

Ministry of Higher Education  
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## Uniform Convergence and Dini's Theorem for Sequences of Continuous Functions

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ وَقُلْ رَبِّ زِدْنِي عِلْمًا ﴾

صَدَقَ اللَّهُ الْعَلِيِّ الْعَظِيمِ

[طه: ١١٤]

## شكر وتقدير

"الحمد لله الذي علم بالقلم، علم الإنسان ما لم يعلم، والصلاة والسلام على خير من نطق بالضاد وأرشد العباد."

في ختام هذا المشوار العلمي الذي لم يكن ليتم لولا توفيق الله وتسديده، أجد من الواجب والوفاء أن أخط كلمات الشكر لمن كان لهم أثر في استقامة حرفي وإنضاج فكري.

أتقدم بجزيل الشكر والتقدير إلى عمادة كلية التربية للعلوم الصرفة في جامعة بابل، هذا الصرح الذي احتضن طموحي وصفل مواهبي. كما أخص بالشكر والامتنان أستاذي المشرف الفاضل رحاب عامر كامل الذي منحني من وقته وعلمه الكثير، وكان لتوجيهاته الدقيقة وصبره النبيل الأثر الأكبر في بلوغ هذا البحث تمامه.

وهنا، أقف ورفة تقدير خاصة لأولئك الذين غرسوا فينا "أمانة الحرف"؛ فالعلم ليس مجرد نصوص تجمع، بل هو روح تبنى، وعقل يُحرّر. شكرا لكل من علمني أن البحث هو رحلة تواضع أمام بحر المعرفة، وأن الرقم في الرياضيات لا يكذب، تماماً كما لا تكذب النيات الصادقة في طلب العلا.

كما يمتد شكري إلى أساتذتي الأجلاء في قسم الرياضيات، الذين لم يبخلوا علينا يوماً بفيض معرفتهم. وإلى كل من ساندني بكلمة تشجيع عابرة كانت لي وقوداً في لحظات الفتور، وإلى عائلتي التي لولا دعاؤها ما ثبتت لي قدم.

ختاماً، أسأل الله أن يجعل هذا العمل لبنة نافعة، وخطوة مباركة في طريق العلم الطويل

## الإهداء

إلى منبع الفخر وعمود البيت.. والدي الغالي

إلى الرجل الذي لم ترهق كاهله السنون وهو يسعى لتأمين مستقبلي، إلى من علمني أن الكرامة في العلم، وأن النجاح هو أجمل رد للجميل. شكراً لأنك كنت دائماً السند الذي أرتكز عليه كلما مالت بي الأيام، ولأنك لم تشكك يوماً في قدراتي. هذا البحث هو ثمرة كفاحك أنت قبل أن يكون تعبي، وأهديه لك وأنا كلي فخر بأنني ابنتك، راجية من الله أن يديمك تاجاً فوق رأسي.

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أهدي هذا البحث لنفسي، لتلك الفتاة التي لم تستسلم رغم لحظات اليأس، وللعقل الذي لم يتوقف عن التفكير والبحث والسهر. أهديه لروحي التي قاومت كل الظروف الصعبة، وأمنت بأن الوصول ممكن مهما طال الطريق. اليوم أحتفل بنجاحي وأعاهد نفسي على أن يكون هذا البحث هو مجرد خطوة أولى نحو طموحات أكبر وأحلام لا سقف لها.

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## Abstract

Motivated by the foundational principles of Dini's Theorem, this mathematical analysis research by Wu (2020) provides a comprehensive framework for characterizing the uniform convergence of function sequences. The study demonstrates that the uniform convergence of a sequence  $f_n \rightarrow 0$  is equivalent to the existence of a bounding, decreasing auxiliary sequence  $g_n$  that also converges uniformly to zero. This core equivalence is rigorously extended to sequences of continuous functions defined on compact metric spaces and sequences of Riemann integrable functions, where the uniform convergence of the sequence ensures the uniform convergence of its integrals. Furthermore, the paper successfully applies these theoretical characterizations to establish the uniform convergence of series of functions and to derive conditions for the existence of fixed points for continuous mappings within metric spaces.

