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كلية التربية للعلوم الصرفة / قسم الرياضيات

Spatial Functions in Fractional Calculus

البحث مقدم الى مجلس كلية التربية للعلوم الصرفة

وهو جزء من متطلبات نيل درجة البكالوريوس في الرياضيات

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يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ

صدق الله العلي العظيم

سورة المجادلة / الآية 11

شكر وتقدير

الحمد لله رب العالمين والصلاة والسلام على معلم البشرية وهادي الإنسانية وعلى اله ومن تبعهم بإحسان إلى يوم الدين ......
بعد رحلة بحث وجهد واجتهاد تكللت بإنجاز هذا البحث نحمد الله عز وجل على نعمه التي منّ بها علينا فهو العلي القدير.......
الى من زرعت التفاؤل في دربي وقدمت لي المساعدات والمعلومات لا يسعني ألا أن اخصها بأسمى عبارات الشكر والتقدير و الامتنان فهي وسام الإنسانية في حياتي " م.م. فاطمة علي عبد الحسين البياتي "
وأيضاً وفاءً وتقديراً و اعترافاً مني بالجميل أتقدم بجزيل الشكر لأولئك المخلصين الذين لم يألو جهداً في مساعدتنا في مجال البحث العلمي.
واخص بالذكر الأساتذة الأفاضل على هذه الدراسة فجزأهم الله كل خير وأن يمدهم بالصحة و الخير والتوفيق

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الي الذي زين إسمي بأجمل الألقاب، من دعمني بلا حدود وأعطاني بلا مقابل

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الي من جعل الله الجنة تحت أقدامها واحتضنني قلبها قبل يدها وسهلت لي الشدائد بدعائها الي القلب الحنون والشمعة التي كانت لي في الليالي المظلمات

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الي ضلعي الثابت وأمان ايامي الي ملهمي نجاحي الى من شددت عضدي بهم فكانوا لي ينابيع ارتوي منها الي خيرة ايامي وصفوتها الي قرة عيني (أخي و أخواتي)

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CHAPTER I

INTRODUCTION

Fractional calculus is a branch of mathematics that generalizes the ordinary calculus you might already know (like finding slopes and areas) by allowing the order of differentiation or integration to be any real or fractional number (like 0.5 or 1.3) instead of just whole numbers (like 1, 2, or 3). In simple terms, while regular calculus asks "what is the slope of a function?" (first derivative) or "what is its acceleration?" (second derivative), fractional calculus lets us ask "what is the half-derivative of a function?" The most important feature of fractional calculus is that it has a **memory** or **non-local** property, meaning the result at a given point depends not only on the function's immediate neighborhood but on its entire past or values over a wide range. This makes it incredibly useful for modeling real-world phenomena that exhibit "strange" behavior, such as the movement of pollutants through porous soil, the stretching of rubber-like materials (viscoelasticity), the spread of diseases, and even fluctuations in stock markets. In short, fractional calculus is a powerful tool that helps scientists and engineers describe complex systems where ordinary calculus falls short.

It is important to introduce some special functions. Functions like the gamma, beta, and Meteg-Leffler functions are to be used in fractional calculus.

1.1.1 Gamma Function

The Gamma function is extremely important in fractional calculus because it acts as a replacement for the ordinary factorial, which only works for whole numbers. In regular calculus, when you take multiple derivatives or integrals of a simple power function, you get factorials like three factorial or four factorial. But in fractional calculus, where you work with orders like one-half or one-third, ordinary factorials do not make sense. The Gamma function solves this problem because it gives meaningful values for fractional and real numbers, and it matches the ordinary factorial for whole numbers. You will find the Gamma function hidden inside almost every definition of a fractional derivative or integral. For example, when you define a fractional integral, you always divide by the Gamma function of the fractional order to make the definition work properly. Without the Gamma function, fractional calculus would simply be impossible to define or calculate, because there would be no way to handle the non-integer orders. In short, the Gamma function is the mathematical tool that makes fractional calculus a consistent and useful theory.

