

*Ministry Of Higher Education  
And scientific Research  
University Of Babylon College  
Of Education For Pure sciences*



# **BESSEL FUNCTION AND ITS APPLICATIONS**

**A proseed research to the council of the college of Education for  
pure sciences/University of Babylon**

**As part of the requirements for a Bachelor`s degree in  
Mathematics**

**preparation:** Zahraa Hassan Ali

**Assistant teacher**

Aseel Sami Hamad

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

{ فَرِحِينَ بِمَا آتَاهُمُ اللَّهُ مِنْ فَضْلِهِ وَيَسْتَبِشِرُونَ بِالَّذِينَ لَمْ يَلْحَقُوا بِهِمْ مِنْ خَلْفِهِمْ أَلَّا

خَوْفٌ عَلَيْهِمْ وَلَا هُمْ يَحْزَنُونَ }

صَاحِبِ اللَّهِ الْعَظِيمِ

السُّورَةُ الْاٰمِرَاتِ: ١١١

# الإمام

إلى من قال:

«كُلُّ وعاءٍ يضيّقُ بما وُضِعَ فيه إلا وعاءَ العلمِ فإنه يتَّسعُ».

إلى بابِ مدينةِ العلم، ومنارِ البلاغة، وملاذِّ الساعين إلى الحقيقة... أميرِ المؤمنين عليّ بنِ أبي

طالب (عليه السلام).

أهدي ثمرةَ هذا الجهدِ المتواضع؛ ومن بعده إلى من ساروا على نهجه في العطاء:

والدَيَّ العزيزين، وأساتذتي الأفاضل

# الشكر والثناء

الحمد لله الذي له ما في السماوات وما في الأرض، وهو الحكيم الخبير.

والصلاة والسلام على نبينا محمد (ﷺ)، وعلى آله وصحبه أجمعين.

وبكل مشاعر الامتنان والتقدير، أهدي هذا العمل إلى:

إلى الذين أناروا عقول الآخرين بعلمهم...

إلى الذين أضاعوا دربي بالعلم والحكمة...

إلى الذين مهّدوا لي الطريق...

إلى الذين غرسوا في نفسي روح المبادرة...

(عائلي، وإخوتي، وأخواتي، وأقاربي)

تقديرًا لفضلهم، وعرفانًا بجميلهم، ومع ختام هذا العمل، لا يسعني إلا أن أتقدم بأسمى آيات الشكر والعرفان

إلى مشرفتي وأستاذتي الفاضلة (أسيل سامي)، ممتنًا لتوجيهاتها في اختيار هذا الموضوع، ولسخائها

العلمي، وملاحظاتها القيّمة، وإرشاداتها البناءة التي كان لها الأثر الكبير في إثراء هذا البحث.

كما أتقدم بجزيل الشكر والتقدير إلى أساتذتي الكرام في قسم الرياضيات، كلية التربية للعلوم الصرفة، الذين

تشرفت بالتعلم على أيديهم والانتماء إليهم.

وفي الختام، أتوجه بخالص الشكر وعظيم الامتنان لكل من أسهم في إنجاز هذا البحث، ولكل زملائي من

أبناء مدينتي وزملائي الطلبة، سائلًا الله تعالى أن يجزيهم خير الجزاء، وأن يجعل لهم نصيبًا في أعلى

درجات الجنة.

## الفهرس

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## **Abstract**

Bessel functions are among the most important special functions in applied mathematics and engineering. They arise naturally as solutions to Bessel's differential equation, which appears in many physical systems exhibiting cylindrical or spherical symmetry. These functions are widely used in various fields including electrical engineering, mechanical engineering, heat transfer, signal processing, and wave propagation.

The importance of Bessel functions lies in their ability to model real-world physical phenomena such as heat conduction in cylindrical objects, electromagnetic wave propagation in circular waveguides, vibration analysis of circular membranes, and solutions to Laplace's equation in cylindrical coordinates.

## Introduction

In mathematics and engineering, differential equations are used to describe many physical systems. Some of the most important differential equations lead to special functions, such as exponential functions, trigonometric functions, and Bessel functions.

