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كلية التربية للعلوم الصرفة // قسم الرياضيات

كثيرات حدود تشيبيشيف

Chebyshev polynomials

البحث مقدم الى مجلس كلية التربية للعلوم الصرفة

وهو جزء من متطلبات نيل درجة البكالوريوس في الرياضيات

اعداد

حيدر جعفر حيدر

اشراف

م.م. فاطمة علي عبد الحسين البياتي

بسم الله الرحمن الرحيم

((وقل ربي زدني علما"))

صدق الله العلي العظيم

شكر وتقدير

الحمد لله رب العالمين والصلاة والسلام على أطهر وأشرف الانبياء والمرسلين سيد الكائنات (محمد ابن عبد الله)وآلة الطيبين الطاهرين يشرفني وقد شارف هذا الجهد على الانتهاء ان اتقدم بجزيل الشكر وعظيم الامتنان لأستاذتي الفاضلة (فاطمه علي عبد الحسين) المشرفة على البحث الذي كان بجهودها المميزة ودقتها العلمية ومتابعتها المستمرة من دون ملل الاثر الكبير في انجاز هذا البحث فلها اسمى آيات الشكر والعرفان.

الاهداء الى

أمين الله وشمس الهدى امام الخلق وبحر الندى الامام المهدي (عج). الى من سعى وشقى لأنعم بالراحة والهناء الذي لم يبخل بشيء من اجل دفعي في طريق النجاح الذي علمن ان ارتقي سلم الحياة محكمه وصبر الى (والدي العزيز) الى الينبوع الذي لا يمل العطاء الى من حاكت سعادتي بخيوط منسوجة من قلبها الى(والدتي العزيزة رحمها الله) الى من علموني حروفا من ذهب وكلمات من درر وعبارات من أسمى وأجلى عبارات العلم (أساتذتنا الكرام) الى من سرنا سوياً ونحن نشق الطرق معا نحو النجاح والابداع الى من تكاتفنا يدا بيد ونحن نقطف ثمرة هذا العمل زوجتي الغالية

Xi	الفهرست

Chapter one

Chebyshev polynomials

His courses were not voluminous, and he did not consider the quantity of knowledge delivered; rather, he aspired to elucidate some of the most important aspects of the problems he spoke on. These were lively, absorbing lectures; curious remarks on the significance and importance of certain problems and scientific methods were always abundant. A. M. Liapunov who attended Chebyshev’s courses in late 1870, the underlying principles of the Chebyshev theory. The polynomials whose properties and applications will be discussed were introduced more than a century ago by the Russian mathematician P. L. Chebyshev (1821-1894). Chebyshev was the most eminent Russian mathematician of the nineteenth century. He was the author of more than 80 publications, covering approximation theory, probability theory, number theory, theory of mechanisms, as well as many problems of analysis and practical mathematics. His interest in mechanisms (as a boy he was fascinated by mechanical toys!) led him to the theory of the approximation of functions. Their importance for numerical analysis was rediscovered around the middle of the last century by C. Lanczos.

1.1 General properties

Let $\mathcal{P}_N$ be the space of algebraic polynomials of degree at most $N \in \mathbb{N},\ N > 0$, and the *weight function* $\omega : I = [-1, 1] \to \mathbb{R}_+$ defined by

$$\omega(x) := \frac{1}{\sqrt{1-x^2}}.$$

Let us introduce the fundamental space $L^2_\omega(I)$ by

$$L^2_\omega(I) := \left\{ v : I \to \mathbb{R} |\ v \text{ Lebesgue measurable and } \|v\|_{0,\omega} < \infty \right\},$$

where the norm

$$\|v\|_\omega := \left(\int_{-1}^{1} |v(x)|^2 \omega(x)\, dx \right)^{\frac{1}{2}},$$

is induced by the *weighted scalar (inner) product*

$$(u, v)_\omega := \left(\int_{-1}^{1} u(x)\, v(x)\, \omega(x)\, dx \right). \qquad (1.1)$$

Definition 1 *The polynomials $T_n(x)$, $n \in \mathbb{N}$, defined by*

$$T_n(x) := \cos(n\ \arccos(x)),\quad x \in [-1.1],$$

are called the Chebyshev polynomials of the first kind

Remark 2 [150] To establish a relationship between algebraic and trigonometric po

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n =$$
$$= \cos^n\theta + i\binom{n}{1}\cos^{n-1}\theta\cdot\sin\theta + i^2\binom{n}{2}\cos^{n-2}\theta\cdot\sin^2\theta + \ldots\ .$$

The terms on the right hand side involving even powers of sinθ are real while those with odd powers sinθ are imaginary. Besides, we know that sin2m θ = ¡ 1 − cos2 θ ¢m , m ∈ N. Consequently, for an

Tn (cos θ) := cos (nθ),

where $T_n(x) := \cos(n\ \arccos(x)) = \alpha_0^{(n)} + \alpha_1^{(n)}x + \ldots + \alpha_n^{(n)}x^n$ *is the Chebyshev's polynomial of order (degree)* n *which is an algebraic polynomial of degree* n *with real coefficients. Obviously,*

$$T_0(x) = 1,\quad T_1(x) = x,\quad T_2(x) = 2x^2 - 1,\quad T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1, \ldots\ .$$

It follows that every even trigonometric polynomial

$$Q_n(\theta) := \frac{\alpha_0}{2} + \sum_{k=1}^{n} \alpha_k \cos(k\theta),$$

is transformed, with the aid of substitution θ = arcos x, into the corresponding algebraic polynomial of degree n

$$P_n(x) := Q_n(\arccos x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} \alpha_k \cos(k\ \arccos(x)).$$