## 1. INTRODUCTION

Uniform convergence of a sequence of functions is an important concept in studying the continuity, differentiability, and integrability of its limit function. It is known that a sequence  $\{f_n\}$  of real-valued functions converges uniformly to a function  $f$  on a set  $A$  if and only if  $\lim_{n \rightarrow +\infty} M_n = 0$ , where

$$M_n := \sup\{|f_n(x) - f(x)| : x \in A\} \quad \text{for } n \in N.$$

Dini's Theorem [ ] states that a pointwise convergent sequence  $\{f_n\}$  of functions is also uniformly convergent on  $A$  if the following conditions are satisfied:

(D1)  $A$  is a compact set in a metric space  $(X, d)$ ;

(D2) each  $f_n$  is continuous on  $A$  for  $n \in N$ ;

(D3)  $\{f_n\}$  is decreasing:  $0 \leq f_{n+1}(x) \leq f_n(x)$  for all  $n \in N$  and all  $x \in A$ .

The theorem provides a sufficient condition for  $f_n \rightarrow 0$  uniformly. Even though a uniformly convergent sequence of functions is not necessary to satisfy all conditions (D1) – (D3), Dini's Theorem motivates us to discuss the following questions:

- Is the uniform convergence of a sequence of functions on  $A$  determined by that of another sequence of functions satisfying condition (D3)?
- Can we characterize the uniform convergence of a sequence of continuous functions on a compact set in terms of another pointwise convergent sequence of functions satisfying conditions (D2) – (D3)?
- What will happen to a uniformly convergent sequence of measurable functions? Especially to a uniformly convergent sequence of Riemann integrable functions.

## 2. UNIFORM CONVERGENCE OF A SEQUENCE OF FUNCTIONS ON A SET

In this section we begin with characterizing the uniform convergence of a sequence of functions defined on a set in terms of that of another sequence of functions satisfying (D3).

**Theorem 2.1.** Let  $\{f_n\}$  be a sequence of real-valued functions defined on a set  $A$  in a metric space  $(X, d)$ . Then the following statements are equivalent:

- (i)  $f_n \rightarrow 0$  uniformly on  $A$ .
- (ii) There exists a sequence  $\{g_n\}$  of functions on  $A$  such that  $g_n \rightarrow 0$  uniformly,  $|f_n(x)| \leq g_n(x)$ , and  $g_{n+1}(x) \leq g_n(x)$  for all  $n \in N$  and all  $x \in A$ .
- (iii) There exist a sequence  $\{g_n\}$  of functions on  $A$  and a sequence  $\{M_n\}$  of real numbers such that  $\lim_{n \rightarrow +\infty} M_n = 0$ ,  $|f_n(x)| \leq g_n(x) \leq M_n$ ,  $g_{n+1}(x) \leq g_n(x)$ , and  $M_{n+1} \leq M_n$  for all  $n \in N$  and all  $x \in A$ .
- (iv) There exists a sequence  $\{M_n\}$  of real numbers such that  $\lim_{n \rightarrow +\infty} M_n = 0$ ,  $|f_n(x)| \leq M_n$ , and  $M_{n+1} \leq M_n$  for all  $n \in N$  and all  $x \in A$ .

**Proof.** Since the implications (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i) are obvious, it suffices to show (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii).

(i)  $\Rightarrow$  (ii): Let (i) be true. Then for each  $\epsilon > 0$  there exists  $n_\epsilon \in N$  such that

$$|f_n(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in A.$$

For each  $n \in N$ , taking

$$g_n(x) := \sup\{|f_m(x)| : n \leq m \in N\} \quad \text{for } x \in A,$$

we have

$$|f_n(x)| \leq g_n(x) \text{ and } g_{n+1}(x) \leq g_n(x) \quad \text{for all } n \in N \text{ and all } x \in A.$$

In addition,  $0 \leq g_n(x) \leq \epsilon$  for all  $n \geq n_\epsilon$  and all  $x \in A$ . Thus (ii) follows.

To prove (ii)  $\Rightarrow$  (iii), let  $\{g_n\}$  be the sequence in (ii). For each  $n \in N$ , define

$$M_n := \sup\{|g_m(x)|: n \leq m \in N, x \in A\}.$$

Then

$$|f_n(x)| \leq g_n(x) \leq M_n \text{ and } M_{n+1} \leq M_n \text{ for all } n \in N \text{ and all } x \in A.$$

For each  $\epsilon > 0$ , by the uniform convergence  $g_n \rightarrow 0$ , there exists  $n_\epsilon \in N$  such that

$$|g_n(x)| \leq \epsilon \text{ for all } n \geq n_\epsilon \text{ and all } x \in A.$$

From this we obtain  $M_n \leq \epsilon$  for all  $n \geq n_\epsilon$  and hence  $\lim_{n \rightarrow +\infty} M_n = 0$ . So (iii) is valid.