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function when z is not an integer. Particularly, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. For $z > 0$, it is defined as,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt.$$

Some noteworthy examples of the Gamma function are:

$$\Gamma(1) = 1,$$

$$\Gamma(z+1) = z\Gamma(z), \quad (1.1)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

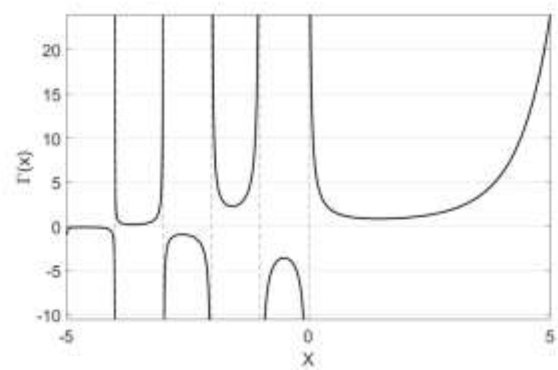

Figure 1.1: The Gamma function $\Gamma(x)$.

1.1.2 The Mittag-Leffler Function

The **Mittag-Leffler function** is the most important special function in fractional calculus because it serves as the natural generalization of the exponential function. In ordinary calculus, the exponential function is the fundamental solution to simple differential equations like the growth or decay equation. However, when we move to fractional calculus, where derivatives have fractional orders like one-half or one-third, the ordinary exponential function no longer works as a solution. Instead, the Mittag-Leffler function appears naturally as the solution to fractional differential equations. It has two parameters that give it great flexibility to model a wide range of behaviors, including slow decay (like in memory materials) or oscillations that fade over time. In fact, the Mittag-Leffler function is to fractional calculus what the exponential function is to ordinary calculus. Without it, we could not solve most fractional differential equations in closed form, and it is essential for applications such as viscoelasticity, anomalous diffusion, fractional control systems, and signal processing. In short, the Mittag-Leffler function is the backbone function that makes fractional calculus useful for describing real-world phenomena with memory and non-locality.

The Mittag-Leffler is a function that generalizes the exponential function. The function can be written as follows,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha \in \mathbb{R}^+, z \in \mathbb{C}, \qquad (1.2)$$

or more generally using two parameters,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta \in \mathbb{R}^+, z \in \mathbb{C}. \qquad (1.3)$$

Note. Let $\alpha, \beta > 0$ and $z \in \mathbb{C}$, and the Mittag-Leffler functions satisfy the equality

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}. \qquad (1.4)$$

Some interesting examples of the Mittag-Leffler function with $\alpha = 0,1,2$ are:

