Ministry of Higher Education

and Scientific Research

University of Babylon

College of Education

Department of Mathematics

Bernstein Polynomials

A graduation thesis submitted to the, Department of Mathematics College, in the academic year 2025-2026 as part of the requirements for obtaining a Bachelor of Science degree in Mathematics

Prepared by:

Atheraa Alaa Mohammed

Supervised by:

Dr. Hawraa Abbas Almurieb

1447HD **2026 AD**

بسم الله الرحمن الرحيم

((يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ))

صدق الله العلي العظيم

(المجادلة:١١

Dedication

To the one who taught me that giving knows no bounds, and illuminated my path in life:

To my beloved mother,

To my support in this life and source of security, to the one from whom I drew my strength and who supported me throughout my studies:

To my dear father,

To my dear brothers and sister... whose faces bring joy to my eyes and whose laughter fills my heart, who gave me strength and guidance, believed in me, and supported me through difficult times, enabling me to reach where I am today.

To my friends... who taught me that life without connection, love, and cooperation is meaningless.

To everyone who loves me sincerely and faithfully, to all those from whom I received advice and support.

The completion of my work would not have been possible without your support, and I hope it meets with your approval.

I dedicate to you the culmination of my efforts.

ACKNOWLEDGMENTS

I offer my deepest gratitude and appreciation first to God,

and secondly to all the professors of the College of Education, especially **Dr. Hawraa Abbas Almurieb**,

my research supervisor.

I also extend my thanks to everyone who supported me throughout my academic journey.

I thank God profusely.

Today, my name is Ms. Atheraa Alaa,

and I have achieved my ambition of earning a Bachelor's degree in Mathematics.

Contents

Introduction

Polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions. They can be differentiated and integrated easily, and can be pieced together to form spline curves that can approximate any function to any accuracy desired.

Bernstein polynomial basis is started to review the historical progress together with contemporary state of theory, algorithms and applying the method of polynomials for f inite domains. Initially introduced by $S.N.$ Bernstein to ease a useful proof of the Weierstrass Approximation Theorem, the slow convergence rate of Bernstein polynomial approximations to continuous functions results in them to fade in obscurity, till the arrival of digital computers.

The Bernstein form started to enjoy common use as a multifaceted means of intuitively creating and working on geometric shapes. At the same time, inciting further development of basic theory, identification of its excellent numerical stability properties and an increasingly variegation of its repertoire of applications, simple and efficient recur Sive algorithms, with the wish for utilizing power of computers for geometric design applications.

Karl Weierstrass gave the first proof of his (fundamental) theorem on approximation by algebraic and trigonometric polynomials, in 1885. This was important for development of Approximation Theory. It was a long and complicated proof and leaded a kind of mathematicians to find simpler and more useful proofs. In 1912, the Russian mathematician Sergei $N.$ Bernstein formulated a sequence of polynomials namely the Bernstein Polynomials. [1]

Chapter one

BERNSTEIN POLYNOMIALS

Definition 1.1(Bernstein polynomial)[2]

A Bernstein polynomial of degree n is defined on the interval [0, 1] as:

$$B_{k,n}(t) = \binom{n}{k} t^k (1 - t)^{n-k} \qquad (1)$$

For $k = 0,1, \cdots, n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example 1.2[2]

The Bernstein polynomials of degree 1 are

$$B_{0,1}(t) = (1 - t), \qquad B_{1,1}(t) = t$$

and can be plotted $0 \leq t \leq 1$ as

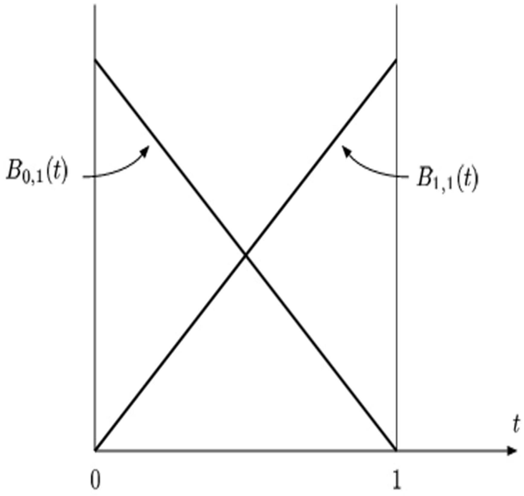

Figure 1.1. $B_{0,1}$ and $B_{1,1}$

1.2. Properties of Bernstein Polynomials

Property 2.1.1(Bernstein Polynomials Recursive Definition) [3]

The Bernstein Polynomial of degree n can be introduced by combining two $(n-1)$st degree Bernstein polynomials with each other. That is, the kth nth- degree Bernstein polynomial can be formulated by

$$B_{n,k}(t) = (1-t)B_{n-1,k}(t) + tB_{n-1,k-1}(t) \qquad (2)$$

Proof.

To prove this, we will use the Definition 1.1 of the Bernstein polynomials (1)

$(1-t)B_{k,n-1}(t) + tB_{k-1,n-1}(t)$

$= (1-t)\binom{n-1}{k} t^k (1-t)^{n-1-k} + \mathrm{t}\binom{n-1}{k-1} t^{k-1} (1-t)^{n-k-(k-1)}$

$= \binom{n-1}{k} t^k (1-t)^{n-k} + \binom{n-1}{k-1} t^k (1-t)^{n-k}$

$= \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right) t^k (1-t)^{n-k}$

$= \binom{n}{k} t^k (1-t)^{n-k}$

$= B_{k,n}(t)$.

Property 1.2.2(Non-negative Bernstein Polynomials) [3]

Bernstein polynomial is a non-negative function over the closed interval $[0,1]$ if $f\ (t) \ \geq\ 0$ for $t \ \in\ [0,1]$.

Proof.

To prove this, we use the mathematical induction with the recursive property **1.2.1** of Bernstein polynomials (2).

1. It is shown that the functions

$$B_{0,1}(t) = 1 - t \text{ and } B_{1,1}(t) = t$$

   are both non-negative over the interval [0,1].

