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Reliability Evaluation of Engineering System using exponential distribution

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By

Mohammed Bazoon Muhaisen

Supervised By

Dr: Nada Mohammed Abbas

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بسم الله الرحمن الرحيم

يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات

صدق الله العظيم

الاهداء

إلى من وضع المولى - سبحانه وتعالى - الجنة تحت قدميها، ووقرها في كتابه العزيز...
(أمي الحبيبة).

إلى خالد الذكر، الذي وفاته المنيَّة ، وكان خير مثال لرب ألأسرة، والذي لم يتهاون يوم في توفير سبيل الخير والسعادة لي..

(أبي الموقر).

إلى من أعتمد عليهم في كل كبيرة وصغيرة..

(أخوتي المحترمون).

إلى أصدقائي ومعارفي الذين أجلهم وأحترمهم..

إلى أساتذتي في كلية التربية للعلوم الصرفة

أهدي لكم بحثي ....

اهداء شكر

في نهاية بحثي اشكر كل شخص ساندني ووقف بجواري حتى أصل إلى ما أنا عليه الآن،واهدي شكري الى الهيئة التدريسية لكلية التربية للعلوم الصرفة وبالخصوص الى دكتورتي

(الدكتورة ندى محمد عباس المحترمة)

التي من دونها لما كنت وصلت الى اي شئ

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Abstract

Growing concern about the increasing number of disasters, safety is the main criterion to design every system. In this paper a complex bridge system is considered which includes five independent components whose longevity follows exponential distribution function. The main aim of this paper is to enhance the system reliability by path tracing method.

Introduction

In modern society safety has become a key concept to design every system. In real world applications system reliability plays important role as system reliability is directly proportional to system safety, therefore, it has become necessary to work on system safety and hence on reliability theory. In literature various kinds of systems have been studied and the reliabilities of the systems have been improved by various methods. This paper consists of two chapters , in the first chapter some basic definition of probability with some continuous probability distributions . in this chapter definition of reliability and simple system reliability are introduced . In the second chapter , study complex system with review some methods used to determine a reliable network. finally we will study one complex system and treat them by path tracing method

Chapter one

Basic Definitions and Concepts

1.1 Some Basic Definitions of Probability

Definition1.1.1 [10]

the probability P is a function $P: S \longrightarrow [0,1]$ which satisfies the following axioms:

Axiom 1: $0 \leq P(A) \leq 1$, for each event A in S

Axiom 2: $P(S) = 1$

Axiom 3: If A and B are mutually exclusive events in S, then

$$P(A \cup B) = P(A) + P(B)$$

Axiom 4: $P(\emptyset) = 0$

Axiom 5: $P(A^c) = 1 - P(A)$

Definition 1.1.2. [10]

For a continuous random variable X a probability density function is a function such that:

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\ dx = 1$
3. $P(a \leq x \leq b) = \int_a^b f(x)\ dx$, for any a and b .

The cumulative distribute function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\ du \qquad -\infty < x < \infty$$

$P(a < x < b) = F(b) - F(a)$ *and* $f(x) = \frac{dF(x)}{dx}$ *if the derivative exists* .

Definition 1.1.3 [11]

The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function or probability distributes of the discrete random variable X , if for each possible outcome x,

1) $f(x) \geq 0$
2) $\sum_x f(x) = 1$
3) $P(X = x) = f(x)$.

Definition 1.1.4 [11]

The cumulative distribution function F(x) of discrete random variable X probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), for \ -\infty < x < \infty$$

1.2 Some Continuous Probability Distributions

Definition1.2.1 [10]

A random variable X is said to have an Exponential distribution with parameter λ if its probability density function is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & For \ x \geq 0 \\ 0 & For \ x < 0 \end{cases}$$

The cumulative distribution function is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; \ if \ x > 0 \\ 0 & ; \ otherwise \end{cases}$$

where $E(X) = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$

Theorem 1.2.2 [10]

Suppose X has Exponential distribution with parameter λ. Them for any $t, s \geq 0 p(X > t + s / X > t) = p(X > s)$ this is called the memoryless property. Further Exponential is the only continuous distribution with this property.