$$E_{\alpha,1}(z) = E_{\alpha}(z)$$

$$E_{0,1}(z) = \frac{1}{1-z};\ \text{if } |z| < 1$$

$$E_{1,1}(z) = e^z \qquad (1.5)$$

$$E_{2,1}(z) = \cosh(z) (1.5)$$

$$E_{2,2}(z) = \frac{\sinh(z)}{z} (1.5)$$

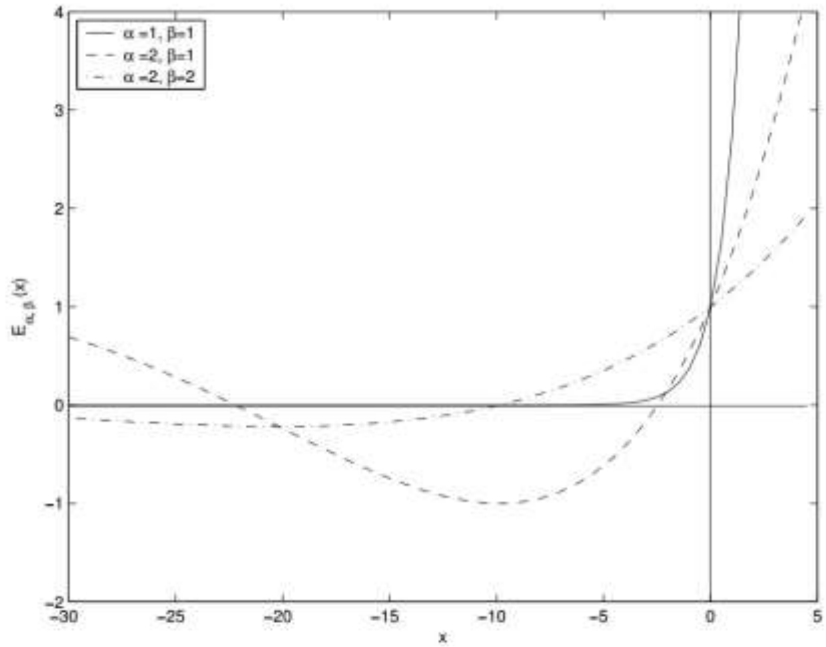

Figure 1.2: The Mittag-Leffler function at $\alpha = 1{,}2$ and $\beta = 1{,}2$

1.2 Fractional Calculus and Differential Equations

In 1675, Gottfried Leibniz achieved an amazing breakthrough by discovering differential and integral calculus. Since then, many mathematicians, namely Isaac Barrow, James Gregory, and Isaac Newton, explored this field even further. The concept of change in time through calculus has been a major part of mathematics since then. Such research opened up many fields of mathematics, like complex analysis, functional analysis, differential geometry, measure theory, and abstract algebra. In 1695 the idea of fractional calculus was sparked by L'Hopital and Leibniz; however, it

remained somewhat dormant until the past couple of decades when new fields of studies found this idea to be very useful. These two mathematicians sent letters to one another regarding the notation of differentiation of order $\frac{1}{2}$ and it's visual representation [7]. With the progression in the field of calculus, many other mathematicians have been contributors to this study like Laplace, Fourier, Abel, Liouville, Reimann, Grunwald, Letnikov, Heaviside, Weyl, Erdelyi, and even more [5]. From this, one might ask, "What makes fractional differentiation so important and different from integer order differentiation?" The answer can be seen through comparing the effect both approaches have on the study. In medicine, finances or economics, physics, engineering, or describing polymers, arbitrary-ordered derivatives are extremely useful because of their added effect of memory and hereditary properties. These properties are not evident when using integer-order differentiation.

1.2.1 The Caputo Derivative

Through modification of the Riemann-Liouville derivative, the Caputo derivative was found with the goal of forming an extension from integer differential equations to fractional differential equation without the need to define fractional initial conditions. In this section, D^n will be used as the standard integer-order differential operator.

$$D^n f = \frac{d^n f}{dt^n} = f^{(n)}$$

To show the following operators, we define J^n to be our integration operator of integer order given by,

$$J^n f(t) = \frac{1}{n!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau \qquad (1.6)$$

where $n \in Z^+$. For fraction-order integrals, we use

$$J^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha+1)}\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau \qquad (1.7)$$

where $n-1 < \alpha \leq n$.

Remark 1. *When using the integration operator for a fractional order n ∈ Q.*

Thus, the representation for the Caputo fractional differential operator can now be defined as,

$$D_*^\alpha f(t) = J^{n-\alpha}D^n f(t)$$

where $n-1 < \alpha \leq n$, for $n \in \mathbb{N}$.

We also introduce the Riemann-Liouville fractional derivative as,

$$D^\alpha f(t) = D^n J^{n-\alpha} f(t)$$

The Riemann-Liouville fractional derivative seems similar to the Caputo fractional operator but since

$$D^n J^{n-\alpha} f(t) \neq J^{n-\alpha} D^n f(t)$$

then,

$$D^\alpha f(t) \neq D_*^\alpha f(t)$$

Lemma 1 ([5]). Let $n-1 < \alpha \leq n$, for any $n \in \mathbb{N}$ and $f(t)$ be such that $D_*^\alpha f(t)$ exists. Then the following properties for the Caputo operator hold ,

$$\lim_{\alpha \to n} D_*^\alpha f(t) = f^{(n)}(t) \qquad (1.8)$$
$$\lim_{\alpha \to n-1} D_*^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0) \qquad (1.8)$$

Proof. First,

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau$$
$$= \frac{1}{\Gamma(n-\alpha)}\left(-f^{(n)}(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\bigg|_{\tau=0}^t + \int_0^t f^{(n+1)}(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}d\tau\right) (1.9)$$
$$= \frac{1}{\Gamma(n-\alpha+1)}\left(f^{(n)}(0)t^{n-\alpha} + \int_0^t f^{(n+1)}(\tau)(t-\tau)^{n-\alpha}d\tau\right) \qquad (1.9)$$