1.1. GENERAL PROPERTIES

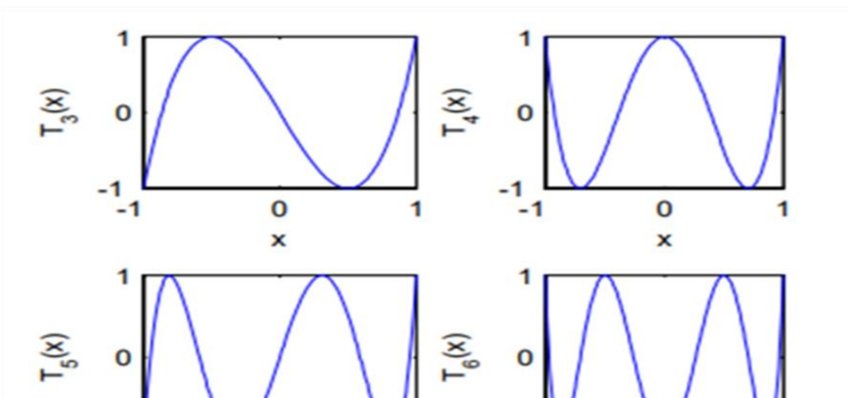

Figure 1.1: Some Chebyshev polynom

This substitution specifies in fact a homeomorphic, continuous and one-to-one mapping of the closed interval [0, π] onto [−1, 1] . It is important that, conversely, the substitution, x = cosθ, transforms an arbitrary algebraic polynomial

$$P_n(x) := a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

of degree n into an even trigonometric polynomial

$$Q_n(\theta) = P_n(\cos\theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} \alpha_k \cos(k\theta),$$

where the coefficients αk depend on Pn. Indeed, we have

$$\begin{aligned} \cos^m x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right) = \frac{1}{2^m}\left(e^{imx} + \binom{m}{1} e^{i(m-2)x} + \ldots + e^{-imx}\right) = \\ &= \frac{1}{2^m}\left(\cos mx + \binom{m}{1}\cos(m-2)x + \ldots + \cos(-mx)\right). \end{aligned}$$

Here we should take into account that cosm x is a real function and therefore the last term in this chain of equalities is obtained from the preceding term by taking its real part. The imaginary part of cosm x is automatically set to zero. Some Chebyshev polynomials are depicted in Fig. 1.1. Proposition 3 (Orthogonality) The polynomials Tn (x) are orthogonal, i.e.,

$$(T_n, T_m)_{0,\omega} = \frac{\pi}{2} c_n \delta_{n,m}, \;\; m, n \in \mathbb{N},$$

Some other properties of Chebyshev polynomials are available, for instance, in the well known monographs Atkinson [10] and Raltson and Rabinowitz [171]. In [171], p.301, the following theorem is proved

Theorem 5 Of all polynomials of degree r with coefficient of xr equal to 1, the Chebyshev polynomial of degree r multiplied by 1/2r−1 oscillates with minimum maximum amplitude on the interval [−1, 1] .

ue to this property the Chebyshev polynomials are sometimes called equalripple polynomials. However, their importance in numerical analysis and in general, in scientific computation, is enormous and it appears in fairly surprising domains. For instance, in the monograph [39] p.162, a procedure currently in use for accelerating the convergence of an iterative method, making use of Chebyshev polynomials is considered.

1.2. FOURIER AND CHEBYSHEV SERIES

Remark 6 Best approximation with Chebyshev polynomials V. N. Murty shows in his paper [147] that there exists a unique best approximation of T1 (x) with respect to linear space spanned by polynomials of odd degree ≥ 3, which is also a best approximation of T1 (x) with respect to the linear space spanned by

$\{T_j(x)\}_{j=0, j\neq 1}^{n}$. *If* $n = 4k$ *or* $n = 4k - 1$, *the extreme points of the deviation of* $T_1(x)$ *from its best approximation are* $2k$ *in number, whereas if* $n = 4k + 1$ *or* $n = 4k + 2$, *this number is* $2k + 2$.

In the next section we try to introduce the Chebyshev polynomials in a more natural way. We advocate that the Fourier series is intimately connected with the Chebyshev series, and that some known convergence properties of the former provide valuable results for the latter

Chapetr Two

2.1 Fourier and Chebyshev Series

rtant feature of Chebyshev series is that their convergence properties are not affected by the values of f (x) or its derivatives at the boundaries x = ±1 but only by the smoothness of f (x) and its derivatives throughout $-1 \le x \le 1$. Gottlieb and Orszag, [90], P. 28

2.1.1 The trigonometric Fourier series

t is well known that the ’trigonometric polynomial’

$$p_N(x) := \frac{1}{2}a_0 + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx), \qquad (1.6)$$

with

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx dx, \;\; b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx dx,$$

can be thought of as a least square approximation to f (x) with respect to the unit weight function on [−1, 1] (see Problem 6). The Fourier series, obtained by letting $n \to \infty$ in (1.6), is apparently most valuable for the approximation of functions of period 2π. Indeed, for certain classes of such functions the series will converge for most values of x in the complete range $-\infty \leq x \leq +\infty$. However, unless f (x) and all its derivatives have the same values at $-\pi$ and π, there exists a 'terminal discontinuity' of some order at these points. The rate of convergence of the Fourier series, that is the rate of decrease of its coefficients, depends on the degree of smoothness of the function, measured by the order of the derivative which first becomes discontinuous at any point in the closed interval $[-\pi, \pi]$. Finally, we might be interested in a function defined only in the range $[0, \pi]$, being then at liberty to extend its definition to the remainder of the periodic interval $[-\pi, 0]$ in any way we please. It is worth noting that, integrating by parts in the expressions of ak and bk over $[0, \pi]$ we deduce that cosine series converge ultimately like k−2, and sine series like k−1, unless f(x) has some special properties. If f (0) = f (π)=0, we can show that sine series converges like k−3, in general, the fastest possible rate for Fourier series.

2.2.2 The Chebyshev series

The terminal discontinuity of Fourier series of a non-periodic function can be avoided with the Chebyshev form of Fourier series. We consider the range $-1 \leq x \leq 1$ and make use of the change of variables

$$x = \cos\theta,$$

so that $f(x) = f(\cos\theta) = g(\theta)$. (1.7)

The new function g (θ) is even and genuinely periodic, since $g(\theta) = g(\theta + 2\pi)$. Moreover, if f (x) has a large numbers of derivatives in $[-1, 1]$, then g (θ) has similar properties in $[0, \pi]$. We should then expect the cosine Fourier series

$$g(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{N} a_k \cos k\theta,\ a_k = \frac{2}{\pi c_k}\int_{-1}^{1} g(\theta)\cos k\theta d\theta \qquad (1.8)$$

to converge fairly rapidly. Interpreting (1.8) in terms of original variable x, we produce the following Chebyshev series

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k T_k(x), \; a_k = \frac{2}{\pi c_k} \int_{-1}^{1} \omega(x) f(x) T_k(x)\, dx, \; \omega(x) := \left(1 - x^2\right)^{-1/2}. \tag{1.9}$$

This series has the same convergence properties as the Fourier series for f(x), with the advantage that the terminal discontinuities are eliminated. Elementary computations show that, for sufficiently smooth functions, the coefficients ak have the order of magnitude 1/2k−1 (k!), considerably smaller for large k than the k−3 of the best Fourier series.

