Ministry Of Higher Education and Scientific Research University Of Babylon/The College of Education For Pure Science Department Of Mathematics

Integral Equations

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بسم الله الرحمن الرحيم

﴿ وَفَوْقَ كُلِّ ذِي عِلْمٍ عَلِيمٌ ﴾

صدق الله العلي العظيم

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الحمد لله الذي بنعمته تتم الصالحات

الى سيد الخلق .... وحبيب الحق ......

الرسول الاسوة والقدوة الحسنة محمد (صلى الله عليه واله وسلم)

الى من زرع في نفسي حب العلم وبذل جهد السنين لكي يضعني على طريق الحياة الزاهرة ..

اهديه علمي وعملي (والدي العزيز)

الى من احاطتني بسور حنانها وذللت لي الصعاب بدعائها ....

اهديك عمري كله بلا ندم

(امي الغالية)

الى **الشموع التي تنير لي طريقي وانتظروا هذه اللحظة كثيراً**

ليفتخروا بي كما افخر بهم وبوجودهم ..

اخوتي واخواتي ....

الى من تحمل الاعباء معي وضل يساعدني كنت خير سند

الى زوجي الحبيب ....

شكر وامتنان ....

لا يسعني بعد الانتهاء من اعداد هذا البحث الا ان اتقدم بجزيل الشكر وعظيم الامتنان الى ....

أ . م. سحر محسن جبار

التي تفضلت بالأشراف على هذا البحث حيث قدمت كل النصح والارشاد طيلة فترة الاعداد فلها مني كل الشكر والامتنان ......

كما لا يسعني ان اتقدم بجزيل الشكر والعرفان الى اخوتي واصدقائي الذين ساعدوني كثيراً لإتمام بحثي هذا ....

والى جميع من قدم يد المساعدة لي ....

Contents

Abstract

The theory of integral equations constitute an important topic in mathematics as this is one of the most useful mathematical tools in both pure and applied mathematics . in this research ,the classifications of integral equations and types of kernels are discussed the methods of solutions of integral equations with some examples are given .

Introduction

Integral equations arise in a natural way in course of solving the initial and boundary value problems associated with mathematical modeling of physical phenomena , the solutions of integral equations play on important role to understand the qualitative features of the physical phenomena in the natural sciences. Fourier (1768-1830) is the initiator of the theory of integral equations . A term integral equations first suggested by Du-Reymond in 1888 . Du Bois-Reymond define an integral equations is understood an equation in which the unknown function occurs under one or more signs of definite integration , late eighteenth and early ninetieth century laplace , Fourier , Poission , Liouville and Able studies some special type of integral equation , the pioneering systematic investigations goes back to late nineteenth and early twentieth century work of volterra , Fredholm and Hilbert . In 1887 , Volterra published a series of famous papers in which he singled out the notion of a functional and pioneered in the development of a theory of functional in theory of linear integral equation of special type . Fredholm presented the fundamentals of the Fredholm integral equation theory in a paper published in 1903 in the Acta Mathematica . this paper become famous almost overnight and soon took its rightful place among the gems of modern Mathematics . Hilbert followed Fredholm's famous paper with a series of papers in the Nachrichten of the GUttingen Academy .The subject of integral equation is one of the most useful Mathematical tools in both pure and applied Mathematics . it has enormous applications in many physical problems , Many initial and boundary value problems associated with ODE (ordinary differential equations) and PDE (partial differential equations) can be transformed into problems of solving some approximate integral equations . integral equations were first encountered in the theory of Fourier integral . In 1826 , another integral equation was obtained by Abel. Actual development of the theory of integral equation began with the works of the Italian Mathematician V.Volterra (1896) and he Swedish Mathematician I.Fredholm (1900).As we discussed in the first chapter the basic definition and concepts , we also discussed in the second chapter the Fredholm integral equations and the adomain decomposition .

CHAPTER ONE

BASIC DEFINITION AND CONCEPTS

1.1 Introduction [1]

Many physical , biological and engineering processes involve rates of change of various quantities according to physical or other principles .

The mathematical expression of these processes can be formulated in two distinct but related ways .

