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Sequences in Topological Spaces

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بسم الله الرحمن الرحيم

وَقُل رَّبِّ زِدْنِي عِلْمًا

من سورة طه- آية (١١٤)

Dedication

To whom do I prefer it over myself, and why not; You sacrificed

for me

She spared no effort to keep me happy

(My beloved mother).

We walk the paths of life, and those who control our minds remain in every path we take

His kind face, and good deeds.

He did not skimp on me all his life

(My dear father) .

To my friends, and all those who stood by me and helped me with whatever they had, in many ways

I present to you this research, and I hope it will be to your satisfaction.

Acknowledgments

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Contents

Abstract

Topological space is, roughly speaking, a geometrical space in which closeness is defined but cannot necessarily be measured by a numeric distance. More specifically, a topological space is a set whose elements are called points, along with an additional structure called a topology, which can be defined as a set of neighborhoods for each point that satisfy some axioms formalizing the concept of closeness. In Topological spaces, some concepts are different from those in metric spaces, sequences, their convergence/ divergence are the main topic in this research.

Through our research, we addressed the following topics (Sequences and convergence, Open versus sequentially open, Basis for a topology, Compactness and sequential compactness, Sequential spaces and Fr´echet-Urysohn spaces.

Introduction

A sequence, in a non-empty set X, is universally defined as a mapping from the set of natural numbers N to the set X, therefore this concept is also valid if X being a topological space but the concept of convergence changes.[1]

In a non-empty set X in R , a convergent sequence has unique limit , but in a topological space , it is not so.

Since convergence is a basic property of a sequence which can arise many differences in properties of the set X and a topological space, hence we require to have an analogous theory of convergence for an arbitrary topological space.

Furthermore, topology of a space can be described completely in terms of convergence of sequences.

Differ from the concept in general sets, in which a convergent sequence has a unique limit but in a topological space a convergent sequence has one limit or more than one limit, but on addition a property like Hausdorff or metric in topological spaces, all sequences have unique limit.

This article covers the concept of sequences, and analyses their convergence and equivalency.

1. Sequences and convergence [2]

Definition 1.1 .

Let X be a topological space. A **sequence** in X is a family $\{x_n : n \in \boldsymbol{N}\}$ of points in X.

Example, Let X = {0,1} with the topology $T = \{\emptyset, X, \{0\}, \{0,1\}\}$. The following are examples of sequences from X

(1) Constant sequence 0,0,0,…

(2) Constant sequence ١,1,1,…

(3) Alternating sequence 0,1,0,1,…

Definition 1.2

We say that a sequence $\{x_n : n \in \boldsymbol{N}\}$ **converges** to a point $x \in X$, if for any neighbourhood U of x, there is an $N \in \boldsymbol{N}$ so that $x_n \in U$ for all $n > N$, and in this case, we write $x_n \to x$.

In previous example,

(a) 0,0,0 … → 0 and 1

(b) 1,1,1 … → 1

(c) 0,1,0,1, … → 1

Definition 1.3

A **subsequence** $\{y_n : n \in \boldsymbol{N}\}$ is a sequence such that $y_i = x_{n_i}$ for some $n_1 < n_2 < \ldots$.

From the definition, one immediately obtains the following result.

$$1,1,1, \ldots \text{ is a subequece } of \ 0,1,0,1, \ldots.$$

$$0,0,0, \ldots \text{ is a subequece } of \ 0,1,0,1, \ldots.$$

Theorem 1.4.

If $\{y_n\}$ is a subsequence of $\{x_n\}$ and $x_n \rightarrow x$, then $y_n \rightarrow x$.

The next example shows that the concept of convergence depends on the topology of the underlying topological space.

Example 1.5.