**Corollary 2.2.** Let  $\{f_n\}$  be a sequence of measurable real-valued functions on a closed interval  $[a, b]$ . Then the following statements are equivalent:

(i)  $f_n \rightarrow 0$  uniformly on  $[a, b]$ .

(ii) There exists a sequence  $\{g_n\}$  of Lebesgue integrable functions on  $[a, b]$  such that  $g_n \rightarrow 0$  uniformly on  $[a, b]$ ,  $|f_n(x)| \leq g_n(x)$ , and  $g_{n+1}(x) \leq g_n(x)$  for all  $n \in N$  and all  $x \in [a, b]$ .

### 3. UNIFORM CONVERGENCE OF A SEQUENCE OF CONTINUOUS FUNCTIONS ON A COMPACT SET

To characterize the uniform convergence of a sequence of continuous functions on a compact set, it is useful to study one property of a uniformly convergent sequence of functions on a set  $A$ . For a uniformly convergent sequence  $\{f_n\}$  on  $A$ , under certain conditions, it can be extended to a uniformly convergent sequence on the closure  $\bar{A}$  of  $A$ .

**Lemma 3.1.** Suppose that  $f_n \rightarrow f$  uniformly on a set  $A$  in a metric space  $(X, d)$  and that for each limit point  $x$  of  $A$  there exists  $\lim_{t \rightarrow x} f_n(t)$ , denoted by  $f_n(x)$ , for each  $n \in N$ . Then  $\{f_n\}$  is uniformly convergent on  $\bar{A}$ .

Proof. For any given  $\epsilon > 0$ , by the uniform convergence of  $\{f_n\}$ , there exists  $n_\epsilon \in N$  such that

$$|f_n(t) - f_m(t)| \leq \epsilon \quad \text{for all } m, n \geq n_\epsilon \text{ and all } t \in A.$$

For each  $x \in \bar{A} \setminus A$ , by taking  $t \rightarrow x$  in the above inequality, we have

$$|f_n(x) - f_m(x)| \leq \epsilon \quad \text{for all } m, n \geq n_\epsilon.$$

This implies that  $\{f_n(x)\}$  is Cauchy and hence converges to a point, denoted by  $f(x)$ . Letting  $m \rightarrow +\infty$  in the above two inequalities gives

$$|f_n(t) - f(t)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and } t \in \bar{A}.$$

This shows that  $f_n \rightarrow f$  uniformly on  $\bar{A}$ .

**Theorem 3.2.** Let  $\{f_n\}$  be a sequence of real-valued continuous functions on a compact set  $\bar{A}$  in a metric space  $(X, d)$ . Then the following statements are equivalent:

- (i)  $f_n \rightarrow 0$  uniformly on  $A$ .
- (ii)  $f_n \rightarrow 0$  uniformly on  $\bar{A}$ .

(iii) There exists a sequence  $\{g_n\}$  of continuous functions on  $\bar{A}$  such that, for each  $x \in \bar{A}$ ,  $\lim_{n \rightarrow +\infty} g_n(x) = 0$ ,  $|f_n(x)| \leq g_n(x)$ , and  $g_{n+1}(x) \leq g_n(x)$  for all  $n \in N$ .

(iv) There exists a sequence  $\{g_n\}$  of continuous functions on  $\bar{A}$  such that  $g_n \rightarrow 0$  uniformly on  $\bar{A}$ ,  $|f_n(x)| \leq g_n(x)$ , and  $g_{n+1}(x) \leq g_n(x)$  for all  $n \in N$  and all  $x \in \bar{A}$ .

(v) There exists a sequence  $\{g_n\}$  of continuous functions on  $A$  such that  $g_n \rightarrow 0$  uniformly,  $|f_n(x)| \leq g_n(x)$ , and  $g_{n+1}(x) \leq g_n(x)$  for all  $n \in N$  and all  $x \in A$ .

(vi) There exist a sequence  $\{g_n\}$  of continuous functions on  $A$  and a sequence  $\{M_n\}$  of real numbers such that  $\lim_{n \rightarrow +\infty} M_n = 0$ ,  $|f_n(x)| \leq g_n(x) \leq M_n$ ,  $g_{n+1}(x) \leq g_n(x)$ , and  $M_{n+1} \leq M_n$  for all  $n \in N$  and all  $x \in A$ .

