B(x, \varepsilon, t) := \{y \in X : M(x, y, t) > 1 - \varepsilon\}$

is called the *open ball* with center x and radius ε with respect to t.

The collection $\{B(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$

is a base of a topology on X; denote this topology by τ_M. Notice that the collection $\left\{ B\left(x, \frac{1}{n}, \frac{1}{n}\right) : x \in X, n \in IN \right\}$

is also a base for τ_M. The topology τ_M is Hausdorff and metrizable.

3. Definitions and examples

The following definitions are motivated by definitions of the classical Menger, Rothberger and Hurewicz covering properties (for details see the survey papers [9] and [10]) and considerations in [1] and [8].

Recall that a topological space has the Menger (Rothberger) [Hurewicz] covering property if for each

sequence $(Un : n \in N)$ of open covers of X there is a sequence $(Vn : n \in N)$ $((Un : n \in N))$ $[(Wn : n \in N)]$ such that for each $n \in N, Vn$ is a finite subset of

$Un\ (Un \in Un)[Wn\ is\ finite\ subset\ of\ Un] and\ X = U_{n \in N} UVn$

$(X = U_{n \in N} u_n)[each\ x$

$\in X\ belongs\ to\ UW_n\ for\ all\ but\ finitely\ many\ n]$

Definition 3.1. A fuzzy metric space $(X, M, *)$ is said to be:

FM : F-*Menger-bounded* (or FM-bounded);

FR : F-*Rothberger-bounded* (or FR-bounded);

FH : F-*Hurewicz-bounded* (or FH-bounded)

if for each sequence $(\varepsilon_n: n \in \mathrm{N})$ of elements of $(0, 1)$ and each $t > 0$ there is a sequence

FM : $(A_n : n \in N)$ of finite subsets of X such that $X = U_{n \epsilon N} U_{a \epsilon A_n} B(a, \varepsilon_n, t)$;

FR : $(x_n : n \in \mathrm{N})$ of elements of X such that $X = U_{n \in N} B(x_n, \varepsilon_n, t)$;

FH : $(A_n : n \in \mathrm{N})$ of finite subsets of X such that for each $x \in X$ there is $n_0 \in N$ such that $x \in U_{a \epsilon A_n} B(a, \varepsilon_n, t)$

for all $n \geq n_0$.

Recall that a fuzzy metric space is said to be *precompact* (respectively, *pre-Lindelo¨f*) if for every $\varepsilon \in (0, 1)$ and every $t > 0$ there is a finite (respectively, countable) set $A \subset X$ such that $X = U_{a \epsilon A} B(a, \varepsilon, t)$.

Evidently,

F−precompact → FH−bounded → FM−bounded → F−pre-Lindelo¨f

and

FR−bounded → FM−bounded.

Example 3.2. Let (X, d) be a metric space with the Menger (Rothberger, Hurewicz) property. Consider the standard fuzzy metric M_d on X induced by d defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \ , \qquad (x, y \in X, t > 0)$$

and denote

$a * b = ab$,$(a,b \in [0,1])$.

Then the fuzzy metric space $(X, M_d, *)$ is FM-bounded (FR-bounded, FH-bounded).

Consider only the FM-bounded case because the other two are shown quite similarly.

Let $(\varepsilon_n : n \in N)$ be a sequence in $(0, 1)$ and let $t > 0$. As (X, d) has the Menger covering property, there is a sequence $(A_n : n \in N)$ of finite subsets of X such that

$$X = \bigcup_{n \in N} \bigcup K(a\ , \varepsilon_n)$$

where $K(a, \varepsilon) = \{y \in X : d(a, y) < \varepsilon\}$.

