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Markov Chains Applications at Weather and Marketing

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

{ إِنَّمَا يَخْشَى اللَّهَ مِنْ عِبَادِهِ الْعُلَمَاءُ إِنَّ اللَّهَ عَزِيزٌ غَفُورٌ }

"سورة فاطر، آية: ٢٨".

صدق الله العلي العظيم

Dedication

The search locomotive went through many obstacles, and yet I tried to overcome it steadfastly, with the grace and grace of God.

To his fragrant biography, and enlightened thought؛

He was the first credit for my attainment of higher education

(My beloved father),

may God prolong his life .

To the one who put me on the path of life, and made me calm،

She nursed me until I was big

(My dear mother)

, may God bless her soul.

to my brothers; Those who had a great impact on many obstacles and hardships.

To all my honorable teachers; Who did not hesitate to extend a helping hand to me

I dedicate to you my thesis

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Dr. Huda Amer Hadi

support and guidance with advice, education, correction and follow-up. I also thank the members of the distinguished debate committee, all friends and those who accompanied me during my study trip.

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Abstract

The study of random processes whose condition in the future does not depend on their states in the past, provided that their state is known in the present. This type of random process is called Markov process. In this research we will introduce the basic symbols and technical terms for these stochastic processes. The theory of Markov processes occupies a very large and important place in the theory of random processes. This status reinforces the multiplicity of applications that Markov processes have in physical and biological models, sociology, engineering and management science, in addition to their multiple applications in many statistical and engineering models and in reliability theory.

In the first chapter, we talked about the Concept of Markov Chains, uses of Markov Chains model, and also examples of Markov Chains.

In the second chapter, we talked about application areas of Markov chains, application Markov Chains in weather, as well as application of Markov Chains in marketing.

Introduction

The study of random processes whose condition in the future does not depend on their states in the past, provided that their state is known in the present. This type of random process is called Markov process. In this chapter we will introduce the basic symbols and technical terms for these stochastic processes. The theory of Markov processes occupies a very large and important place in the theory of random processes. This status reinforces the multiplicity of applications that Markov processes have in physical and biological models, sociology, engineering, and management science, in addition to their multiple applications in many statistical and engineering models and in reliability theory. Markov chain usually is interpreted as a sequence of states in which a system can be at any point in time7, or a sequence of positions in which a moving particle is located. Here we present the mathematical definition of the Markov chain, and the mathematical models used in this case are the games, services, and others. The location of the process is a simple example of (continuous time) or Markov chains. In general, a Markov chain is described as a set of states or the rate of movement from state to state, and the state of the chain changes at random moments. The transition from the current state to the new state corresponds to the transition rates [1].

A Markov chain is a type of Markov process that contains either a discrete state space or a discrete index group (often representing time), but the exact definition of a Markov chain varies. It is common to define a Markov chain as a Markov process either in discrete or continuous time with a countable state space (thus regardless of the nature of time), but it has also been common to define a Markov chain as having discrete time in either a countable or continuous state space (thus regardless of the state space). Many now tend to use the first definition of a Markov chain, as it has discrete time, although the second definition has been used by researchers such as

Joseph Dube and Kai Lai Chung [1]

Chapter One

Definition Markov Chain

1.1. Definition Markov Chain

(Markov Processes) The concept of Markov processes

It is an analysis represented in the image of the transition matrix [2], that is, the probability of transition from one state to another during a certain period of time called transition probabilities and is represented by a matrix called the transition matrix or Markov matrix [3]

1.2. The Concept of Markov Chains.

Markov chains are a random, stochastic process with certain characteristics, in which the system of events occurs with respect to time or space according to the laws of probabilities [4]. It is a broader and more comprehensive concept than Markovian processes, as it expresses separate states in separate times and includes continuous Markovian processes. Then Markov chains can be defined as follows :Levin and Rubin [5] defined Markov chains as: a method of analyzing some current variables in an attempt to predict the behavior of the same variables in the future.