**Proof.** Since  $(i) \Rightarrow (ii)$  is immediate from Lemma 3.1 and the implications  $(iv) \Rightarrow (v)$  and  $(vi) \Rightarrow (i)$  are obvious while  $(v) \Rightarrow (vi)$  can be obtained as  $(ii) \Rightarrow (iii)$  in Theorem 2.1, we only need to show  $(ii) \Rightarrow (iii) \Rightarrow (iv)$ .

$(ii) \Rightarrow (iii)$ : Let  $(ii)$  be true. Then for each  $\epsilon > 0$  there exists  $n_\epsilon \in N$  such that

$$|f_n(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in \bar{A}.$$

For each  $n \in N$ , taking

$$g_n(x) := \sup\{|f_m(x)| : n \leq m \in N\} \quad \text{for } x \in \bar{A},$$

we have  $|f_n(x)| \leq g_n(x) \leq g_{n+1}(x) \leq g_n(x)$  for all  $n \geq n_\epsilon$  and all  $x \in \bar{A}$ .

Thus  $\lim_{n \rightarrow +\infty} g_n(x) = 0$  for  $x \in \bar{A}$ .

Next, we prove that each  $g_n$  is continuous on  $\bar{A}$ . Let  $c \in \bar{A}$ . Then  $g_n(c) \geq 0$ . If  $g_n(c) = 0$ , then  $g_m(c) = 0$  and hence  $|f_m(c)| = 0$  for all  $m \geq n$ . As we stated above, for each  $\epsilon > 0$  there exists  $n_\epsilon \geq n$  such that

$$|f_m(x)| \leq \epsilon \quad \text{for all } x \in \bar{A} \text{ and all } m \geq n_\epsilon.$$

For each  $n \leq m < n_\epsilon$ ,  $f_m$  is continuous at  $c$ , so there exists  $\delta_\epsilon > 0$  such that

$$|f_m(x)| \leq |f_m(c)| + \epsilon = \epsilon \quad \text{for all } x \in \bar{A} \text{ with } d(x, c) < \delta_\epsilon \text{ and all } n \leq m < n_\epsilon.$$

Hence

$$0 \leq g_m(x) - g_m(c) \leq \epsilon \quad \text{for all } x \in \bar{A} \text{ with } d(x, c) < \delta_\epsilon \text{ and all } m \geq n.$$

This shows that  $g_m$  is continuous at  $c$  for each  $n \leq m \in N$ .

If  $g_n(c) > 0$ , then for  $\epsilon_c := \frac{g_n(c)}{3}$  there exists  $n_c > n$  such that

$$|f_m(x)| \leq \epsilon_c \quad \text{for all } x \in \bar{A} \text{ and all } n_c \leq m \in N.$$

From this it follows that  $g_m(x) \leq \epsilon_c$  for all  $x \in \bar{A}$  and all  $n_c \leq m \in N$ . Thus

$$g_n(c) = \max\{|f_m(c)| : n \leq m < n_c\}.$$

Denote  $I(n) = \{m \in N : |f_m(c)| = g_n(c), n \leq m < n_c\}$ . Then there exists  $\delta_c > 0$  such that

$$2\epsilon_c = |f_m(c)| - \epsilon_c \leq |f_m(x)| \leq |f_m(c)| + \epsilon_c = 4\epsilon_c$$

for all  $x \in \bar{A}$  with  $d(x, c) < \delta_c$  and all  $m \in I(n)$ . This shows that

$$g_n(x) = \max\{|f_m(x)| : n \leq m < n_c\} \text{ for all } x \in \bar{A} \text{ with } d(x, c) < \delta_c.$$

Now, for each  $\epsilon \in (0, \epsilon_c)$ , by the continuity of  $f_m$ , there exists  $\delta \in (0, \delta_c)$  such that

$$|f_m(c)| - \epsilon \leq |f_m(x)| \leq |f_m(c)| +$$

for all  $x \in A$  with  $d(x, c) < \delta$  and all  $n \leq m < n_c$ . Thus

$$g_n(c) - \epsilon \leq g_n(x) \leq g_n(c) + \epsilon \quad \text{for all } x \in \bar{A} \text{ with } d(x, c) < \delta.$$

This proves that  $g_n$  is continuous at  $c$ .