I . Differential equations

I I . integral equations

In the case of Differential Equations , the unknowns function is differentiated and the boundary, conditions are imposed after general solution has been found .

In the case of integrated and the boundary conditions are incorporated within the formulation .

An integral equation is an equation in which the unknown function u(x) to be determined appears under the integral sign , A typical form of an integral equation in u(x) is of the form :

$$Cu(x) = F(x) + \lambda \int_a^{b(x)} k\left(x, t, u(t)\right) dt \ldots\ldots\ldots\ldots (1.1)$$

Where the forcing function f(x) and the kernel function k(r,t) are prescribed , while u(x) is unknown function to be determined , and C is constant . the parameter λ is often omitted it is , however , of importance in certain .

1.2 Classification of integral Equations .

Definition 1.2.1 [4]

The integral equation (1.1) is called linear integral equation if the kernel

k(x ,t, u(t)) = k(x ,t) u(t) , otherwise it is called non linear .

$$cu(x) = f(x) + \int_a^{b(x)} k\left(x, t\,, u(t)\right) dt \qquad (linear\ integral\ equation)$$

$$cu(x) = f(x) + \int_a^{b(x)} k\left(x, t, u(t)\right)dt \qquad (nonlinear\ integral\ equation)$$

Definition 1.2.2 [3]

The linear integral equation (1.1) is called homogeneous ,

If f(x) =0 , otherwise it is called nonhomogeneous .

$$cu(x) = \int_a^{b(x)} k(x, t)u(t)dt \qquad (homogeneous\ integral\ equation\ )$$

$$cu(x) = f(x) + \int_a^{b(x)} k(x, t)u(t)dt \qquad (nonhomogeneous\ integeral\ equation\ )$$

Definition 1.2.3 [3]

The integral equation (1.1) is said to be an equation of the first kind if C=0

$$f(x) = \int_a^{b(x)} k\ (x, t)u(t)\ dt$$

Definition 1.2.4 [3]

The integral equation (1.1) is said to be an equation of the second kind if C=1

$$u(x) = f(x) + \int_a^{b(x)} k(x, t)u\ (t)dt$$

Definition 1.2.5 [4]

The integral equation (1.1) is called volterra integral equation (V I E) when b(x)=x

$$u(x) = f(x) + \int_a^{x} k(x, t)u\ (t)dt$$

Definition 1.2.6 [4]

The integral equation (1.1) is called fredholm integral equation (F I E), if b(x)=b , where b is constant such that b≥a

$$u(x) = f(x) + \int_a^{x} k(x, t)u(t)dt$$

Definition 1.2.7 [1]

An integro -differential equation is an equation that involves one (or more) of an unknown function u (x), together with differential and integral operations on x .

The following sre examples of integro -differential equation:

1. $u''(x) = -x + \int_0^x (x-t)u(t)dt \, , u(o) = 0 \, , u'(o) = 1$

(2nd order volterra integro -differential equation)

2. $u'(x) = 1 - \frac{1}{3}x + \int_o^1 xt \; u(t)dt \quad , u(o) = 1$

1st order fredholm integro -differential equation

Definition 1.2.8 [1]

The integral equation is called singular if the lower limit , the upper limit or both limits of integration are infinite . in addition , the integral equation is also called a singular integral equation if the kernel k (x,t) becomes infinite at one or more point in the domain of integration .

Examples of the second kind of singular integral equations given by :

$$u(x) = 2x + 6\int_o^{\infty} sin \; (x-t)u(t)dt$$

$$u(x) = x + \frac{1}{3}\int_{-\infty}^{o} cos\,(x+t)u(t)dt$$

$$u(x) = 1 + x^2 + \frac{1}{6}\int_{-\infty}^{\infty} (x+t)\,u(t)dt$$

Examples of the first kind of singular integral equations are given by :

$$x^2 = \int_o^x \frac{1}{\sqrt{x-t}} u(t)dt$$

$$x = \int_o^x \frac{1}{(x-t)^{\propto}} u(t)dt \, , o < \propto < 1$$

Definition 1.2.9 (Taylor series) [2]

Let f(x) be a function that is infinitely differentiable in an interval (b ,c) that contains an interior point a the Taylor series of f(x) generated at x=a is given by the sigma notation .

