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وزارة التعليم العالي والبحث العلمي  
جامعة بابل  
كلية التربية للعلوم الصرفة  
قسم الرياضيات

بحث تخرج مقدم إلى مجلس كلية التربية للعلوم الصرفة  
كجزء من متطلبات نيل شهادة البكالوريوس في الرياضيات

## **Coloring Graph and Four-Color Theorem**

من قبل الطالبة  
زهراء محمد كاظم  
بإشراف  
م. م سناء كاظم كامل

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

( وَلَسَوْفَ يَعْطِيكَ رَبُّكَ فَتَرْضَى )

صدق الله العلي العظيم

سورة الضحى ايه ٥

## الأهداء

الحمد لله حبا وشكرا وامتنانا ع البدء والختام

(واخر دعواهم ان الحمد لله رب العالمين)

لم تكن الرحلة قصيرة ولا الطريق محفوفاً بالتسهيلات لكنني فعلتها فالحمد لله

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## الشكر والتقدير

ومن حق النعمة الذكر، وأقل جزاء للمعروف الشكر  
فبعد شكر المولى عز وجل المتفضل بجليل النعم، وعظيم الجزاء .  
يجدر بي أن أتقدم ببالغ الأمتنان وجزيل العرفان إلى كل من وجهني  
وعلمني، وأخذ بيدي في سبيل إنجاز هذا البحث وأخص بذلك مشرفتي  
الأستاذة سناء كاظم كامل  
وأتقدم بخالص الشكر والتقدير إلى رئاسة قسم الرياضيات  
كلية التربية للعلوم الصرفة  
وأحمل الشكر والعرفان إلى كل من أمدني بالعلم والمعرفة  
وأسدى لي النصح، والتوجيه

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## abstract

This study examines the concepts of graph coloring and the Four-Color Theorem, which are fundamental topics in graph theory. The study focuses on the process of assigning colors to elements of a graph—either vertices or edges—under specific constraints. In particular, it emphasizes proper coloring, where adjacent vertices or edges are assigned different colors to avoid conflicts. Furthermore, the research highlights the significance of the Four-Color Theorem, which states that any planar graph can be colored using no more than four colors such that no two adjacent regions share the same color. These concepts have important applications in areas such as scheduling, map coloring, and network design.

## 1- Introduction

Graph theory is considered one of the most important topics in modern mathematics, which has great credit for exploring methods of proof in mathematics. It appeared at the beginning of the eighteenth century, and credit for its appearance goes to Swiss scientist Leonhard Euler, through his famous problem (the Seven Bridges Problem), which is known as the oldest document researched in graph theory in 1936.

After that, graph theory witnessed great development and this development expanded in the past thirty years, as this development was embodied in many fields, whether scientific or economic fields, through graphs, and because the verbal graph indicates the structure of Specific mathematics can be expressed in the form of a diagram (such as a diagram) that contains points connected by lines. These points may indicate chemical atoms, cities, electrical limbs, or anything that can be described as pairs.

The lines may represent chemical bonds, roads, or wires. There are theoretical applications for graph theory in communications, structures, mechanisms, electrical networks, transportation systems, computer science, as well as psychology because it describes relationships between individuals or common characteristics.

A graph  $G$  has a collection of vertices, denoted by the finite, non-empty set  $V(G)$ , of elements. Unordered pairs of vertices from the set  $V(G)$  that are not necessarily distinct are also part of the graph  $G$  and make up what is termed the graph family. Graphs may be described in terms of the ordered pair of vertices and the set of edges using the notation  $G = (V, E)$ , where  $V$  is the set of vertices and  $E$  is the set of edges. Gary, others (2016)

## Previous Studies

Graph theory is a topic rich in information and concepts. Therefore, several previous studies have been conducted, and we will mention whatever is available, as follows:

In the study Robin (1998) presented a historical overview of the origin of the four-color theorem and its developments at that time. He stated that coloring the countries of a planar map using four colors such that areas with common borders are colored in a different color from the other, meaning that it is not just a point only, as such a result resisted. The evidence lasted for a century and a quarter of a century and also provided some generalizations related to this theorem.

In the study Sarah (2020) addressed in her study some problems related to chromatic polynomials and talked about coloring graphs and the greedy algorithm. She also presented colors for split vertices of the graph, presented an application to the tabulation problem, and gave an application about coloring maps.