As defined by Hiller and Lieberman [6] as: the behavior of some system operations in some periods of time often leads to the analysis of the random processes that follow them.

While Al-Atoum [7], defined Markov chains as: a method by which past changes and fluctuations are analyzed in order to predict future changes and take appropriate decisions.

Greenwell and et al. [8] defined that it is one of the types of random processes that enjoys the Markovian property if its state in the future is affected only by its state in the present and that any state in the past has no effect on its state in the future.

Hence, Markov chains can be defined as: one of the quantitative methods that support the decision-making process by relying on analyzing the behavior of system variables at the present time in order to predict the behavior of the same variables, but in subsequent periods, i.e. predicting the future based on the present without the need to know the past.

1.3. Markov Chin History

Andrei Andreevich Markov was born June 14th (June 2nd, old style), 1856, in Ryazan, Imperial Russia, and died on July 20, 1922, in Petrograd, which was – before the Revolution, and is now again – called Sankt Peterburg (St.Petersburg). In his academic life, totally associated with St. Petersburg University and the Imperial Academy of Science, he excelled in three mathematical areas: the theory of numbers, mathematical analysis, and probability theory. What are now called Markov chains first appear in his work in a paper of 1906 [9], when Markov was 50 years old. It is the 150th anniversary of his birth, and the 100th anniversary of the appearance of this paper that we celebrate at the Markov Anniversary Meeting, Charleston, South Carolina, June 12 - 14, 2006.

Markov's writings on chains occur within his interest in probability theory. On the departure in 1883 of his mentor, Pafnuty Lvovich Chebyshev (1821 - 1894) from the university, Markov took over the teaching of the course on probability theory and continued to teach it yearly, even in his capacity of a Privat-Dozent (lecturer) after his own retirement from the university as Emeritus Professor in 1905. His papers on Markov chains utilize the theory of determinants (of finite square matrices), and focus heavily on what are in effect finite stochastic matrices.

How- ever, explicit formulation and treatment in terms of matrix multiplication, properties of powers of stochastic matrices, and more generally of inhomogeneous products of stochastic matrices, and of associated spectral theory, are somewhat hidden, even though striking results, rediscovered by other authors many years later, follow from ideas in [9]. Our mathematical focus is an exploration of the contractivity ideas of that paper in the context of finite stochastic matrices, and specifically of the structure and usage of the Markov-Dobrushin coefficient of ergodicity.

Markov's motivation in writing the Markov chain papers was to show that the two classical theorems of probability theory, the Weak Law of Large Numbers and the Central Limit Theorem, could be extended to sums of dependent random variables. Thus he worked very much in terms of probabilistic quantities such as moments and expectations, and particularly with positive matrices.

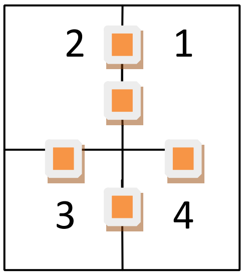
Markov chains were introduced by Andrei Andreevich Markov (1856-1922) and is named in his honor. He was a talented college student who won a gold medal Thesis at St. Petersburg University. Besides being an active research mathematician and teacher, he was also active in politics and Participated in the liberal movement in Russia at the beginning of the Twenty Centuries. Markov led to the development of Markov chains as an extension of natural sequences of independent random variables. In his first paper, in 1906, he proved that for his travel series with probabilities of transmission and numerical states, random outcomes of value converge to a single probability. A. A. Markov began studying a new type of chance process. This experience is in experiment only. This is a Markov list lawsuit type [10].

1.4. Uses of Markov Chains Model

1. Banks and financial institutions [11]
   - Current funding sources for the project
   - Studying and analyzing the negative phenomena facing the financial activity associated with customers in order to predict the necessary financing, repayment rates and bad debts in the future [12].
   - Funding sources at the end of a certain period of time.
   - Capital structure at the end of the fiscal year.
2. Studying and analyzing the negative phenomena facing institutions and organizations, such as [13]:
   - Managing technical activities and maintenance in industrial facilities and predicting the state of machines in terms of downtime and production, such as the number of machines that operate according to the requirements of the production process, the number of machines needed by the establishment, the current number of workers.
   - Analysis of demographic phenomena such as migration between governorates during specific time periods.
   - Predicting the graduation rates of students from different universities
3. Analyzing consumer behavior and its transition from one product to another during a certain period of time and predicting its behavior in the future.