$$f(x) = \sum_{n=o}^{\infty} \frac{f^{(n)}(a)}{n!}(x \quad - a)^n$$

Which can be written as

$$f(x) = f(a) + \frac{f(a)}{1!}(x - a) + \frac{f(a)}{2!}(x - a)^2 \frac{f(a)}{3!}(x - a)^3 + \cdots$$

The Taylor series of the function f(x) at a=o is given by :

$$f(x) = \sum \frac{f^{(n)}(o)}{n!} x^n$$

Definition 1.2.10 (Infinite Geometric Series) [2]

A Geometric Series is a Series with constant ratio between successive terms , the standard form of an infinite geometric series is given by :

$$s_n = \sum_{k=0}^{n} a_1 r^k$$

An infinite geometric series converges if and only if |r|<1 , otherwise it diverges , the sum of infinite geometric series , for |r|<1 , is given by :

$$s_n = \frac{a_1}{1-r}$$

Examples 1.2.11 [2]

1)find the sum of infinite geometric series :

$$1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} + \cdots$$

It is obvious that the first term is $a_1 = 1$

And the common ratio is $r = \frac{3}{5}$

The sum is therefore given by : $s = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$

2) find the sum of infinite geometric series:

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

It is obvious that the first term $a_1 = 1$ and the common ratio is

$r = -\frac{1}{3}$, $|r| < 1$

The sum is therefore given by :

$$s = \frac{1}{1 + \frac{1}{3}} = \frac{3}{4}$$

Definition 1.2.12 (Leibniz Rule) [2]

To differentiate the integral $\int_{\propto(x)}^{\beta(x)} G\,(x,t)dt$

With respect to x , we usually apply the useful Leibniz rule given by :

$$\frac{d}{dx}\int_{\propto(x)}^{\beta(x)} G\,(x,t)dt = G\big(x,\beta(x)\big)\frac{d\beta}{dx} - G\big(x\,,\propto(x)\big)\frac{d\propto}{dx} + \int_{\propto(x)}^{\beta(x)} \frac{\partial G}{\partial(x)}dt(2.1)$$

Where G(x,t) and $\frac{\partial G}{\partial X}$are continuous function in the domain D in the xt – plan that contains the rectangular region R , a≤x ≤b , to ≤t≤t_1,and the limit of integration $\propto(x)$and $\beta(x)$are defined functions having continuous derivatives for a <x <b . we note that the Leibniz rule is usually presented in most calculus books , and our concern will be on using the rule rather than its theoretical proof , the following examples are illustrative and will be used to convert volterra integral equations to differential equations .

Particular case : if $\propto(x)$and $\beta(x)$ are absolute constants , then (2.1) reduces to :

$$\frac{d}{dx}\int_{\propto(x)}^{\beta(x)} G\,(x,t)dt = \int_{\propto(x)}^{\beta(x)} \frac{\partial G}{\partial x} dt$$

Examples 1.2.13 [1]

1)$\int_o^x (x-t)^3\, u(t)dt$

$\propto (x) = o$, $\beta(x) = x$

$\frac{d \propto}{dx} = o$, $\frac{d\beta}{dx} = 1$

$\frac{\partial G}{\partial x} = 3(x-t)^2\ \ u(t)$

$$\frac{d}{dx}\int_o^{x^2} (x-t)^3\, u(t)dt = 1 + \int_o^x 3\,(x-t)^2 u(t)dt$$

2) $\int_x^{x^2} e^{xt}\, dt$

$\propto (x) = x$, $\beta(x) = x^2$

$\frac{d \propto}{dx} = 1$, $\frac{d\beta}{dx} = 2x$

$\frac{\partial G}{\partial x} = te^{xt}$

$$\frac{d}{dx}\int_x^{x^2} e^{xt}\, dt = 2x - 1 + \int_x^{x^2} te^{xt}$$

3) $\int_o^x (x-t)^4\, u(t)dt$

$\propto (x) = o$, $\beta(x) = x$

$\frac{d \propto}{dx} = o$, $\frac{d\beta}{dx} = 1$

$\frac{\partial G}{\partial x} = 4(x-t)^3 u(t)$

$$\frac{d}{dx}\int_o^x (x-t)^4 u(t)dt = 1 + \int_o^x 4\,(x-t)^3 u(t)dt$$

1.3 Special Types of Kernels [1]

The following special cases of the Kernel of on integral equation are of main interest :