Proof:

$$P(X > t + s / x > t) = \frac{P(X > t + s, X > t)}{p(X > t)}$$

$$= \frac{p(X > t + s)}{p(X > t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s}$$

$$= P(X > s)$$

Theorem 1.2.3 [11]

If $X_1, X_2, X_3, \dots\dots, X_n$ are identical independent distribution $Exp(\lambda)$, then , $S_n = \sum_{i=1}^{n} X_i$ has following probability density function

$$f(s) = \begin{cases} \frac{\lambda^n}{\Gamma(n)} S^{n-1} e^{\lambda s} & ; \quad if\ S>0 \\ 0 & ; \quad othwise \end{cases}$$

Proof:-

We will prove it by mathematical induction:

True for n=1 : $\Gamma(n) = (n-1)!\ then\ \Gamma(1) = 0! = 1$, So $f(s) = \lambda e^{-\lambda s}, \lambda > 0$.

We assume that it is true for n=k-1

Fact: If X and Y are two independent continuous random variable with probability density f and g, respectively, then X+Y is again a continuous random variable with probability density function.

$$h(s) = \int_{-\infty}^{\infty} f(s-t)\ g(t)\ dt\ ,this\ is\ called\ the\ convolution\ formula.$$

So for n=k using the fact and induction hypothesis we have for s > 0

$$f_{S_k}(s) = \int_{-\infty}^{\infty} f_{S_{k-1}}(s-t)\ f_{X_k}(t)\ dt$$

$$= \int_0^s \frac{\lambda^{k-1}}{\Gamma(k-1)} (s-t)^{k-2}\ e^{-\lambda(s-t)} (\lambda e^{-\lambda t})\, dt$$

$$= \frac{\lambda^k}{\Gamma(k-1)} e^{-\lambda s} \int_0^s (s-t)^{k-2}\ dt$$

$$= \frac{\lambda^k\ e^{-\lambda s}}{\Gamma(k-1)} \frac{s^{k-1}}{\Gamma(k-1)}$$

$$Since\ \Gamma(k-1) = \Gamma(k-2)! = (k-1)(k-2)! = \Gamma(k)$$

So,

$$f_{S_k}(s) = \frac{\lambda^k\ e^{-\lambda s}}{(k-1)\Gamma(k-1)} s^{k-1}$$

$$= \frac{\lambda^k\ e^{-\lambda s}}{\Gamma(k)} \cdot s^{k-1}$$

$$= \frac{\lambda^k}{\Gamma(k)} \cdot s^{k-1} e^{-\lambda s} = f(s)$$

Theorem 1.2.4. [10]

If $X_1, X_2, X_3, \dots \dots, X_n$ are independent $Exp(\lambda_i)$, then

$min_i \ X_i \sim Exp(\sum \lambda_i)$

Proof:-

For t >0, Then $P(min_i \ X_i > t) = P(X_1 > t, X_2 > t, \dots \dots, X_n > t$

$$= \prod_{i=1}^{n} P(\ X_i > t)$$

$$= \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\sum_{i=1}^{n} \lambda_i)t} = 1 - F_X(t)$$

Theorem 1.2.5. [11]

If $X_1, X_2, X_3, \dots \dots, X_n$ are independent $Exp(\lambda_i)$, then

$P(\ X_i = min_j X_j) = \frac{\lambda_i}{\sum_j \lambda_j}$

Proof:-

For n=2 , $P(\ X_1 = \min\{X_1, X_2\} = P(X_1 < X_2))$

$$= \int_0^{\infty} P(X_1 < X_2 \ / \ X_1 = x) \lambda_1 e^{\lambda_1 x} dx$$

$$= \int_0^{\infty} P(X_2 > x \ ) \lambda_1 e^{-\lambda_1 x} dx$$

$$= \int_0^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Similarly if $P(\ X_2 = \min\{x_1, x_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.

.

Definition 1.2.6. [10]

The continuous random variable X has a Weibull distribution, with parameters $\propto and\ \beta$, if its density function is given by:

$$f(x,\propto,\beta) = \begin{cases} \frac{\beta\, x^{\beta-1}}{\propto^{\beta}} e^{-(\frac{x}{\propto})^{\beta}} & ; \quad x \geq 0 \\ 0 & ; \quad else\ where \end{cases}$$

Where $\propto > 0$ is scale parameters and $\beta > 0$ is shape parameters.

The cumulative distribution function is $F(x,\propto,\beta) = 1 - e^{-(\frac{x}{\propto})^{\beta}}$.

The expected value or mean is

$$E(X) = \propto \Gamma\left(1 + \frac{1}{\beta}\right)$$

And its variance is

$$\propto^2 [\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)]$$

1.3 Basic Concept of Reliability

Definition 1.3.1. [1]

System is the overall structure being considered , which in turn consists of sub-ordinate structures called sub system are called "**components"** .