### 1-1 Some basic concepts in graph theory

**Definition (1-1-1):-** If  $G=(V, E)$  is a graph that has an edge with two identical ends, then this edge is called a loop[6].

**Definition (1-1-2):-** If  $G=(V, E)$  is a graph in which two or more edges are identical at both ends, then these edges are called multiple edges[3].

**Definition (1-1-3):-** If  $G=(V, E)$  is a graph without loops and multiple edges then  $G$  is called a simple graph. Ali (1983) .

**Definition (1-1-4):-** If  $G=(V, E)$  is a graph and the set of edges  $E$  is empty, then  $G$  is called a null graph and is denoted by  $N_n$ . Rami & Suhail (2011)

**Definition (1-1-5):-** If  $G=(V, E)$  is a graph, then it is called a bipartite graph if it is possible to divide the set of vertices  $V$  into two separate subsets  $X$  and  $Y$  such that every edge in the graph connects a vertex of  $X$  and vertex in  $Y$ . Robin (1996)

**Note(1-1-1):-** The number of edges passing through the vertex is called the degree of the vertex denoted by  $\text{deg}(v)$ . Douglas (2002)

**Definition (1-1-6):-** If  $G=(V, E)$  is a graph of the degrees of its vertices that are equal, then  $G$  is called a regular graph. That is, if every vertex in  $G$  has a degree of  $r$ , then the graph  $G$  is called a regular graph of degree  $r$ , or  $r$ -regular. Frank (1971)

**Definition (1-1-7):-** If  $G=(V, E)$  is a graph, then it is called a graph  $G$  is called a connected graph if and only if there exists a route between every pair of vertices in  $G$ . Otherwise, the graph is called disconnected. Robin (1996)

**Definition (1-1-8):-** If  $G=(V, E)$  is a connected and regular graph of degree 2, then  $G$  is called a circle, and the circle of order  $n$  is symbolized by the symbol  $C_n$ . Robin(1996)

**Definition (1-1-9):-** If  $G=(V, E)$  is a connected graph that does not contain a circle, then  $G$  is called tree with  $n$  vertices is symbolized by  $T_n$ . In the tree, there is only one path between each pair of vertices. SUDEV NADUVATH (2017)

**Definition (1-1-10):-** If  $G=(V, E)$  is a graph that can be obtained from a circle after deleting one of its edges, the path is a tree consisting of a single path through all vertices and the path of order  $n$  is symbolized by the symbol  $P_n$ . Robin (1996)

**Definition (1-1-11):-** If  $G=(V, E)$ , a graph is formed by adding a vertex to the circle  $C_n$  and  $n \geq 3$  and connecting that vertex with all the vertices of  $C_n$  with new edges. The wheel  $G$ , which has the number of vertices  $n$ , is symbolized by  $W_n$ . Robin (1996)

**Definition (1-1-12):-** If  $G=(V,E)$  is a simple graph, then  $L(G)$  is called a line graph for  $G$  if the set of vertices of the  $L(G)$  represents the set of edges of  $G$ , that is,  $V(L(G))=E(G)$  Two vertices of  $L(G)$  are adjacent if and only if their corresponding edges in  $G$  share a vertex. John M (2008)

## **1-2 Coloring Graph**

Coloring graph is a special case of graph naming, that is, assigning labels traditionally called colors, to graph elements that are subject to certain restrictions. Its purpose is to improve the graph by coloring its vertices so that no two adjacent vertices share the same color. This is called vertex coloring. Through Planer duality, the vertices are colored and in this way, they are generalized to all graphs in mathematical and computer representations. Coloring graph has several practical applications, including the famous puzzle game Sudoku and the graph coloring game a graph coloring game played by two players one player tries to complete the graph, while the other tries to prevent him from achieving that.

### **1-2-1 Vertices coloring**

Vertex coloring represents the main gateway to the concept of coloring and its types because it is considered one of the most important problems in graph theory. After all, certain conditions are required to be relied upon when coloring.

If  $G=(V,E)$  is a graph without loops and  $\alpha$  is a positive integer, coloring  $\alpha$  is a function from the set of vertices to the set of positive integers that are less than

or equal to  $\alpha$ . This means that  $\alpha:V(G)\rightarrow\{1,2,3,\dots,\alpha\}$  and  $\{1,2,3,\dots, \alpha\}$  is a set of colors and that  $\alpha$  is one color for each vertex. Gary, others (2016)

**Definition (1-2-1-1):-** The coloring  $\alpha$  of a vertex is the assignment of colors  $1,2,3,\dots, \alpha$  to the vertices of the graph  $G$ , and that coloring is called (proper coloring) if every two adjacent vertices in  $G$  do not share the same color.