1)the kernel k(x ,t) is called difference kernel , if k(x ,t)=k(x-t) , and the linear integral equation is called an integral equations of convolution type .

$$u(x) = f(x) + \int_a^b k\,(x - t)u(t)dt$$

2)the kernel k(x,t) is called the separable or degenerate kernel of rank n if it is of the form :

$$k(\,x, t) = \sum_{j=1}^{n} a_j\,(x)\;\; b_j(t)$$

Where n is finite and the function (a_j) and (b_j) are sufficiently smooth functions .

1.4 Examples [4]

1) $u(x) = x + \int_0^1 xt\,u(t)dt$

Fredholm integral equation .

Linear integral equation.

Nonhomogeneous integral equation

Equation of the second kind

2)$u(x) = 1 + x^2 + \int_0^x (x - t) \quad u(t)dt$

Volterra integral equation.

Linear integral equation

Nonhomogeneous integral equation.

Equation of the second kind.

3)$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{e}^{\boldsymbol{x}} + \int_{\boldsymbol{0}}^{\boldsymbol{x}} (\boldsymbol{t}\boldsymbol{u}^{2}(\boldsymbol{t}))\, \boldsymbol{dt}$

Volterra integral equation.

non Linear integral equation.

Nonhomogeneous integral equation

Equation of the second kind.

4)$\boldsymbol{u}(\boldsymbol{x}) = \int_{\boldsymbol{0}}^{\boldsymbol{1}} (\boldsymbol{x} - \boldsymbol{t})^{2}\, \boldsymbol{u}(\boldsymbol{t})\boldsymbol{dt}$

Fredholm integral equation

Linear integral equation

homogeneous integral equation

Equation of the first kind

1.5 Solution of Integral Equation [2]

A Solution of Integral Equation or integro-differential equation on the interval of integration is a function u(x) such that it satisfies the given equation , in other words , if the given solution is substituted on the right-hand side of the equation , the output of this direct substituted must yield on the left-hand side , we should verify that the given function u (x) satisfies the integral equation or the

integro-differential equation under discussions .

this important concept will be illustrated first by examining the following example .

Examples 1.5.1 [2]

1)show that $u(x) = e^{x}$ is a solution of the volterra integral equation :

$$u(x) = 1 + \int_0^x u(t)\,dt$$

Substituting $u(x) = e^x$ in the right-hand side (R H S) of the above integral equation yields :

R H S= $1 + \int_0^x u\ (t)$**dt**

=$1 + \int_0^x e^t dt$

=$1 + \int_0^x \{e^t\}_o^x$

=$1 + e^x - e^o$

=e^x

=u (x)

= L H S

2)show that u(x)=x is solution of the following fredholm integral equation :
$u(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)\ u(t)\,dt$

R H S= $\frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)\ \ u(t)dt$

= $\frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)\ \ u(t)dt$

= $\frac{5}{6}x - \frac{1}{9} + \frac{1}{3}[x\ \frac{t^2}{2} + \frac{t^3}{3}]_0^1$

= $\frac{5}{6}x - \frac{1}{9} + \frac{1}{3}[\frac{x}{2} + \frac{1}{3}]$

= x

= u(x)

= L H S

3) verify that the given function is a solution of the corresponding integral equation :

$$u(x) = \frac{2}{3}x + \int_0^1 xt\ u(t)dt\,, \quad u(x) = x$$

R H S = $\frac{2}{3}x + \int_0^1 x\,t^2\,dt$

$= \frac{2}{3}x + [\frac{t^3}{3}x]_0^1$

$= \frac{2}{3}x + [\frac{1}{3}x - 0]$

$= \frac{2}{3}x + \frac{1}{3}x$

$= \frac{3}{3}x$

= x

= u (x)

= L H S

1.6 converting volterra Equation to ODE [4]

In this section , we will present the technique that converts volterra integral equation of the second kind to equivalent differential equations.

this may be easily achieved by applying the important Leibniz Rule for differentiating an integral .

it seems reasonable to review the basic outline of the rule .