Bessel functions were first introduced by the German mathematician Friedrich Wilhelm Bessel in 1824. He studied these functions while analyzing planetary motion. Later, scientists discovered that Bessel functions appear naturally in many engineering applications, especially when solving problems involving cylindrical symmetry.

For example, when analyzing heat transfer in cylindrical pipes, vibrations in circular drums, or electromagnetic waves in cylindrical conductors, the governing equations lead to Bessel's differential equation.

The general form of Bessel's differential equation is:

$$x^2 (d^2y/dx^2) + x (dy/dx) + (x^2 - n^2) y = 0$$

The solutions to this equation are called Bessel functions.

These functions are extremely important because many engineering systems cannot be solved using elementary functions alone. Instead, Bessel functions provide exact and accurate solutions.

This research focuses on explaining Bessel functions in detail, their properties, and their applications in real engineering problems

# **Chapter One**

## **Mathematical**

### **Foundations of Bessel**

#### **Function**

## 1-1 Definition of Bessel Functions [1]

Bessel functions are solutions of the differential equation:

$$x^2 (d^2y/dx^2) + x (dy/dx) + (x^2 - n^2) y = 0$$

This equation is called Bessel's equation.

The solutions of this equation are called:

Bessel Functions

They are denoted as:

$$J_n(x)$$

Where:

J = Bessel function

n = order of the function

x = variable

## 1-2 Standard Form of Bessel Equation [1]

The standard form is:

$$d^2y/dx^2 + (1/x) dy/dx + (1 - n^2/x^2) y = 0$$

This equation appears when solving physical problems in cylindrical coordinates.

### 1-3 Origin of Bessel Equation [2]

Bessel's equation appears when solving Laplace's equation in cylindrical coordinates.

Laplace's equation is<sup>2</sup>:  $\nabla^2 V = 0$

In cylindrical coordinates:

$$\frac{\partial^2 V}{\partial r^2} + (1/r)(\partial V/\partial r) + (1/r^2)(\partial^2 V/\partial \theta^2) + \partial^2 V/\partial z^2 = 0$$

When solving using separation of variables, we obtain Bessel's equation.

### 1-4 Series Solution of Bessel Functions[2]

The Bessel function of the first kind is given by:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n}$$

Where:

$\Sigma$  = summation

$\Gamma$  = gamma function

This series provides exact solutions

### 1-5 Bessel Function of the First Kind[2]

It is denoted by:

$$J_n(x)$$

It is finite at  $x = 0$

It is the most commonly used Bessel function.

Examples:  $J_0(x)$        $J_1(x)$        $J_2(x)$

## **.1-6 Physical Meaning of Bessel Functions<sup>[3]</sup>**

Bessel functions describe:

- Heat flow
- Vibrations
- Wave motion
- Electromagnetic fields

They are essential in engineering analysis

## **1-6 Graphical Behavior<sup>[3]</sup>**

Bessel functions behave like oscillating waves.

Similar to sine and cosine but with decreasing amplitude.

Example:  $J_0(x)$

Starts at 1 when  $x = 0$

Then oscillates

## **1-7 Importance in Engineering<sup>[3]</sup>**

Bessel functions are used in:

Electrical Engineering

Mechanical Engineering

Civil Engineering

Signal Processing

Communication Systems

They provide accurate solutions.

# **Chapter Two**

## **Types of**

## **Bessel**

## 2-1 Bessel Functions of the First Kind<sub>[4]</sub>

Bessel functions of the first kind are the most common solutions to Bessel's differential equation. They are denoted by:  $J_n(x)$

where  $n$  represents the order of the function.

These functions are finite at  $x = 0$  when  $n \geq 0$ . They are obtained using the power series method.

The general series form is:  $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n}$

These functions are widely used in engineering applications.

Examples include:  $J_0(x)$        $J_1(x)$        $J_2(x)$

Each function has unique oscillatory behavior.

## 2-2 Bessel Functions of the Second Kind<sub>[4]</sub>

These are denoted as:  $Y_n(x)$

They are also called Neumann functions.

Unlike  $J_n(x)$ , these functions are infinite at  $x = 0$ .