This means that if  $u, v \in V$  are adjacent, then  $\alpha(v) \neq \alpha(u)$ , meaning they have a different color. Gary, others (2016)

**Note (1-2-1-1):** - The proper coloring of the vertex  $\alpha$  in the graph with loops is the partitioning of the vertices  $v_1, v_2,\dots, v_\alpha$  into  $\alpha$  independent sets. J.A& U.S(1976)

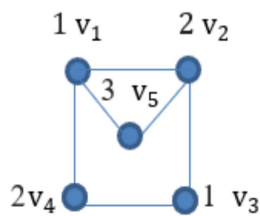
**Definition (1-2-1-2):** The subset of vertices that are assigned the same color is called color classes. Gary, others(2016)

**Example (1-2-1-1):** - If  $G=(V,E)$  is a graph and  $V=\{v_1,v_2,v_3,v_4,v_5\}$  and the color class is  $\{1,2,3\}$  then

$\alpha(v_1)=\alpha(v_3)=1$  color class 1

$\alpha(v_2)=\alpha(v_4)=2$  color class 2

$\alpha(v_5)=3$  color class 3



**Figure (1) G is colored by 3 colors**

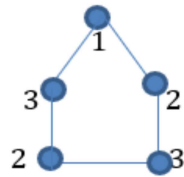
We note that the set of vertices is divided into 3 independent sets

**Definition (1-2-1-3):-** If  $G$  is a simple graph, then it is said to be a graph that can be  $\alpha$  colorable for vertices. If  $G$  has a proper coloring  $\alpha$  for vertices, that is,  $G$  can be  $\alpha$  colorable of different colors if its vertices can be divided into

independent sets that include  $\alpha$  of independent subsets. For ease, we will denote the graph as to color  $\alpha$  the vertices with  $\alpha$  chromatic. J.A& U.S(1976)

**Note (1-2-1-2):-** If  $G=(V, E)$  is  $\alpha$  colorable but not  $(\alpha-1)$  colorable then  $G$  is called  $\alpha$  chromatic. Robin (1996)

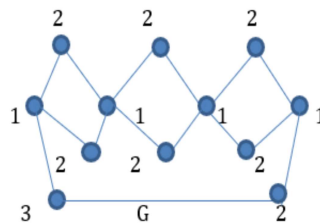
**Example (1-2-1-2):-** If  $G = C_5$  then  $G$  can be colored in 3 colors, but not in 2 colors as in the following figure



**Figure (2)  $G$  is a 3-chromatic graph**

**Definition (1-2-1-4):-** If  $G$  is a graph, then the chromatic number is the smallest number of colors that is sufficient to color the graph. Rinhard (2017)

**Example (1-2-1-3):-** If  $G$  is a graph as shown in the following figure then



**Figure(3)  $G$  is colored by 3 colors**

$X(G)=3$  and a graph  $G$  can be called the Crown graph.

**Theorem (1-1):-** If  $G=$ , then the chromatic number  $X(G)=\{2$  when  $n$  is an even number 3 when  $n$  is an odd number. J.A& U.S(1976)

Below is a set of notes that can be proven:

**Notes (1-2-1-3):**

1. If  $X(G) = \alpha$  then  $G$  is called  $\alpha$  chromatic.
2.  $X(G)=1$  if and only if  $G$  is an empty graph.
3.  $X(G)=2$  if and only if  $G$  is a bipartite graph.

4. For any graph of order  $n$  then  $X(G) \leq n$ . J.A& U.S(1976)
5. If  $G$  is a complete graph then  $X(G)=n$ . Gary, others(2016)
6. If  $G$  is a wheel with  $n$  vertices then  $X(G)=\begin{cases} 4 & \text{when } n \text{ is an even number} \\ 3 & \text{when } n \text{ is an odd number.} \end{cases}$

**Proof:(1)**

Let  $G=(V, E)$  be a graph and let  $X(G)=\alpha$  we must prove that  $G$  is  $\alpha$  chromatic graph

Since  $X(G)=\alpha$  then the least number of colors needed to color the graph  $G$  is equal to  $\alpha$ . Therefore,  $G$  is the  $\alpha$

colorable and cannot be colored with  $\alpha-1$  color From notes (1-2-1-2), we find that  $G$  is  $\alpha$  chromatic ■

