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Collectionwise Normal Topological Spaces via α-open Sets Structures

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الايه القرانيه

(يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات والله بما تعملون خبير)

سورة المجادله.. الايه رقم 11

الاهداء

وصلت رحلتي الجامعية إلى نهايتها بعد تعب ومشقَّة..

وها أنا ذا أختم بحث تخرُّجي بكل همَّة ونشاط،

وأمتنُّ لكل من كان له فضل في مسيرتي،

وساعدني ولو باليسير،

إلى من أُفضِّلها على نفسي، ولِمَ لا؛ فلقد ضحَّت من أجلي

ولم تدَّخر جُهدًا في سبيل إسعادي على الدَّوام

(أُمِّي الحبيبة).

نسير في دروب الحياة، ويبقى من يُسيطر على أذهاننا في كل مسلك نسلكه

صاحب الوجه الطيب، والأفعال الحسنة.

فلم يبخل عليَّ طيلة حياته

(والدي العزيز).

الى (اخوتي) ونور عيوني ومصدر قوتي واحبائي

الى من وقف بجانبي وساندني بجميع الصعاب

صديقي ورفيق دربي صاحب القلب الطيب

(زوجي العزيز)

، الى اساتذتي المُبجَّلين...

أُهديكم بحث تخرُّجي.............

Abstract:

In this search, the notion of collectionwise normal topological spaces of class α introduced and several properties of these spaces studied. A comparison between this class and the class of collectionwise topological spaces is presented

الخلاصه:

في هذا البحث قمنا بدراسه الفضاءات التوبولوجيه(الطبيعيه ثنائية اللفيف)مع بعض من بديهيات الفصل والمعرفه من خلال المجموعه المفتوحه كذلك تم دراسة بعض العلاقات الجديده بين هذه البديهيات من جهه وبينها وبين الفضاءات الباراكومباكتيه التوبولوجيه من جهة اخرى.

1. Introduction

Let A be a subset of a topological space X. Any point $x \in A$ is said to be interior of A, if x belongs to an open set G contained in A, i.e. $x \in G \subseteq A$. The set of interior points of A is denoted by int (A) or A°, which is called the interior of A. Th closure of A is defined as the intersection of all closed sets containing A.The Closure of A is denoted by Cl(A) .

A subset A of a topological space (X, τ) is said to be α-open [10] if A ⊆ int(cl(int(A))). The complement of a α-open set in a space (X, τ) is said to be α-closed [10]. The family of all α-open sets in a topological space (X, τ) is a topology on X finer than τ denoted by τα.

Many athures like [1,2,3,5,6,7,8,9,11,12] use this notion to introduce more general definitions using this concepts. The collection of all $\alpha - open$ set is denoted by αO(X) and the pair (X,αO(X)) is called the α − topological space associated with (X,Ƭ). We remark that (X, αO(X)) is a topological space. The complement of all $\alpha - open$ is called $\alpha - closed$ and the intersection of all $\alpha - closed$ set in X containig A is called $\alpha - clouser$ of A and is denoted by Cl_{α}(A).

CHAPTER ONE

BASIC DEFINITIONS AND PRILIMINARIES

1-1Basic definitions and preliminaries:

In this chapter we will display the basic definitions and the main concepts of our work like the definitions of a topological space and the cover, star refinement cover and barycentric cover of a topological space and what meaning by compact, paracompct, collectionwise normal topological spaces and many other related spaces also discussed and showed.

1.1.1. Definition:[10]

A subset A of a topological space (X, τ) is said to be α-open , if A $\subseteq$int(cl(int(A))).

1.1.2. Exammple

As given, X = {a, b, c, d, e}
And τ = {φ, X, {a, b, c}, {d, e}, {c}, {d, e, c}}
So that we have
Open sets:{ φ, X, {a, b, c}, {d, e}, {c}, {d, e, c}
Closes sets: {X, φ, {d, e}, {a, b, c}, (a, b, d, e}, {a, b}
Now as per definition of α -open set, here we have

X, φ,{c},{d,e},{a,c}.{a,b,c}, {c,d,e}, {b, c, d, e}, {c, e, d, a} and {c, e, d, a} are α-open sets with respect to this topology.

1.1.3. Definition:[5]

Let V be a topological space. A family $\{A_s\}_{s\in S}$ of subset of V is called a cover of V if $\bigcup_{s\in S} A_s = V$. If all the sets A_s are open (closed), we say that the cover $\{A_s\}_{s\in S}$ is open (closed).

1.1.4. Definition: [5]

Let V be a topological space, a collection F=$\{F_\alpha : \alpha \in I\}$ of subsets of V is said to be locally finite if for each $v \in V$, $\exists$ open set U in V containing v and $U \cap F_\alpha \neq \emptyset$.