Remark 7 (Continuous least square approximation) The expansion

$$p_n(x) := \sum_{k=0}^{n} a_k T_k(x), \; a_k = \frac{2}{\pi c_k} \int_{-1}^{1} \omega(x) f(x) T_k(x)\, dx,$$

has the property that the error en(x) := f (x) − pn(x) satisfies the ’continuous’ least square condition

$$S := \int_{-1}^{1} \omega(x) e_n^2(x) dx = \min.$$

The minimum value is given by

$$S_{\min} = \int_{-1}^{1} \omega(x) f^2(x) dx - \frac{1}{2}\pi \left(\sum_{k=0}^{n} c_k a_k^2 \right).$$

As n → ∞, we produce the Chebyshev series, which has the same convergence properties as the Fourier series, but generally with a much faster rate of convergence.

2.2.3.Discrete least square approximation

We now move on to the discrete case of least square approximation in which the integrated mean square error over I, from the classical least square approximation, is replaced by a sum over a finite number of nodes, say x0, x1, ..., xN ∈ I. The function f (x), f : I → R is approximated by a polynomial p (x) with the error e (x) := f (x) − p (x) and find the polynomial p(x) such that the sum

$$S := \sum_{k=0}^{N} \omega\left(x_k\right) e^2\left(x_k\right),$$

attains its minimum with respect to the position of the nodes xk in [−1, 1] and for a specified class of polynomials. We seek an expansion of the form

$$p_N(x) := \sum_{r=0}^{N} a_r \psi_r\left(x\right),$$

s ψr (x) are, at this stage, arbitrary members of some particular system (should that consist of polynomials, trigonometric functions, etc.). Conditions for a minimum are now expressed with respect to the coefficients

ar, S = S (a0, a1, ..., ar). They are

∂S/∂ai = 0, i = 0, 1, 2, .., N

and they produce a set of linear algebraic equations for these quantities. The matrix involved is diagonal if the functions are chosen to satisfy the discrete orthogonality conditions

$$\sum_{k=0}^{N} \omega\left(x_k\right) \psi_r\left(x_k\right) \psi_s\left(x_k\right)=0, \ r \neq s.$$

The corresponding coefficients ar are then given by

$$a_r=\frac{\sum_{k=0}^{N} \omega\left(x_k\right) \psi_r\left(x_k\right) f\left(x_k\right)}{\sum_{k=0}^{N} \omega\left(x_k\right) \psi_r^2\left(x_k\right)}, \ r=0,1,2, \ldots, N,$$

and the minimum value of S is

$$S_{\min }=\sum_{k=0}^{N} \omega\left(x_k\right)\left\{f^2\left(x_k\right)-\sum_{k=0}^{N} a_r^2 \psi_r^2\left(x_k\right)\right\}.$$

2.2.4 Chebyshev discrete least square approximation

Let's consider a particular case relevant for the Chebyshev theory. In the trigonometric identity

$$\frac{1}{2}+\cos \theta+\cos 2 \theta+\ldots+\cos (N-1) \theta+\frac{1}{2} \cos N \theta=\frac{1}{2} \sin N \theta \cot \frac{\theta}{2}, \quad (1.10)$$

the right-hand side vanishes for θ = kπ/N, k ∈ Z. Since

2 cos rθ cos sθ = cos (r + s) θ + cos (r − s) θ, (1.11)

it follows that the set of linearly independent functions ψr (θ) = cos rθ satisfy the discrete orthogonality conditions

$$\sum_{k=0}^{N} \frac{1}{\overline{c}_k} \psi_r\left(\theta_k\right) \psi_s\left(\theta_k\right)=0, \ r \neq s, \ \theta_k=k \pi / N, \quad (1.12)$$

where, throughout in this work, the coefficients ck are defined by

$$\overline{c}_k := \begin{cases} 2, & k = 0, N, \\ 1, & 1 \le k \le N-1. \end{cases}$$

Further, we find from (1.10) and (1.11) that the normalization factors for these orthogonal functions are

$$\sum_{k=0}^{N} \frac{1}{\overline{c}_k} \psi_r^2(\theta_k) = \begin{cases} N/2, & k = 0, N, \\ N, & 1 \le k \le N-1. \end{cases} \qquad (1.13)$$

Consequently, for the function g (θ), θ ∈ [0, π], a trigonometric (Fourier) discrete least square approximation, over equally spaced nodes

θk = kπ/N, k = 0, 1, 2, ..., N,

is given by the ’interpolation’ polynomia

$$p_N(\theta) = \sum_{r=0}^{N} \frac{1}{\overline{c}_r} a_r \cos r\theta, \;\; a_r = \sum_{k=0}^{N} \frac{2}{N\overline{c}_k} g(\theta_k) \cos r\theta_k, \;\; \theta_k = \frac{k\pi}{N}. \qquad (1.14)$$

The corresponding Chebyshev discrete least square approximation follows immediately using (1.7). It reads

$$p_N(x) = \sum_{r=0}^{N} \frac{1}{\overline{c}_r} a_r T_r(x), \;\; a_r = \sum_{k=0}^{N} \frac{2}{N\overline{c}_k} f(x_k) T_r(x_k), \;\; x_k = \cos\left(\frac{k\pi}{N}\right). \quad (1.15)$$

Let us observe that the nodes xk are not equally spaced in [−1, 1] . The nodes θk = kπ\ N , k = 1, 2, ..., N − 1,

are the turning points (extrema points) of TN (x) on [−1, 1] and they are called the Chebyshev points of the second kind.

Remark 8 For the expansion (1.15), the error eN (x) := f (x)−pN (x) satisfies the ’discrete’ least square condition

$$S := \sum_{k=0}^{N} \frac{1}{\overline{c}_k} e_N^2 (x_k) = \min,$$

and

$$S_{\min} = \sum_{k=0}^{N} \frac{1}{\overline{c}_k} \left\{ f^2 (x_k) - \sum_{r=0}^{N} a_r^2 T_r^2 (x_k) \right\}.$$