**(1)**→ Let  $G=(V, E)$  be a graph and let  $X(G)=\alpha$  we must prove that  $G$  is a null graph.

Since  $X(G)=\alpha$  then  $G$  contains either one or  $n$  vertices that are not connected by edges, because if they are connected by edges, then  $X(G) \neq 1$  both cases,  $G$  is a null graph■

**(2)**← Let  $G$  be a null graph we must prove that  $X(G)=1$ . Since  $G$  is a null graph, the number of its vertices is either one or  $n$  vertices. It is possible to color the vertices with  $\alpha$  color. In both cases, the least number of colors needed to proper coloring the graph is one color, and therefore  $X(G)=1$  then the proof is complete.

**(3)**→ Let  $G=(V,E)$  be a graph and let  $X(G)=2$ . We must prove that  $G$  is a non-empty bipartite graph. Since  $X(G)=2$ ,  $G$  contains at least two vertices connected by an edge, and therefore  $G$  is a bipartite graph■

**(4)**← Let  $G$  be a bipartite graph. We must prove that  $X(G)=2$

Since  $G$  is a bipartite graph, the set of vertices can be divided into two sets, each set colored in a different color from the other. Therefore, the minimum number

of colors needed to color graph  $G$  is two colors, and therefore  $X(G) = 2$  then the proof is complete ■

(5) Let  $G$  be a complete graph with  $n$  vertices. We must prove that the chromatic number of the graph is  $n$ . Since  $G$  is a complete graph, all the vertices are adjacent to each other, and therefore the minimum number of colors needed for proper coloring of the graph is exactly  $n$ . Therefore,  $X(G) = n$ , and thus the proof is completed ■

(6) Let  $G=(V,E)$  be a wheel with  $n$  vertices and  $n \geq 4$ . We will prove, using mathematical induction on the number of vertices  $n$ .

When  $n=4$  then  $W_4 \cong K_4$  so  $X(K_4) = X(W_4) = 4$

When  $n = 5$ , the central vertex in the wheel will be colored 1, while the remaining vertices represent a circle with the number of vertices 4. Therefore, from the theorem (1-1),  $X(C_4) = 2$ , and therefore  $X(G) = 3$

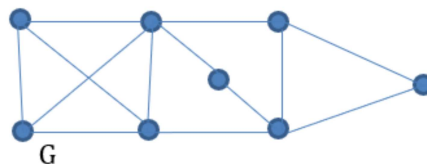
Now we assume that  $n = k$ , where  $k$  is an even number, so the central vertex of the wheel is the first color or 1. As for the remaining vertices of the circle  $C_{k-1}$ , the number of colors is  $C_{k-1} = 3$  and thus  $X(W_k) = 4$  the proof is complete ■

**Definition (1-2-1-6):** - If  $G=(V, E)$  is a graph, then the clique number is the order of the largest complete subgraph of the graph  $G$ , and the set number is symbolized by the symbol  $\omega(G)$ . John M (2008)

**Theorem (1-2):**- If  $G=(V,E)$  is a graph, then  $X(G) \geq \omega(G)$ . J.A& U.S(1976)

Above theorem help us to find the chromatic number in the graph which contain clique.

**Example (1-2-4):** If  $G=(V,E)$  be a graph as follows:



**Figure (4)  $G$  contains two complete sub-graph**

Where ( $G$ ) contains ( $H_1$ ) and ( $H_2$ ) two complete sub-graphs.



Then  $X(G)=4$  ,  $\omega(G)=4$ ,  $X(H1)=4$  and  $X(H2)=3$ .

**Definition (1-2-1-7):** - If  $G=(V, E)$  is a graph and  $X(G)=\alpha$  then  $G$  is said to be a critical graph if a vertex is deleted from  $G$ , then  $G$  has a lower chromatic number. John M (2008)

**Example (1-2-5):-** If  $G$  is a graph as follows then  $G$  is a critical graph



**Figure (5)  $G$  is a critical graph**

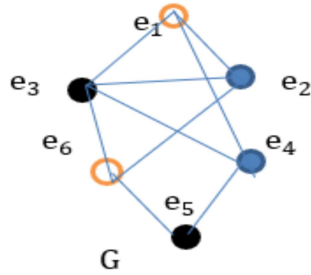
Because  $X(G)=3$  and  $X(G-v1)=2$  in this case  $G$  is called 3 critical graph.

