Solving Nonlinear Two Point Boundary Value Problems

Using Exponential Finit Difference Method

بحث مقدم من الطالب

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الى كلية التربية للعلوم الصرفة جامعة بابل كجزء من متطلبات نيل شهادة البكالوريوس في الرياضيات

بأشراف

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

(فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ وَلَا تَعْجَلْ بِالْقُرْآنِ مِنْ قَبْلِ أَنْ يُقْضَى إِلَيْكَ وَحْيُهُ وَقُلْ رَبِّ زِدْنِي عِلْمًا)

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الإهداء

الحمد لله رب العالمين والصلاة والسلام على خاتم الأنبياء والمرسلين الهي لا يطيب
الليل إلا بشكرك ولا يطيب النهار إلا بطاعتك ولاتطيب اللحظات إلا بذكرك
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CONTENTS

1-INTRODUCTION

Two point boundary value problems for ordinary differential equations arise in many branches of sciences and engineering. The existence of the solutions of the two point boundary value problems, either associated with system of linear or nonlinear ordinary differential equations and boundary conditions are specified at two points of the domain, depends on the domain considered for the solution of the problems. In most case it is impossible to obtain solutions of these problems using analytical methods which satisfy the given specified boundary conditions. In these cases we resort to approximate solutions and the last few decades have seen substantial progress in the development of approximate solutions of these problems.

In the literature, there are many different methods and approaches such as method of integration and discretization which be used to derive the approximate solutions in the domain of these problems [1,2,3,4].

In this article we proposed a method for the numerical solution of the boundary value problems of the form

$$y''(x) = f(x,y,y'),\ a < x < b, \quad (1)$$

subject to the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta,$$

where α and β are real constants and f is continuous on (x, y, y') for all $x \in [a,b]$ $y,y' \in \Re$.

2-THE EXPONENTIAL DIFFERENCE METHOD

We defined $N+1$ finite numbers of nodal points of the domain [a,b], in which the solution of the problem (1) is desired,

as $x_i = a + ih$, $i = 0,1,2,........,N$ using uniform step length where. $h = \frac{b-a}{N} = x_0 = a \text{ and } x_N = b$. Suppose we wish to determine numerical approximation of the theoretical solution $y(x)$ of the problem (1) at the nodal point x_i, $i = 1,2,.....,N-1$ and denote as y_i. Let f_i denotes the approximation of the theoretical value of the source function $f(x,y(x),y'(x))$ at node $x = x_i$, $i = 0,1,2,.....,N$. We can define other notations $f_i \pm 1$, $y_i \pm 1$, in the similar way used in this article. To develop the exponential difference method for the numerical solution of the problem (1), we need the following definitions:

$$y'_i = \frac{y_i+1 - y_{i-1}}{2h}, \quad (2)$$

$$y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, \quad (3)$$

$$y'_{i-1} = \frac{-y_{i+1} - 4y_i + 3y_{i-1}}{2h}, (4)$$

Define

$$f_{i+1} = f(x_{i+1},\ y_{i+1},\ y'_{i+1}),(5)$$

$$f_{i-1} = f(x_{i-1},\ y_{i-1},\ y'_{i-1}),(6)$$

$$y'_i = y_i + ch(f_{i+1} - f_{i-1}),(7)$$

$$y'_i = y_i + dh(f_{i+1} - f_{i-1}),(8)$$

$$f_i = f(x_i,\ y_i,\ y'_i),(9)$$

and

$$f_i = f(x_i,\ y_i,\ y'_i).(10)$$

We note that c and d from equations (7) and (8) respectively, are finite parameters to be determined. We proposed the exponential difference method for solving problem (1) numerically as,

$$y_{i+1} - 2y_i + y_{i-1} = h^2\, f_i\, exp\,\left(\frac{h^2 f_i^{''}}{12 f_i}\right)\ , fi \neq 0, i = 1, 2, \ldots\ldots N\ (11).$$

3-DERIVATION OF THE METHOD

By the Taylor series expansion about node $x = x_i$, from (3) we have:

$$y_{i+1} = y'_{i+1} - \frac{h^2}{3} y_i^{(3)} - \frac{h^3}{12} y_i^{(4)} + O\,(h^4). \qquad (12)$$

Let us define $G_{i\pm1}^1 = \left(\frac{\partial f}{\partial y'}\right)_{i\pm1}$*, so from (5)we have*

$$f_{i+1} = f_{i+1} - \frac{h^2}{3}\left(y_i^{(3)} + \frac{h^3}{4} y_i^{(4)}\right) G_{i+1}^1 + O\,(h^4). \qquad (13)$$

Similarly from (4) and (6), we have

$$f_{i-1} = f_{i-1} - \frac{h^2}{3}\left(y_i^{(3)} - \frac{h^3}{4} y_i^{(4)}\right) G_{i-1}^1 + O\,(h^4). \qquad (14)$$