Let $x \in X$. There is $n \in N$ and a point $a_n \in A_n$ satisfying $d(x, a_n) < \varepsilon_n$. Then

$$M_d(x, a_n, t) = \frac{t}{t + d(x, a_n)} > \frac{t}{t + \varepsilon_n} = 1 - \frac{\varepsilon_n}{t + \varepsilon_n} > 1 - \varepsilon_n$$

Therefore we have $x \in B(a_n, \varepsilon_n, t)$, i.e. $X = X = U_{n \in N} U_{a \in A_n}\ B(a, \varepsilon_n, t)$ which means that $(X, M_d, *)$ is FM-bounded.

Example 3.3. Let $X = R$ and $d = |\cdot|$, and let $*$ be as in Example 3.2. Then the fuzzy metric space $(X, M_d, *)$ is FM-bounded by the previous example and the fact that $(R, |\cdot|)$ has the Menger property [9], [10]. On the other hand, $(X, M_d, *)$ is not FR-bounded.

Indeed, if we take the sequence $(2^{-n} : n \in N) \subset (0, 1)$ and $t = 2^{-1}$, then X cannot be covered by the open balls $B(x_n, 2^{-n}, 2^{-1})$ for any choice of elements x_n, $n \in N$, from X. Otherwise, we would have that for every $x \in R$ there is $n \in N$ such that

$$\frac{2^{-1}}{2^{-1}+|x-x_n|} > 1-2^{-n},$$

i.e.

$$\frac{2^{-1}+|x-x_n|}{2^{-1}} < \frac{2^n}{2^2-1}$$

From here we obtain that for each $x \in X$ there is $n \in$ N with

$$|x-x_n| < \frac{1}{2(2^n-1)} < \frac{1}{2^n},$$

which means

$$R = \bigcup_{n\in N}(x_n - 2^{-n}, x_n + 2^{-n})$$

However, it is impossible.

Example 3.4. Let $X = (0, \infty)$ with the fuzzy metric $(M, *)$ defined by

$$a * b = ab, (a, b \in [0,1]); \quad M(x, y, t) = \begin{cases} \frac{x}{y}, & if\ x \le y\,; \\ \frac{y}{x} & if\ x \le y\ , for\ all\ x, y\ \in X, t > 0 \end{cases}$$

It is easy to see that for $x \in X, \varepsilon \in (0,1)$, t > 0.

$$B(x, \varepsilon, t) = ((1-\varepsilon)x, x/(1-\varepsilon)) \subset X,$$

i.e. the open balls in $(X, M,*)$ are the usual open intervals in X equipped with the metric topology on R.

Thus $(X, M,*)$ is not FR-bounded. Consider the sequence $(2^{-n} : n \in N) \subset (0, 1)$, and $t = 2^{-1}$. For any sequence (x_n: n ∈ N) of points of X the open balls B(x_n, 2^{-n}, 2^{-1}), $n \in N$, cannot cover X because the sum of lengths of intervals $((1-2^{-n})\, x_n, x_n/(1-2^{-n}))$, $n \in N$, is finite.

Notice that if X = N, and M and $*$ are as above, then $(N, M,*)$ is FR-bounded.

4.Results

4.1. Subspaces

If $(X, M,*)$ is a fuzzy metric space and $Y \subset X$, then $(Y, M_Y,*)$, where $M_Y = M \upharpoonright Y^2 \times (0, \infty)$, is also a fuzzy metric space and it is called the fuzzy metric subspace (or shortly fm-subspace) of $(X, M,*)$.

Theorem 4.1. Every fm-subspace of an FM-bounded space $(X, M,*)$ is also FM-bounded.

Proof. Let $(Y, M_Y,*)$ be an fm-subspace of $(X, M,*)$ and let $(\varepsilon_n : n \in N)$ be a sequence of elements of (0, 1) and $t > 0$. Because of continuity of $*$ for each $n \in N$ there is $\delta_n \in (0,1)$ such that $(1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$. Apply now to the sequence $(\delta_n : n \in N)$ and t the assumption on (X, M, $*$). There is a sequence $(P_n : n \in N)$
of finite subsets of X such that

$$X = \bigcup_{n \in N} \bigcup_{P \in P_n} B(p, \delta_n, t/2)$$

For each $n \in N$ let

$Q_n = \{q \in P_n : \exists y \in Y \text{ with } y \in B(q, \delta, t/2)\}$.