Let $X = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

- In the trivial topology $T_{ID} = \{\emptyset, X\}$, all sequences converge to a, b and c.
- In the discrete topology,

$$\mathrm{T_D} = \{\emptyset, \{\mathrm{a}\}, \{\mathrm{b}\}, \{\mathrm{c}\}, \{\mathrm{a}, \mathrm{b}\}, \{\mathrm{a}, \mathrm{c}\}, \{\mathrm{b}, \mathrm{c}\}, \mathrm{X}\}$$

for a sequence $\{x_n\}$ to converge to a point x, it has to be constantly equal to x for all large enough n. That is

The sequence $a, a, a, \dots \dots .. \to a$

and $b, b, b, \dots \dots .. \to b$

Example 1.6.

The constant sequence is convergent, regardless of the topology on the space.

The constant function $f(x) = c$ converges to every point in X in the indiscrete topology $T_{ID} = \{ \emptyset, X\}$, but $f(x) = c$ converges to only c in the discrete topology T_D

Let us show that we recover the possibly well-known definition of convergence for metric spaces and in particular for $\boldsymbol{R}$

Theorem 1.7.

Let (X, d) be a metric space with the metric topology. Then a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if

$$\forall \varepsilon > 0, \exists N > 0 : n > N \Rightarrow d(x_n, x) < \varepsilon.$$

1. If $f : X \to Y$ is a continuous function and $x_n \to y$ in X, then also $f(x_n) \to f(y)$ in Y (We say that continuous functions preserve convergence of sequences.)
2. Convergence in a product space is pointwise, i.e. a sequence (x_n) in $\prod_{i \in I} X_i$ converges to y if and only if

$$x_n(i) \to y(i) \text{ in } Xi \text{ for all } i \in I.$$

Definition1.8

A topological space is **Hausdorff** if for every two distinct points x and y, we can find a neighbourhood of x and a neighbourhood of y that are disjoint. That is

$$\forall x \neq y : \exists u_x, uy : u_x \cap uy = \theta$$

NOTE:

Sequences in general can have more than one limit, but in a Hausdorff space limits (if they exist at all) are unique. Indeed, a sequence cannot be eventually in two disjoint neighbourhoods at once

Theorem 1.9.

Let X be Hausdorff. If $x_n \to x$ and $x_n \to y$ in X, then $x = y$.

2. Open versus sequentially open[3]

Definition 2.1

In a topological space X, a set

1. A is open if and only if every $a \in A$ has a neighbourhood contained in A.
2. A is sequentially open if and only if no sequence in $X \setminus A$ has a limit in A, i.e. sequences cannot converge out of a sequentially closed set.

The implication from open to sequentially open is true in any topological space.

Theorem 2.2

In any topological space $X, if\ A$ is open, then A is sequen tially open.

We can just copy the proof for metric spaces, it remains valid in any topological space.

Definition 2.3 .

A topological space is sequential when any set A is open if

and only $if\ A$ is sequentially open.

However, importantly, not every space is sequential.

Theorem 2.4

There is a topological space that is not sequential.

Example 2.5:

Let X be an uncountable set, such as the set of real numbers. Consider (X, τ_{cc}), the countable complement topology on X.

Thus $A \subseteq X$ is closed if and only if $A = X$ of A is countable.

Suppose that a sequence (x_n) has a limit y. Then the neighbourhood

$$(R \setminus \{x_n \mid n \in \boldsymbol{N}\}) \cup \{y\}$$

of y must contain x_n for n large enough. This is only possible if

$$x_n = y$$

for n large enough. Consequently a sequence in any set A can only converge to an element of A, so every subset of X is sequentially open.

But as X is uncountable, not every subset is open. So (X, τ_{cc}) is not sequential.

Example 2.6:

Consider the order topology on the ordinal $\omega_1 + 1 = [0, \omega_1]$.

Because ω_1 has cofinality ω_1, every sequence of countable ordinals

has a countable supremum. Hence no sequence of countable ordinals

converges to ω_1, so $\{\omega_1\}$ is sequentially open. However, $\{\omega_1\}$ is not open as ω_1 is a limit ordinal. So the order topology on $[0, \omega_1]$ is not sequential.