### 1-2-2 Edges coloring

If it is a graph without loops and  $\alpha$  is a positive integer, then the coloring  $\alpha$  of the edges represents a function from the set of edges  $E$  to the set of positive integers that are less than or equal to  $\alpha$ . This means that  $C: E \rightarrow \{1,2,\dots,\alpha\}$  where  $\{1, 2,\dots,\alpha\}$  a set of colors for edges such that each edge in the graph has only one color. Rinhard(2017)

**Definition (1-2-2-1):** - If  $G$  is a graph, then the graph is called  $\alpha$  edge coloring if its edges can be colored with  $\alpha$  colors such that each two adjacent edges have a different color from the other. Robin (1996)

**Example (1-2-2-1):** - If  $G=(V,E)$  is a line graph for  $G$  in figure (1)

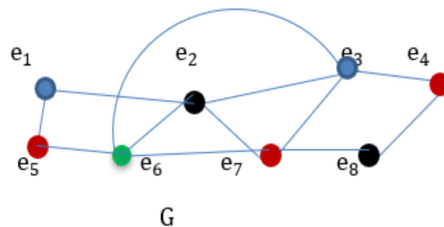


**Figure (6) G is 3 colored edges**

So, G is a graph with 3 colored edges because the graph G has its edges colored with 3 colors.

**Definition (1-2-2-2):** - If G is colored edge graph but not  $(\alpha-1)$  colored edge graph, then the chromatic index of G is  $\alpha$  and is symbolized by  $\chi'(G) = \alpha$ . Robin (1996)

**Example (1-2-2-2):** - If  $G=(V,E)$  is a line graph as in the following figure:



**Figure (7) G is a line graph**

Then  $\chi'(G)=4$

**Notes (1-2-2-1)**

1. If  $G=(V, E)$  is a null graph, then  $\chi'(G)=0$ .
2. If  $G=(V, E)$  is a circle, then  $\chi'(G)= \chi'(C_n)$ .
3. If  $G=(V, E)$  is a path, then  $\chi'(G)=n-1$ .
4. If  $G=(V, E)$  is a star, then  $\chi'(G)= \chi'(K_n)$ .
5. If  $G=(V, E)$  is a complete graph then  $\chi'(G)= \begin{cases} n & \text{when } n \text{ is an odd number} \\ n-1 & \text{when } n \text{ is an even number.} \end{cases}$  Blakrishnan & Rananathan (2012)

The following is one of the important topics in graph theory

### **1-2-3 Map coloring**

Historically, the issue of colouring maps with just four colours arose; now, we would wonder how many colours would be needed to colour a map such that no two nations or areas sharing a border would be coloured the same. This seemingly straightforward issue has emerged, although it is same in nature. The assertion that any map may be coloured in four colours is a staple of four-color theory and a solution to the four-color conjecture, one of the most difficult and well-known issues in graph theory. Robin (1996)

Francis Guthrie, when colouring a map of English counties, theorised that a globe map might be coloured with only four colours while yet allowing for clear distinction between neighbouring nations. In 1852, his sibling De Morgan spread this hypothesis. To say that every plane graph can satisfy Guthrie's hypothesis is similar to saying that any plane graph Every flat graph may be coloured with four vertices, according to a hypothesis proved by colouring it with four faces. Bennett & others(2008)

Many researchers have laboured over a solution to this question since its first publication in 1852. Numerous formula equivalents were discovered throughout the course of proving this hypothesis. Algebra, number theory, and finite geometry were only some of the areas of mathematics that were drawn upon. In 1976, three mathematicians from the University of Illinois used computers to verify this theorem. The proof required an incredible 1,200 hours of processing time on the fastest computers of the day. Blakrishnan & Rananathan (2012)

Before delving into how to color the faces of a graph (or maps), we must know what is a map. Robin (1996) defined a map as a planar graph that is 3-connected, meaning that the map does not contain a group of pieces from one or two edges, that is, in other words, it does not contain a vertex of degree 1 or 2, and John M (2008)] A map is defined as a plane graph that can be created by representing each region or country with a vertex and connecting each two vertices with an edge.