Further, for each $q \in Q_n$ pick an element $y_q \in Y$ such that $y_q \in B(q, \delta, t/2)$ and set

$Sn = \{y_q : q \in Q_n\}$.

Let us show that the sequence $(S_n : n \in N)$ of finite subsets of Y witnesses for $(\varepsilon_n : n \in N)$ and t that $(Y, M_Y, *)$ is FM-bounded.

Let y be an arbitrary element of Y. There exist $n \in N$ and $p \in P_n$ such that $y \in B(p, \delta_n, t/2)$, and from the definition of Q_n it follows $p \in Q_n$. Therefore, there exists $y_p \in S_n$ such that $y_p \in B(p, \delta_n, t/2)$, hence $p \in B(y_p, \delta_n, t/2)$. So, we have

$M(p, y, t/2) > 1 - \delta_n$ and $M(p, y_p, t/2) > 1 - \delta_n$.

According to (M.4) we have

$M(y, y_p, t) \geq M(y, p, t/2) * M(p, y_p, t/2) > (1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$,

which means $y \in B(y_p, \varepsilon_n, t)$. As $y \in Y$ was arbitrary we conclude

$$Y = \bigcup_{n \in N} \bigcup_{y_p \in S_n} B(y_p, \varepsilon_n, t)$$

i.e. Y is FM-bounded.

Theorem 4.2. Every fm-subspace of an FH-bounded space $(X, M, *)$ is also FH-bounded.

Theorem 4.3. Let a fuzzy metric space $(Y, M_Y, *)$ be dense in a fuzzy metric space $(X, M, *)$. If Y is F-Hurewicz- bounded, then X is also F-Hurewicz-bounded.

Proof. Let a sequence $(\varepsilon_n: n \in N)$ of elements from (0, 1) and $t > 0$ be given. Choose a sequence $(\delta_n : n \in N)$ of elements of (0, 1) such that $(1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$ for each $n \in N$. Applying to $(\delta_n : n \in N)$ and t/2 the fact that $(Y, M_Y, *)$ is FH-bounded we find a sequence $(A_n : n \in N)$ of finite subsets of Y such that each $y \in Y$ belongs to $a \in A_n$ $B(a, \delta_n, t/2)$ for all but finitely many n. We claim that $(A_n : n \in N)$ witnesses for $(\varepsilon_n: n \in N)$ and t that $(X, M, *)$ is FH-bounded.

Let $x \in X$. Since Y is dense in x the intersection $Y \cap B(x, \delta_n, t/2)$ is non-empty for each $n \in N$; let $y_n \in Y \cap B(x, \delta n, t/2)$. But $(Y, M_Y, *)$ is FH-bounded, so that there is $n_0 \in N$ such that $y_n \in a \in A_n$ $B(a, \delta_n, t/2)$ for each $n \geq n_0$, i.e. for each $n \geq n_0$ there is $a_n \in An$ with $y_n \in B(a_n, \delta_n, t/2)$. Therefore, for each $n \geq n_0$ we have

$$M(x, a_n, t) \geq M(x, y_n, t/2) * M(y_n, a_n, t/2) > (1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n,$$

i.e. for each $n \geq n_0$, $x \in S_a \in A_n$ $B(a, \varepsilon_n, t)$ which means that X is FH-bounded.

4.2. Products

Let $(X, M_X, *)$ and $(Y, M_Y, *)$ be fuzzy metric spaces and let $Z = X \times Y$. For $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2) \in Z$ and $t > 0$
define $M_Z(z_1, z_2, t) = M_X(x_1, y_1, t) * M_Y(x_2, y_2, t)$.

Then $(M_Z, *)$ is a fuzzy metric on Z, and the triple $(Z, M_Z, *)$ is called the product metric space of X and Y.

