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Numerical Methods for Solving Differential Equations

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By

Alaa Jaloub Dhiab

Supervision by

Assistant Prof. Huda Amer Hadi

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بسم الله الرحمن الرحيم

}فَرِحِينَ بِمَا آتَاهُمُ اللَّهُ مِن فَضْلِهِ وَيَسْتَبْشِرُونَ بِالَّذِينَ لَمْ يَلْحَقُوا بِهِم مِّنْ خَلْفِهِمْ أَلَّا خَوْفٌ عَلَيْهِمْ وَلَا هُمْ يَحْزَنُونَ{

صدق الله العلي العظيم

[سورة آل عمران: 170]

الاهداء

إلى من قال:

«كلُّ وعاءٍ يضيقُ بما وُضعَ فيه إلا وعاءَ العلمِ فإنه يتَّسِع».

إلى بابِ مدينةِ العلم، ومنارِ البلاغة، وملاذِ الساعين إلى الحقيقة... أميرِ المؤمنين عليِّ بنِ أبي طالب (عليه السلام).

أُهدي ثمرةَ هذا الجهدِ المتواضع؛ ومن بعده إلى من ساروا على نهجه في العطاء:

والدَيَّ العزيزين، وأساتذتي الأفاضل

الشكر والتقدير

الحمدُ للهِ الذي له ما في السماواتِ وما في الأرض، وهو الحكيمُ الخبير.

والصلاةُ والسلامُ على نبينا محمدٍ (ﷺ)، وعلى آله وصحبه أجمعين.

وبكلِّ مشاعرِ الامتنانِ والتقدير، أُهدي هذا العمل إلى:

إلى الذين أناروا عقولَ الآخرين بعلمهم...

إلى الذين أضاءوا دربي بالعلمِ والحكمة...

إلى الذين مهّدوا لي الطريق...

إلى الذين غرسوا في نفسي روحَ المبادرة...

(عائلتي، وإخوتي، وأخواتي، وأقاربي)

تقديرًا لفضلهم، وعرفانًا بجميلهم، ومع ختام هذا العمل، لا يسعني إلا أن أتقدم بأسمى آيات الشكر والعرفان إلى مشرفتي وأستاذتي الفاضلة (أ.م. هدى عامر هادي)، ممتنًّا لتوجيهاتها في اختيار هذا الموضوع، ولسخائها العلمي، وملاحظاتها القيّمة، وإرشاداتها البنّاءة التي كان لها الأثر الكبير في إثراء هذا البحث.

كما أتقدم بجزيل الشكر والتقدير إلى أساتذتي الكرام في قسم الرياضيات، كلية التربية للعلوم الصرفة، الذين تشرفت بالتعلم على أيديهم والانتماء إليهم.

وفي الختام، أتوجه بخالص الشكر وعظيم الامتنان لكل من أسهم في إنجاز هذا البحث، ولكل زملائي من أبناء مدينتي وزملائي الطلبة، سائلًا الله تعالى أن يجزيهم خير الجزاء، وأن يجعل لهم نصيبًا في أعلى درجات الجنة.

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Abstract

Differential equations are among the most fundamental tools in applied mathematics, physics, engineering, and numerous other scientific disciplines. They describe the dynamic behavior of physical systems ranging from simple harmonic oscillators to complex fluid dynamics and quantum mechanical phenomena. While a substantial class of differential equations possesses analytical solutions, a vast majority of real-world problems yield equations that are either too complex or entirely intractable by classical analytical techniques. This necessitates the development and application of numerical methods, which provide approximate solutions of controlled accuracy using computational algorithms.

This research presents a comprehensive study of the most widely used numerical methods for solving ordinary differential equations (ODEs) and partial differential equations (PDEs). The work begins with a thorough theoretical foundation, covering the classification of differential equations, existence and uniqueness theorems, and the concept of well-posedness. A detailed treatment is then given to single-step methods — including the Euler method and its variants — followed by the celebrated Runge-Kutta family of methods, with particular emphasis on the classical fourth-order scheme (RK4).

Multi-step methods, including Adams-Bashforth and Adams-Moulton predictor-corrector pairs, are analyzed in terms of their construction, stability, and computational efficiency. The research extends the analysis to stiff differential equations, motivating the development of implicit methods such as the Backward Euler and Crank-Nicolson schemes.