Example 2.7:

Let X be an uncountable set and let $\{0, 1\}$ have the discrete topology.

Consider $\mathcal{P}(X) = \{0, 1\}^x$ with the product topology. Let $A \subseteq \mathcal{P}(X)$ be the collection of all uncountable subsets of X.

A is not open; indeed every basic open contains finite sets.

However, we claim that A is sequentially open.

Let (X_n) be a sequence of countable subsets of X and suppose that $X_n\ Y$.

Then for every $x \in X$ we must have $x \in Y$ if and only if $x \in X_n$ for n large enough. In particular

$Y \subseteq \bigcup_{n \in N} X_n$.

But $\bigcup_{n \in N} X_n$ is a countable union of countable sets.

Hence a sequence of countable sets can only converge to countable sets, so A is sequentially open.

Still, a large class of topological spaces is sequential.

3.Basis for a topology[1]

Definition 3.1.

Let X be a set, and let $\boldsymbol{B} \subseteq \mathcal{P}(X)$ be any collection of subsets of X.

Then $\boldsymbol{B}$ is called a basis for a topology on X if

(B1) for each $x \in X$, there is a $B \in \boldsymbol{B}$ such that $x \in B$, and

(B2) if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \boldsymbol{B}$, then there is a $B_3 \in \boldsymbol{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Theorem 3.2.

This collection $T_{\boldsymbol{B}} \subset P(X)$ is a topology.

Theorem 3.3.

Let $\boldsymbol{B}$ be the basis for a topology on a set X. Then $U \in \mathrm{T}_{\boldsymbol{B}}$ if and only if $U = \bigcup_{i \in I} B_i$ for some sets $B_i \in \boldsymbol{B}$. That is, $\mathrm{T}_{\boldsymbol{B}}$ consists of all unions of elements from $\boldsymbol{B}$.

Theorem 3.4.

Let (X, T) be a topological space. Let $C \subset \mathrm{T}$ be a collection of open sets on X with the following property: for each set $U \in \mathrm{T}$ and each $x \in U$ there is a $C \in C$ so that $x \in C \subset U$. Then C is a basis for T.

Example 3.5.

If $X = \{a, b\}, then\ B = \{\{a\}, \{b\}\}$ is a basis for a topology on X. The topology $\mathrm{T}_{\boldsymbol{B}}$ is exactly the discrete topology, $\mathrm{T}_{\boldsymbol{B}} = \mathcal{P}(X)$. More generally, let X be any set, and let $\boldsymbol{B}$ consist of those sets that contain only a single element, that is

$$\boldsymbol{B} = \{\{x\} \mid x \in X\}.$$

Then $\boldsymbol{B}$ is a basis for a topology, and $\mathrm{T}_{\boldsymbol{B}}$ is the discrete topology: clearly, every set U in X is a union of sets from this collection since $\mathrm{U} = \bigcup_{x \in U} \{x\}$ so it follows that $\mathrm{T}_{\boldsymbol{B}}$ consists of all subsets of X.

So far, we have been dealing with abstract sets and topological spaces, but at the end of the day, we will be interested in particular topologies on concrete spaces, so at this point, let us use the notion of a basis for a topology to show how we can easily describe a topology on $\boldsymbol{R^n}$ that agrees with the one we know from analysis.

For $x \in R^n \; and \; r > 0$, let $B(x,r) = \{y \in R^n | \parallel x - y \parallel < r\}$

be the open ball centered in x with radius r.

Theorem 3.6.

The collection

$\boldsymbol{B} = \{B(x,r) \mid x \in R^n, r > 0\}$ is the basis for a topology on R^n .

The resulting topology $\mathrm{T}_{\boldsymbol{B}}$ is called the standard topology and its open sets are exactly the open sets that one will have encountered in a course on analysis or calculus.

This result will follow from the more general Proposition [2] below. While the standard topology is the most interesting one to consider, below we introduce certain other topologies on R.