Theorem 4.4. The product (Z, M_Z, ∗) of two FH-bounded spaces (X, M_X, ∗) and (Y, M_Y, ∗) is also FH-bounded.

Proof. Suppose the sequence (ε_n : n ∈ N) ⊂ (0, 1) and t > 0 are given. Choose for each n ∈ N an element δn in (0, 1) such that $(1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$. B_y assumption on X and Y there are sequences (S_n : n ∈ N) and (T_n : n ∈ N) of finite sets of X and Y, respectively and natural numbers n_1 and n_2 such that each x ∈ X belongs to $\bigcup_{a \in S_n} B(a, \delta_n, t/2)$ for all $n \geq n_1$, and each y ∈ Y belongs to $\bigcup_{c \in T_n} B(c, \delta_n, t/2)$ for all $n \geq n_2$. We claim that the sequence ($S_n \times T_n$: n ∈ N) of finite subsets of Z witnesses for (ε_n : n ∈ N) and t that (Z, M_Z, ∗) is FH-bounded.

Let $z = (x, y) \in Z$. Pick n_1, n_2 ∈ N such that for each $n \geq n_1$ and each $k \geq n_2$ $x \in B(a_n, \delta_n, t/2)$ for some $a_n \in S_n$ and

$y \in B(c_k, \delta_k, t/2)$ for some $c_k \in T_k$.

Then for each $n \geq n_0 = \max\{n_1, n_2\}$ there is $z_n = (a_n, c_n) \in S_n \times T_n$ such that

$$M_Z(z, z_n, t) = M_X(x, a_n, t) * M_Y(y, c_n, t) > (1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n.$$

This means that $z \in B(z_n, \varepsilon_n, t)$ as required to be shown.

Theorem 4.5. The product (Z, M_Z, ∗) of an FM-bounded space (X, M_X, ∗) and an FH-bounded space (Y, M_Y, ∗) is FM-bounded.

Example 4.6. The product of an FR-bounded fuzzy metric space and a (pre)compact fuzzy metric space need not be FR-bounded.

Let X = N with the fuzzy metric (M_X, ∗) defined by

$$a * b = a.b;\ M_x(x, y, t) = \begin{cases} \frac{x}{y}, & if\ x \leq y\,; \\ \frac{y}{x} & if\ x \leq y\ , \end{cases}$$

and Y = [0, 1] with the fuzzy metric (M_Y, ∗)

$$M_x(x, y, t) = \frac{1}{t + |x - y|'}$$

for all x, y ∈ Y and t > 0, We observed that X is FR-bounded; also, Y is (pre)compact. However, the product space Z = X × Y is not FR-bounded.

Take for each n ∈ N, $\varepsilon_n = 2^{-n}$ and consider the sequence (2^{-n}: n ∈ N) ⊂ (0, 1) and t = 1. An easy calculation shows that for $z_n = (n, y_n) \in \{n\} \times [0, 1]$

we have B(z_n, 2^{-n}, 1) = {(n, y) : y ∈ [0, 1], |y−y_n| < $(2^n-1)^{-1}$}. Therefore, for any sequence (z_n : n ∈ N) of elements of Z the balls B(z_n, 2^{-n}, 1), n ∈ N, do not cover Z.

4.3.Two theorems more

Let S be a subset of X, ε ∈ (0, 1), t > 1. Then we denote

$$B(s,\varepsilon,t) := \bigcup_{x\in s} B(x,\varepsilon,t)$$

Theorem 4.7. For a fuzzy metric space (X, M, ∗) the following are equivalent:

(a) For each sequence (ε_n : n ∈ N) ⊂ (0, 1) and each t > 0 there is a sequence (S_n : n ∈ N) of finite subsets of X such that each finite subset E ⊂ X is contained in B(S_n, ε_n, t) for some S_n;

(b) For each sequence (ε_n : n ∈ N) ⊂ (0, 1) and each t > 0 there is a sequence (S_n : n ∈ N) of finite subsets of X and an increasing sequence $n_1 < n_2 < \cdots$ of natural numbers such that each finite subset E ⊂ X is contained in nk $\bigcup_{n_k\le i<n_{k+1}} \mathrm{B}(\mathrm{s}_i, \varepsilon_i, \mathrm{t})$ for some k ∈ N.