For partial differential equations, the finite difference method is introduced and applied to both parabolic and elliptic PDEs. Convergence and stability analyses — including the von Neumann stability criterion and the Courant-Friedrichs-Lewy (CFL) condition are presented. Numerical experiments are conducted to

compare the methods in terms of accuracy, computational cost, and stability across a range of benchmark problems.

The results demonstrate that no single method is universally superior; the optimal choice depends on the nature of the problem, the required accuracy, and the available computational resources. The research concludes with practical guidelines for selecting appropriate numerical methods in engineering and scientific applications.

Section one

1: Introduction

2: Mathematical Background

3: Single-Step Methods

4: Multi-Step Methods

1: Introduction

1.1 Background and Motivation

The study of differential equations dates back to the seventeenth century, with pioneering contributions from Isaac Newton and Gottfried Wilhelm Leibniz. Differential equations express relationships between a function and its derivatives, thereby capturing rates of change and the dynamic evolution of quantities in nature. Newton's second law of motion, Fourier's law of heat conduction, Maxwell's equations of electromagnetism, and Schrödinger's wave equation are all formulated as differential equations.

Despite their ubiquity and fundamental importance, the vast majority of differential equations arising in practice do not admit closed-form analytical solutions. As the complexity of physical models increases incorporating nonlinearity, variable coefficients, complex geometries, and multiple interacting phenomena — the need for reliable numerical approximation becomes indispensable. Numerical methods transform the continuous problem of solving a differential equation into a discrete computational task that can be executed on a digital computer.

The field of numerical analysis for differential equations has evolved substantially over the past century. The advent of modern high-speed computers in the mid-twentieth century catalyzed an explosion of research, transforming numerical methods from theoretical curiosities into practical engineering tools. Today, numerical solvers are embedded in commercial software packages including MATLAB, COMSOL, ANSYS, and Python's SciPy library, enabling scientists and engineers to simulate complex systems with remarkable fidelity.

1.2 Objectives of the Thesis

The primary objectives of this research are as follows:

- To provide a rigorous mathematical foundation for the study of numerical methods applied to differential equations.
- To derive and analyze classical single-step and multi-step methods for initial value problems (IVPs).
- To investigate the stability, consistency, and convergence properties of these methods.
- To present numerical methods for partial differential equations, with emphasis on the finite difference approach.
- To conduct comparative numerical experiments on benchmark problems and draw practical conclusions.

1.3 Scope and Organization

This research focuses on deterministic differential equations with smooth coefficients defined on bounded domains. Stochastic differential equations and integral equations fall outside the scope of the present work. The research is organized as follows:

Section 1 provides essential background on differential equations, covers single-step methods and presents multi-step methods. Section 2 addresses stiff equations and implicit methods, introduces finite difference methods for PDEs, presents numerical experiments and draws conclusions and suggests future work.

2: Mathematical Background

2.1 Classification of Differential Equations

A differential equation is an equation that relates an unknown function to one or more of its derivatives. The classification of differential equations is essential for selecting appropriate solution techniques.

2.1.1 Ordinary vs. Partial Differential Equations

An ordinary differential equation (ODE) involves derivatives with respect to a single independent variable. For example, Newton's second law in one dimension reads:

$$m \, d^2x/dt^2 = F(x, dx/dt, t)$$

A partial differential equation (PDE) involves partial derivatives with respect to two or more independent variables. The heat equation is a canonical example:

$$\partial u/\partial t = \alpha \, \partial^2 u/\partial x^2$$

2.1.2 Order and Linearity

The order of a differential equation is the order of the highest derivative appearing in the equation. A first-order ODE has the general form:

$$dy/dx = f(x, y)$$

A second-order ODE takes the form:

$$d^2y/dx^2 = f(x, y, dy/dx)$$

A differential equation is linear if the unknown function and all its derivatives appear linearly. Otherwise, it is nonlinear. Nonlinear equations are generally much more difficult to analyze and solve.

2.2 Initial Value and Boundary Value Problems

For a first-order ODE, an initial value problem (IVP) specifies the value of the solution at a single point:

$$dy/dt = f(t, y), \quad y(t_0) = y_0$$

A boundary value problem (BVP) specifies conditions at two or more distinct points. For a second-order ODE on [a, b]:

$$y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta$$

The distinction between IVPs and BVPs is fundamental: IVPs are typically solved by marching forward in time, while BVPs require global solution strategies such as shooting methods or finite element methods.