(iii)  $\Rightarrow$  (iv): Suppose that (iii) is true. For each  $n \in N$ , by the continuity of  $g_n$  on the compact set  $\bar{A}$ , there exists  $x_n \in \bar{A}$  such that

$$g_n(x_n) = \max\{g_n(x) : x \in \bar{A}\}.$$

For  $x \in \bar{A}$ , since  $\{g_n(x)\}$  is nonincreasing,

$$g_n(x_n) \geq g_n(x_{n+1}) \geq g_{n+1}(x_{n+1}) \geq 0.$$

In addition, we must have  $g_n(x_n) \rightarrow 0$  as  $n \rightarrow +\infty$ . Otherwise, suppose that  $g_n(x_n) \not\rightarrow 0$  as  $n \rightarrow +\infty$ . Then there exists  $\epsilon_0 > 0$  and a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\epsilon_0 < g_{n_k}(x_{n_k}) \quad \text{for all } k \in N.$$

Since  $\bar{A}$  is compact, there exists a subsequence  $\{x_{n_{k_i}}\}$  of  $\{x_{n_k}\}$  and a point  $\bar{x} \in \bar{A}$  such that

$$\lim_{i \rightarrow +\infty} x_{n_{k_i}} = \bar{x}.$$

Now, since each  $g_{n_k}$  is continuous on  $\bar{A}$ ,

$$g_{n_k}(\bar{x}) = \lim_{i \rightarrow +\infty} g_{n_k}(x_{n_{k_i}}) \geq \liminf_{i \rightarrow +\infty} g_{n_{k_i}}(x_{n_{k_i}}) \geq \epsilon_0.$$

This contradicts  $\lim_{k \rightarrow +\infty} g_{n_k}(\bar{x}) = 0$ . Therefore  $g_n(x_n) \rightarrow 0$ , which implies  $g_n \rightarrow 0$  uniformly on  $\bar{A}$ , and hence (iv) follows.

**Example 3.3.** Consider the functions  $f_n(x) = x^n$  on  $A = (0,1)$  for  $n \in N$ .

Since

$$\sup\{x^m : x \in (0,1), n \leq m \in N\} = 1 \quad \text{for all } n \in N,$$

(vi) in Theorem 3.2 is not satisfied,  $\{f_n\}$  is not uniformly convergent on  $A$ .

**Example 3.4.** Recall that, for a Lebesgue integrable function  $f$  on  $[a, b]$ , the function

$$F(x) := \int_a^x f \, dt$$

is continuous on  $[a, b]$  [ ]. For a sequence  $\{f_n\}$  of real-valued Lebesgue integrable functions on  $[a, b]$ , the uniform convergence of the sequence of functions  $\int_a^x |f_n| \rightarrow 0$  on  $[a, b]$  can be characterized based on Theorem 3.2 just by replacing  $|f_n(x)|$  in it with  $\int_a^x |f_n|$ . In particular, if also  $0 \leq f_{n+1}(x) \leq f_n(x)$  for all  $n \in N$  and all  $x \in [a, b]$ , then  $\int_a^x f_n \rightarrow 0$  uniformly on  $[a, b]$  if  $\int_a^x f_n \rightarrow 0$  for each  $x \in [a, b]$ .

**Example 3.5.** Note that a sequence  $\{f_n\}$  converges uniformly to a function  $f$  on  $A$  if and only if  $(f_n - f) \rightarrow 0$  uniformly on  $A$ . If each  $f_n$  is continuous on a compact set, then  $f$  is also continuous on it. So, Theorem 3.2 can equivalently be stated by replacing  $f_n$  with  $f_n - f$ .

From the proof of (ii)  $\Rightarrow$  (iii) in Theorem 3.2 we see that if  $f_n \rightarrow 0$  uniformly on  $[a, b]$  and all functions  $f_n$  are continuous at  $c \in [a, b]$ , then each function  $g_n$  is continuous at  $c \in [a, b]$ . So, if each  $f_n$  is Riemann integrable on  $[a, b]$ , then  $f_n$  is continuous almost everywhere in  $[a, b]$  and hence  $g_n$  is Riemann integrable since it is measurable and continuous almost everywhere in  $[a, b]$ . Thus, the following result is valid.