1.5. Examples of Markov Chains

Example: The diagram on the bottom shows four compartments with doors leading from one to another. A mouse in any compartment is equally likely to pass through each of the doors of the compartment. Find the transition matrix of the Markov Chain.



The Solution: The transition matrix of the Markov Chain is:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array}$$

1/ If the mouse starts to move from the second compartment, what is the probability that it will be in the same compartment after three steps?
the initial probability distributed is $\pi_0 = (0\ 1\ 0\ 0)$, so we must compute:

$\pi_1=\pi_0 P$, $\pi_2=\pi_1 P$, $\pi 3=\pi_2 P$

$$\pi_1 = \pi_0 P = (0 \quad 1 \quad 0 \quad 0)\begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} = (2/3 \quad 0 \quad 1/3 \quad 0)$$

$$\pi_2 = \pi_1 P = (⅔ \quad 0 \quad ⅓ \quad 0)\begin{pmatrix} 0 & ⅔ & 0 & ⅓ \\ ⅔ & 0 & ⅓ & 0 \\ 0 & ½ & 0 & ½ \\ ½ & 0 & ½ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{11}{18} & 0 & \frac{7}{18} \end{pmatrix}$$

$$\pi_3 = \pi_2 P = \begin{pmatrix} 0 & \frac{11}{18} & 0 & \frac{7}{18} \end{pmatrix}\begin{pmatrix} 0 & ⅔ & 0 & ⅓ \\ ⅔ & 0 & ⅓ & 0 \\ 0 & ½ & 0 & ½ \\ ½ & 0 & ½ & 0 \end{pmatrix} = \begin{pmatrix} \frac{65}{108} & 0 & \frac{43}{108} & 0 \end{pmatrix}$$

∴ the probability that it will be in the second compartment after three steps is 0.

2/ In the long time, what is the probability that it will be in the second compartment?

$$\alpha\, \mathrm{p} = \alpha$$

$$\alpha = (x \quad y \quad z \quad m)$$

$$(x \quad y \quad z \quad m)\begin{pmatrix} 0 & ⅔ & 0 & ⅓ \\ ⅔ & 0 & ⅓ & 0 \\ 0 & ½ & 0 & ½ \\ ½ & 0 & ½ & 0 \end{pmatrix} = (x \quad y \quad z \quad m)$$

$$\frac{2}{3}y + \frac{1}{2}m = x$$

$$\frac{2}{3}x + \frac{1}{2}z = y$$

$$\frac{1}{3}y + \frac{1}{2}m = z$$

$$\frac{1}{3}x + \frac{1}{2}z = m$$

$$Let\ x = 1 \xrightarrow{from\ \ 1\&2} y = 1$$

$$\xrightarrow{from\ \ 1} m = \frac{2}{3}$$

$$\xrightarrow{from\ \ 3} z = \frac{2}{3}$$

$$\alpha = (x \quad y \quad z \quad m) = (1 \quad 1 \quad \frac{2}{3} \quad \frac{2}{3}\ )$$

$$1+1 + \frac{2}{3} + \frac{2}{3} = \frac{10}{3}$$

$$\frac{\alpha}{\frac{10}{3}} = (0.3 \quad 0.3 \quad 0.2 \quad 0.2)$$

Then the probability that the mouse will be in the second compartment, in the long time is 0.3.