We now turn to our main goal to convert a volterra integral equation to an equivalent differential equations , noting that the Leibniz Rule should be used in differentiating the integral as stated above .

the differentiating process should be continued as a many times as needed until we obtain a pure differential equations with the integral sign removed , moreover , the initial conditions needed can be obtained by substituting x=0 in the integral equation

and the resulting integro-differential equations will be shown . we are now ready to given the following illustrative examples .

Examples 1.6.1 [4]

1)find the initial value problem equation to the volterra integral equation :

$$u(x) = 1 + \int_0^x u\,(t)dt$$

Differentiating both sides of the integral equation and using the Lebibniz rule we find : ù(x) = u(x)

The initial condition can be obtained by substituting x=0 into both sides of the integral equation , hence we find u (o)=1 consequently ,

The corresponding initial value problem of the first order is given by: $\dot{u}(x) - u(x) = 0$ **, u(o)=1**

2)convert the following volterra integral equation to initial value problem :
$u(x)x + \int_0^x (t - x)\ \ u(t)dt$

Differentiating both sides of the resulting integro-differential equations to remove the integral sign , there for, we obtain :

ũ̃ (x)=-u(x)=0 or equivalently ũ̃(x)+u(x)=0

the related initial conditions are obtained by substituting x=0 in u(x) and in ù(x) in the equations above , and as a result we find u(0)=0

and ù(0)=1 . combining the above results yields given by :

ũ̃(x)+ u(x) =0 , u(0)=0 , ù(0)=1

1.7 converting IVP to Volterra Equation [4]

In this section , we will study the method that converts an initial value problem to an equivalent volterra integral equation . before outlining the method needed , we wish to recall the useful transformation formula :

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_{n-1}} f(xn)\,dxn \,\dots .\,dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt \qquad (2.2)$$

That converts any multiple integral to a single integral .

This is an essential and useful formula that will be employed in the method that will be used in the conversion technique .

We point out that this formula appears in most calculus texts for practical considerations , the formulas :

$$\int_0^x \int_0^x f(t)\,dt\,dt = \int_0^x (x-t)\ f(t)dt \qquad (2.3)$$

$$\int_0^x \int_0^x \int_0^x f(t)\ dt\,dt\,dt = \frac{1}{2}\int_0^x (x-t)^2\ f(t)dt \qquad (2.4)$$

Are two special cases of the formula given above , and the most used formula that will transform double and triple integrals respectively to a single integral . noting that the right-hand side of (2.3) is a function of x allows us to set she equation :

$$I(x) = \int_0^x (x-t)\ f(t)dt \qquad (2.5)$$

Differentiating both sides of (2.5) and using the Leibnize rule we obtain :

$$\grave{I}\,(x) = \int_0^X f(t)\ \ dt \qquad (2.6)$$

Integrating both sides of (2.6) from o to x , noting that I(0)=0

From (2.5) , we find : $I\,(x) = \int_0^x \int_0^x f(t)\ dt\,dt$

Examples 1.7.1[4]

1)convert the following volterra integral equation to an initial value problem : $u(x) = x + \int_0^x (t-x)\ u(t)dt$

Differentiating both sides of the integral equation . we obtain :

$\grave{u}(x) = 1 - \int_0^x u(t)\ dt$ **or equivalently** $\tilde{\tilde{u}}(x) + u(x) = 0$

The related initial conditions are obtained by substituting x=0 in u(x) and in ù(x) in the equations above , and as a result we find u(0)=0 and ù(0)=1 . combining the above results yields the equivalent initial value problem of the second order given by : ũ̃(x)+u(x)=0 ,u(0)=0,ù(0)=1

2)find the initial value problem equivalent to the volterra integral equation : $u(x) = x^3 + \int_0^x (x-t)^2\ u(t)dt$