Proof. Evidently (a) implies (b). We prove (b) → (a). Let (ε_n : n ∈ N) be a sequence of elements from (0, 1) and t > 0. For each n ∈ N let $\delta_n = \min\{\varepsilon_i : i \le n\}$ and apply (b) to (δ_n : n ∈ N) and t. There is an increasing sequence $n_1 < n_2 < \cdots$ in N such that each finite set S ⊂ X is contained in $\bigcup_{n_k\le i<n_{k+1}} \mathrm{B}(\mathrm{s}_i, \delta_i, \mathrm{t})$ for some k ∈ N. Define now

$T_n = \bigcup_{i<n_1} \mathrm{s}_i$ *for each* $n > n_1$,

$T_n = \bigcup_{n_k\le i<n_{k+1}} \mathrm{s}_i$ *for each n such that* $n_k \le n_{k+1}$.

We claim that the sequence (T_n : n ∈ N) of finite subsets of X witnesses for (ε_n : n ∈ N) and t that (a) is satisfied.

Let F be a finite subSset of X. Choose k ∈ N such that

F ⊂ $\bigcup_{n_k\le i<n_{k+1}} \mathrm{B}(\mathrm{s}_i, \delta_i, \mathrm{t})$. For (each) n with $n_k \le n < n_{k+1}$

put T_n= $\bigcup_{i<n_1} \mathrm{s}_i$, we have that for each $x \in F$ there is j , $n_k \le j < n_{k+1}$ and $y \in \mathrm{s}_i$ *with* $x \in B(y, \delta j, t)$. Further , we have B(y, ε_j, t) and since y ∈ T_n we have x ∈ B(T_j, ε_j, t), and thus F ⊂ B(T_j, ε_j, t).

Theorem 4.8. For a fuzzy metric space $(X, M, *)$ the following are equivalent:

(a) For each sequence $(\varepsilon_n : n \in N) \subset (0, 1)$ and each $t > 0$ there is a sequence $(S_n : n \in N)$ of finite subsets of X such that each finite subset $E \subset X$ is contained in $B(S_n, \varepsilon_n, t)$ for all but finitely many n;

(b) For each sequence $(\varepsilon_n : n \in N) \subset (0, 1)$ and each $t > 0$ there is a sequence $(S_n : n \in N)$ of finite subsets of X and an increasing sequence $n_1 < n_2 < \cdot\cdot\cdot$ of natural numbers such that each finite subset $E \subset X$ is contained in $\bigcup_{n_k \le i < n_{k+1}} B(s_i, \varepsilon_i, t)$ for all but finitely many $k \in N$.

5.Fixed Point Theorems

Grabiec proved a fuzzy Banach contraction theorem whenever fuzzy metric space was considered in the sense of Kramosil and Michalek and was complete in Grabiec's sense. Meanwhile, Gregori and Sapena gave fixed point theorems for complete fuzzy metric space in the sense of George and Veeramani and also for Kramosil and Michalek's fuzzy metric space which are completexd in Grabiec's sense. Recently, Zikic proved that the fixed point theorem of Gregori and Sapena holds under general conditions (theory of countable extension of a t-norm).

We begin with the definition of contraction mappings in fuzzy metric Spaces.

Definition 5.1. Let (X, M, *) be a fuzzy metric space. A mapping f: $X \to X$ is said to be fuzzy contraction if there exists a $k \in (0,1)$ such that

$$M(fx, fy, t) \geq M(x, y, t/k) \text{ for all } x,y \in X.$$

Theorem 5.2: Let (X, M, *) be a complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Let f: $X \to X$ be a contractive mapping. Then f has a unique fxed point.