**Definition (1-2-3-1):-** Let  $G=(V,E)$  be a plane graph which drawing in the surface without crossing), then the

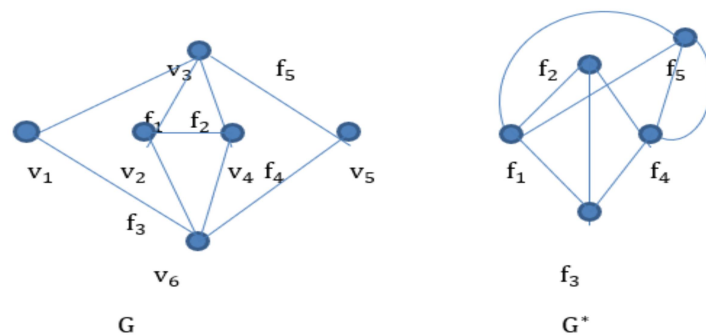
dual graph of  $G$ , which is symbolized by the symbol  $G^*=(V^*,E^*)$ , can be formed by placing a new vertex inside each face of  $G$  and connecting these vertices. The new vertices to form the vertices of the graph  $G^*$ , and for each edge  $e$  in the graph, we connect the two new vertices on the adjacent faces with  $e$  to an edge  $e^*$  and cut the edge  $e$ . If the edge  $e$  is adjacent to one face, then we node  $e^*$  to the new vertex on that face again through the edge. Rinhard (2017)

**Definition(1-2-3-2):-**Face coloring of a graph  $G$  is the colouring of edges in a planar graph  $G$  such that no two adjacent faces share a border and a single edge is coloured the same. Colouring the faces of a graph  $G$  is equivalent to coloring the vertices of a dual graph  $G^*$ . Robin(1996)

**Definition (1-2-3-3):-**If  $G=(V, E)$  is a plane graph, then  $G$  is called an  $\alpha$ -colored graph for faces if it is possible to color the faces with an  $\alpha$  color such that any two faces that have the same common edges have a different color from the other. Robin(1996)

**Definition(1-2-3-4):-**If  $G = (V, E)$  is a plane graph, then the face chromatic number is the smallest number of colors needed to color the faces of the graph and is symbolized by the symbol  $\chi^*(G) = \alpha$ . Robin(1996)

**Example (1-2-3-1): -** Let  $G=(V, E)$  be a graph and  $G^*$  be a dual graph for  $G$  as in the following figure



**Figure (8)  $G$  is a 3-chromatic graph for vertices and  $G^*$  is a 3-chromatic graph for faces**

That is,  $X(G)=3$  and  $X^*(G^*)=3$ .

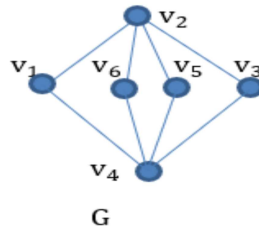
**Notes (1-2-3-1)**

1. If  $G = N_n$  then  $X^*(G)=1$ .
2. If  $G = C_n$  then  $X^*(G)=2$ .
3. If  $G = T_n$  then  $X^*(G)=1$ .
4. If  $G = K_4$  then  $X^*(G)=4$ .

**Theorem (1-3):** If  $G=(V, E)$  is a map, then a graph is 2 chromatic if and only if  $G$  is Euler's graph. Robin(1996)

By relying on the above theorem, it is possible to test the graph whether 2chromatic graph is colored for faces or not.

**Example(1-2-3-2):**-If  $G=(V,E)$  is a graph as in the following figure:



**Figure (9) Euler's graph and a 2-colored graph for the faces**

$G$  satisfies the above theorem.

Below is one of the most important theorems of the graph theory, which is the four-color theorem