We now take the limit as α approaches n and $n-1$, respectively,

$$\lim_{\alpha \to n} D_*^\alpha f(t) = f^{(n)}(0) + \left[f^{(n)}(\tau)\right]_{\tau=0}^{t} = f^{(n)}(t) \qquad (1.10)$$

and

$$\lim_{\alpha \to n-1} D_*^\alpha f(t) = \left(f^{(n)}(0) + f^{(n)}(\tau)(t-\tau)\right)\Big|_{\tau=0}^{t} + \int_0^t f^{(n)}(\tau)d\tau$$

$$= f^{(n-1)}(\tau)\Big|_{\tau=0}^{t} \qquad (1.11)$$

$$= f^{(n-1)}(t) - f^{(n-1)}(0)(1.11)$$

Now, we present some properties of the Caputo derivative.

- Linearity of Caputo derivative

The **Caputo derivative** is one of the most popular definitions used to compute fractional derivatives, such as half-derivatives or third-derivatives, and it is especially valued because it handles initial conditions in a natural and straightforward way, similar to ordinary calculus. To calculate the Caputo fractional derivative of a function, you first take the ordinary derivative of that function an integer number of times (the smallest integer greater than the fractional order), and then you apply a fractional integral to the result. The major advantage of this definition is that when you apply the Laplace transform to a Caputo derivative, the initial conditions of the function, such as its value and the values of its ordinary derivatives at zero, appear explicitly just like they do in standard differential equations. This makes the Caputo derivative the preferred choice in practical applications such as modeling viscoelastic materials, heat diffusion in heterogeneous media, control systems, and fluid dynamics, because engineers and physicists can easily input the real initial conditions of their systems. In short, the Caputo derivative is the most convenient and practical tool for turning fractional calculus problems into solvable models that directly relate to measurable real-world conditions.

Lemma 2. Let $n-1 < \alpha \leq n$, for $n \in \mathbb{N}, c \in \mathbb{R}$ and functions $f(t)$ and $g(t)$ be such that $D_*^\alpha f(t)$ and $D_*^\alpha g(t)$ exist. Then the Caputo fractional derivative is a linear operator defined as,

$$D_*^\alpha(cf(t) + g(t)) = cD_*^\alpha f(t) + D_*^\alpha g(t)$$

- Caputo is non-commutative

Lemma 3. Let $n-1 < \alpha \leq n$, with $n, m \in \mathbb{N}$ and function $f(t)$ be such that $D_*^\alpha f(t)$ exists. Then,

$$D_*^\alpha D^m f(t) = D_*^{\alpha+m} f(t) \neq D^m D_*^\alpha f(t) \qquad (1.12)$$

1.3 Laplace Transform

The Laplace Transform is an extremely powerful tool in fractional calculus because it converts difficult fractional differential equations into much simpler algebraic equations. In fractional calculus, when you have an equation containing fractional derivatives like a half-derivative or a third-derivative, solving it directly using ordinary methods is very complicated. However, when you apply the Laplace Transform to such an equation, it transforms those fractional derivatives into simple algebraic terms that involve the Gamma function and the initial conditions of the problem, similar to how the Laplace Transform handles ordinary derivatives in traditional calculus. This allows you to solve the algebraic equation easily, and then by applying the inverse Laplace Transform, you obtain the final solution to the original fractional problem. Without the Laplace Transform, solving most fractional calculus problems—especially those involving initial conditions, control systems, or engineering physics—would be extremely difficult or nearly impossible. In short, the Laplace Transform is an indispensable bridge that turns complex fractional problems into manageable algebra.

The Laplace transform can be used to solve differential equations. It reduces a linear differential equation to an algebraic one which can then be solved by rules of algebra. The transform is written as follows:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (1.13)$$

Where as Laplace transform of the Caputo fractional derivative is given by

$$\mathcal{L}[D_*^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0)s^{\alpha-k-1} \qquad (1.14)$$

The Laplace transform of the Mittag-Leffler function is given by

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha)\right] = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda} \qquad (1.15)$$

1.3.1 Laplace Transform Examples

On Table 1.1 examples of the Laplace transform can be seen. On top of these.