**Theorem 3.6.** Let  $\{f_n\}$  be a sequence of real-valued Riemann integrable functions on  $[a, b]$ . Then the following statements are equivalent:

- (i)  $\{f_n\}$  converges uniformly to a function  $f$  on  $[a, b]$ .
- (ii) There exist a Riemann integrable function  $f$  on  $[a, b]$  and a sequence  $\{g_n\}$  of real-valued Riemann integrable functions on  $[a, b]$  such that  $g_n \rightarrow 0$  uniformly on  $[a, b]$ ,  $|(f_n - f)(x)| \leq g_n(x)$ , and  $g_{n+1}(x) \leq g_n(x)$  for all  $n \in N$  and all  $x \in [a, b]$ .
- (iii) There exist a Riemann integrable function  $f$  on  $[a, b]$ , a sequence  $\{g_n\}$  of real-valued Riemann integrable functions on  $[a, b]$ , and a sequence  $\{M_n\}$  of real numbers such that  $\lim_{n \rightarrow +\infty} M_n = 0$ ,  $|(f_n - f)(x)| \leq g_n(x) \leq M_n$ ,  $g_{n+1}(x) \leq g_n(x)$ , and  $M_{n+1} \leq M_n$  for all  $n \in N$  and all  $x \in [a, b]$ .

**Proof.** Since the implication  $(ii) \Rightarrow (iii)$  is immediate from Theorem 2.1 and  $(iii) \Rightarrow (i)$  is trivial, it suffices to show  $(i) \Rightarrow (ii)$ .

$(i) \Rightarrow (ii)$ : For each  $n \in N$ , let

$$D_n := \{x \in [a, b]: f_n - f \text{ is not continuous at } x\}.$$

As we know,  $(i)$  implies that the function  $f$  is Riemann integrable on  $[a, b]$ , so  $f_n - f$  is Riemann integrable on  $[a, b]$  and hence  $D_n$  has measure zero. Thus the union  $\cup_{n \in N} D_n$  has measure zero.

For each  $\epsilon > 0$ , by the uniform convergence  $f_n \rightarrow f$ , there exists  $n_\epsilon \in N$  such that

$$|(f_n - f)(x)| \leq \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and all } x \in [a, b].$$

For each  $n \in N$ , taking

$$g_n(x) := \sup\{|(f_m - f)(x)|: n \leq m \in N\} \text{ for } x \in [a, b],$$

we have

$$|(f_n - f)(x)| \leq g_n(x) \text{ and } g_{n+1}(x) \leq g_n(x) \leq \epsilon \text{ for all } n \geq n_\epsilon \text{ and all } x \in [a, b].$$

From this we see that  $g_n \rightarrow 0$  uniformly. By definition, the above inequalities (without  $\epsilon$ ) are also valid for all  $n < n_\epsilon$  and  $x \in [a, b]$ . In addition, each  $g_n$  is continuous in  $[a, b] \setminus (\cup_{n \in N} D_n)$  and hence is Riemann integrable on  $[a, b]$ . So we obtain  $(ii)$ .

For a sequence  $\{f_n\}$  of real-valued Riemann integrable functions on  $[a, b]$ , if  $f_n \rightarrow 0$  uniformly on  $[a, b]$ , then  $\int_a^x |f_n| \rightarrow 0$  uniformly on  $[a, b]$ . Therefore the following statements are valid.

**Corollary 3.7.** Let  $\{f_n\}$  be a sequence of real-valued Riemann integrable functions on  $[a, b]$ . If  $f_n \rightarrow 0$  uniformly on  $[a, b]$ , then there hold

(i)  $\int_a^x |f_n| \rightarrow 0$  uniformly on  $[a, b]$ ;

(ii) there exists a sequence  $\{g_n\}$  of Riemann integrable functions on  $[a, b]$  such that, for each  $x \in [a, b]$ ,  $\lim_{n \rightarrow +\infty} \int_a^x g_n = 0$ ,

$$\int_a^x |f_n| \leq \int_a^x g_n, \text{ and } g_{n+1}(x) \leq g_n(x) \text{ for all } n \in N;$$

(iii) there exist a sequence  $\{g_n\}$  of Riemann integrable functions on  $[a, b]$  and a sequence  $\{M_n\}$  of real numbers such that  $\lim_{n \rightarrow +\infty} M_n = 0$ ,

$$\int_a^x |f_n| \leq \int_a^x g_n \leq M_n(x - a), g_{n+1}(x) \leq g_n(x), \text{ and } M_{n+1} \leq M_n$$

for all  $n \in N$  and all  $x \in [a, b]$ .