2.2.5 Orthogonal polynomials least square approximation

We have to notice that, so far, we have not used the orthogonality properties of the Chebyshev polynomials, with respect to scalar product (1.1). Similar particular results can be found using this property. For the general properties of orthogonal polynomials we refer to the monographs [43] or [187]. Each and every set of such polynomials satisfies a three-term recurrence relation

$$\phi_{r+1} (x) = (\alpha_r x + \beta) \phi_r (x) + \gamma_{r-1} \phi_{r-1} (x), \tag{1.16}$$

with the coefficients

$$\alpha_r = \frac{A_{r+1}}{A_r}, \ \gamma_{r-1} = -\frac{A_{r+1}}{A_r} \frac{A_{r-1}}{A_r} \frac{k_r}{k_{r-1}},$$

where A_r is the coefficient of x^r in $\phi_r (x)$, and

$$k_r = \int_{-1}^{1} \omega (x) \phi_r^2 (x) \, dx.$$

Following Lanczos [126], we choose the normalization kr = 1, and write (1.16) in the for

$$p_{r-1}\phi_{r-1}(x) + (-x + q_r)\phi_r(x) + p_r\phi_{r+1}(x) = 0, \qquad (1.17)$$

with

$$p_r = \frac{A_r}{A_{r+1}}, \quad q_r = -\beta_r p_r.$$

If we define p−1 (x) := 0 and choose the N + 1 nodes xi, i = 0, 1, 2, ..., N so that they are the zeros of the orthogonal

polynomial φN+1 (x), we see that they are also the eigenvalues of the tridiagonal matrix diag (pk−1 qk pk). The eigenvector corresponding to the eigenvalue xk has the components φ0 (xk), φ1 (xk), ..., φN (xk) and from the theory of symmetric matrices we know that the set of these vectors forms an independent orthonormal system. Each and every vector is normalized to be a unit vector, i.e

$$\lambda_k \sum_{r=0}^{N} \phi_r^2(x_k) = 1,$$

and the matrix

$$X = \begin{pmatrix} \lambda_0^{1/2}\phi_0(x_0) & \lambda_1^{1/2}\phi_0(x_1) & \dots & \lambda_N^{1/2}\phi_0(x_N) \\ \lambda_0^{1/2}\phi_1(x_0) & \lambda_1^{1/2}\phi_1(x_1) & \dots & \lambda_N^{1/2}\phi_1(x_N) \\ \dots & \dots & \dots & \dots \\ \lambda_0^{1/2}\phi_N(x_0) & \lambda_1^{1/2}\phi_N(x_1) & \dots & \lambda_N^{1/2}\phi_N(x_N) \end{pmatrix}$$

is orthogonal. It means X ·X0 = X0 ·X = IN+1, which implies two more discrete conditions in addition to the normalization one. i.e.,

$$\begin{aligned} \textstyle\sum_{k=0}^{N} \lambda_k\phi_r^2(x_k) = 1, \ r = 0, 1, 2, ..., N \\ \textstyle\sum_{k=0}^{N} \lambda_k\phi_r(x_k)\phi_s(x_k) = 0, \ r \neq s. \end{aligned} \qquad (1.18)$$

It follows that a solution of the least square problem in this case, with weights ω (xk) = λk, and the nodes taken as the

$N + 1$ zeros of $\varphi_{N+1}(x)$, is given by

$$p_N(x) = \sum_{r=0}^{N} a_r \phi_r(x), \quad a_r = \sum_{k=0}^{N} \lambda_k f(x_k) \phi_r(x_k). \tag{1.19}$$

For the Chebyshev case, using weight function $\omega(x) = \left(1 - x^2\right)^{-1/2}$, we find

$$\phi_0(x) = \pi^{-1/2} T_0(x), \quad \phi_r(x) = \left(\tfrac{1}{2}\pi\right)^{-1/2} T_r(x), \ r = 0, 1, 2, \ldots$$

$$\lambda_k^{-1} = \tfrac{2}{\pi} \textstyle\sum_{r=0}^{N} \tfrac{1}{c_k} T_r^2(x_k) = \tfrac{2}{\pi} \textstyle\sum_{r=0}^{N} \tfrac{1}{c_k} \cos^2 r\theta_k,$$

$$\theta_k = \tfrac{2k+1}{N+1} \tfrac{\pi}{2}, \ k = 0, 1, 2, \ldots, N.$$

The trigonometric identity (1.10) leads to a very simple form of λ_k, namely

$$\lambda_k = \pi / (N + 1),$$

and finally to

$$\begin{gathered} p_N(x) = \textstyle\sum_{r=0}^{N} \tfrac{1}{c_r} b_r T_r(x), \\ b_r = \tfrac{2}{N+1} \textstyle\sum_{k=0}^{N} f(x_k) T_r(x_k), \ x_k = \cos\left(\tfrac{2k+1}{N+1} \tfrac{\pi}{2}\right), \ k = 0, 1, 2, \ldots, N. \end{gathered} \tag{1.20}$$

Remark 9 For the expansion (1.20), the error $e_N(x) := f(x) - p_N(x)$ satisfies the 'discrete' least

$$S := \sum_{k=0}^{N} e_N^2(x_k) = \min,$$

and

$$S_{\min} = \sum_{k=0}^{N} \left\{ f^2(x_k) - \sum_{r=0}^{N} a_r^2 T_r^2(x_k) \right\}.$$

square

It can be shown that the error eN (x) satisfies the following minmax criterion for sufficiently smooth functions

$$\max \left| e_N(x) / f^{(N+1)}(\xi) \right| = \min, \quad \xi \in (-1,1).$$

Remark 10 The least square approximation polynomial pN (x) from (1.20) must agree with the Lagrangian interpolation polynomial

$$p_N(x) = \sum_{k=0}^{N} l_k(x) f(x_k),$$

(see Appendix 1) which uses as nodes the Chebyshev points of the first kind xk = cos ³ 2k+1 N+1 π 2 ´ , k = 0, 1, 2, ..., N. These nodes are in fact the zeros of TN+1(x). Remark 11 In [71] it is shown that for sufficiently well-behaved functions f (x) the approximation formula (1.20) is slightly better than (1.15).

2.2.6 Orthogonal polynomials and Gauss-type quadrature formulas

There exists an important connection between the weights λk of the orthogonal polynomial discrete least square approximation and the corresponding Gauss type quadrature formulas. First, we notice that Lagrangian quadrature formula (see Appendix 1) reads

$$\int_{-1}^{1} \omega(x) f(x)\, dx = \sum_{k=0}^{N} \mu_k f(x_k), \tag{1.21}$$

where

$$\mu_k = \int_{-1}^{1} \omega(x)\, l_k(x)\, dx.$$

The polynomial

$$p_N(x) = \sum_{k=0}^{N} l_k(x) f(x_k), \tag{1.22}$$

fits f (x) exactly in the N + 1 zeros of Π (x) and has degree N. The formula (1.21) is exact for polynomials of degree N or less. A Gauss quadrature formula has the form

$$\int_{-1}^{1} \omega(x) f(x)\, dx = \sum_{k=0}^{N} \nu_k f(x_k), \qquad (1.23)$$

where the weights ν_k and abscissae x_k (quadrature nodes) are to be determined such that the formula should be exact for polynomials of as high a degree as

possible. Since there are 2N + 2 parameters in the above formula, we should expect to be able to make (1.23) exact for polynomials of degree ≤ 2N + 1. To this end, we consider a system of polynomials $\varphi_k(x)$, k = 0, 1, 2, ..., N which satisfy the "continuous" orthogonality conditions

$$\int_{-1}^{1} \omega(x)\, \phi_r(x)\, \phi_s(x)\, dx = 0, \quad r \neq s. \qquad (1.24)$$