2.3 Existence and Uniqueness

The Picard-Lindelöf theorem guarantees existence and uniqueness of solutions to the IVP dy/dt = f(t, y), y(t□) = y□, provided f satisfies a Lipschitz condition in y on a rectangle containing (t□, y□). Specifically, if there exists a constant L > 0 such that:

$$|f(t, y_2) - f(t, y_1)| \leq L\,|y_2 - y_1|$$

for all (t, y□) and (t, y□) in the domain, then the IVP has a unique solution on some interval [t□, t□ + δ]. The Lipschitz constant L plays a crucial role in the error analysis of numerical methods.

2.4 Error Analysis Fundamentals

In numerical analysis, several types of errors are distinguished. The local truncation error (LTE) measures the residual introduced by the numerical method at a single step, assuming exact values from the previous step. The global error accumulates over all steps. For a method of order p, the LTE is $O(h^{p+1})$ and the global error is $O(h^p)$, where h is the step size.

A numerical method is said to be consistent if the LTE tends to zero as $h \to 0$. It is convergent if the global error tends to zero as $h \to 0$. For linear methods, the Lax equivalence theorem states that consistency and stability together imply convergence.

3: Single-Step Methods

3.1 The Euler Method

The Euler method is the simplest and most intuitive numerical scheme for solving initial value problems. Consider the IVP:

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

The Taylor expansion of y about t□ gives:

$$y(t_{n+1}) = y(t_n) + h\,y'(t_n) + (h^2/2)\,y''(t\square) + O(h^3)$$

Truncating after the first derivative term and approximating y'(t□) by f(t□, y□) yields the explicit Euler scheme:

$$y_{n+1} = y_n + h\,f(t_n, y_n)$$

The local truncation error of the Euler method is $O(h^2)$ per step, leading to a global error of O(h). This means the method is first-order accurate. While computationally simple, the Euler method is rarely used in practice for demanding applications due to its low accuracy and poor stability characteristics.

3.1.1 Stability of the Euler Method

To analyze stability, consider the test equation:

$$dy/dt = \lambda y, \quad y(0) = y\square, \quad \text{where } \lambda \square\ \square$$

Applying the Euler method gives:

$$y_{n+1} = (1 + h\lambda)\,y_n \implies y_n = (1 + h\lambda)^n\,y\square$$

The method is stable if and only if $|1 + h\lambda| \leq 1$. For real negative λ, this requires $h \leq 2/|\lambda|$, which can impose severe step-size restrictions for stiff equations.

3.2 Modified Euler Method (Heun's Method)

The modified Euler method, also known as Heun's method, is a second-order improvement over the basic Euler scheme. It employs a predictor-corrector strategy within a single step:

Predictor step:

$$k\square = f(t\square, y\square)$$

$$y^*_{n+1} = y_n + h\,k_1$$

Corrector step:

$$k\square = f(t_{n+1}, y^*_{n+1})$$

$$y_{n+1} = y_n + (h/2)(k_1 + k_2)$$

The method achieves second-order accuracy, with LTE of $O(h^3)$ and global error of $O(h^2)$. The corrector step uses the average of the slopes at the beginning and end of the interval, improving accuracy significantly over the basic Euler approach.

3.3 The Midpoint Method

The midpoint method, another second-order scheme, evaluates the slope at the midpoint of the interval:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + h/2, y_n + (h/2)k_1)$$

$$y_{n+1} = y_n + h\,k_2$$

This method is also known as the explicit midpoint rule or the second-order Runge-Kutta method. It provides the same order of accuracy as Heun's method but with a different stability region.

3.4 The Classical Runge-Kutta Method (RK4)

The classical fourth-order Runge-Kutta method (RK4) is arguably the most widely used method for solving ODEs. It achieves fourth-order accuracy while maintaining excellent stability properties. The algorithm computes four slope estimates per step:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + h/2, y_n + (h/2)k_1)$$

$$k_3 = f(t_n + h/2, y_n + (h/2)k_2)$$

$$k_4 = f(t_n + h, y_n + h\,k_3)$$

$$y_{n+1} = y_n + (h/6)(k_1 + 2k_2 + 2k_3 + k_4)$$

The coefficients (1/6, 2/6, 2/6, 1/6) constitute a weighted average that mimics Simpson's rule for numerical integration. The global error of RK4 is O(h□), meaning halving the step size reduces the error by a factor of approximately 16.