Example: Let $\{x_n, n \geq 0\}$ be a Markov Chain with state space (1,2), and initial distribution $\pi_0 = (\frac{1}{3}, \frac{2}{3})$ and the Transition Probability Matrix is:

$$\begin{array}{cc} & \begin{array}{cc} 1 & \quad 2 \end{array} \\ p = \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

Compute:1) $p\left[x_1 = 2, x_4 = 1, x_6 = 1, x_{18} = \frac{1}{x_0} = 1\right] = p_{12} * p_{21}^3 * p_{11}^2 \ * p_{11}^{12}$

$$p^2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} * \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 * 0.5 + 0.5 * 0.3 & 0.5 * 0.5 + 0.5 * 0.7 \\ 0.3 * 0.5 + 0.7 * 0.3 & 0.3 * 0.5 + 0.7 * 0.7 \end{pmatrix} = \begin{pmatrix} 0.40 & 0.60 \\ 0.36 & 0.64 \end{pmatrix}$$

$$p^3 = p^2 * p = \begin{pmatrix} 0.40 & 0.60 \\ 0.36 & 0.64 \end{pmatrix} * \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.40 * 0.5 + 0.60 * 0.3 & 0.40 * 0.5 + 0.60 * 0.7 \\ 0.36 * 0.5 + 0.64 * 0.3 & 0.36 * 0.5 + 0.64 * 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.38 & 0.62 \\ 0.372 & 0.628 \end{pmatrix}$$

Chapter Two

Markov Chain Applications

2. Application Areas of Markov chains Introduction

Since Markov chains can be designed to model many real-world processes, they are used in a wide variety of situations. These fields range from the mapping of animal life populations to search-engine algorithms, music composition and speech recognition. In economics and finance, they are often used to predict macroeconomic situations like market crashes and cycles between recession and expansion. Other areas of application include predicting asset and option prices, and calculating credit risks. When considering a continuous-time financial market Markov chains are used to model the randomness. The price of an asset, for example, is set by a random factor – a stochastic discount factor – which is defined using a Markov chain [14].

2.1. Application Markov Chains in Weather

Markov chains have been researched heavily for predicting weather. In the paper, \Weather Forecasting Using Hidden Markov Model", Khaitani, Diksha and Ghose, Udayan, 2017 [15] trained a Markov model with weather data from past 21 years. The authors used Viterbi Algorithm and MATLAB software to calculate predicted data. The study found Markov model performed well predicting weather for the next 5 days based on current day's weather. Jordan and Talkner in their paper [16] investigated daily weather types of the Alpine region using seasonal Markov chain models. The study compared a 1st and 2nd order Markov chain model and found the two yield similar results.

The predictions by Markov models were found to coincide with actual observations for different types of time periods.

However, the predictive power of the models were not found to be very high, which was due to high randomness of the data and not due to weakness of the model.

Here in our for example we have observed the weather of Bemidji, Minnesota and calculated the probabilities using Markov model. For simpli cation assuming the weather can only be in one of 3 possible states, \sunny",\snowy" or \cloudy". In the context of Markov chains the probability of the weather being sunny, snowy or cloudy tomorrow, only depends on whether it is sunny, snowy or cloudy today.

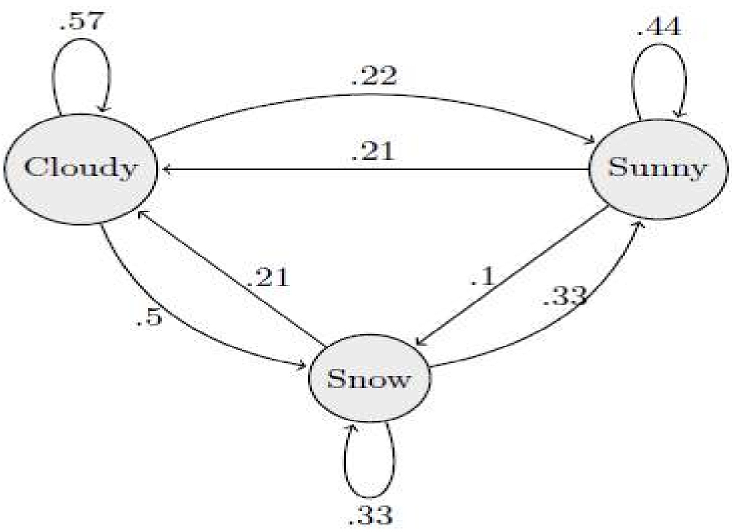


Figure 2.1(The state diagram using a directed graph)