2. If we suppose that all Bernstein polynomials of degree less than k are non-negative, the other case we can use the recursive definition of the Bernstein polynomial (2) and it is written by:

$$B_{n,k}(t) = (1 - t)B_{n,k-1}(t) + tB_{n-1,k-1}(t)$$

and prove that $B_{n,k}(t)$ is also non-negative over the interval [0,1].

Definition 1.2.3.(Partition Unity) [3]

If the summation of all values of t is one, then $f_n(t)$ is a called a partition unity.

Property 1.2.4 (Partition of Unity) [3]

The kth-degree $k + 1$ Bernstein polynomials for a partition of unity in that they all sum to one.

Proof.

If we assume that this is true, it is easy to show an undistinguished different fact: for each k, the sum of the $k + 1$ of degree k is equal to the sum of the k Bernstein polynomial so f degree $k - 1$.That is,

$$\sum_{k=0}^{n} B_{n,k}(t) = \sum_{k=0}^{n-1} B_{n,k-1}(t)$$

This computation is clear, using the recursive definition of Bernstein polynomial and rearranging the sums:

$$\sum_{k=0}^{n} B_{n,k}(t) = \sum_{k=0}^{n} \left[(1-t)B_{n,k-1}(t) + tB_{n-1,k-1}(t)\right]$$

$$= (1-t)\left[\sum_{n=0}^{k-1} B_{n,k-1}(t) + B_{k,k-1}(t)\right] + t\left[\sum_{n=1}^{k} B_{n-1,k-1}(t) + B_{-1,k-1}(t)\right]$$

$$= (1-t)\sum_{n=0}^{k-1} B_{n,k-1}(t) + t\sum_{n=1}^{k} B_{n-1,k-1}(t)$$

$$= (1-t)\sum_{n=0}^{k-1} B_{n,k-1}(t) + t\sum_{n=0}^{k-1} B_{n,k-1}(t)$$

$$= \sum_{n=0}^{k-1} B_{n,k-1}(t)$$

where we have utilized $B_{k,k-1}(t) = B_{-1,k-1}(t) = 0$

Once we have established this equality, it is simple to write

$$\sum_{n=0}^{k} B_{n,k-1}(t) = \sum_{n=0}^{k-1} B_{n,k-1}(t) = \sum_{n=0}^{k-2} B_{n,k-2}(t) = \cdots = \sum_{n=0}^{1} B_{n,1}(t)$$

$$= (1-t) + t = 1$$

Property 1.2.4 (Degree Raising) [3]

In this case, any $(n-1)$th-degree Bernstein polynomial can be written as a linear combination of nth-degree Bernstein polynomials.

Proof.

Firstly, we note that

$$tB_{k,n}(t) = \binom{n}{k} t^{k+1}(1-t)^{n-k} = \binom{n}{k} t^{k+1}(1-t)^{(n+1)-(k+1)}$$

$$= \frac{\binom{n}{k}}{\binom{n+1}{k+1}} \binom{n+1}{k+1} t^{k+1}(1-t)^{(n+1)-(k+1)}$$

$$= \frac{\binom{n}{k}}{\binom{n+1}{k+1}} B_{k+1,n+1}(t)$$

$$= \frac{k+1}{n+1} B_{k+1,n+1}(t)$$

And

$$(1-t)B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n+1-k}$$

$$= \frac{\binom{n}{k}}{\binom{n+1}{k}} \binom{n+1}{k} t^k (1-t)^{n-k}$$

$$= \frac{\binom{n}{k}}{\binom{n+1}{k}} B_{k,n+1}(t)$$

$$= \frac{n-k+1}{n+1} B_{k,n+1}(t)$$

And finally

$$\frac{1}{\binom{n}{k}} B_{k,n}(t) + \frac{1}{\binom{n}{k+1}} B_{k+1,n}(t) = t^k (1-t)^{n-k} + t^{k+1}(1-t)^{n-(k+1)}$$

$$= t^k (1-t)^{n-k-1} ((1-t)+t)$$

$$= t^k (1-t)^{n-k-1}$$

$$= \frac{1}{\binom{n-1}{k}} B_{k,n-1}(t)$$

Using this final equation, it can be written as

$$B_{k,n-1}(t) = \binom{n-1}{k}\left[\frac{1}{\binom{n}{k}}B_{k,n}(t) + \frac{1}{\binom{n}{k+1}}B_{k+1,n}(t)\right]$$

$$= \left(\frac{n-k}{n}\right)B_{k,n}(t) + \left(\frac{k+1}{n}\right)B_{k+1,n}(t)$$

Which expresses a Bernstein polynomial of degree $n-1$ in terms of a linear combination of Bernstein polynomials of degree n.

Property2.1.5 (Power Basis) [3]

Any nth degree Bernstein polynomial can be written innermost the power basis which is expressed by $\{1, t, t^2, \dots, t^n\}$.

Proof.

This can be directly computed by using the definition of the Bernstein polynomials (1) and the binomial theorem, as follows:

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k} = \binom{n}{i} t^i \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} t^k$$

$$= \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} t^{k+i}$$

$$= \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k-i} t^k$$

$$= \sum_{k=0}^{n-i} (-1)^{k-i} \binom{n}{k}\binom{k}{i} t^k$$

Property 1.2.7 (Matrix Representation) [3]

The Matrix Representation of Bernstein polynomials A matrix representation is useful for the Bernstein polynomials. The linear combination of Bernstein basis functions for a given polynomial is given by

$$B(t) = c_0B_{0,k}(t) + c_1B_{1,k}(t) + \cdots + c_kB_{k,k}(t).$$

It is easy to write this as a dot product of two vectors

$$B(t) = \begin{bmatrix} B_{0,k}(t) & B_{1,k}(t) & \cdots & B_{k,k}(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{bmatrix}$$

We can trans form this to

$$B(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^k \end{bmatrix} \begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{bmatrix}$$

Where the $b_{m,n}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials. We note that the matrix in this case in lower triangular matrix.