Suppose that f(x) is a polynomial of degree 2N + 1 and write it in the form

$$f(x) = q_N(x)\, \phi_{N+1}(x) + r_N(x), \qquad (1.25)$$

where the suffices indicate the degrees of the polynomial involved. Since $q_N(x)$ can be expressed as a linear combination of orthogonal polynomials $\varphi_k(x)$, k = 0, 1, 2, ..., N, the orthogonality relations imply

$$\int_{-1}^{1} \omega(x) f(x)\, dx = \int_{-1}^{1} \omega(x)\, r_N(x)\, dx,$$

which by (1.21) is exactly, i.e.,

$$\int_{-1}^{1} \omega(x) f(x)\, dx = \sum_{k=0}^{N} \mu_k r_N(x_k),$$

for specified xk and corresponding μk. If we choose xk to be the zeros of φN+1 (x), it follows from (1.25) that we obtained formally the required Gauss be represented exactly, due to (1.19), in the form quadrature formula (1.23) with νk = μk. Now rN (x), as a polynomial of degree N can

$$r_N(x) = \sum_{k=0}^{N} a_r \phi_r(x).$$

Consequently, we can write

$$\int_{-1}^{1} \omega(x) r_N(x)\, dx = \int_{-1}^{1} \omega(x) \left(\sum_{k=0}^{N} a_r \phi_r(x) \right) dx = a_0 \phi_0 \int_{-1}^{1} \omega(x),$$

due to (1.24) with r = 0. Moreover, the general solution of the least square problem (1.19) and in particular, the normalization condition, imply

$$a_0 \phi_0 \int_{-1}^{1} \omega(x) = \sum_{k=0}^{N} \lambda_k f(x_k) \int_{-1}^{1} \omega(x) \phi_0^2 dx = \sum_{k=0}^{N} \lambda_k f(x_k),$$

or, more explicitly

$$\int_{-1}^{1} \omega(x) f(x)\, dx = \sum_{k=0}^{N} \lambda_k f(x_k).$$

It follows that the weights in Gauss quadrature formula (1.23), which is exact for polynomials of order 2N + 1, equal the weights λk of the discrete least square solution (1.19), and the nodes xk are the zeros of the relevant orthogonal polynomial φN+1 (x). If, in particular,

$$\phi_0(x) := (\pi)^{-1/2} T_0(x), \quad \phi_r(x) := \left(\frac{1}{2}\pi\right)^{-1/2} T_r(x), \; r = 1, 2, \ldots$$

we get the *Gauss-Chebyshev quadrature formula, i.e.,*

$$\int_{-1}^{1} \omega(x) f(x)\, dx = \frac{\pi}{N+1} \sum_{k=0}^{N} f(x_k), \; x_k = \cos\left(\frac{2k+1}{N+1}\frac{\pi}{2}\right). \qquad (1.26)$$

2.2 Chebyshev projection

Let us introduce the map $P_N : L^2_\omega(I) \to \mathcal{P}_N,\ I = [-1,1]$,

$$P_N u(x) := \sum_{k=0}^{N} \widehat{u}_k \cdot T_k(x), \qquad (1.27)$$

where the coefficients ubk, k = 1, 2,...,N are defined in (1.5). Due to the orthogonality properties of Chebyshev polynomials, PN u (x) represents the orthogonal projection of function u onto PN with respect to scalar product (1.1). Consequently, we can write

$$(P_N u(x), v(x))_\omega = (u(x), v(x))_\omega, \quad \forall v \in \mathcal{P}_N. \qquad (1.28)$$

More than that, due to the completeness of the set of Chebyshev polynomials, the following limit holds:

$$\|u - P_N u\|_\omega \to 0 \quad as \quad N \to \infty.$$

Remark 12 A lot of results concerning the general theory of approximation by polynomials are available in Chapter 9 of [33]. We extract from this source only the results we strictly use.

The quantity u − PN u is called truncation error and for it we have the following estimate.

Lemma 13 For each and every u ∈ Hs ω(I), s ∈ N, one has

$$\|u - P_N u\|_\omega \leq C N^{-s} \|u\|_{s,\omega}, \qquad (1.29)$$

Remark 14 There exists a more general result which reads

$$\|u - P_N u\|_\omega \leq C\, \sigma_N(p)\, N^{-m} \sum_{k=0}^{m} \left\| u^{(k)} \right\|_\omega ,$$

for a function u that belongs to $L^2_\omega(-1,1)$ along with its distributional derivatives of order m and $\sigma_N(p) = \begin{cases} 1, & 1 < p < \infty, \\ 1 + \log N, & p = 1 \text{ and } p = \infty. \end{cases}$

Remark 15 Unfortunately, the approximation using the Chebyshev projection is optimal only with respect to the scalar product (·, ·)0,ω . This statement is confirmed by the estimation

$$\|u - P_N u\|_\omega \leq C N^{2l-s-\frac{1}{2}} \|u\|_{s,\omega}, \;\; s \geq l \geq 1,$$

in which a supplementary quantity ¡ l − 1 2 ¢ appears in the power of N. To avoid this inconvenient Canuto et al. [33] [1988, Ch. 9,11] introduced orthogonal projections with respect to other scalar products.

Remark 16 If (1.5) is the Chebyshev series for u (x), the same series for the derivative of u ∈ H1 ω(I), has the form

$$u'(x) = \sum_{k=0}^{\infty} \hat{u}_k^{(1)} \cdot T_k(x), \tag{1.30}$$

where (see (1.56) in the Problem 10)

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{p = k+1 \\ p+k = odd}}^{\infty} p\hat{u}_k.$$

Consequently,

$$P_N(u') = \sum_{k=0}^{N} \hat{u}_k^{(1)} \cdot T_k(x),$$

but in applications is sometimes used the derivative of the projection, namely (PN u) 0 , which is called the ‘Chebyshev-Galerkin derivative’ . We end this section with some ‘inverse inequalities’ concerning summability and differentiability for algebraic polynomials.