Definition1.3.2. [1]

Reliability is The probability that for a certain period of time the system will survive is established This can be expressed as the time to system failure in terms of random variable T. The probability of survival or reliability R(t) at time t, has the following properties

1- $R(t) \in [0,1]$

2- Since $F(0) = 0, F(\infty) = 1,$ therefore

$R(0) = 1\ and\ R(\infty) = 0\ this\ implies\ that\ 0 \leq R(t) \leq 1$

3- R(t) is a decreasing function of time t.

The probability of failure of a given system in a particular time interval $[t_1, t_2]$ can be written in terms of the reliability function as:

$$\int_{t_1}^{t_2} f(x)\ dx = \int_{t_1}^{\infty} f(x)\ dx - \int_{t_2}^{\infty} f(x)\ dx$$

sing the exponential distribution, the pdf can be written in the form:

$$f(t) = \lambda e^{-\lambda t}$$

here λ is a parameter of the exponential distribution.

Therefore, the reliability function of the exponential distribution can be derived as follows:

$$R(t) = 1 - \int_0^t \lambda e^{-\lambda x} dx$$

So, the reliability function becomes as follows:

$$R(t) = e^{-\lambda t}$$

here λ is a failure rate.

1.4 **Simple System Reliability :**

The reliability function of simple system includes series , parallel , series – parallel , parallel - series, combination of series and parallel and K-out of – n .

1) Series system [2].

The series system works only if all of its components are working. This system depends on both of them[57] .Device elements are executed properly.

Figure (1)The series system

If R_1 , R_2 , … , R_n are the reliabilities of the individual components , then the reliability of the system is giving by :

$$R_{system} = R_1 . R_2 . \dots . R_n$$

2) Parallel System [2].

For a parallel system to succeed, at least one of the components must succeed

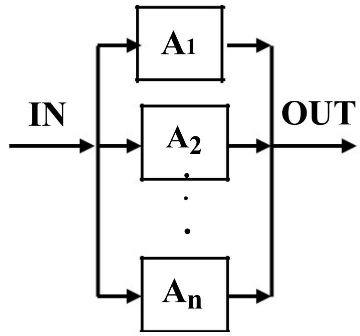

Figure (2) Parallel system

the reliability of the system (assuming independent failures) is :

$R_{system} = 1 - p$ (all fail)

$= 1 - [\, p(A_1 \text{ fail}) \times p(A_2 \text{ fail}) \times \ldots \times p(A_n \text{ fail}) \,]$

$= 1 - (1 - R_1)(1 - R_2) \ldots (1 - R_n)$

$$R_{system} = 1 - \prod_{i=1}^{n} (1 - R_i)$$

3)Parallel - Series Systems [3]:-

A system in which "m" subsystem are connected in series where each subsystem has "n" components connected in parallel as shown in the figure(4) is said to be in parallel series configuration

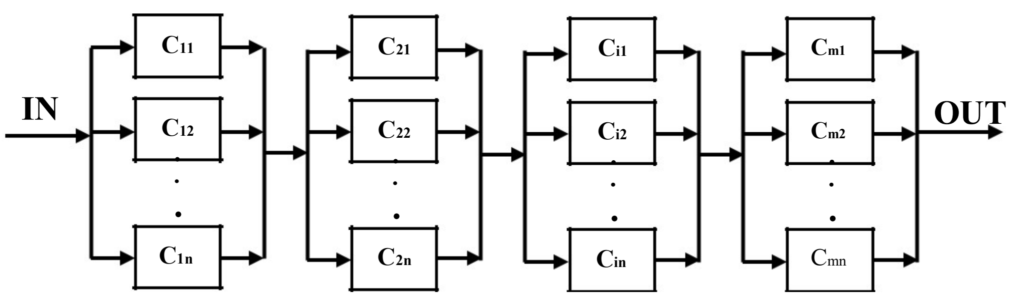

Figure (3) **Parallel - Series Systems**

If R is the reliability of the individual component then the reliability of each of The subsystem with i.i.d. components equal to $1-(1-R)^n$.

Therefore the reliability of the whole system is

$R_s = \{1-(1-R)^n\}^m$

For example: - Suppose that the component lifetimes are i.i.d. exponentially distributed with mean λ , then

$R_s(t) = \{1 - (1 - e^{-\frac{t}{\lambda}})^n\}^m$.

4) Series-Parallel Systems" [3].