1.1.5. Definition: [5]

Let $\{A_s\}_{s\in S}$ be cover of V and let $\{B_t\}_{t\in T}$ be another cover we say that $\{B_t\}_{t\in T}$ is a refinement of $\{A_s\}_{s\in S}$ if $\forall\ t \in T, \exists\ s \in S \ni B_t \subseteq A_s$.

1.1.6. Definition: [5]

Let V be a topological space, then V is called a Paracompact space if it is hausdorff and every open cover of V has a locally finite open refinement cover.

1.1.7. Definition: [5]

Let V be a topological space, then V is called a compact space if it is hausdorff space with the property that every cover by open sets contains a finite sub cover.

1.1.8. Definition: [5]

Let Ѵ be a topological space, then it is called a regular space if and only if for each $v \in$ Ѵ and closed set F in Ѵ with $v \notin F$, there are open sets U, V such that $v \in U , F \subseteq V \; and \; U \cap V \neq \emptyset$.

1.1.9. Definition: [5]

A topological space Ѵ is called T_1 if for any two distinct points v and u of Ѵ there exist two disjoint open set U and V such that $u \in U , v \notin U \; and \; v \in V , u \notin V$.

1.1.10. Definition: [5]

Let Ѵ be a topological space, then it is called a normal if and only if F_1 and F_2 are two disjoint closed subset of Ѵ, then there exists set G , H, $\ni F_1 \subset G , F_2 \subset H \; and \; G \cap H = \emptyset$.

1.1. 11. Definition: [5]

Let $\mathcal{A} = \{A_s\}$ s ∈ S be a cover of a set X; the star of a set $M \subset X$ with respect to $\mathcal{A}$ is the set $St(M, \mathcal{A}) = \bigcup\{A_s : M \cap A_s \neq \emptyset\}$. The star of a one-point set {x} with respect to a cover $\mathcal{A}$ and is denoted by $t(x, \mathcal{A})$.

1.1. 12. Definition: [5]

We say that a cover $\mathcal{B} = \{B_t\}$t ∈ T of a set X is a star refinement of another cover $\mathcal{A} = \{A_s\}$ s ∈ S of the same set X if for every $t \in T$ there exists an $s \in S$ such that $St(B_t, \mathcal{B}) \subset A_s$; if for every $x \in X$ there exists an $s \in S$ such that $St(x, \mathcal{B}) \subset A_s$, then we say that $\mathcal{B}$ is a barycentric refinement of $\mathcal{A}$. Clearly , every star refinement is a barycentric refinement and every barycentric refinement is a refinement .

1.1. 13.Definition: [5]

Let (X, T) be a topological space, then X is called a collectionwise normal of if X is a T_1 – space and for every discrete family $\{F_s\}_{s\in S}$ of closed subset of X there exists a discrete family $\{V_s\}_{s\in S}$ of open subset of X such that $F_s \subset V_s$ for every $s \in S$. Clearly , every collectionwise is normal.

1-2 Some important theorems

In this part we give some important theorem which we shall use and generalized in the second chapter.

1.2.1. Theorem: [4]

Every compact space is paracompact space.

1.2.2. Theorem: [4]

Every open cover of a Lindelof space has locally finite open refinement cover.

1.2.3. Theorem: [4]

Any Lindelof space is pararacompact .

1.2.4. Lemma: [4]

Let V be pararacompact space and A, B a pair of closed subsets of V. If for every $v \in B$ there exists open set U_v, V_v such that $A \subseteq U_v, v \in V_v$ and $U_v \cap V_v = \emptyset$, then there also exists open set U, V such that $A \subset U , B \subset V$ and $U \cap V = \emptyset$.

1.2.5. Theorem: [4]

Every paracompact space is normal.

1.2.6. Lemma: [4]

If Ѵ is T_1 –space and for every closed set $F \subset$ Ѵ and every open $W \subset$ Ѵ that contains F threr exists a sequence $W_1, W_2, \ldots$ of open subset of Ѵ such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for i=1,2,.., then the space Ѵ is normal.

1.2.7. Lemma: [4]

If every open cover of a regular space Ѵ has a locally finite refinement (consisting of arbitrary sets), then for every open cover $\{U_s\}_{s \in S}$ of the space Ѵ there exists a closed locally finite cover $\{F_s\}_{s \in S}$ of Ѵ such that $F_s \subset U_s$ for every $s \in S$.

1.2.8.Lemma: [4]

If an open cover $\mathcal{U}$ of a topological space X has a closed locally finite refinement , then$\mathcal{U}$ has also an open barycentric refinement.

1.2.9.Lemma: [4]

If $\mathcal{A} = \{A_s\} s \in S$ of a set X is barycentric refinement of a cover $\mathcal{B} = \{B_t\} t \in T$ of X and $\mathcal{B}$ is a barycentric refinement of a cover $\mathcal{C} = \{C_z\} z \in Z$ of the same set , then $\mathcal{A}$ is a star refinement of $\mathcal{C}$.