Theorem 5.3. Let (X, M*) be a fuzzy metric space with a * b = Min(a, b). Let f_i: $X \to X$ be a function with at least one fixed point x_i for each i = 1,2,…, and f_0: $X \to X$ be a fuzzy contraction mapping with fxed point x_0. If the sequence (f_i) converges uniformly to f_0, then the sequence (x_i) converges to x_0 .

Proof. Let $k \in (0,1)$ and choose a positive number $N \in N$ such that $i \geq N$ implies.

$$M(f_i x, f_0 x, (1 - k)t) > 1-r$$

where $r \in (0,1)$ and $x \in X$. Then, if $i \geq N$, we have

$$M(x_i, x_0, t) = M(f_i x_i, f_0 x_0, t)$$

$$\geq M(f_i x_i, f_0 x_i, (1 - k)t) * M(f_0 x_i, f_0 x_0, kt)$$

$$\geq \text{Min}(1 - r, M(x_i, x_0, t)).$$

Hence, $M(x_i, x_0, t) \rightarrow 1$ as $i \rightarrow \infty$. This proves that (x_i) converges to x_0 .

In what follows $\pi_1: X \times Y \rightarrow X$ will denote the first projection mapping defined by $\pi_1(x, y) = x$, while $\pi_2 : X \times Y \rightarrow Y$ will denote the second projection mapping defined by $\pi_2(x, y) = y$.

Definition 5.4. Let $(X, M, *)$ be a fuzzy metric space and Y be any space. A mapping $f: X \times Y \rightarrow X \times Y$ is said to be locally fuzzy contraction in the first variable if and only if for each $y \in Y$ there exists an open ball $B_y(e)$, $e \in (0,1)$ containing y and a real number $k(y) \in (0,1)$ such that

$M(\pi_1 \circ f(x_1,y), \pi_1 \circ f(x_2,y),t) \geq M(x_1, x_2, t/k(y))$ for all $x_1, x_2 \in X$.

A mapping $f: X \times Y \rightarrow X \times Y$ is called fuzzy contraction in the first variable if and only if there exists a real number $k \in (0,1)$ such that for any $y \in Y$

$M(\pi_1 \circ f(x_1,y), \pi_1 \circ f(x_2, y), t) \geq M(x_1,x_2, t/k)$ for all $x_1, x_2 \in X$.

It is obvious that every fuzzy contraction mapping is locally fuzzy contraction in the first variable.

We define a fuzzy contraction mapping in the second variable in an analogous fashion.

Definition 5.5. The fuzzy metric space $(X, M, *)$ has fixed point property (f.p.p) if every continuous mapping $f: X \rightarrow X$ has fixed point.

Theorem 5.6. Let (X, M_x, *) be a complete fuzzy metric space, (Y, M_y, *) be a fuzzy metric space with the f.p.p., and let f: X×Y→X×Y be uniformly continuous and a fuzzy contraction mapping in the first variable. Then, f has a fixed point.

Proof. For y ∈ Y, let f_y: X→X be defined by $f_{y,}(x) = \pi_1$ o f (x , y) for all x ∈ X. Since, for every y ∈ Y, f_y is a fuzzy contraction mapping, therefore f_y has a unique fixed point (see, Theorem 4.2). Let G: Y→X be given by G(y) = f_y(G(Y)) is the unique fixed point of f_y. Now, let y_0 ∈ Y and let (y_n) be a sequence of points of Y which converges to y_0. Since f is uniformly continuous, the sequence (f_{yn}) converges uniformly to f_{y0}. Hence, by Theorem 4.3, the sequence (G(y_n)) converges to G(y_0). This shows that the function G is continuous on Y. Now, let g: Y → Y be a continuous function defined via g(y) = π_2 o f(G(y), y) for each y ∈ Y. Since, (Y, My, *) has f.p.p., there is a point z ∈Y such that g(z) = z, i.e., z = g(z) = π_2 o f(G(z),z). It follows that (G(z),z) is a fixed point of f. This completes the proof.