#### 4. SOME APPLICATIONS

As the limit function of a sequence of partial sums of functions, the uniform convergence of a series of functions can similarly be discussed. For a series of continuous functions on a compact set, we can apply Theorem 3.2 to derive the following result:

**Theorem 4.1.** Let  $\{f_n\}$  be a sequence of real-valued continuous functions on a compact set  $A$  in a metric space  $(X, d)$ . Then the following statements are equivalent:

- (i)  $\sum_{k=1}^{\infty} |f_k|$  is uniformly convergent to a function  $f$  on  $A$ .
- (ii) There exists a continuous function  $f$  on  $A$  such that, for each  $x \in A$ ,

$$\sum_{k=1}^{\infty} |f_k(x)| = f(x).$$

If also there holds  $|f_{n+1}(x)| \leq |f_n(x)|$  for all  $n \in N$  and all  $x \in A$ , then (i) is equivalent to

- (iii) For any  $1 < q \in N$  and each  $x \in A$  the series  $\sum_{k=1}^{\infty} q^k |f_{q^k}(x)|$  converges.

Furthermore, if there exists some  $1 < q \in N$  such that

$$\lim_{n \rightarrow +\infty} \frac{|f_{q^{n+1}}(x)|}{|f_{q^n}(x)|} < \frac{1}{q},$$

then  $\sum_{k=1}^{\infty} |f_k|$  is uniformly convergent on  $A$ .

**Proof.** Statement (i) is valid if and only if there exists a continuous function  $f$  on  $A$  such that  $g_n := f - \sum_{k=1}^n |f_k| \rightarrow 0$  uniformly on  $A$ . It is easy to see that

$$0 \leq g_{n+1}(x) \leq g_n(x) \quad \text{for all } n \in N \text{ and all } x \in A.$$

Thus, by Theorem 3.2,  $g_n \rightarrow 0$  uniformly on  $A$  if and only if, for each  $x \in A$ ,  $\lim_{n \rightarrow +\infty} g_n(x) = 0$ , that is, (ii) is true.

Next, if also there holds  $|f_{n+1}(x)| \leq |f_n(x)|$  for all  $n \in N$  and all  $x \in A$ , then, by [ ], (ii)  $\Leftrightarrow$  (iii) and hence (i)  $\Leftrightarrow$  (iii).

Finally, if for some  $1 < q \in N$  there holds

$$\lim_{n \rightarrow +\infty} \frac{|f_{q^{n+1}}(x)|}{|f_{q^n}(x)|} < \frac{1}{q},$$

then, by [4, Theorem 3],  $\sum_{k=1}^{\infty} |f_k(x)|$  is convergent at each  $x \in A$ . So  $\sum_{k=1}^{\infty} |f_k|$  is uniformly convergent to a function  $f$  on  $A$ .

**Example 4.2.** Consider the series

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{for } x \in [-1,1],$$

where  $a_n \geq a_{n+1} > 0$  for all  $n \in N$ . If there exists some  $1 < q \in N$  such that

$$\lim_{n \rightarrow +\infty} \frac{a_{q^{n+1}}}{a_{q^n}} < \frac{1}{q},$$

then

$$\lim_{n \rightarrow +\infty} \frac{|a_{q^{n+1}} x^{q^{n+1}}|}{|a_{q^n} x^{q^n}|} = \lim_{n \rightarrow +\infty} \frac{a_{q^{n+1}} |x|^q}{a_{q^n}} < \frac{|x|^q}{q} \leq \frac{1}{q} \quad \text{for } x \in [-1,1],$$

it follows from Theorem 4.1 that  $\sum_{n=1}^{\infty} a_n |x|^n$  is convergent for each  $x \in [-1,1]$ . By Theorem 4.1,  $\sum_{n=1}^{\infty} a_n |x|^n$  is uniformly convergent on  $[-1,1]$ . Therefore  $\sum_{n=1}^{\infty} a_n x^n$  is uniformly convergent on  $[-1,1]$ .

**Theorem 4.3.** Let  $A$  be a subset in a metric space  $(X, d)$  and let  $f : A \rightarrow X$  be a continuous mapping. For  $x \in A$ , denote

$$f^1(x) := f(x), \quad f^{n+1}(x) := f(f^n(x)), \quad \text{and} \quad f_n(x) := d(f^n(x), f^{n+1}(x))$$

provided that  $f^n(x) \in A$  for  $n \in N$ . Then the following statements are equivalent:

(i) There exists  $a \in A$  such that  $f(a) = a$ .