Lemma 17 For each and every u ∈ PN , we have

$$\begin{aligned} \|u\|_{L^q_\omega(-1,1)} &\leq CN^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p_\omega(-1,1)}, \; 1 \leq p \leq q \leq \infty, \\ \left\|u^{(r)}\right\|_{L^p_\omega(-1,1)} &\leq CN^{2r} \|u\|_{L^p_\omega(-1,1)}, \; 2 \leq p \leq \infty, \; r \geq 1. \end{aligned} \tag{1.31}$$

2.3 Chebyshev interpolation

We re-write the results from Fourier and Chebyshev Series Section in a more formal way. First, we observe that the quadrature formulas represent a way to connect the space L2 ω (−1, 1) with the space of polynomials of a specified degree. For the sake of precision, the interpolation nodes will be furnished by following Chebyshev-Gauss quadrature formula (rule

$$\int_{-1}^{1} f(x)\,\omega(x)\,dx := \sum_{j=0}^{N} f(x_j)\,\omega_j,$$

where the choices for the nodes xj and the weights ωj lead to rules which have different orders of precision. The most frequently encountered rules are: 1. the Chebyshev-Gauss formula (CGauss)

$$x_j := \cos\frac{(2j+1)\pi}{2N+1} \;\; and \;\; \omega_j = \frac{\pi}{N+1}, \;\; j = 0, 1, 2, \ldots, N \tag{1.32}$$

The quadrature nodes are the roots of the Chebyshev polynomial TN+1 and the formula is exact for polynomials in P2N+1. 2. the Chebyshev-Gauss-Radau formula (CGaussR)

$$x_j := \cos\frac{2j\pi}{2N+1} \quad j = 0, 1, 2, \ldots, N \;\; and \;\; \omega_j = \begin{cases} \frac{\pi}{N+1}, & j = 0, \\ \frac{\pi}{2N+2}, & j = 1, 2, \ldots, N. \end{cases} \tag{1.33}$$

In this case, the order of precision is only 2N.

3. the Chebyshev-Gauss-Lobatto formula (CGaussL)

$$x_j := \cos\frac{j\pi}{N} \quad j = 0, 1, 2, \ldots, N \;\; and \;\; \omega_j = \begin{cases} \frac{\pi}{2N}, & j = 0 \; and \; j = N, \\ \frac{\pi}{N}, & j = 1, 2, \ldots, N-1. \end{cases} \tag{1.34}$$

In this case, the order of precision diminishes to 2N − 1. Corresponding to each and every formula above we introduce a discrete scalar (inner) product and a norm as follows:

$$(u, v)_N := \sum_{j=0}^{N} \omega_j u(x_j) v(x_j), \tag{1.35}$$

$$\|u\|_N := \left(\sum_{j=0}^{N} \omega_j u^2(x_j) \right)^{\frac{1}{2}}. \tag{1.36}$$

The next result is due to Quarteroni and Vali [169], Ch.5

Lemma 18 For the set of Chebyshev polynomials, there holds

$$\|T_k\|_N = \|T_k\|_\omega, \; k = 0,1,2,...,N-1, \; \|T_N\|_N = \begin{cases} \|T_N\|_\omega, & for\ CGauss \\ \sqrt{2}\,\|T_N\|_\omega, & for\ CGaussL. \end{cases}$$

Proof. The first two equalities are direct consequences of the order of precision of quadrature formulas. For the third, we can write

$$\|T_N\|_N^2 = \frac{\pi}{2N}\left(\cos^2 0 + \cos^2 \pi\right) + \frac{\pi}{N}\sum_{j=1}^{N-1} \cos^2 j\pi = \pi = 2\,\|T_N\|_\omega^2 .$$

nomial of order (degree) N corresponding to one of the above three sets of nodes xk. It has the form

$$I_N u = \sum_{k=0}^{N} u_k T_k(x), \qquad (1.37)$$

where the coefficients are to be determined and are called the 'degrees of freedom' of u in the transformed space (called also "phase" space). For the (CGaussL) choice of nodes, using the discrete orthogonality and normality conditions (1.12), (1.13) we have

$$(I_N u, T_k)_N = \sum_{p=0}^{N} u_p (T_p, T_k)_N = \frac{\overline{c}_k \pi}{2} u_k. \qquad (1.38)$$

But interpolation means

$$I_N u(x_j) = u(x_j), \; j = 0,1,2,...,N,$$

which implies

$$(I_N u, T_n)_N = (u, T_n)_N = \sum_{j=0}^{N} \frac{\pi}{\overline{c}_j N} u(x_j) \cos\frac{nj\pi}{N}. \qquad (1.39)$$

The identities (1.38) and (1.39) lead to the 'discrete Chebyshev transform'

$$u_k = \frac{2}{\overline{c}_k N}\sum_{j=0}^{N} \frac{1}{\overline{c}_j} u(x_j)\cos\frac{kj\pi}{N}, \; k = 0,1,2,...,N. \qquad (1.40)$$

ormation, we can pass from the set of values of the function u in the nodes (CGaussL), the so-called physical space, to the transformed space. The inverse transform reads

$$u(x_j) = \sum_{j=0}^{N} u_j \cos \frac{kj\pi}{N}, \ j = 0, 1, 2, ..., N. \tag{1.41}$$

Due to their trigonometric structure, these two transformations can be carried out using FFT (fast Fourier transform-see [33] Appendix B, or [40] and [41]). A direct consequence of the last lemma is the equivalence of the norms k·kω

and $\|\cdot\|_N$. Thus, in the $(CGaussL)$ case, for $u^N = \sum_{k=0}^{N} u_k T_k$ we can write

$$\|u^N\|_N^2 = \sum_{k=0}^{N} (u_k)^2 \|T_k\|_N^2 = \sum_{k=0}^{N-1} (u_k)^2 \|T_k\|_\omega^2 + 2(u_N)^2 \|T_N\|_\omega^2,$$

and

$$\|u^N\|_\omega^2 = \sum_{k=0}^{N} (u_k)^2 \|T_k\|_\omega^2 .$$

Consequently, we get the sequence of inequalities

$$\|u^N\|_\omega \leq \|u^N\|_N \leq \sqrt{2} \|u^N\|_\omega .$$

For the Chebyshev interpolation, in each and every case, (CG), (CGR), (CGL), we have the following result (see [33], Ch. 9 and [169] Ch. 4):