Example

If we want to give an example, we can give the quadratic case ($n = 2$) with the matix expression

$$B(t) = [1 \quad t \quad t^{2}]\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

Previously, we define the Bernstein polynomials in equation (1) for each positive integer. It will be shown that if f is continuous over the interval[0,1],its sequence of Bernstein polynomials converges uniformly to f over thein terval[0,1], thus giving a useful proof of Weier strass' stheorem. For the proof of Weierstrass Theorem, Bernstein composed incoming polynomials in place of the known polynomials. For instance, Taylor polynomials are not useful for all continuous functions, it can be applicable only infinitely differentiable functions.

It is clear from equation (1) that for all $n \geq 1$,.

$$B_n(f,0) = f(0) \text{ and } B_n(f,1) = f(1) \quad (3)$$

so that a Bernstein polynomial of f interpolates f at both endpoints of the interval [0,1].

Besides, from the binomial expansion it follows that

$$B_n(1\,;\,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n (4)$$

Thus, the Bernstein polynomial of the constant function 1 is also 1. In addition, the Bernstein polynomial of the function $f(t) = t$ is x. In fact, since

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$

for $1 \leq k \leq n$, the Bernstein polynomial of the function t is

$$B_n(t;x) = \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k}$$

$$= x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1}(1-x)^{n-k}$$

$$= x \sum_{S=1}^{n} \binom{n-1}{s} x^{s}(1-x)^{n-1-S} = x \quad (5)$$

The Bernstein operator B_n maps a function f, defined over the interval $[0,1]$ to $B_n f$, which is the function $B_n f$ computed at x represented by $B_n(f;x)$. The Bernstein oper ator is clearly linear, since it comes from equation (1.0.1) that

$$B_n(\lambda f + \mu g) = \lambda B_n f + \mu B_n g \qquad (6)$$

For all functions f and g defined over the interval $[0,1]$ and all real λ and μ.

B_n is a monotone operator from the equation (1), then it follows from the monotonicity of B_n and equation (2) that

$$p \leq f(x) \leq P, x \in [0,1] \Rightarrow p \leq B_n(f;x) \leq P, x \in [0,1] \qquad (7)$$

In this case, letting $p = 0$ in equation (7), we get

$$f(x) \geq 0, x \in [0,1] \Rightarrow B_n(f,x) \geq 0, x \in [0,1] (8).$$

1.3. Bernstein polynomials Derivatives

Property 1.3.1 (Derivatives Polynomial) [3]

Derivatives Polynomial of degree $n-1$ are derivatives of the nth degree of the Bernstein polynomials. Also, these derivatives can be written as a linear combination of Bernstein polynomials using the definition of Bernstein polynomials. In this case,

$$\frac{d}{dt} B_{i,n}(t) = n\,(B_{i-1,n-1}(t) - B_{i,n-1}(t))$$

Proof.

For $0 \leq i \leq n$.This can be written by direct differentiation

$$\frac{d}{dt} B_{i,n}(t) = \frac{d}{dt} \binom{n}{i} t^{i} (1-t)^{n-i}$$

$$= \frac{in!}{i!\,(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-i)n!}{i!\,(n-i)!} t^{i} (1-t)^{n-i-1}$$

$$= \frac{n(n-1)!}{(i-1)!\,(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{n(n-1)!}{i!\,(n-i-1)!} t^{i} (1-t)^{n-i-1}$$

$$= n\left(\frac{(n-1)!}{(i-1)!\,(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-1)!}{i!\,(n-i-1)!} t^{i} (1-t)^{n-i-1}\right)$$

$$= n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)$$

Theorem 1.3.2(Bernstein polynomial Derivative Formulation) [3]

The Bernstein polynomial can be written in the following form

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{k} f(0) x^{k} \qquad (9)$$

Where Δ is the forward difference operator, shown as

$$\Delta f(x_j) = f(x_{j+1}) - f(x_j) = f(x_j + h) - f(x_j) \qquad (10)$$

with step size $h = \frac{1}{n}$.

Proof.

Beginning with equation (1) and extending the term $(1-x)^{n-k}$, we have

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^{k} \sum_{s=0}^{n-k} (-1)^{s} \binom{n-k}{s} x^{s}$$

Let us put $t = k + s$. We might write

$$\sum_{k=0}^{n} \sum_{s=0}^{n-k} \cdots = \sum_{t=0}^{n} \sum_{k=0}^{t} \cdots$$

Also, we have

$$\binom{n}{k}\binom{n-k}{s} = \binom{n}{t}\binom{t}{k}$$

And so, we might write the double summation as

$$\sum_{t=0}^{n} \binom{n}{t} x^t \sum_{k=0}^{t} (-1)^{t-k} \binom{t}{k} f\left(\frac{k}{n}\right) = \sum_{t=0}^{n} \binom{n}{t} \Delta^t f(0) x^t$$

On using the expansion fora higher-order for ward difference.

Theorem 1.3.3. (Bernstein polynomial Derivative) [3]

The derivative of the Bernstein polynomial $B_{n+1}(f, x)$ can be written in the following form

$$B'_{n+1}(f, x) = (n + 1) \sum_{k=0}^{n} \Delta f\left(\frac{k}{n+1}\right) \binom{n}{k} x^k (1 - x)^{n-k} \quad (11)$$

For $n \geq 0$, where Δ is applied with step size $h = \frac{1}{(n+1)}$. Otherwise, if f is monotonically increasing or monotonically decreasing over the interval $[0, 1]$, so are all its Bernstein polynomials.