To prove the following theorem, we require:

Lemma5.7. Let (X, M, *) be a fuzzy metric space with a * a ≥a for every a ∈ [0,1] and Y be a fuzzy topological space with f.p.p. Let f: X×Y→ X×Y be locally fuzzy contraction in the first variable. Let x_0 ∈ X and y ∈ Y. Define the sequence (p_n(y)) in X as follows:

$$Po(y) = x_0 \text{ and } P_n = P_n\,(y) = \pi_1 \text{ o } f(P_n\text{-}1(y), y).$$

Then,

(i) (P_n(y)) is a Cauchy sequence in X.

(ii) If P_n→P_y, then π_1 of (p_y,y) = Py

(iii) Define g: Y→Y as g(y) = π_2 of (P_y, y). Then, g is a continuous function.

Proof. (i) Since f is a locally fuzzy contraction mapping in the first variable, there exists a real number k ∈ (0, 1) such that

$M(P_n, P_n+1, t) = M(\pi_1 \text{ o } f(P_{n-1}, Y), \pi_1 \text{ o } f(P_n, Y), t) \geq M(P_{n1}:P_n, t/k)$ $n \geq 1$.By a simple induction we get

$$M(P_n, P_{n+1}, t) \geq M(P_0. P_1, t/k^n)$$

for all n and t > 0. We note that, for every positive integer m, n with m > n and k ∈ (0,1), we have

$$(1- k)(1+k+k^2+\ldots + k^{m-n-1}) =1-k^{m-n} <1.$$

Therefore, $t > (1-k)(1+k+k^2+...+k^{m-n-1})t$. Since M is no decreasing, we have

$$M(P_n, P_m,t) \geq M(P_n , P_m, (1 - k)(1 +k+k^2+...+k^{m-n-1})t).$$

Thus, by (M4), we notice that, for m > n،

$$M (P_n, P_m, (1 - k)(1 +k+k^2+...+k^{m-n-1})t)$$

$$\geq M(P_n, P_{n+1}, (1 - k)t) *... * M(P_{m-1}, P_m (1 - k)k^{m-n-1}t)$$

$$\geq M(P_0.P_1, (1 - k)t/k^n) *\ldots* M(P_0,P_1, (1 - k)k^{m-n-1}t/k^{m-1})$$

$$=M(P_0.P_1, (1 - k)t/k^n) * \ldots * M(P_0.P_1, (1 - k)t/k^n)$$

Since $a * a \geq a$, we conclude that

$$M(P_n, P_m,t) \geq M(P_0,P_1, (1 - k)t/k^n).$$

By letting $n \to \infty$ and m > n, we get

$$\text{Lim}_{n,m\to\infty} M(P_n, P_m,t) = \lim_{n\to\infty} M(P_o , P_1, (1-k)t/k^n) = 1.$$

This implies that $(p_n (y))$ is a Cauchy sequence in X.

(ii) Let $u = \pi_1 \text{ o } f(P_y, y)$. By contradiction, suppose that $u \neq P_y$: Then $M(u, P_y, t) = e < 1$ for every t > 0. Since $f: X \times Y \to X \times Y$ is continuous, there exists an open set U× V in X×Y and a real number $\lambda \in (0,e)$ such that $(P_y, y) \in U \times V$, $U \subseteq B_{py}(\lambda, t)$ and $f(U \times V) \subseteq B_u(\lambda, t) \times Y$. Since $p_n \to P_y$ there is a positive number N ∈ N such that $p_n \in U$ for all $n \geq N$. But π_1 of$(p_k,y) = P_{k+1} \in U$. Therefore $f(P_k ,Y) \notin B_u(\lambda, t) \times Y$ which contradicts the fact that $f(U \times V) \subseteq B_u(\lambda,t) \times Y$. Therefore our assumption is incorrect.

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