By the Taylor series expansion of $G_{i\pm1}^1$ about node x = xi and from (13) and (14), we have

$f_{i+1} - f_{i-1} = f_{i+1} - f_{i-1} + O(h^4)$

On expanding (1) in Taylor series about x = xi, then substitute in (7) together with (15), we have

$$y'_i = y'_i + h^2\left(2c + \frac{1}{6}\right) y_i^{(3)} + O(h^4) \qquad (16)$$

y'_i will provide fourth order approximation for y' if we choose parameter c in (16) such that

$$2c + \frac{1}{6} = 0$$

$$c = -\frac{1}{12} \qquad (17)$$

Thus from (16) and (17) we have find , a fourth order approximation for yi' i.e

$$y'_i = y'_i + O(h^4) \qquad (18)$$

So from (9) and (18), we have

$$f_i = f_i + O(h^4) \qquad (19)$$

Let us define

$$f''_i = \frac{f_{i+1}+f_{i-1}-2f_i}{h^2} \qquad (20)$$

Using the approximations defined above, we can prove that $f_{i+1} + f_{i-1} - 2f_i$ *will provide a fourth order approximation for* $f_{i+1} + f_{i-1} - 2f_i$ *if we choose parameter* $d=\frac{-1}{4}$ *in (8)* i.*e*

$$y'_i = y'_i - \frac{1}{4}\, h\, (f_{i+1} - f_{i-1}) \qquad (21)$$

Thus

$$\frac{f_{i+1}+f_{i-1}-2f_i}{h^2} = \frac{f_{i+1}+f_{i-1}-2f_i}{h^2} + O(h^2) \qquad (22)$$

Finally, following the idea. for the source function f(x,y), from (11), we proposed our fourth order exponential difference method for solving problem (1) numerically as,

$$y_{i+1} - 2y_{i-1} = h^2 f_i \exp\left(\frac{f_{i+1}+f_{i-1}-2f_i}{12f_i}\right),$$

$f_i \neq 0$, $I = 1,2,\ldots..N - 1$

For each nodal point $x = xi$, $i = 1,2,\ldots.,N - 1$, *we will obtain a system of nonlinear equations given by (23).*

4-LOCAL TRUNCATION ERROR

from equations (19),(20) and (22), by Taylor series expansion of f on each node x = xi ,we have

$$\frac{h^2 f''_i}{12 f_i} = \frac{f_{i+1} + f_{i-1} - 2f_i}{12 f_i}$$

$$\frac{f_{i+1} + f_{i-1} - 2f_i}{12 f_i}$$

$$= \frac{h^2 y_i^{(4)} + \frac{h^4}{12} y_i^{(6)}}{12 y''_i}$$

From (23) and (24), the truncation error Ti at the nodal point x = xi may be written as [8,12,13],

$$T = y_{i+1} - 2y_{i-1} - h^2 f_i \exp\left(\frac{h^2 y_1^{(4)} + f_{i-1} - 2f_i}{12 f_i}\right),$$

By the Taylor series expansion of y at nodal point x = xi and second order expansion of exponential function, we have

$$T_i = -\frac{h^6}{240}\{y_i^{(6)} + \frac{5}{6}\frac{(y_i^{(4)})^2}{y_i''}\} + O(h^7) \qquad (25)$$

5-Exponential Finite Difference Method

A numerical comparison with existing finite difference method of same order is made and this comparison indicates the efficiency of the exponential finite difference method for model problems. Comparison of maximum absolute errors in solution in Table 1-4, show that the exponential finite difference method has less discretization error and is definitely better than the finite difference method . Table 5 shows the accuracy and efficiency of the exponential finite difference method for solving linear problem numerically. Note that for small N, method yield good results except in model problems 2 and 4. However, as the N becomes larger, the exponential finite difference method shows less error than the method . It is an advantage of the exponential finite difference method over existing method .

6-NUMERICAL RESULTS

To illustrate our method and demonstrate its computationally efficiency, we consider some model problems. In each case, we took uniform step size h. In Table 1 - Table 5, we have shown the maximum absolute error (MAY), computed for different values of N and is defined as

$$MAY = \min_{1 \le j \le N-1} |y(x_j) - y_j|.$$

We have used Newton-Raphson iteration method to solve the system of nonlinear equations arised from equation (23). All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC. The solutions are computed on N-1 nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10(−10) or the number of iteration reached 10^3.

Problem 1.

The first model problem is a nonlinear problem given by

$$y''(x)= \frac{\exp(2y)+ (y')^2}{2}, y(0) =0,\ y(1) =\log \frac{1}{2},\ x\in[0,1].$$

The analytical solution is y(x)= log $\frac{1.0}{1+x}$. For comparison purpose, we computed the MAY . The MAY computed by both methods for different values of N are presented in Table 1.