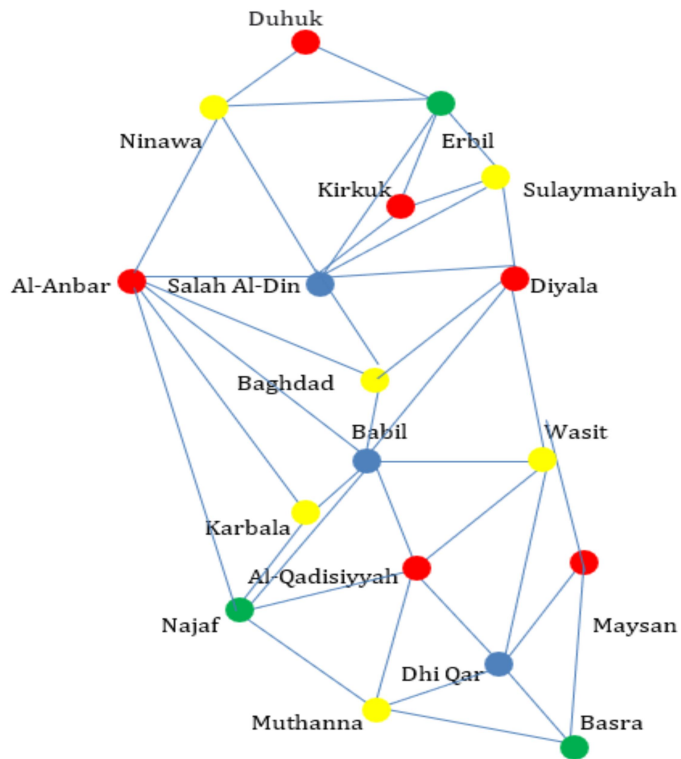
**Theorem (1-4):** - Each map can be colored in at least

The four-color theorem can be applied to the map of Iraq, as it can be colored with at least four colors so that no two neighboring regions have the same color.



**Figure (10) Map of Iraq to which the four-color theorem has been applied**

The map of Iraq can be converted into a plane graph, as in the following figure blow:-



**Figure (11) Show the map of Iraq after it was converted into a graph**

We can conclude from above that every flat map of regions or countries can be colored using at least four colors so that each two adjacent regions on the map has a different color from the other. The map is a 4-colored graph for the faces, and after converting it to a 4-colored graph for the vertices.

## **2- CONCLUSIONS**

1. The chromatic number was obtained for some graphs, such as the null graph, the complete graph, the bipartite graph, the wheel, the regular of degree  $r$ , and the crown graph.
2. The chromatic index was obtained for the null graph, circle, path, and tree.
3. The face chromatic number was obtained for the null graph, circle, tree, and  $K_4$ .
4. Use the four-color theorem and apply it to the map of Iraq and transform the map of Iraq into a graph.

## **3- Future Works**

Future research will focus on determining key coloring parameters for special cog-graphs and their dual graphs. In particular, this includes finding the chromatic number, chromatic index, and face chromatic number under various structural conditions. Further investigation may also explore efficient algorithms for computing these parameters and analyzing how graph transformations affect their coloring properties.

## REFERENCES:

- [1] Shaheen, Rami and Suhail Mahfouz, (2011), "graph Theory", Tishreen University, Syrian Arab Republic.
- [2] Ali, Aziz Ali, (1983), "Introduction to graph Theory", Mosul Press.
- [3] Balakrishnan. R., and Ranganathan. K., (2012); " A Textbook of Graph Theory " ,2nd ed., Springer Science and Business Media/ New york.
- [4] Bondy J. A., and Murty, U. S. R., (1976); " Graph Theory with Applications ", The Macmillan press Ltd,Great Britain
- [5] Chartrand, G. Lesniak, L. and Zhang, P., (2016); "Graph and digraph", 6thed., Taylor and French group/ CRC. Prees.
- [6] C. VASUDEV, (2006): "Graph Theory with Application" (Common to all P.G. &U.G. Courses: M.C.A.; M.TECH; M.I.T.; M.SC.; B.C.A.; B.C.S.; B.C.S.; B.I.T.; B.E., B. Tech.; B.SC.).
- [7] Diestel, R. (2017): "Graph theory".5th ed. Siproinger, Germany.
- [8] Harary, Frank, (1971),: "GRAPH THEORY" ,Addison-weley
- [9] Harris, John M., Jeffry L. Hirst, Michael J. Mossinghoff, (2008): "Combinatorics and Graph Theory" 2nd, Siproinger Science+Business Media, LLC, New York/USA.
- [10] Thomas, R. (1998): " An Update on the Four-Color Theorem ". Notices of the AMS, volume 45,Number 7,848-859.
- [11] Sudev Naduvath, (2017): " Lecture Notes on Graph Theory " Centre for Studies in Discrete Mathematics Vidya Academy of Science & Technology Thrissur-680501, Kerala, India.
- [12] WEST, D.B., (2002);"Introduction to Graph Theory ",2nded., Indian Branch, India.
- [13] Wilson, R, (1996): "Introduction to graph theory".4Ed, Addison Wesley Longman Limited.

- [14] Sara, S. (2020). Master of Arts in Pure Mathematics, "Chromatic Polynomials for Graphs with Split Vertices".