A system in which "m" subsystem are connected in parallel where each subsystem has "n" components connected in series as shown in the figure (5) is said to be in series-parallel configuration

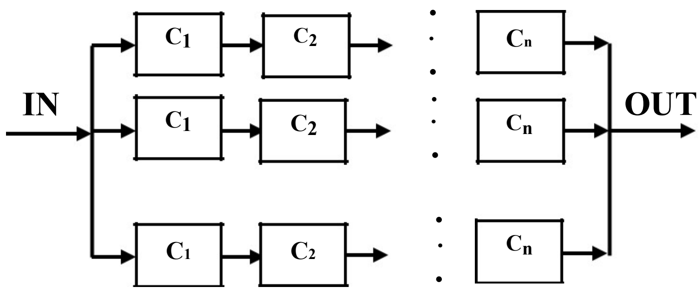

Figure (4) Series-Parallel Systems

If R is the reliability of the individual component then the reliability of each of the subsystem equal to$(R(t))^n$.
Therefore the reliability of the whole system is $R_s = 1 - (1 - (R(t))^n)^m$
For example: - Suppose that the component lifetimes are i.i.d. exponentially distributed with mean λ , then$R_s = 1 - (1 - e^{-\frac{nt}{\lambda}})^m$

5) System k – out – of – n [4].

This system is considered a special case of parallel redundancy where this method needs to succeed in working on at least the k components of the total n components. Al though this system is a special case of parallel redundancy in some cases it is a general configuration because the number of units required to maintain the success of the system is close to the total number of units in the system and the behavior of the system is close to the series system if the number of units required equals the number the units in the system is thus a series system .

The system will failed if n – k + 1 or more components are failed simultaneously.

For a k – out – of – n system, this system has n parallel components, however at least k components must survive if the system is to continue operating. The reliability for k – out – of – n system is given by:
$R_s = \sum_{i=k}^{n} \binom{n}{i} P^i$(1-P)$^{n-i}$
Where p is the component reliability and is assumed that the components are identical.

Chapter Two

Complex Reliability System

2.1 Introduction

If the system structure is not one of the simple forms, accurate reliability cannot be calculated. To address the more general issue, we provide a graphical network model in which a system can be verified as operational by determining whether a successful path exists through the system. If no such path exists, the system fails. **[7].**

2.2 Some Methods of Solving Complex Systems: [5].

A complex system's reliability can be determined using many different kinds of techniques, such as:

1. Path Tracing Method (Tie-Set Method)
2. Minimal Cut Method
3. Reduction to Series Element Method.
4. Decomposition Method
5. Event Space Method
6. Composite method.
7. Boolean Truth Table Method
8. Pivotal Decomposition
9. The Inclusion-Exclusion Method
10. Sum of Disjoint Products Method

Definition 2.3 [8].

A path is a collection of components when they function, join the start node and the end node through other parts that are function, ensuring that the system is in a functioning state.

Definition 2.4 [9].

The minimal path set connects the source and sink nodes as long as it does not contain any cycles, the minimal path set cannot be reduced, as it has no redundant elements, and removing any of the edges from the path means that the source and sink nodes are no longer connected, as an example.

Definition 2.5. [9].

Path Tracing Method In this method, we convert the complex system into a series- parallel system to facilitate the calculation of its reliability, by Using the following steps.

1. We extract the minimal path sets, as each of these sets causes the system to success.

2. We link the elements of each of the groups of the minimal path with a series connection, because the failure of Any component causes the failure of the path.

3. We link parallel between these sets. Fourthly, we will have a series

4. We calculate its reliability according To its law.

Example 2.6 [1].

considered the system in which five Components are connected in complex bridge configuration shown below in figure-5

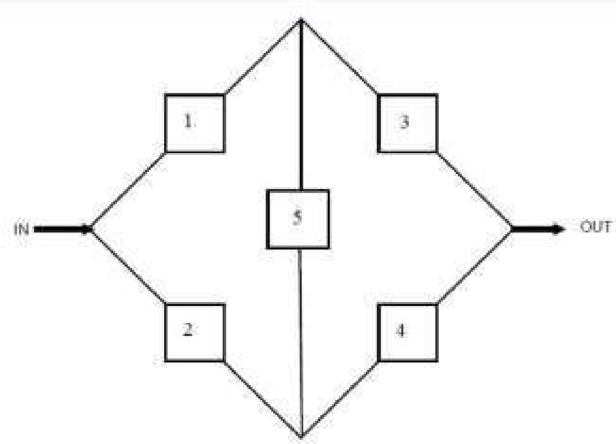

Figure (5)

Complex bridge configuration

Each rectangular block in the diagram denotes a component with reliability R_i, $i=1,2,3,4,5$. The reliability function[3] of the system can be computed by using minimal path and can expressed as.