$f(t)$	Df	$D_*^\alpha f$	$\mathcal{L}(f)$	$\mathcal{L}(Df)$	$\mathcal{L}(D_*^\alpha f)$
c	0	0	$\frac{c}{s}$	0	0
t	1	$\frac{t^\alpha}{\Gamma(\alpha+1)}$	$\frac{1}{s^2}$	$\frac{1}{s}$	$\frac{1}{s^{\alpha+1}}$
e^t	e^t	$t^{n-\alpha}E_{1,n-\alpha+1}(t)$	$\frac{1}{s-1}$	$\frac{1}{s-1}$	$\frac{s^{2\alpha-n+1)}}{s^\alpha-\alpha}$
t^n	nt^{n-1}	$nt^{n-\alpha}$	$n!\,s^{-n-1}$	$n!\,s^{-n}$	$n!\,s^{-n-\alpha}$
n^t	$n^t\log(n)$	$n^t\log(n)^\alpha e^{t\log(n)}$	$\frac{1}{s-\log(n)}$	$\frac{\log(n)}{s-\log(n)}$	$\frac{\log(n)^\alpha}{s-\log(n)}$

Table 1.1: Laplace transform of various functions, their derivative, and their derivative of fractional order α.

1.3.2 Inverse Laplace Transform

If we have some integrated function $F(s)$ and wish to retrace back to its pre-integrated form, $f(s)$, then we can use the inverse Laplace transform, defined as,

$$f(t) = \mathcal{L}^{-1}\{F(s);t\} := \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{st}F(s)ds, c = \mathrm{Re}(s) > c_0$$

Corollary 1. Let $n-1 < \alpha \leq n, n \in \mathbb{N}, 0 < \beta = \alpha - (n-1) \leq 1$, and function $f(t)$ be such that $D_*^\alpha f(t)$ exists. Then,

$$D_*^\alpha f(t) = D_*^\beta D^{n-1} f(t)$$

Proof. Using (1.12), substitute β for α and $n-1$ for m, then

$$D_*^\beta D^{n-1} f(t) = D_*^{\beta+n-1} f(t) = D_*^\alpha f(t)$$

Lemma 4. The Taylor series of a real function $f(t)$ which is algebraic at a real or complex value a is the power series,

$$f(t) = f(a) + f'(a)(t-a) + \frac{f''(a)}{2!}(t-a)^2 + \frac{f'''(a)}{3!}(t-a)^3 + \cdots,$$

which can be rewritten as,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(t-a)^n$$

where $f^{(n)}(a)$ denotes the nth order derivative of the function f.

Proposition 1. For $0 < \alpha \leq 1$,

$$D_*^{\alpha} f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, \text{where } k = 1,2,3 \ldots \qquad (1.16)$$

Example 1. For $f(t) = t^k$ and $k \in N$,

$$D_*^{\alpha} f(t) = D_*^{\alpha} t^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} \qquad (1.17)$$

$$= \frac{k!}{\Gamma(k-\alpha+1)} t^{k-\alpha} \qquad (1.17)$$

Example 2. Fix $\lambda \in \mathbb{R}$,

$$f(t) = e^{\lambda t}, \text{for } t \in \mathbb{R} \qquad (1.18)$$

From Proposition 1,

$$D_*^{\alpha} f(t) = D_*^{\alpha} e^{\lambda t} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} D_*^{\alpha} t^k$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} D_*^{\alpha} t^k$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{k!}{\Gamma(k-\alpha+1)} t^{k-\alpha} (1.19)$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda t)^{k-\alpha}}{\Gamma(k-\alpha+1)} t^{\alpha}$$

$$= t^{\alpha} E_{1,1-\alpha}(\lambda t)$$

Now we turn to solving fractional differential equations.

Theorem 1. Consider the initial value problem

$$D_*^\alpha x(t) = f\big(t, x(t)\big), \qquad (1.20)$$
$$x(0) = x_0.$$

Let

$$g(v, x_*(v)) = f\left(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}, x\left(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}\right)\right). \qquad (1.21)$$

Then, a solution of (1.20) is given by

$$x(t) = x_*\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \qquad (1.22)$$

where $x_*(v)$ is a solution of the integer order differential equation

$$\frac{d(x_*(v))}{dv} = g(v, x_*(v)) \qquad (1.23)$$

with initial condition

$$x_*(0) = x_0$$

Proof. Let $\tau = t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}$. Then we can rewrite (1.6)[2]

$$x(t) = x_0 + \int_0^{\frac{t^\alpha}{\Gamma(\alpha+1)}} f\left(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}, x\left(t - (t^\alpha - v\Gamma(\alpha+1))^{\frac{1}{\alpha}}\right)\right) dv$$

$$= x_0 + \int_0^{\frac{t^\alpha}{\Gamma(\alpha+1)}} g(v, x_*(v)) dv \qquad (1.24)$$