Theorem 1.3.4(Bernstein polynomial m^{th} derivative) [3]

For any integer $m \geq 0$, the m^{th} derivative of $B_{n+m}(f, x)$ can be expressed in terms of m^{th} differences of f as

$$B_{n+m}^{(m)}(f,x) = \frac{(n+m)!}{n!}\sum_{k=0}^{n} \Delta^m f\left(\frac{k}{n+m}\right)\binom{n}{k} x^k (1-x)^{n-k} \quad (12)$$

For all $n \geq 0$. Here Δ is applied with step size $h = \frac{1}{(n+k)}$

Proof:

We write

$$B_{n+m}(f,x) = \sum_{k=0}^{n+m} f\left(\frac{k}{n+m}\right)\binom{n+m}{k} x^k (1-x)^{n+m-k}$$

And differentiate m times to get

$$B_{n+m}^{(m)}(f,x) = \sum_{k=0}^{n+m} f\left(\frac{k}{n+m}\right)\binom{n+m}{k} P(x)(13)$$

where

$$P(x) = \frac{d^m}{dx^m} x^k (1-x)^{n+m-k}$$

Now, we use the Leibniz rule which is

$$\frac{d^m}{dx^m}(f(x)g(x)) = \sum_{k=0}^{m}\binom{m}{k}\frac{d^k}{dx^k} f(x)\frac{d^{m-k}}{dx^{m-k}} g(x).$$

To differentiate the product of x^k and $(1-x)^{n+m-k}$. First, we find that

$$\frac{d^s}{dx^s}x^k = \begin{cases} \frac{k!}{(k-s)!}x^{k-s}, & k-s \geq 0 \\ 0, & k-s < 0 \end{cases}$$

And

$$\frac{d^{m-s}}{dx^{m-s}}\,(1-x)^{n+m-k}$$

$$=\begin{cases}(-1)^{m-s}\dfrac{(n+m-k)!}{(n+s-k)!}\,(1-x)^{n+m-s}, & k-s\le n\\ 0, & k-s>n\end{cases}$$

Accordingly, the mth derivative of $x^{k}(1-x)^{n+m-k}$ is

$$P(x)=\sum_{s}(-1)^{m-s}\binom{m}{s}\frac{k!}{(k-s)!}\,\frac{(n+m-k)!}{(n+s-k)!}\,x^{k-s}(1-x)^{n+s-k} \quad (14)$$

Where the last summation is overalls from 0 to m, with the limitations $0\le k-s\le n$. Now, were place l with $k-s$, such that

$$\sum_{k=0}^{n+m}\sum_{s}\cdots=\sum_{t=0}^{n}\sum_{s=0}^{m}\cdots \qquad (15)$$

We also note that

$$\binom{n+m}{k}\frac{k!}{(k-s)!}\,\frac{(n+m-k)!}{(n+s-k)!}=\frac{(n+m)!}{n!}\binom{n}{k-s} \quad (16)$$

It then follows form equations that the mth derivative of $B_{n+m}(f,x)$ is

$$\frac{(n+m)!}{n!}\sum_{\iota=0}^{n}\sum_{s=0}^{n-k}(-1)^{m-s}\binom{m}{s}f\left(\frac{\iota+s}{n+m}\right)\binom{n}{\iota}x^{\iota}(1-x)^{n-\iota}$$

Finally, we note that

$$\sum_{s=0}^{n-k}(-1)^{m-s}\binom{m}{s}f\left(\frac{\iota+s}{n+m}\right)=\Delta^{m}f\left(\frac{\iota}{n+m}\right)$$

where the operator Δ is applied with step size $h=\frac{1}{n+m}$.Whence the result.

Theorem 1.3.5.[3]

Let $f \in C^m[0,1]$ for some $m \geq 0$ then

$$p \leq f^{(m)}(x) \leq P , x \in [0,1] \implies c_m \, p \leq B_n^{(m)}(f;x) \leq c_m P , \forall x \in [0,1],$$

for all $n \geq m$, where $c_0 = c_1 = 1$

$$c_m = (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots \left(1 - \frac{m-1}{n}\right), 2 \leq m \leq n$$

Proof.

The former relation in the theorem can also be seen in equation (7) if we let $m = 0$. For $m \geq 1$ we begin with equation (12) and replace n by $n-m$. Then, using

$\frac{\Delta^p f(x_0)}{h^p} = f^{(p)}(\xi)$, where $\xi \in (x_0, x_p), x_p = x_0 + ph$

with $h = \frac{1}{n}$, we write

$$\Delta^m f(\frac{k}{n}) = \frac{f^{(m)}(\xi_k)}{n^m} \qquad (17)$$

where $\frac{k}{m} < \xi_k < \frac{k+m}{n}$ .Thus

$$B_n^{(m)}(f,x) = \sum_{k=0}^{n-m} c_m f^{(m)}(\xi_k) x^k (1 - x)^{n-m-k} .$$

and the theorem follows easily from the latter equation.

1.4. Uniform Convergence of Bernstein polynomials

Theorem 1.4.1 (Convergence of Bernstein polynomials) [4]

If $f \in C[0,1]$ and $\varepsilon > 0$ is arbitrary, then there exists an integer N such that

$$| f(x) - B_n(f,x) | < \varepsilon, \ 0 \leq x \leq 1,$$

for all $n \geq N$.

Proof.

We start with the identity

$$\left(\frac{k}{n} - x\right)^2 = \left(\frac{k}{n}\right)^2 - 2\left(\frac{k}{n}\right)x + x^2$$

Multiply each term by $\binom{n}{k} x^k (1-x)^{n-k}$ and sum from $k = 0$ to n, to give

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1 - x)^{n-k}$$

$$= B_n(t^2; x) - 2xB_n(t; x) + x^2 B_n(1; x).$$

It then follows from equations (4), (5), and

$$B_n(t^2; x) = x^2 + \frac{1}{n} x(1-x)$$

that

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1 - x)^{n-k} = \frac{1}{n} x(1-x). \quad (18)$$

For any fixed $x \in [0,1]$, let us approximate the sum of the polynomials $p_{n,k}(x)$ over all values of k for which $\frac{k}{n}$ is not close to x.