Differentiating both sides of the above equation three times , we find :

$$\grave{u}(x) = 3x^2 + 2\int_0^x (x-t)\ u(t)dt$$

$$\tilde{\ddot{u}}(x) = 6x + 2\int_0^x u\ (t)dt$$

$$\dddot{u} = 6 + 2u(x)$$

The proper initial conditions can be easily obtained by substituting x=0 in u(x) , ù(x) and ȕ̀(x) in the obtained equations above . consequently , we obtain the nonhomogeneous initial value problem of third order given by :

ȕ̀(x)-2u(x)=6 , u(0)=0 , ù(0)=0 , ũ̃(0)=0

1.8 Converting BVP to Fredholm Equation [4]

So far we have discussed how an initial value problem can be transformed to an equivalent volterra integral equation . in this section , we will present the technique that will be used to convert boundary value problem to an
the technique is similar that are related equivalent fredholm integral equation to boundary conditions , it is important to point out here that the procedure of reducing the boundary value problem to the fredholm integral equation is complicated are rarely used the method is similar to the technique discussed above , which reduces the initial value problem to volterra integral equation , with the exception that we are given boundary condition .

Special attention should be taken to define ẏ(0) since it is not always given , as will be seen later . this can be easily determined from the resulting equations , it seems useful and practical to illustrate this method by applying it to an example rather than proving it .

Example 1.8.1 [4]

We want to derive equivalent fredholm integral equation to the following boundary value problem :

ỹ̃(x)+y(x)=x , o<x<π

subject to the boundary conditions :

$$\tilde{\tilde{y}}(x) = u(x)$$

$$\dot{y}(x) - \dot{y}(o) = \int_0^x u\,(t)dt$$

$$\dot{y}(x) = \dot{y}(o) + \int_0^x u\,(t)dt$$

$$y(x) - y(o) = \dot{y}(o)x + \int_o^x \int_o^x u(t)\, dt\, dt$$

$$y(x) = 1 + \dot{y}(o)x + \int_o^x (x-t)\, u(t)dt$$

$$x = \pi \rightarrow y(\pi) = 1 + \dot{y}(o)\pi + \int_o^x (\pi - t)\, u(t)dt$$

$$\dot{y}(o) = \frac{1}{\pi}[\pi - 1 - 1 - \int_o^x (\pi - t)\, u(t)dt$$

$$\dot{y}(o) = \frac{1}{\pi}[\pi - 2 - \int_o^\pi (\pi - t)\, u(t)dt$$

$$y(x) = 1 + \frac{x}{\pi}\left[\pi - 2 - \int_o^\pi (\pi - t)\, u(t)dt\right] + \int_o^x (x-t)\, u(t)dt$$

$$\tilde{\tilde{y}}(x) + y(x) = x$$

$$u(x) + 1 + \frac{x}{\pi}\left[\pi - 2 - \int_o^\pi (\pi - t)\, u(t)dt\right] + \int_o^x (x-t)\, u(t)dt = x$$

$$u(x)x - 1 - x + \frac{2x}{\pi} + \frac{x}{\pi}\int_o^\pi (\pi - t)\, u(t)dt - \int_o^x (x-t)\, u(t)dt$$

$$= \frac{-\pi + 2x}{\pi} + \frac{x}{\pi}\int_o^x (\pi - t)\, u(t)dt + \frac{x}{\pi}\int_x^\pi (\pi - t)\, u(t)dt - \int_o^x (x-t)\, u(t)dt$$

$$= \frac{2x - \pi}{\pi} + \int_o^x [\frac{x}{\pi}(\pi - t) - (x - t)]u(t)dt + \frac{x}{\pi}\int_x^\pi (\pi - t)\, u(t)dt$$

$$= \frac{2x - \pi}{\pi} + \int_o^x \frac{x(\pi - t) - \pi(x - t)}{\pi} u(t)dt + \int_x^\pi \frac{x(\pi - t)}{\pi} u(t)dt$$

$$= \frac{2x - \pi}{\pi} + \int_o^x \frac{-xt + \pi t}{\pi} u(t)dt + \int_x^\pi \frac{x(\pi - t)}{\pi} u(t)dt$$

$$u(x) = \frac{2x - \pi}{\pi} + \int_o^\pi k\,(x,t)u(t)dt$$

$$k(x,t) = [\begin{matrix} \frac{t(\pi - x)}{\pi}, o \le t \le x \\ \frac{x(\pi - t)}{\pi}, x \le t \end{matrix}$$