CHAPTER TWO

COLLECTIONWISE NORMAL TOPOLOGICAL SPACES of Class α

2 On Collectionwise Normal Topological Spaces of class α

2.1. Definition:

Let Ѵ be a topological space. A family $\{A_s\}_{s\in S}$ of subset of Ѵ is called a cover of Ѵ if $\bigcup_{s\in S} A_s = Ѵ$. If all the sets A_s are α −open (α −closed), we say that the cover $\{A_s\}_{s\in S}$ is α −open (α −closed).

2.2. Definition:

Let Ѵ be a topological space, a collection F=$\{F_i : i \in I\}$ of subsets of Ѵ is said to be α −locally finite if for each $v \in Ѵ$, ∃ α − open set U in Ѵ containing v and $U \cap F_i \neq \emptyset$.

2.3. Definition:

Let $\{A_s\}_{s\in S}$ be cover of Ѵ and let $\{B_t\}_{t\in T}$ be another cover we say that $\{B_t\}_{t\in T}$ is a refinement of $\{A_s\}_{s\in S}$ if $\forall\ t \in T, \exists\ s \in S \ni B_t \subseteq A_s$.

2.4. Definition:

Let Ѵ be a topological space, then Ѵ is called a paracompact space of class α if it is hausdorff and every open cover of Ѵ has a locally finite α −open refinement.

2.5. Definition:

Let Ѵ be a topological space, then it is called a regular space of class αif and only if for each $v \in$ Ѵ and α −closed set F in Ѵ with $v \notin F$, there are α −open sets U, V such that $v \in U , F \subseteq V \text{ and } U \cap V \neq \emptyset$.

2.6. Definition:

A topological space (Ѵ, α O(Ѵ)) is called α $- T_1$if for any two distinct points v and u of Ѵ there exist two disjoint− α − open set U and V such that $u \in U , v \notin U \text{ and } v \in V , u \notin V$.

2.7. Definition:

Let Ѵ be a topological space, then it is called a normal of class α if and only if F_1 and F_2 are two disjoint closed subset of Ѵ, then there exists α − open set G , H,∋ $F_1 \subset G , F_2 \subset H \text{ and } G \cap H = \emptyset$.

2.8. Definition:

Let $\mathcal{A} = \{A_s\}$ s ∈ S be a cover of a α − open subsets of X; the α −star of a set $M \subset X$ with respect to $\mathcal{A}$ is the set α $- St(M, \mathcal{A}) = \bigcup\{A_s : M \cap A_s \neq \emptyset\}$. The star of a one-point set {x} with respect to a cover $\mathcal{A}$ and is denoted by $t(M, \mathcal{A})$.

We say that a cover $\mathcal{B} = \{B_t\}$t ∈ T of a set X is a α −star refinement of another cover $\mathcal{A} = \{A_s\}$ s ∈ S of the same set X if for every $t \in T$ there exists an $s \in S$ such that α $- St(B_t, \mathcal{B}) \subset A_s$; if for every $x \in X$ there exists an $s \in S$ such that α $- St(x, \mathcal{B}) \subset A_s$, then we say that $\mathcal{B}$ is a α −barycentric refinement of $\mathcal{A}$. Clearly, every α − star refinement is a α − barycentric refinement and every α −barycentric refinement is a α −refinement .

2.9. Theorem:

For every α −T$_1$- space X the following conditions are equivalent

(i) The space X is paracompact of class α

(ii) Every α −open cover of the space X has an α − barycentric refinement.

(iii) Every α −open cover of the space X has a α − star refinement.

(iv) The space X is α −regular and every α −open cover of X has an open α −discrete refinement.

2.10. Lemma:

If an α − open cover $\mathcal{U}$ of a topological space X has a α − closed locally finite refinement , then$\mathcal{U}$ has also an α − barycentric refinement.

Proof:

Let $\mathcal{F} = \{F_t\}$t ∈ T be a α −closed locally refinement of = $\{U_s\}$ s ∈ S . For every $t \in T$ choose an $s(t) \in S$ such that$F_t \subset U_{s(t)}$. It follow the local finiteness of $\mathcal{F}$ that the set $T(x) = \{t \in T: x \in F_t\}$ is finite for every$x \in X$, and this implies that the set

(2) $$V_x = \bigcap_{t\in T(x)} U_{s(t)} \cap (X \setminus \bigcup_{t\notin T(x)} F_t)$$

Is α −open for every ∈ X . As $x \in V_x$, the family $\mathcal{V} = \{V_x\} x \in X$ is an α −open cover of X . Let x_o be a point of X and t_o an element of T(x_o) ; it follows from (2) that if $x_o \in V_x$, then $t_o \in T(x)$, and thus $V_x \subset U_{s(t_0)}$. Hence we have α $- St(x_o, \mathcal{V}) \subset U_{s(t_o)}$ which shows that $\mathcal{V}$ is a α − barycentric refinement of.