To make this notation exact, we take a number $\delta > 0$ and let S_δ indicate the set of all values of k satisfying $| \frac{k}{n} - x | \geq \delta$. For such k, we have

$$\frac{1}{\delta^2} \left(\frac{k}{n} - x\right)^2 \geq 1 \qquad (19)$$

Then, using equation (19), we obtain

$$\sum_{k\,\in\,S_\delta}\binom{n}{k}x^k(1-x)^{n-k} \le \frac{1}{\delta^2}\sum_{k\,\in\,S_\delta}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}$$

The last-mentioned sum is not greater than the sum of the same expression over all k. Using equation (18), we have

$$\frac{1}{\delta^2}\sum_{k\,\in\,S_\delta}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{x(1-x)}{n\delta^2}$$

Since $0 \le x(1-x) \le \frac{1}{4}$ on $[0,1]$, we have

$$\sum_{k\,\in\,S_\delta}\binom{n}{k}x^k(1-x)^{n-k} \le \frac{1}{4n\delta^2} \qquad (20)$$

Let us write

$$\sum_{k=0}^{n}\cdots = \sum_{k\,\in\,S_\delta}\cdots + \sum_{k\,\notin\,S_\delta}\cdots$$

where the last-mentioned sum is therefore over all k such that $|\frac{k}{n}-x| < \delta$. Having separated the summation into two parts, which depend on a choice of δ that we still have to make. Now we are ready to approximate the difference between $f(x)$ and its Bernstein polynomial. Using equation (2), we have

$$f(x) - B_n(f;x) = \sum_{k=0}^{n}\left(f(x)-f\left(\frac{k}{n}\right)\right)\binom{n}{k}x^k(1-x)^{n-k}$$

And hence

$$f(x) - B_n(f;x) = \sum_{k\,\in\,S_\delta}\left(f(x)-f\left(\frac{k}{n}\right)\right)\binom{n}{k}x^k(1-x)^{n-k}$$

$$+\sum_{k\notin S_\delta}\left(f(x)-f\left(\frac{k}{n}\right)\right)\binom{n}{k}x^k(1-x)^{n-k}$$

Thus we obtain the inequality

$$|f(x)-B_n(f;x)|\le\sum_{k\in S_\delta}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}$$

$$+\sum_{k\notin S_\delta}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}.$$

Since $f\in C[0,1]$, it is bounded over [0, 1] and $|\,f(x)\,|\le M$ for some $M>0$. Hence, we can denote

$$\left|f(x)-f\left(\frac{k}{n}\right)\right|\le 2M$$

For all k and all $x\in[\,0\,,1\,]$, and thus

$$\sum_{k\in S_\delta}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}\le 2M\sum_{k\in S_\delta}\binom{n}{k}x^k(1-x)^{n-k}.$$

Using equation (2٠), we obtain

$$\sum_{k\in S_\delta}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}\le\frac{M}{2n\delta^2}\qquad(21)$$

Since f is continuous, it is also uniformly continuous over the interval $[\,0\,,1\,]$. Hence, for any choice of $\varepsilon>0$ there exists a number $\delta>0$ depending on ε and f such that

$$|\,x-x'\,|<\delta\Longrightarrow|\,f(x)-f(x')\,|<\tfrac{\varepsilon}{2}$$

for all $x,x'\in[0,1]$. Hence, for some $k\notin S_\delta$, we have

$$\sum_{k\notin S_\delta}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}<\frac{\varepsilon}{2}\sum_{k\notin S_\delta}\binom{n}{k}x^k(1-x)^{n-k}$$

$$< \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} ,$$

Using equation (2) once more, we find that

$$\sum_{k \notin S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1 - x)^{n-k} < \frac{\varepsilon}{2} \qquad (22)$$

On combining the equations (21) and (22) , we get

$$|f(x) - B_n(f; x)| < \frac{M}{2n\delta^2} + \frac{\varepsilon}{2}$$

It follows from the above that if we select $N > \frac{M}{\varepsilon\delta^2}$, then

$$|f(x) - B_n(f; x)| < \varepsilon$$

for all $n \geq N$, and hence the result.

Theorem 1.4.2. (Convergence of mth Bernstein polynomials' Derivatives) [4]

If $f \in C^m[0,1]$ for some integer $m \geq 0$, then $B_n^{(m)}(f; x)$ converges uniformly to $f^{(m)}(x)$ on $[0, 1]$.

Proof.

The case $m = 0$ holds by Theorem (14). For $m \geq 1$, we begin with the expression for $B_{n+m}^{(m)}(f; x)$ given in equation (12) and write

$$\Delta^m f\left(\frac{k}{n + m}\right) = \frac{f^{(m)}(\xi_k)}{(n + m)^t}$$

Where $\frac{k}{n+m} < \xi_k < \frac{k+m}{n+m}$

As we do computations in equation (17). We then approximate $f^{(m)}(\xi_k)$ by writing

$$f^{(m)}(\xi_k) = f^{(m)}\left(\frac{k}{n}\right) + \left(f^{(m)}(\xi_k) - f^{(m)}(\frac{k}{n})\right).$$

$$\frac{n!(n+m)^m}{(n+m)!} B_{n+m}^{(m)}(f;x) = S_1(x) + S_2(x), \qquad (23)$$

Where

$$S_1(x) = \sum_{k=0}^{n} f^{(m)}\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}$$

And

$$S_2(x) = \sum_{k=0}^{n}\left(f^{(m)}(\xi_k) - f^{(m)}\left(\frac{k}{n}\right)\right)\binom{n}{k} x^k(1-x)^{n-k}$$

In $S_2(x)$, we can ensure $| \xi_k - \frac{k}{n} | < \delta$ for all k , for any chosen $\delta > 0$, by taking n sufficiently large. Thus, given any $\varepsilon > 0$,

we can select $\delta > 0$ such that, by the uniform continuity of $f^{(m)}$, we have

$| f^{(m)}(\xi_k) - f^{(m)}\left(\frac{k}{n}\right) | < \varepsilon$ for all k.

Hence $S_2(x) \rightarrow 0$ Suniformly over [0 , 1] as $n \rightarrow \infty$.

We can justify similarly that as $n \rightarrow \infty$, and from Theorem (14) with $f^{(m)}$ in place of f, $S_1(x)$ converges uniformly to $f^{(m)}(x)$ over [0,1].