CHAPTER TWO

FREDHOLM INTEGRAL EQUATIONS

2.1 Introduction [2]

We sall be concerned with the nonhomogeneous fredholm integral equations of the second kind of the from :

$$u(x) = f(x) + \lambda \int_a^b k\,(x,t)u(t)dt \quad ,a \le x \le b \qquad (2.1)$$

Where k (x,t) is the kernel of the integral equation , and λ is a parameter , A considerable amount of discussion will be directed toward the various methods and techniques that are used for solving this type of equation starting with the most recent methods that proved to be highly reliable and accurate . to do this we will naturally focus our study on the degenerate or separable kernels all through this chapter . the standard from the degenerate or separable kernels is given by :

$$k(x,t) = \sum_{j=1}^{n} g_j\,(x)h_j(t) \qquad (2.2)$$

The expressions $x - t$, $x + t$, xt , $x^2 - 3xt + t^2, etc$. are examples of separable kernels . for other well-behaved non- separable kernels , we can convert them to separable in the form (2.2) simply by expanding these kernels using Taylor's expansion .

Definition (2.1) [2]

The kernel k(x,t) is defined to be square integrable in both x and t in the square a≤x ≤b , a≤ t ≤ b if the following regularity condition :

$$\int_a^b \int_a^b k\ (x,t)dx\,dt \ < \propto \qquad (2.3)$$

Is satisfied

This condition gives rise to the development of the solution of the fredholm integral equations (2.1) . it is also convenient to state , without proof , the so-called fredholm Alternative theorem that relates the soultions of homogenous and nonhomogenous fredholm integral equations .

2.1.1 Fredholm Alternative Theorem [1]

The nonhomogenous fredholm integral equations (2.1) has one and only one solution if the only solution to the homogenous fredholm integral equations :

$$u(x) = \lambda \int_a^b k\,(x,t)u(t)dt \qquad (2.4)$$

Is the trivial solution u(x)=0.

We end this section by introducing the necessary condition that will guarantee a unique solution to the integral equations (2.1.1) in the integral of discussion .

Considering (2.2) , if the kernel k(x,t) is real continuous , and bounded in the square a≤ t ≤ b , and a≤ t ≤b , if :

$$|k(x,t)| \leq M, a \leq x \leq b \quad and \quad a \leq t \leq b \qquad (2.5)$$

And if f(x)≠0 , and continuous in a ≤ x ≤b , then the necessary condition that will guarantee that (2.2) has only a unique solution is given by :

$$|\lambda| M(b-a) < 1 \qquad (2.6)$$

It is important to note that a continuous solution to fredholm integral equations may exist , even though the condition (2.6) is not satisfied . this may be seen by considering the equation :

$$u(x) = -4 + \int_0^1 (2x+3t)\,u(t)dt \qquad (2.7)$$

In this example , $\lambda = 1, |k(x,t)| \leq 5 \;\; and \;\; (b-a) = 1_j \;\; therefore: \;\; |\lambda| M(b-a) = 5 \nless 1$

Accordingly , the necessary condition (2.6) fails to hold , but the integral equation (2.7)has an exact solution given by : u(x)=4x (2.9)

And this can be justified through direct substitution . in the following , we will discuss several methods that handle successfully the fredholm integral equation of the second kind .

2.2 The A domain Decomposition Method [1]

A domain developed the so-called A domain Decomposition Method or simply the decomposition method (ADM) . the method proved to be reliable and effective for a wide class of equation , differential and integral equation , and linear and nonlinear

models , the method was applied mostly to ordinary and partial differential equations and was rarely used for integral equation .