2.11. Remark:

The same proof shows that if a locally finite α −open cover of a topological vector space has a α −closed locally finite refinement then it has also a locally finite α −open barycentric refinement ; indeed , if the cover $\mathcal{U}$ is locally finite , then the family of all sets of the from (2) is a locally finite α −open barycentric refinement of $\mathcal{U}$.

2.12.Lemma:

If $\mathcal{A} = \{A_s\}$s ∈ S of a set X is α −barycentric refinement of a cover $\mathcal{B} = \{B_t\}$t ∈ T of X and $\mathcal{B}$ is a α −barycentric refinement of a cover $\mathcal{C} = \{C_z\} z \in Z$ of the same set , then $\mathcal{A}$ is a α −star refinement of $\mathcal{C}$

Proof :

Let us take an $s_o \in S$ and for every $x \in A_{so}$ let us choose a t(x) such that

(3) $$St(x, \mathcal{A}) \subset B_{t(x)}.$$

Thus we have

(4) $$\alpha - St(A_{so}, \mathcal{A}) = \bigcup_{x \in A_{so}} \alpha - St(x, \mathcal{A}) \subset \bigcup_{x \in A_{so}} B_{t(x)}.$$

Let x_o be a fixed element of A_{so}; from (3) it follows $x_o \in B_{t(x)}$that for every $x \in A_{so}$, so that

$$\bigcup_{x \in A_{so}} B_{t(x)} \subset \alpha - St(x_o, \mathcal{B})$$

Since for $\alpha - St(x_o, \mathcal{B}) \subset C_z$ a $\in Z$, the last inclusion , along with (4) , implies that $\mathcal{A}$ is a α −star refinement of $\mathcal{C}$.

2.13. Lemma:

If every α −open cover of a topological space X has an open star refinement, then every α −open cover of X has also an open α −discrete refinement.

Proof of Theorem 2.9.

By virtue of the last three lemmas, It suffices to show that every $\alpha - T_1$ space X satisfying(iii) is α −regular.

Consider a point $x \in X$ and a closed set $F \subset X$ such that $x \notin F$ and take α −star refinement $\mathcal{U}$ of the α −open cover {X\F, X\{x}}of the space X. Let U be a member of $\mathcal{U}$ that contains x . As $St(U, \mathcal{U}) \subset X\backslash F$ we have $\overline{U} \cap F = \emptyset$, so that the space X is α −regular.

2.14. Definition:

Let (X, α O(X)) be a topological space, then X is called a collectionwise normal of class α if X is a α −T_1 – space and for every discrete family $\{F_s\}_{s\in S}$ of α −closed subset of X there exists a discrete family $\{V_s\}_{s\in S}$ of α −open subset of X such that $F_s \subset V_s$ for every $s \in S$. Clearly , every collectionwise normal of class α is normal of class α.

4.31. Theorem:

A α −T_1 – space is collectionwise normal of class α if and only if for every discrete family $\{F_s\}_{s\in S}$ of α −closed subset of X there exists a discrete family $\{U_s\}_{s\in S}$ of α −open subset of X such that $F_s \subset V_s$ for every $s \in S$ and $U_s \cap U_{s\prime} = whenever\ s \neq s'$.

Proof :

It suffices to prove that any α −T_1- space X satisfying the condition in the theorem is α − collectionwise normal. Clearly, the space X is normal of class α, so that for a discrete family $\{F_s\}_{s\in S}$ of α −closed subset of X and the family $U_s\}_{s\in S}$ of pairwise disjoint α −open set , the α −closed set are respectively contained in disjoint α −open sets U and V . One easily check that the family $\{V_s\}_{s\in S}$, where $V_s = U_s \cap U$ is discrete .

4.32.Theorem:

Every paracompact space of class α is collectionwise normal of class α

Proof:

Let $\{F_s\}_{s\in S}$ be a discrete family of α −closed subset of paracompact space X of class . For every $x \in X$ choose a neighbourhood H_x of the point X whose closure meets at most one set F_s , consider an α −open locally finite refinement $\mathcal{W}$ of the cover $\{H_s\}_{s\in S}$, and for every $s \in S$, let $V_s = X \cup\{\overline{W}: W \in \mathcal{W} \; and \; \overline{W} \cap F_s = \emptyset\}$. Clearly $F_s \subset V_s$, so that to conclude the proof it suffice to show that every $W \in \mathcal{W}$ meets at most one element of the family $\{V_s\}_{s\in S}$. This, however, follows from the fact that $\overline{W}$ meets at most one set F_s .

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