The following result allows us to compare the topologies generated by bases if we know how to compare the bases. Notice that so far, all the proofs are similar in spirit:

the spaces in question are so abstract and have so little structure that one is forced to use the few things that one actually knows about the spaces.

Theorem 3.7.

Let X be a set, and let B and $\grave{B}$

be bases for topologies T and Ṫ respectively; both on X. Then the following are equivalent:

(1) The topology Ṫ is finer than T .

(2) For every $x \in X$ and each basis element $B \in \boldsymbol{B}$ satisfying x ∈ B, there is a basis element $B' \in \boldsymbol{B'}$ so that $x \in B' \subset B$.

Example 3.8.

We can define a basis for a topology on R by letting B_l consist of all sets of the form $\{x \in R \mid a \leq x < b\}$, where a, b ∈ R vary. The topology T_l generated by B_l is called the lower limit topology on R, and we write $R_l = (R, T_l)$.

Example 3.9.

Let $K = \{1/n \mid n \in N\} \subset R$ and let B_k consist of all open intervals as well as all sets of the form (a, b) \ K. Then B_k is a basis and the topology T_k that it generates is called the K-topology (sv: K-topologin) on R. We write $R_k = (R, \mathrm{T}_k)$. So, at this point we have introduced three different topologies on R and we can now use our results above to compare them.

Theorem 3.10.

The topologies R_l and R_k are both strictly finer than the standard topol- ogy but are not comparable with each other.

Definition 3.11.

A countable basis at a point x is a countable set

$\{U_n \mid n \in \boldsymbol{N}\}$ of neighbourhoods of x, such that for any neighbourhood V of x there is an $n \in \boldsymbol{N}$ such that $U_n \subseteq V$.

Definition 3.12.

A topological space is **first countable** if every point has a countable basis. Every metric space is first countable, as

$$\left\{ B\left(x, \frac{1}{n+1}\right) |n \in \boldsymbol{N} \right\}$$

is a countable basis at any point x. We can prove that every first count- able space is sequential by generalizing the proof that every metric space is sequential.

Theorem 3.13..

Every first countable space X (and hence every metric space) is sequential.

Sequential spaces are also exactly those spaces X where sequences can correctly define continuity of functions from X into another topological space.

Theorem 3.14.

Let X be a topological space. Then $A \subseteq X$ is sequentially open if and only if every sequence with a limit in A has all but finitely many terms in A.

Theorem 3.15..

The following are equivalent for any topological space X:

1. X is sequential;

2. for any topological space Y and function $f : X \to Y$, f is continuous

if and only if f preserves convergence (i.e. whenever $xn \to y$ in X, also $f(x_n) \to f(y)$ in Y).

4. Compactness and sequential compactness[1]

Definition 4.1 .

X is compact if and only if every open covering of X (i.e. every collection of open sets whose union is X) has a finite subcovering. It is sufficient to consider coverings of basic opens.[3]

Definition 4. 2.

If every sequence in X has a convergent subsequence then X is **sequentially compact**. A metric space is compact if and only if it is sequentially compact.

However, bearing in mind the difference between open and sequentially open, we should be very suspicious of this equivalence holding in general. And indeed, neither direction of the equivalence holds in every topological space.

Theorem 4.3.

There is a topological space that is compact but not sequentially compact.

Theorem 4.4..

There is a topological space that is sequentially compact but not compact.

5. Sequential spaces and Fr´echet-Urysohn spaces[3]

We have considered sequential spaces. Similar to sequential spaces are the Fr´echet-Urysohn spaces.

Definition 5. 1.

A topological space X is Fr´echet-Urysohn if the closure of any $A \subseteq X$ contains exactly the limits of sequences in A.

They are also sometimes simply called Fr´echet spaces, but this might cause confusing as there are other uses for the term Fr´echet space.

References

1. James R. Munkres. 2000 .Topology. Prentice Hall, second edition.
2. Jorgensen ,Soren , 2024, Basic Topology , Uppsala University.
3. Stijn Vermeeren ,2010, Sequences and nets in topology,University of Leeds,