In the decomposition method , we usually express the solution u(x) of the integral equation (2.1) in a series form defined by :

$$u(x) = \sum_{n=0}^{\propto} un(x)$$

Substituting the decomposition (2.10) into both sides of (2.1) yields :

$$\sum_{n-0}^{\propto} un(x) = f(x) + \lambda \int_a^b k\,(x,t)\,(\sum_{n=0}^{\propto} un(t))dt \qquad (2.11)$$

Or equivalently

$$u_o(x) + u_1(x) + u_2(x) + \cdots = f(x)\lambda \int_a^b k\,(x,t)u_0(t)dt + \lambda \int_a^b k\,(x,t)u_1(t)dt + \lambda \int_a^b k\,(x,t)u_2(t)dt + \cdots \qquad (2.12)$$

The components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots.$ Of the unknown function u(x) are completely determined in a recurrent manner , if we set :

$$u_0(x) = \mathrm{f(x)} \qquad (2.13)$$

$$u_1(x) = \lambda \int_a^b k\,(x,t)u_0\,(t)dt \qquad (2.14)$$

$$u_2(x) = \lambda \int_a^b k\,(x,t)u_1(\mathrm{t})\mathrm{dt} \qquad (2.15)$$

$$u_3(x) = \lambda \int_a^b k\,(x,t)u_2(t)dt \qquad (2.16)$$

And so on . the above-discussed scheme for the determination of the components

$u_0(x), u_1(x), u_2(x), u_3(x), \ldots.$ Of the solution u(x) of Eq . (2.1) can be written recursively by :

$$u_0(x) = \mathrm{f(x)} \qquad (2.17)$$

$$u_{n+1}(x) = \lambda \int_a^b k(x,t)\, u_n(t)dt \quad , n \geq 0 \qquad (2.18)$$

In view of (2.17) and (2.18 the components $u_0(x), u_1(x), u_2(x), u_3(x)$, …. Follow immediately . with these components determined , the solution u(x) of (2.1) is readily determined in a series form using the decompositions (2.10) . it is important to note that the obtained series for u(x) converges to the exact solution in a closed form if such a solution exicts as will be seen later . however , for concrete problems , where the exact solution cannot be evaluated , a truncated series $\sum_{n=0}^{\propto} un(x)$ is usually used to approximate the solution u(x) and this can be used for numerical purposes . we point out here that a few terms of the truncated series usually provide a higher accuracy level of the approximate solution if compared with the existing numerical techniques.

In the following , we discuss some examples that illustrate the decomposition method outlined above .

2.2.1 Examples [1]

1)we first consider the Fredholm integral equation of the second kind .

$$u(x) = \frac{9}{10} x^2 + \int_0^1 \frac{1}{2} x^2 t^2\, u(t)dt \qquad (2.19)$$

It is clear that$f(x) = \frac{9}{10} x^2$, $\lambda = 1$. to evaluate the components $u_0(x), u_1(x), u_2(x), u_3(x)$, …. Of the series solution we use the recursive scheme (2.17) and (2.18) to find :

$$u_0(x) = f(x) = \frac{9}{10} x^2$$

$$u_1(x) = \lambda \int_a^b k(x,t) u_0(t) dt = \int_0^1 \frac{1}{2} x^2 t^2 \left(\frac{9}{10} x^2\right) dt = \int_0^1 \frac{9}{20} x^2 t^4 dt = \frac{9}{100} x^2 \quad (2.21)$$

$$u_2(x) = \lambda \int_a^b k(x,t) u_1(t) dt = \int_0^1 \frac{1}{2} x^2 t^2 \left(\frac{9}{100} t^2\right) dt = \int_0^1 \frac{9}{200} x^2 t^4 dt$$

$$= \frac{9}{1000} x^2 \qquad (2.22)$$

And so on , noting that :$u(x) = u_0(x) + u_1(x) + u_2(x) + \ldots \qquad (2.23)$
We can easily obtain the solution in a series form given by :

$$u(x) = \frac{9}{10} x^2 + \frac{9}{100} x^2 + \frac{9}{1000} x^2 + \cdots \qquad (2.24)$$

So that the solution (2.19) in closed form :

$$u(x) = x^2 \qquad (2.25)$$

Follows immediately ufon using formula for the sum of the infinite geometric series

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الخلاصة

نظرية المعادلات التكاملية تشكل موضوع مهم في الرياضيات كواحدة من اهم الادوات في الرياضيات الصرفة والتطبيقية. في هذا البحث تناولنا تصنيف المعادلات التكاملية و طرق حلها مع بعض الامثلة.