The following transition matrix was constructed by taking data from the website [17] at 12 p.m. time for every day in November, 2019, totaling 30 observations.

$$P = \begin{array}{c} \\ \text{cloudy} \\ \text{sunny} \\ \text{snowy} \end{array} \begin{array}{c} \begin{array}{ccc} \text{cloudy} & \text{sunny} & \text{snowy} \end{array} \\ \begin{bmatrix} .57 & .22 & .5 \\ .21 & .44 & .1 \\ .21 & .33 & .33 \end{bmatrix} \end{array}$$

By observation, today's (12 November, 2020) weather at noon is cloudy. This is repre- sented by the vector,

$$x_0 = \begin{array}{c} \text{cloudy} \\ \text{sunny} \\ \text{snowy} \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

To predict the weather of 15 November, can be predicted the following way,

$$x_3 = P^3.x_0 = \begin{bmatrix} .57 & .22 & .5 \\ .21 & .44 & .1 \\ .21 & .33 & .33 \end{bmatrix}^3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 \\ 0.30 \\ 0.23 \end{bmatrix} \begin{array}{c} \text{cloudy} \\ \text{sunny} \\ \text{snowy} \end{array}$$

We followed up and checked that it was cloudy on November 15th.

2.2. Application of Markov Chains in Marketing

In the application of Markov chains to credit risk measurement, the transition matrix represents the likelihood of the future evolution of the ratings. The transition matrix will describe the probabilities that a certain company, country, etc. will either remain in their current state, or transition into a new state. [18]

Markov Chains to Predict Market Trends

Markov chains and their respective diagrams can be used to model the probabilities of certain financial market climates and thus predicting the likelihood of future market conditions [19].

These conditions, also known as trends, are:

● Bull markets: periods of time where prices generally are rising, due to the actors having optimistic hopes of the future.

● Bear markets: periods of time where prices generally are declining, due to the actors having a pessimistic view of the future.

● Stagnant markets : periods of time where the market is characterized by neither a decline nor rise in general prices.

In fair markets, it is assumed that the market information is distributed equally among its actors and that prices fluctuate randomly. This means that every actor has equal access to information such that no actor has an upper hand due to inside-information. Through technical analysis of

historical data, certain patterns can be found as well as their estimated probabilities. [19] For example, consider a hypothetical market with Markov properties where historical data has given us the following patterns: After a week characterized of a bull market trend there is

a 90% chance that another bullish week will follow. Additionally, there is a 7.5% chance that the bull week instead will be followed by a bearish one, or a 2.5% chance that it will be a stagnant one. After a bearish week there's an 80% chance that the upcoming week also will be bearish, and so on. By compiling these probabilities into a table, we get the following transition matrix M:

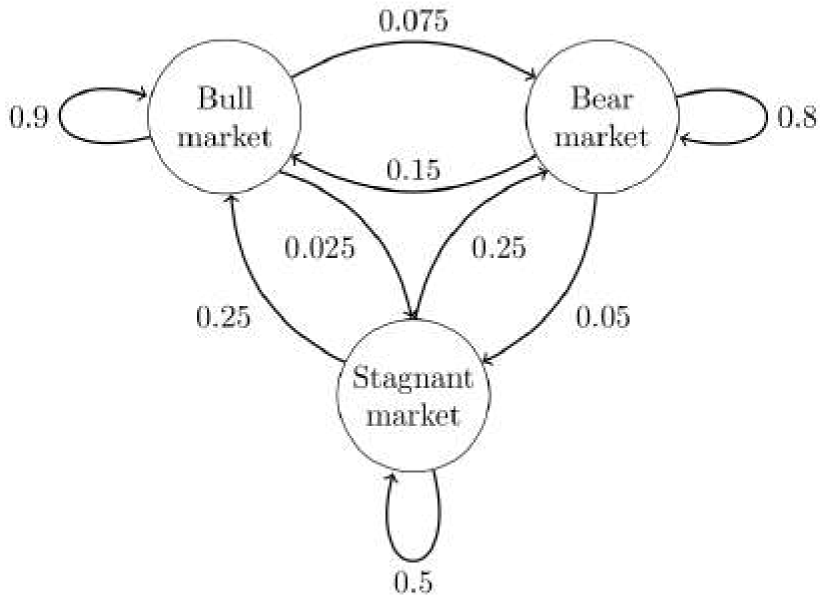


Figure 2.2 Virtual Market Chart Format

Table (2.1) transition matrix M:

To / *From*	Bull	Bear	Stagnant
Bull	0.9	0.075	0.025
Bear	0.15	0.8	0.05
Stagnant	0.25	0.25	0.5

$$= \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = M$$

We then create a 1x3 vector C which contains information about which of the three different states any current week is in; where column 1 represents a bull week, column 2 a bear week and column 3 a stagnant week. In this example we will choose to set the current week as bearish, resulting in the vector C = [0 1 0] .

Given the state of the current week, we can then calculate the possibilities of a bull, bear or stagnant week for any number of n weeks into the future. This is done by multiplying the vector C with the matrix , giving the following:

$$\text{One week from now: } C * M^1 = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^1 = [0.15 \quad 0.8 \quad 0.05]$$

$$\text{5 weeks from now: } C * M^5 = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^5 = [0.48 \quad 0.45 \quad 0.07]$$

$$\text{52 weeks from now: } C * M^{52} = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{52} = [0.63 \quad 0.31 \quad 0.05]$$

$$\text{99 weeks from now: } C * M^{99} = [0 \quad 1 \quad 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{99} = [0.63 \quad 0.31 \quad 0.05]$$

From this we can conclude that as $n \rightarrow \infty$, the probabilities will converge to a steady state, meaning that 63% of all weeks will be bullish, 31% bearish and 5% stagnant. What we also see is that the steady-state probabilities of this Markov chain do not depend upon the initial state [20]. The results can be used in various ways, some examples are calculating the average time it takes for a bearish period to end, or the risk that a bullish market turns bearish or stagnant.

Conclusion

Markov chains are an important concept in stochastic processes. They can be used to greatly simplify processes that satisfy the Markov property, namely that the future state of a stochastic variable is only dependent on its present state. This means that knowing the previous history of the process will not improve the future predictions - which of course significantly reduces the amount of data that needs to be taken into account. Mathematically, Markov chains consist of a state space, which is a vector whose elements are all the possible states of a stochastic variable, the present state of the variable, and the transition matrix. The transition matrix contains all the probabilities that the variable will transition from one state to another, or remain the same. To calculate the probabilities of a variable ending up in certain states after n discrete partitions of time, one simply multiplies the present state vector with the transition matrix raised to the power of n. There are different types of concepts regarding Markov chains depending of the nature of the parameters and application areas. They can be computed over discrete or continuous time. The state space can vary to be finite or countably infinite and depending on which, behave in different ways. A Markov chain with a countably infinite state space can be stationary which means that the process can converge to a steady state. Markov chains are used in a broad variety of academic fields, ranging from biology to economics. When predicting the value of an asset, Markov chains can be used to model the randomness. The price is set by a random factor which can be determined by a Markov chain. Regarding the application to credit risk measurement, calculating the transition matrix is the most important part when applying Markov chains in this subject. One way that has been presented in this paper is to use a combination of empirical data from several years in the past from credible credit rating institutions (Standard & Poor, Moody's, Fitch), and other types of more qualitative data. One should bear in mind that the homogenous Markov chain model probably will be less realistic than a non-homogenous model, but much less complicated. By analyzing the historical data of a market, it is possible to distinguish certain patterns in its past movements. From these patterns, Markov diagrams can then be formed and used to predict future market trends as well as the risks associated with them.

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