Lemma 19 *If* $u \in H_\omega^m(-1,1)$, $m \geq 1$, *then the following estimation holds*

$$\|u - I_N u\|_\omega \leq CN^{-m} \|u\|_{m,\omega}, \tag{1.42}$$

and if $0 \leq l \leq m$, *then a less sharp one holds, namely*

$$\|u - I_N u\|_{l,\omega} \leq CN^{2l-m} \|u\|_{m,\omega} . \tag{1.43}$$

In $L_\omega^\infty(-1,1)$, *we have the estimation*

$$\|u - I_N u\|_{L_\omega^\infty} \leq CN^{2l-m} \|u\|_{m,\omega} . \tag{1.44}$$

2.3.1 Collocation derivative operator

r Associated with an interpolator is the concept of a collocation derivative (differentiation) operator called also Chebyshev collocation derivative or even pseudo spectral derivative. The idea is summarized in [184]. Suppose we know the value of a function at several points (nodes) and we want to approximate its derivative at those points. One way to do this is to find the polynomial that passes through all of data points, differentiate it analytically, and evaluate this derivative at the grid points. In other words, the derivatives are approximated by exact differentiation of the interpolate. Since interpolation and differentiation are linear operations, the process of obtaining approximations to the values of the derivative of a function at a set of points can be expressed as a matrix-vector multiplication. The matrices involved are called pseudo spectral differentiation (derivation) matrices or simply differentiation matrices.

Thus, if u := (u (x0) u (x1) ...u (xN))T is the vector of function values, and u0 := (u0 (x0) u0 (x1) ...u0 (xN))T is the vector of approximate nodal derivatives, obtained by this idea, then there exists a matrix, say D(1), such that

$$u' = D^{(1)}u. \qquad (1.45)$$

We will deduce the matrix D(1) and the next differentiation matrix D(2) defined by

$$u'' = D^{(2)}u. \qquad (1.46)$$

To get the idea we proceed in the simplest way following closely the paper of Solomonoff and Turkel [183]. Thus, if

$$L_N(x) := \sum_{k=0}^{N} u(x_k)\, l_k(x)\,, \qquad (1.47)$$

is the Lagrangian interpolation polynomial, we construct the first differentiation matrix D(1) by analytically differentiating that. In particular, we shall explicitly construct D(1) by demanding that for Lagrangian basis {lk (x)}N k=0 , lk (x) ∈ PN ,

$$D^{(1)} l_k(x_j) = l'_k(x_j), \;\; j, k = 0, 1, 2, ..., N,$$

i.e.

$$D^{(1)} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} l'_k(x_0) \\ \vdots \\ l'_k(x_k) \\ \vdots \\ l'_k(x_N) \end{pmatrix},$$

where 1 stands in the k th row. Performing the multiplication, we get

$$d_{jk}^{(1)} = l'_k(x_j). \tag{1.48}$$

We have to evaluate explicitly the entries d (1) jk in terms of the nodes xk, k = 0, 1, 2, ..., N. To this end, we rewrite the Lagrangian polynomials lk (x) in the form

$$l_k(x) := \frac{1}{\alpha_k} \Pi_{\substack{l=0 \\ l \neq k}}^{N} (x - x_l), \;\; \alpha_k := \Pi_{\substack{l=0 \\ l \neq k}}^{N} (x_k - x_l).$$

Taking, with a lot of care, the logarithm of lk (x) and differentiating, we obtain

$$l'_k(x) = l_k(x) \sum_{\substack{l=0 \\ l \neq k}}^{N} 1/(x - x_l). \tag{1.49}$$

This equality implies the diagonal elements

$$d_{kk}^{(1)} = \sum_{\substack{l=0 \\ l \neq k}}^{N} 1/\left(x_k - x_l\right), \ k = 1, 2, ..., N. \qquad (1.50)$$

In order to evaluate (1.49) at x = xj , j 6= k we have to eliminate the 0/0 indetermination from the right hand side of that. We therefore write (1.49) as

$$l_k'(x) = l_k(x) / (x - x_j) + l_k(x) \sum_{\substack{l=0 \\ l \neq k,j}}^{N} 1/\left(x - x_l\right).$$

Since lk (xj) = 0 for j 6= k, we obtain that

$$l_k'(x_j) = \lim_{x \to x_j} \frac{l_k(x)}{(x - x_j)}.$$

Using the definition of lk (x), we get the off-diagonal elements, i.e.,

$$d_{jk}^{(1)} = \frac{1}{\alpha_k} \Pi_{\substack{l=0 \\ l \neq k,j}}^{N} \left(x_j - x_l\right) = \frac{\alpha_j}{\alpha_k \left(x_j - x_k\right)}. \qquad (1.51)$$

It is sometimes preferable to express the entries of D(1), (1.50) and (1.51), in a different way. Let's denote by φN+1 (x) the product ΠN l=0 (x − xl). Then we have successively

$$\phi_{N+1}'(x) = \sum_{k=0}^{N} \Pi_{\substack{l=0 \\ l \neq k}}^{N} \left(x - x_l\right),$$
$$\phi_{N+1}'(x_k) = \alpha_k,$$
$$\phi_{N+1}''(x_k) = 2\alpha_k \sum_{\substack{l=0 \\ l \neq k}}^{N} 1/\left(x_k - x_l\right),$$

and eventually we can write

$$d_{jk}^{(1)} = \begin{cases} \frac{\alpha_j}{\alpha_k(x_j - x_k)} = \frac{\phi'_{N+1}(x_j)}{\phi'_{N+1}(x_k)(x_j - x_k)}, & j \neq k \\ \sum_{\substack{l=0 \\ l\neq k}}^{N} \frac{1}{(x_k - x_l)} = \frac{\phi''_{N+1}(x_k)}{2\phi'_{N+1}(x_k)}, & j = k. \end{cases} \tag{1.52}$$

Similarly, for the second derivative we write

$$D^{(2)} l_k(x_j) = l''_k(x_j), \; j, k = 0, 1, 2, ..., N,$$

and consequently

$$d_{jk}^{(2)} = \begin{cases} 2d_{jk}^{(1)} \left[d_{jj}^{(1)} - \frac{1}{x_j - x_k} \right], & j \neq k, \\ \left[d_{kk}^{(1)} \right]^2 - \sum_{\substack{l=0 \\ l\neq k}}^{N} \frac{1}{(x_k - x_l)^2}, & j = k. \end{cases} \tag{1.53}$$

Remark 20 In [206], a simple method for computing n × n pseudodifferential matrix of order p in O ¡ pn2 ¢ operations for the case of quasi-polynomial approximation is carried out. The algorithm is based on recursions relations for the generation of finite difference formulas derived in [68]. The existence of efficient preconditioners for spectral differentiation matrices is considered in [72]. Simple upper bounds for the maximum norms of the inverse ¡ D(2)¢−1 , corresponding to (CGaussL) points, are provided in [182]. In [189] it is shown that differentiating analytic functions using the pseudospectral Fourier or Chebyshev methods, the error committed decays to zero at an exponential rate.