Thus, every solution of (1.23) is also a solution of the Volterra integral equation given below and vice versa

$$x_*(v) = x_0 + \int_0^v g\big(s, x_*(s)\big) ds, 0 \le v \le \frac{a^\alpha}{\Gamma(\alpha+1)} \qquad (1.25)$$

Since $0 \le \frac{t^\alpha}{\Gamma(\alpha+1)} \le \frac{a^\alpha}{\Gamma(\alpha+1)}$, the right-hand side of 1.25, is equal to $x_*\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)$. When using this method of solving a fractional differential equation, the substitution of $f(\tau)$ is used to turn the problem into the integer-order differential equation (1.23). From then we solve the problem as an ordinary differential equation and end up with (1.24). The integration from 0 to $\frac{t^{\alpha}}{\Gamma(\alpha+1)}$ is done because, from (1.22), we know the solution to the original equation is found by this final substitution. Thus, we are able to solve fractional differential equations using this method.

1.4.1 Examples of Fractional Differential Equations

Example 3. Consider the fractional order initial value problem given by,

$$D^{\frac{1}{2}}x(t) = t, \tag{1.26}$$
$$x(0) = x_0. \tag{1.26}$$

For this example,

$$g(v) = 2\sqrt{t}\Gamma\left(\frac{3}{2}\right)v - v^2\Gamma^2\left(\frac{3}{2}\right).$$

The solution of the corresponding integer order initial value problem is,

$$x_1(v) = \sqrt{t}\Gamma\left(\frac{3}{2}\right)v^2 - \frac{v^3\Gamma^2\left(\frac{3}{2}\right)}{3} + x_0$$

The solution of the given fractional order initial value problem is

$$x(t) = x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right) = \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}} + x_0$$