Problem2*.*

The second model problem is a nonlinear problem

$$y''(x) = y3\text{-}yy',\ y(1)= \frac{1}{3},\ x \in [1,2].$$

The analytical solution is . For comparison purpose, we computed the MAY. The MAY computed by both methods for different values of N are presented in Table 2.

Problem 3.

The third model problem is a nonlinear problem given by

$$y''(x) = \frac{3}{y}(y')^2,\ y(0)= \frac{1}{\sqrt{2}},\ x \in [0,1].$$

The analytical solution is y(x) = $\frac{1}{\sqrt{1+x}}$. For comparison purpose, we computed the MAY .

The MAY computed by both methods for different values of N are presented in Table 3.

***Problem 4**.*

The fourth model problem is a nonlinear problem given by

$$y''(x) = \frac{x}{\sqrt{1-y}}\, y' + f(x), y(0)=0,\ y(1)=-3,\ \ x \in [0,1].$$

where f(x) is calculated so that $y(x)=1-(x2+1)^2$ is analytical solution. For comparison purpose, we also computed the MAY by the method in [17]. The MAY computed by both methods for different values of N are presented in Table 4. Problem 5. The fifth model problem is a general two points linear problem given by

$$y''(x) = \frac{y+xy}{1+x}\ , y(0)=1,\ y(1)= \exp(1),\ \ x \in [0,1].$$

The analytical solution is y(x) = exp(x). Solving this model problem by method in [17], for each nodal point we obtained a system of linear equations. We applied Gauss-Seidel iterative for the solution of resulting system of linear equations. For comparison purpose, we *computed the MAY. The MAY computed by both methods for different values of N are presented in Table 5.*

Table1: Maximum absolute error in y(x)=log $\frac{1.0}{1+x}$ for Problem 1.

Method	$\frac{MAY}{N}$		
	4	8	16
(23)	14916062 (-4)	86426735 (-6)	14901161(-7)
[17]	94473362 (-5)	59604645 (-6)	59604645 (-7)

***Table2**: Maximum absolute error in y(x)=* $\frac{1}{1+x}$ *for Problem 2.*

Method	$\frac{MAY}{N}$			
	4	8	16	32
(23)	19040373 (-5)	28049245 (-7)	19022758 (-7)	19022758 (-7)
[17]	26351214 (-4)	56655786 (-5)	62916013 (-6)	32939408 (-7)

***Table3**: Maximum absolute error in y(x)=* $\frac{1}{\sqrt{1+x}}$ *for Problem 3.*

Method	$\frac{MAY}{N}$			
	4	8	16	32
(23)	36334609(-5)	15810301(-6)	39880490(-7)	80773226(-7)
[17]	21770765(-6)	74650984 (-7)	14037788(-6)	21254630(-7)

***Table4**: Maximum absolute error in* $y(x)=1-(x^2+1)^2$ *for Problem 4.*

Method	$\frac{MAY}{N}$			
	4	8	16	32
(23)	96634030(-4)	57965517(-5)	25331974(-6)	93132257(-9)
[17]	74594378(-0)	14065456 (-0)	14402390(-1)	10793209 (-7)

Table5: *Maximum absolute error in y(x)=exp(x) for Problem 5.*

Method	$\frac{MAY}{N}$		
	4	8	16
(23)	14901161 (-4)	59604645 (-6)	11920929 (-6)
[17]	12397766 (-4)	83446503 (-6)	23841858 (-6)

Example *Consider the two-point BVP:*

$$Y'' = -y + \frac{2(Y')^2}{Y}, \quad -1 < x < 1,$$

$$Y(-1) = Y(1) = (e + e^{-1})^{-1} = 0.324027137.$$

The true solution is $Y(t) = (et + e^{-t})$-1. We applied the preceding finite-difference procedure to the solution of this BVP. The results are given in Table 6 for successive doublings of N = 2/h. The nonlinear system was solved using Newton's method, as described in The initial guess was

$$y_h^0(xi) = (e + e^{-1})^{-1}, \; i = 0, 1, \ldots, N,$$

based on connecting the boundary values by a straight line. The quantity

$$dh = \max_{0 \le i \le N} |y_i^{(m+1)} - y_i^{(m)}|$$

***Table6**: Maximum absolute error in y(x)=exp(x) for Problem 5.*

N=2/h	E_h	*Ratio*
4	*2.63e – 2*	
8	*5.87e – 3*	*4.48*
16	*1.43e – 3*	*14.11*
32	*355e – 4*	*4.03*
64	*8.86e – 5*	*4.01*

was satisfied, the iteration was terminated. In all cases, the number of iterates com- puted was 5 or 6. For the error, let

$$E_h = \max_{0 \le i \le N} |y(xi) - y_h(xi)|$$

with yn the solution of (11.37) obtained with Newton's method. According to (11.38) Уп and (11.39), we should expect the values Eh to decrease by a factor of approximately 4 when h is halved, and that is what we observe in the table.

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