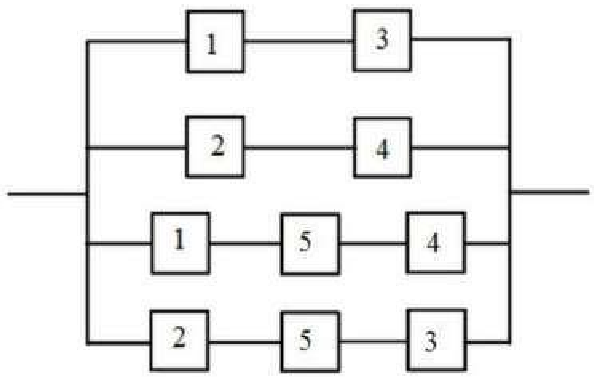

Figure (6)

Minimal path representation for complex bridge configuration

Minimum path set is defined as the minimum set of components which by functioning ensure the functioning of the structure and the minimal path sets of the above complex bridge structure are

P1 = {1,3} , P2 = {2,4} , P3 = {1,4,5} , P4 = {2,3,5}

The minimal path sets represents the bridge configuration as series paralled system illustrated by figure 6.

$$
\begin{aligned}
R(t) = {} & 1 - [(1 - R_1R_3)(1 - R_1R_4)(1 - R_1R_5R_4)\,)(1 - R_2R_5R_3)] \\
= {} & 1 - [(1 - R_2R_4 - R_1R_3 + R_1R_2\,R_3R_4\,) * (1 - R_2R_5R_3 - R_1R_5R_4 \\
& \quad + R_1R_2R_3R_4R_5)] \\
= {} & 1 - [(1 - R_2R_5R_3 - R_1R_4R_5 + R_1R_2R_3R_4R_5 - R_2R_4 \\
& \quad + R_2R_3R_4R_5 + R_1R_2R_4R_5 - R_1R_2R_3R_4R_5 - R_1R_3 \\
& \quad + R_1R_2R_3R_5 - R_1R_2R_3R_4R_5 - R_1R_3 + R_1R_2R_3R_5 \\
& \quad + R_1R_3R_4R_5 - R_1R_2R_3R_4R_5 + R_1R_2R_3R_4 \\
& \quad - R_1R_2R_3R_4R_5 - R_1R_2R_4R_5 + R_1R_2R_3R_4R_5]
\end{aligned}
$$

$= 1 - 1 + R_2R_5R_3 + R_1R_4R_5 - R_1R_2R_3R_4R_5 + R_2R_4 - R_2R_3R_4R_5 - R_1R_2R_4R_5 + R_1R_2R_3R_4R_5 + R_1R_3 - R_1R_2R_3R_5 - R_1R_3R_4R_5 + R_1R_2R_3R_4R_5 - R_1R_2R_3R_4 + R_1R_2R_3R_4R_5 + R_1R_2R_4R_5 - R_1R_2R_3R_4R_5$

$$R_{S(t)} = R_1R_3 + R_2R_4 + R_1R_4R_5 + R_2R_3R_5 - R_1R_3R_4R_5 - R_1R_2R_4R_5 - R_1R_2R_3R_4 - R_1R_2R_3R_5 - R_2R_3R_4R_5 + 2\,R_1R_2R_3R_4R_5$$

The failure rates for they components are as follows :

1=0.1 , 2=0.2 , 3=0.8 , 4=0.5 , 5=0.4

$$R_s = e^{-(0.2+0.5)t} + e^{-(0.1+0.5+0.4)t} + e^{-(0.2+0.8+0.4)t} + e^{-(0.1+0.5+0.4)t} - e^{-(0.1+0.8+0.5+0.4)t} - e^{-(0.1+0.2+0.5+0.4)t} - e^{-(0.1+0.2+0.8+0.5)t} - e^{-(0.1+0.2+0.8+0.4)t} - e^{-(0.2+0.8+0.5+0.4)t} + 2\,e^{-(0.1+0.2+0.8+0.5+0.4)t}$$

$$R_s = e^{-0.7t} + e^{-t} + e^{-1.4t} + e^{-t} - e^{-1.8t} - e^{-1.2t} - e^{-1.6t} - e^{-1.5t} - e^{-1.9t} + 2\,e^{-2t}$$

If t = 0.5 then

$$R_s = e^{-0.35} + e^{-0.5} + e^{-0.7} + e^{-0.5} - e^{-0.9} - e^{-0.6} - e^{-0.8} - e^{-0.75} - e^{-0.95} + 2\,e^{-1}$$

$= 0.886$

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