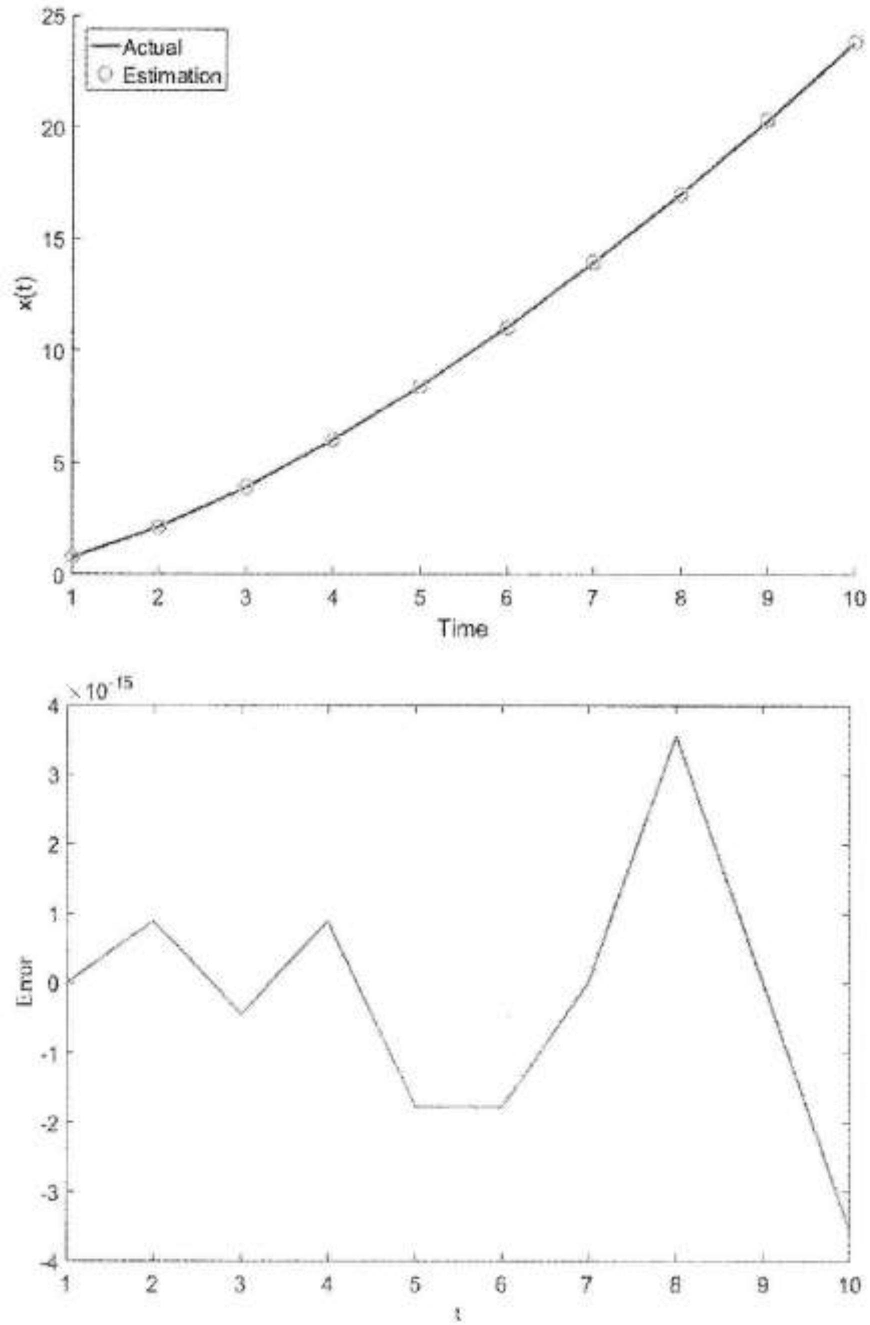

Figure 1.3: Graph of the actual solution compared to the estimate and the error.

From Figure 1.4, it can be seen that the numerical solution overlaps the estimation on the left. This graph shows just how close the numerical solution is to the actual one. Further illustration is seen on the right-side graph, where the error is shown to be extremely low. The error is calculated by subtracting the value of the estimated $x(t)$ by the actual values for each t.

Example 4. Consider the fractional order initial value problem given by,

$$D^{\frac{1}{2}}x(t) = t + x(t),$$
$$x(0) = x_0.$$

The corresponding differential equation of this fractional initial value problem is

$$\frac{dx_1(v)}{dv} = f_1(v) = x_1(v) + 2\sqrt{t}\Gamma\left(\frac{3}{2}\right)v - v^2\Gamma^2\left(\frac{3}{2}\right)$$
$$x(0) = x_0$$

Hence, the solutions of this integer order linear initial value problem is

$$x_1(v) = -2\sqrt{t}\Gamma\left(\frac{3}{2}\right)(v+1) + \Gamma^2\left(\frac{3}{2}\right)(v^2 + 2v + 2) + e^v\left(x_0 + 2\sqrt{t}\Gamma\left(\frac{3}{2}\right) - 2\Gamma^2\left(\frac{3}{2}\right)\right)$$
$$x(0) = x_0$$

Consequently, the solution of the given fractional order initial value problem is

$$x(t) = x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right) = -t + \frac{\pi}{2} + e^{\frac{2\sqrt{t}}{\sqrt{\pi}}}\left(x_0 + \sqrt{t\pi} - \frac{\pi}{2}\right)$$

Example 5. An example of nonlinear fractional differential equations describing the process of cooling of a semi-infinite body by radiation is

$$D^{\frac{1}{2}}(x(t)) - \alpha(u_0 - x(t))^4 = 0$$
$$x(0) = 0$$

The solution of this problem can be found as

$$x(t) = u_0 - \left(\frac{u_0^3\sqrt{\pi}}{6\sqrt{t} + \sqrt{\pi}}\right)^{\frac{1}{3}}$$

Example 6. Consider the fractional order initial value problem given by,

$$D_*^{\alpha}x(t) = \mu x(t) \qquad (1.27)$$
$$x(0) = 1$$

where $\mu \in \mathbb{R}$. The solution to (1.27) is given by

$$x(t) = \sum_{k=0}^{\infty} \frac{\mu^k t^{\alpha k}}{\Gamma(\alpha k + 1)} = E_{\alpha,1}(\mu t^{\alpha}) \qquad (1.28)$$

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