Remark 21 The entries of the Chebyshev first derivative matrix can be found also in [93]. The gridpoints used by this matrix are xj from (1.34), i.e., Chebyshev Gauss Lobato nodes. The entries d(1) jk are

$$\begin{aligned} d_{jk}^{(1)} &= \frac{\overline{c}_j}{\overline{c}_k} \frac{(-1)^{j+k}}{(x_j - x_k)}, \; j \neq k, \\ d_{jj}^{(1)} &= \frac{-x_j}{2\left(1 - x_j^2\right)}, \; j \neq 0, N, \\ d_{00} &= -d_{NN} = \frac{2N^2+1}{6}. \end{aligned} \tag{1.54}$$

Remark 22 The software suite provided in the paper of Weideman and Reddy [204] contains, among others, some codes (MATLAB ∗.m functions) for carrying out the transformations (1.40) and (1.41), as well as for computing derivatives of arbitrary order corresponding to Chebyshev, Hermite, Laguerre, Fourier and sinc interpolators. It is observed that for the matrix D(l), which stands for the l − th order derivative, is valid the recurrence relation

$$D^{(l)} = \left(D^{(1)}\right)^{l}, \ l = 1, 2, 3, ...,$$

which is also suggested by (1.53). The existence of this relation is a consequence of the barycentric form of the interpolator (see P. Henrici [110], P. 252). On the other hand, we have to observe that throughout this work we use standard notations, which means that interpolating polynomials are considered to have order N and sums to have lower limit j = 0 and upper limit N. Since MATLAB environment does not have a zero index the authors of these codes begin sums with j = 1 and consequently their notations involve polynomials of degree N −1. Thus, in formulas (1.54) instead of N they introduce N − 1. However, it is fairly important that, in these codes, the authors use extensively the vectorization capabilities as well as the built-in (compiled) functions of MATLAB avoiding at the same time nested loops and conditionals. Another important source for pseudospectral derivative matrices is the book of L. N. Trefethen [197].

Remark 23 For Chebyshev and for Lagrangian polynomials as well, projection (truncation) and interpolation do not commute, i.e., (PN u) 0 6= PN (u0) and (IN u) 0 6= IN (u0). The Chebyshev-Galerkin derivative (PN u) 0 and the pseudospectral derivative (IN u) 0 are asymptotically worse approximations of u0 than

PN−1 (u0) and IN−1 (u0), respectively, for functions with finite regularity (see Canuto et al. [33] Sect. 9.5.2. and [93]). **Remark 24** (Computational cost) First, we consider the cost associated with the matrix D(l). Thus, N2 operations are requested to compute αj . Given αj , another 2N2 is required to find the off-diagonal elements. N2 operations are required to find all the diagonal elements from (1.52). Hence, it requires 4N2 operations to construct the matrix D(1). Second, a matrix-vector multiplication takes N2 operations and consequently the evaluation of u0 in (1.45) would require 5N2 operations, which means asymptotically something of order O(N2). This operation seems to be a somewhat expensive one because this would take up most of CPU time if it were used in a numerical scheme to solve a typical PDE or ODE boundary value problem (the other computations take only O(N) operations). Fortunately, the matrices of spectral differentiations have various regularities in them. It is reasonable to hope that they can be exploited. It is well known that

certain methods using Fourier, Chebyshev or sinc basis functions can also be implemented using FFT. By applying this technique the matrix-vector multiplication (1.45) can be performed in O (N log N) operations rather than the O ¡ N2¢ operations. However, our own experience, confirmed by [204], shows that there are situations where one might prefer the matrix approach of differentiation in spite of its inferior asymptotic operation count. Thus, for small values of N the matrix approach is in fact faster than the FFT approach. The efficiency of FFT algorithm depends on the fact that the integer N has to be a power of 2. More than that, the FFT algorithm places a limitation on the type of algorithm that can be used to solve linear systems of equations or eigenvalue problems that arise after discretization of the differential equations.

Bibliography

[1] Boyd, J. P., Chebyshev and Fourier Spectral Methods, Second Edition, DOVER Publications, Inc., 2000

[2] Boyd, J. P., A Chebyshev/rational Chebyshev spectral method for the Helmholtz equation in a sector on the surface of a sphere: defeating corner singularities, J. Comput. Phys., 206(2005), 302-310

[3] Butzer, P., Jongmans, F., P. L. Chebyshev (1821-1894) A Guide to his Life and Work, J. Approx. Theory, 96(1999) 111-138

[4] Canuto, C., Boundary Conditions in Chebyshev and Legendre Methods, SIAM J. Numer. Anal., 23(1986) 815-831

[5] Davis, P.J., Interpolation and Approximation, Blaisdell Pub. Co., NewY

[6] Don, W. S., Gottlieb, D., The Chebyshev-Legendre Method: Implementing Legendre Methods on Chebyshev Points, SIAM J. Numer. Anal., 31(1994), 1519-1534

[7] Dongarra, J.J., Straughan, B., Walker, D.W., Chebyshev tau- QZ algorithm for calculating spectra of hydrodynamic stability problems, Appl. Numer. Math. 22(1996), 399-434

[8] Fox, L., Parker, I.B., Chebyshev Polynomials in Numerical Analysis, Oxford Mathematical Handbooks, O U P, 1968

[9] Funaro, D., A Preconditioning Matrix for the Chebyshev Differencing Operator, SIAM J. Numer. Anal., 24(1987) 1024-1031

[10] Funaro, D., A Variational Formulation for the Chebyshev Pseudospectral Approximation of Neumann Problems, SIAM J. Numer. Anal. 27(1990), 695-703

[11] Gheorghiu, C.I., Pop, S.I.,On the Chebyshev-tau approximation for some singularly perturbed two-point boundary value problems, Rev. Roum. Anal. Numer. Theor. Approx., 24(1995), 117-124, Zbl M 960.44077

[12] Gheorghiu, C.I., Pop, S.I., A Modified Chebyshev-tau Method for a Hydrodynamic Stability Problem, Proceedings of I C A O R, vol. II, pp.119-126, Cluj-Napoca, 1997 MR 98g:41002

[13] Hiegemann, M., Chebyshev matrix operator method for the solution of integrated forms of linear ordinary differential equations, Acta Mech. 122(1997), 231-242