Note also that

$$\frac{n!\,(n+m)^m}{(n+m)!} \rightarrow 1 \text{ as } n \rightarrow \infty,$$

Chapter Two

BERNSTEIN POLYNOMIALS Curves

2.1. Bernstein Polynomials Curves

In the following figures, we show curves of Bernstein Polynomials for $n = 1,2,3,4,5$.

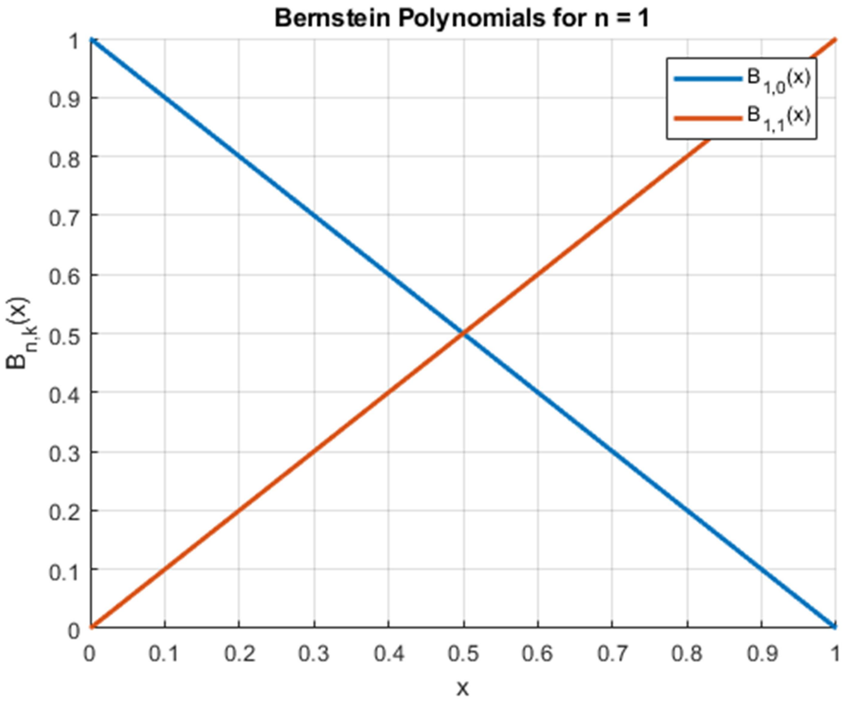

Figure2.1. Bernstein Polynomials for $n = 1$

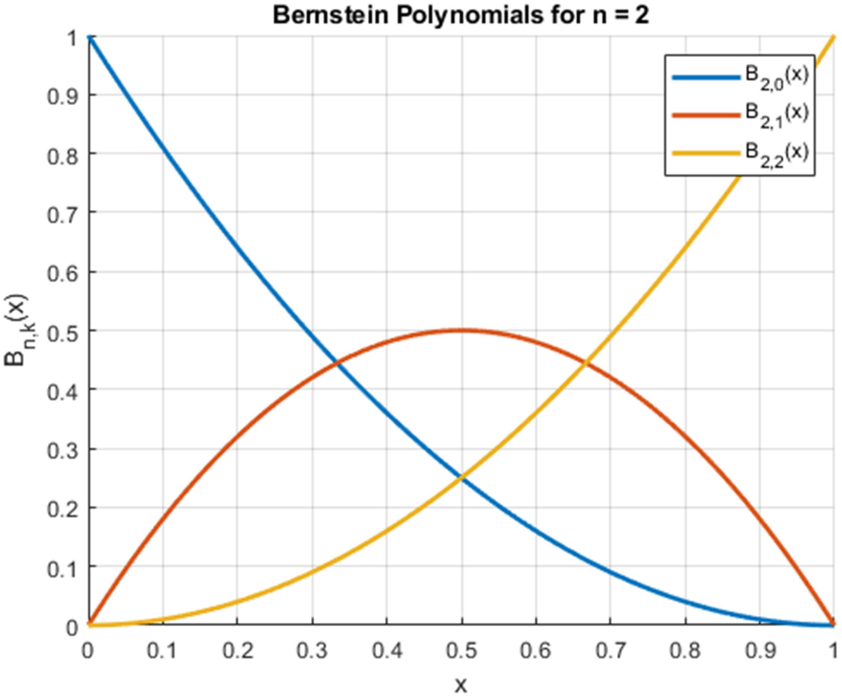

Figure2.2. Bernstein Polynomials for $n = 2$