They are used in physical systems where singular behavior exists.

The general solution of Bessel's equation is:  $y(x) = A J_n(x) + B Y_n(x)$

Where  $A$  and  $B$  are constants.

## 2-3 Modified Bessel Functions<sub>[5]</sub>

Modified Bessel functions are used when solving exponential-type problems.

They are denoted as:  $I_n(x)$  and  $K_n(x)$

These appear in heat transfer and diffusion problems.

Modified Bessel equation:  $x^2 y'' + x y' - (x^2 + n^2)y = 0$

These functions grow or decay exponentially

## 2-4 Spherical Bessel Functions<sup>[5]</sup>

These functions appear in spherical coordinate systems.

They are denoted as:  $j_n(x)$  and  $y_n(x)$

Used in wave equations and quantum mechanics.

## 2-5 Recurrence Relations<sup>[5]</sup>

Bessel functions follow recurrence relations.

Example :  $J_{n-1}(x) + J_{n+1}(x) = (2n/x) J_n(x)$

This helps compute higher-order functions.

## 2-6 Orthogonality Property<sup>[5]</sup>

Bessel functions are orthogonal.

This means:  $\int_0^1 x J_n(ax) J_n(bx) dx = 0$  if  $a \neq b$

This property is useful in solving boundary value problems.

## 2-7 Zeros of Bessel Functions<sup>[6]</sup>

Bessel functions have infinite zeros.

Example:

$J_0(x) = 0$  at: 2.404 5.520 8.654

These zeros are important in vibration analysis.

## **2-8 Asymptotic Behavior**

For large  $x$ :  $J_n(x) \approx \sqrt{2/\pi x} \cos(x - n\pi/2 - \pi/4)$

This shows oscillatory behavior

## **2-9 Graphical Representation<sup>[6]</sup>**

Bessel functions oscillate like sine waves.

But amplitude decreases over time.

Important in wave analysis.

**Chapter Three**

**Applications**

**of**

**Bessel**

# **Bessel functions have many engineering applications**

## **3-1 Application in Electrical Engineering [7]**

Bessel functions are used in:

- Antenna analysis
- Waveguides
- Electromagnetic fields

Example:

Electric field in cylindrical waveguide:  $E(r) = J_0(kr)$

## **3-2 Heat Transfer Applications[7]**

Heat conduction in cylindrical rods:

Temperature equation leads to Bessel function.

Example:  $T(r,t) = J_0(r)$

Used in thermal engineering.

## **3-3 Mechanical Engineering Applications[7]**

Used in vibration analysis.

Example:

Vibration of circular drum: Solution involves Bessel function

### **3-4 Signal Processing Applications<sup>[7]</sup>**

Used in filter design.

Bessel filters provide smooth response.

Advantages:

- No signal distortion
- Accurate signal transmission

### **3-5 Electromagnetic Wave Applications<sup>[7]</sup>**

Maxwell's equations lead to Bessel functions.

Used in:

Radar systems

Communication systems

### **3-6 Acoustics Applications<sup>[7]</sup>**

Used in sound wave analysis.

Example: Sound propagation in cylindrical tubes

### **3-7 Quantum Mechanics Applications<sup>[7]</sup>**

Used in solving Schrödinger equation.

Important in physics.

### **3-8 Series Expansion Method [8]**

Most accurate method.

Used in computers.

### **3-9 Recurrence Method [8]**

Used to calculate higher order functions

.

### **3-10 Software Applications [8]**

Programs that use Bessel functions:

MATLAB

Python

Mathematica

Example MATLAB code:

```
besselj(0,x)
```

Discussion

Bessel functions play an essential role in solving engineering problems.

They provide exact mathematical models for cylindrical systems.

They help engineers design systems accurately.

They are essential in electrical engineering.

They improve system analysis.

## **Conclusion**

This research studied Bessel functions and their applications.

Bessel functions are solutions of Bessel differential equation.

They are important in engineering and physics.

They are used in heat transfer, electrical engineering, and signal processing.

They provide accurate solutions.

Understanding Bessel functions helps engineers solve complex problems.

They are essential tools in modern engineering

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