(ii) There exists a nonempty compact set  $C \subseteq A$  such that  $f(C) \subseteq C$  and for each  $x \in C$  there exists  $n_x$  such that  $(n + 1)f_{n+1}(x) \leq nf_n(x)$  for all  $n \geq n_x$ .

(iii) There exists a nonempty compact set  $C \subseteq A$  such that  $f(C) \subseteq C$  and for each  $x \in C$  there exists a strictly increasing nonnegative function  $g$  on  $N$  such that  $\lim_{n \rightarrow +\infty} g(n) = \infty$  and for some  $n_x \in N$  there holds

$$g(n + 1)f_{n+1}(x) \leq g(n)f_n(x) \quad \text{for all } n \geq n_x.$$

(iv) There exists a nonempty compact set  $C \subseteq A$  such that  $f(C) \subseteq C$  and  $f_n \rightarrow 0$  uniformly on  $C$ .

(v) There exist a nonempty compact set  $C \subseteq A$  and a nonincreasing sequence  $\{g_n\}$  of continuous functions on  $C$  such that  $f(C) \subseteq C$ ,  $f_n(x) \leq g_n(x)$  for all  $n \in N$ , and  $\lim_{n \rightarrow +\infty} g_n(x) = 0$  for each  $x \in C$ .

(vi) There exist a compact set  $C \subseteq A$  such that  $f(C) \subseteq C$  and  $\lim_{n \rightarrow +\infty} f_n(x) = 0$  for some point  $x$  in  $C$ .

(vii) There exists a compact set  $C \subseteq A$  such that  $f(C) \subseteq C$  and for each  $x \in C$  with  $d(x, f(x)) > 0$  there exists  $\bar{x} \in C$  satisfying  $d(\bar{x}, f(\bar{x})) < d(x, f(x))$ .

**Proof.** The implication (i)  $\Rightarrow$  (ii) is immediate by taking  $C := \{a\}$ , (ii)  $\Rightarrow$  (iii) and (v)  $\Rightarrow$  (vi) are obvious, and (iv)  $\Rightarrow$  (v) is direct from Theorem 3.2, so it suffices to show that (iii)  $\Rightarrow$  (iv), (v)  $\Rightarrow$  (vii)  $\Rightarrow$  (i), and (vi)  $\Rightarrow$  (i).

To show (iii)  $\Rightarrow$  (iv), we suppose that (iii) is true. Then, for the compact set  $C$  in (iii) and each  $x \in C$ , the function  $g$  in (iii) satisfies

$$f_n(x) \leq \frac{g(n_x)f_{n_x}(x)}{g(n)} \quad \text{and} \quad f_{n+1}(x) \leq f_n(x) \quad \text{for all } n \geq n_x.$$

It follows that  $\lim_{n \rightarrow +\infty} f_n(x) = 0$  for  $x \in C$ . By Theorem 3.2, (iv) is valid.

Next, to prove the implications (v)  $\Rightarrow$  (vii)  $\Rightarrow$  (i), we suppose that (v) is true. Then, for each  $x \in C$  with  $d(x, f(x)) > 0$ , there exists  $n_x \in N$  such that

$$f_n(x) \leq g_n(x) < d(x, f(x)) \quad \text{for all } n \geq n_x.$$

Taking  $n = n_x$  and  $\bar{x} = f^n(x)$ , we have  $d(\bar{x}, f(\bar{x})) = f_n(x) < d(x, f(x))$ , that is, (vii) is valid.

Now, let (vii) be true. Since the function  $d(x, f(x))$  is continuous on the compact set  $C$ , it attains its minimum at a point  $a$  in  $C$ . By (vii), we must have  $d(a, f(a)) = 0$ , that is, (i) is true. Therefore (vii)  $\Rightarrow$  (i).

Finally, for the compact set  $C$ ,  $x \in C$ , and the function  $f$  in (vi), we have  $f^n(x) \in C$  for all  $n \in N$ , so there exists a subsequence  $\{f^{n_k}(x)\}$  of  $\{f^n(x)\}$  converging to some  $a \in C$ . From this we obtain

$$d(a, f(a)) = \lim_{k \rightarrow +\infty} d(f^{n_k}(x), f(f^{n_k}(x))) = \lim_{k \rightarrow +\infty} f_{n_k}(x) = 0.$$

Thus (vi)  $\Rightarrow$  (i) follows.

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