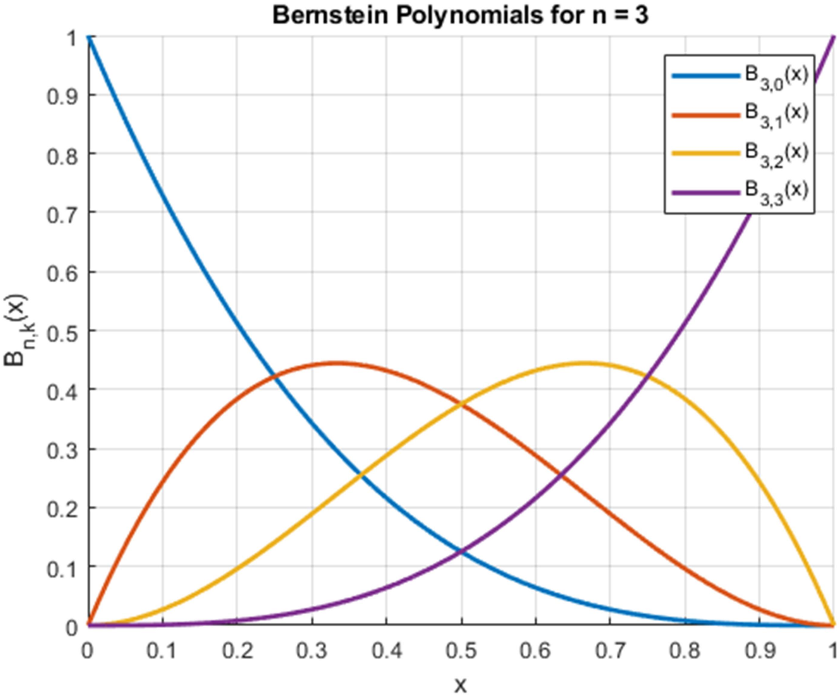

Figure2.3. Bernstein Polynomials for $n = 3$

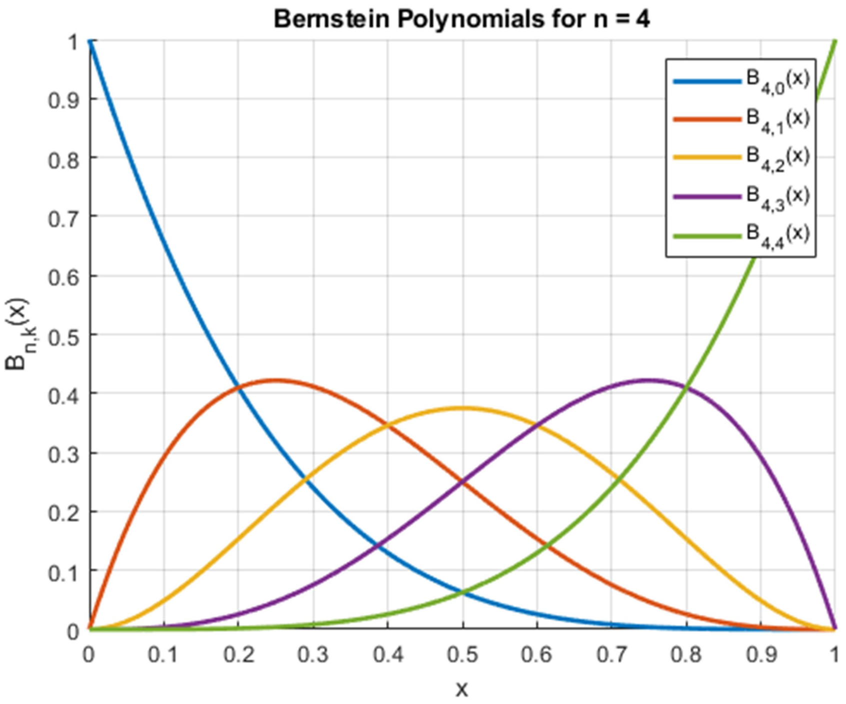

Figure2.4. Bernstein Polynomials for $n = 4$

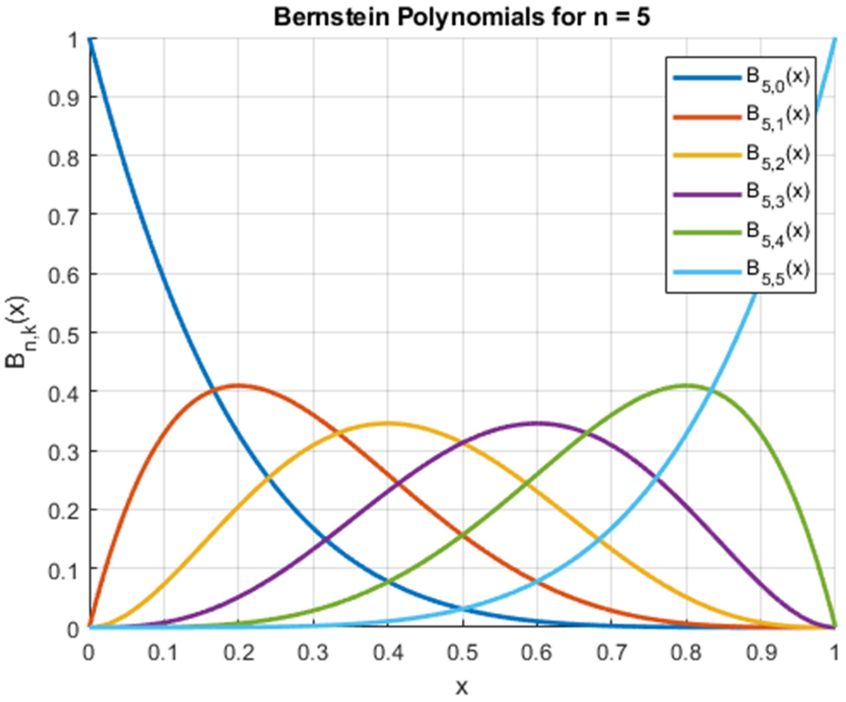

Figure2.5. Bernstein Polynomials for $n = 5$

2.2. Bernstein Polynomials Convergence

In the following figures, we show curves of Bernstein Polynomials that converges to continuous functions for $n = 3,5,10,20$.

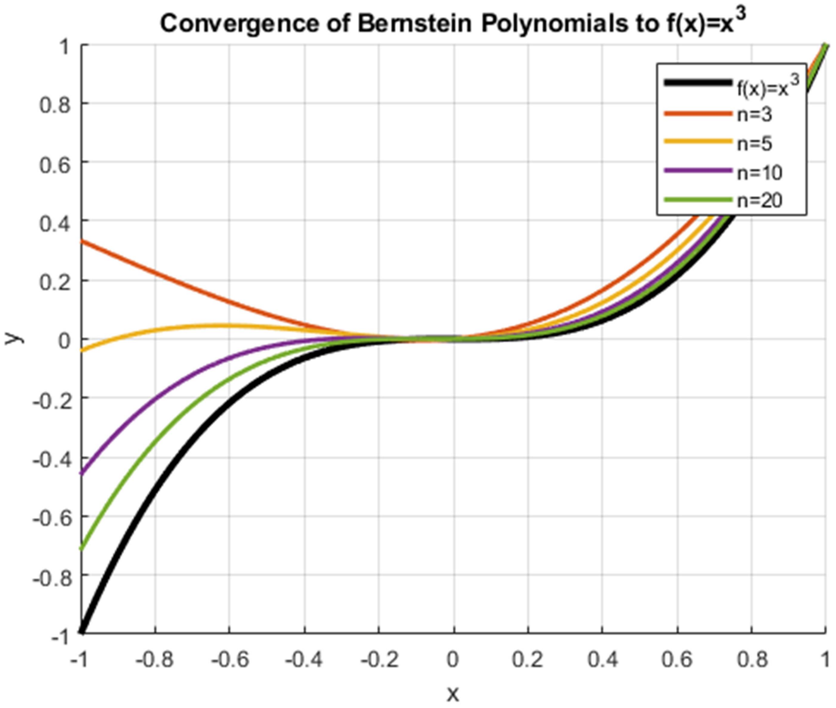

Figure2.6. Bernstein Polynomials Convergence of $f(x) = x^3$

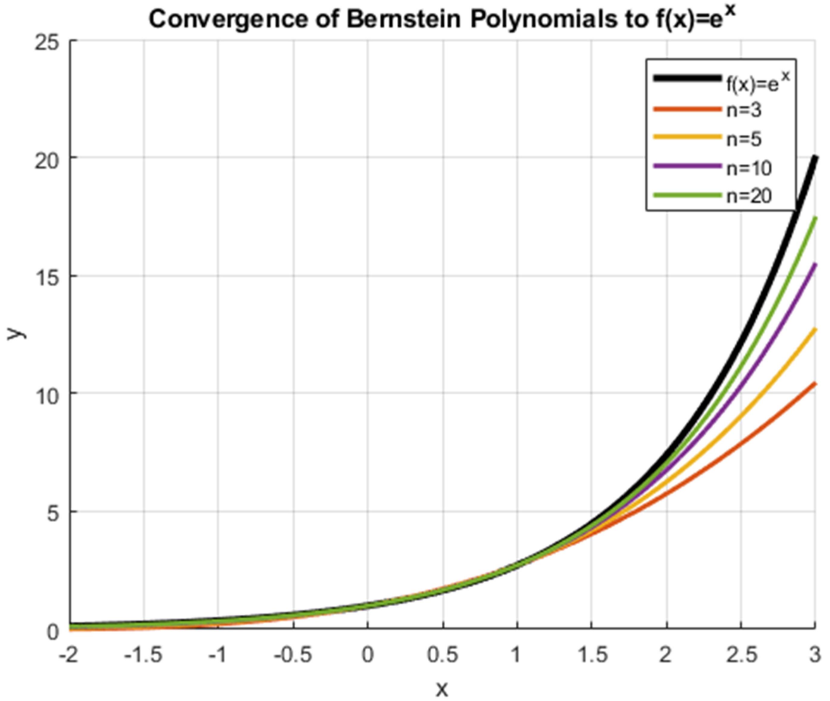

Figure2.7. Bernstein Polynomials Convergence of $f(x) = e^x$

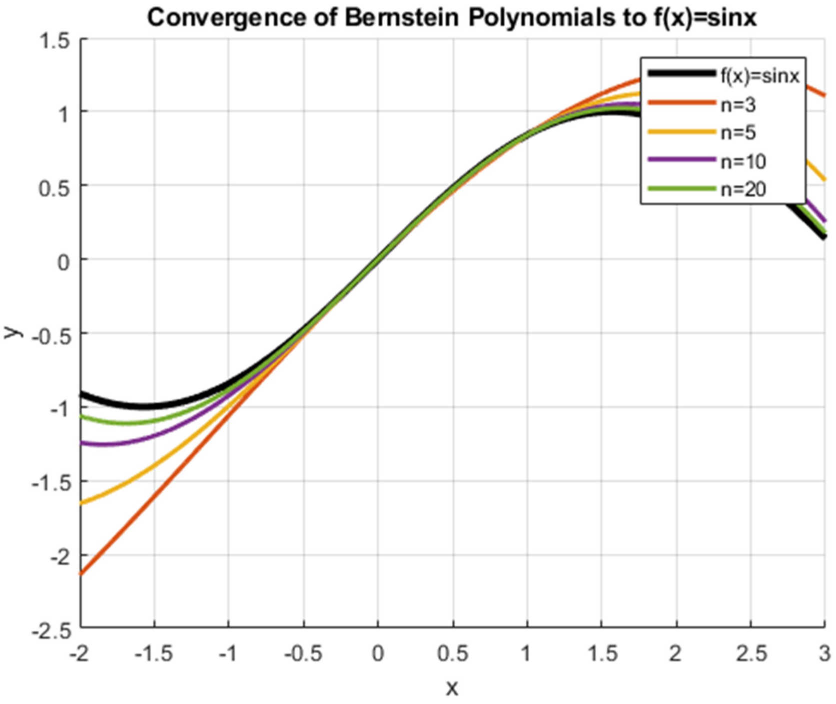

Figure2.8. Bernstein Polynomials Convergence of $f(x) = \sin x$

References

[1] S. N. Bernstein, “Demonstration du théorème de Weierstrass fondée sur le calcul des probabilités,” Communications de la Société Mathématique de Kharkov, vol. 13, pp. 1–2, 1912.

[2] K. J. Joy, “Bernstein Polynomials,” Visualization and Graphics Research Group, Department of Computer Science, University of California, Davis, 1996.

[3] R. T. Farouki, “The Bernstein Polynomial Basis: A Centennial Retrospective,” Department of Mechanical and Aerospace Engineering, University of California, Davis, CA, USA, 2012.

[4] G. M. Phillips, Interpolation and Approximation by Polynomials. New York, NY, USA: Springer, 2003.

جمهورية العراق
وزارة التعليم العالي والبحث العلم
جامعة بابل
كلية التربية للعلوم الصرفة

متعددات الحدود بيرنشتاين

بحث مُقدَّم إلى مجلس كلية التربية للعلوم الصرفة جامعة بابل كجزء من متطلبات الحصول على درجة البكالوريوس في الرياضيات

من قبل:

عذراء علاء محمد

إشراف:

د. حوراء عباس المرعب

١٤٤٧هـ
٢٠٢٦م