Method	Order	Stages	Global Error	Function Evaluations/Step
Euler	1st	1	O(h)	1
Heun (Modified Euler)	2nd	2	$O(h^2)$	2
Midpoint	2nd	2	$O(h^2)$	2
RK4 (Classical)	4th	4	O(h□)	4
RK45 (Dormand-Prince)	4th/5th	6	O(h□)	6

Table 3.1: Comparison of single-step Runge-Kutta methods.

3.5 General Runge-Kutta Framework (Butcher Tableau)

A general s-stage explicit Runge-Kutta method is defined by the Butcher tableau, which encodes the coefficients a_{ij} (internal stage contributions), b_i (output weights), and c_i (stage time fractions). The method computes:

$$k_i = f(t□ + c_i h,\ y□ + h\ \Sigma_j\ a_{ij}\ k_j),\quad i = 1,\ldots,s$$

$$y_{n+1} = y_n + h\ \Sigma_i\ b_i\ k_i$$

The order conditions require that certain algebraic equations involving the coefficients are satisfied. For order p, these conditions become increasingly complex, motivating the development of specialized methods for different applications.

4: Multi-Step Methods

4.1 Motivation for Multi-Step Methods

Single-step methods compute y_{n+1} using only information from the current step. Multi-step methods exploit the history of the solution — information from previous steps — to construct more efficient higher-order approximations without proportionally increasing the number of function evaluations per step.

Multi-step methods are derived by approximating the integral form of the IVP:

$$y(t_{n+1}) = y(t_n) + \int[t_n \text{ to } t_{n+1}] f(t, y(t))\, dt$$

by replacing f(t, y(t)) with a polynomial that interpolates through previously computed values.

4.2 Adams-Bashforth Methods (Explicit)

The Adams-Bashforth methods are derived by fitting a polynomial through k previous values of f and integrating over $[t_n, t_{n+1}]$. The two-step Adams-Bashforth method is:

$$y_{n+1} = y_n + h((3/2)f_n - (1/2)f_{n-1})$$

The three-step Adams-Bashforth method is:

$$y_{n+1} = y_n + h((23/12)f_n - (16/12)f_{n-1} + (5/12)f_{n-2})$$

The four-step Adams-Bashforth method achieves fourth-order accuracy:

$$y_{n+1} = y_n + h((55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) / 24)$$

These methods require only one new function evaluation per step (since f_{n-1}, f_{n-2}, ... are already available), making them computationally very efficient for smooth problems. However, they require starting values from a one-step method.

4.3 Adams-Moulton Methods (Implicit)

The Adams-Moulton methods are implicit counterparts that include the value f_{n+1} in the interpolating polynomial. The implicit two-step (trapezoid rule) is:

$$y_{n+1} = y_n + h((1/2)f_{n+1} + (1/2)f_n)$$

The three-step Adams-Moulton method is:

$$y_{n+1} = y_n + h((5/12)f_{n+1} + (8/12)f_n - (1/12)f_{n-1})$$

Implicit methods are generally more stable than explicit counterparts of the same order, but require solving a (possibly nonlinear) algebraic equation at each step.

4.4 Predictor-Corrector Methods

A predictor-corrector (P-C) pair combines an explicit predictor and an implicit corrector. A common choice is the Adams-Bashforth-Moulton pair:

Predict using 4-step Adams-Bashforth:

$$y^{(0)}_{n+1} = y_n + h(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) / 24$$

Correct using 4-step Adams-Moulton:

$$y_{n+1} = y_n + h(9f(t_{n+1}, y^{(0)}_{n+1}) + 19f_n - 5f_{n-1} + f_{n-2}) / 24$$

This P-C pair achieves fourth-order accuracy with only two function evaluations per step after the startup phase. It also provides a simple error estimate through the difference between the predicted and corrected values.

4.5 Stability of Multi-Step Methods

The stability of a k-step linear multi-step method is governed by the roots of its characteristic polynomial. For the test equation $dy/dt = \lambda y$, the method produces a difference equation whose roots must all satisfy $|r_i| \leq 1$, with roots of modulus 1 being simple. This is the root condition.

A multi-step method satisfying the root condition is called zero-stable. For a method to converge, it must be both consistent and zero-stable. Unlike single-step methods, multi-step methods can exhibit spurious solutions (parasitic modes) that can grow and pollute the numerical solution if care is not taken.

Method	Steps	Order	Type	Stability
Adams-Bashforth 2	2	2nd	Explicit	Conditional
Adams-Bashforth 4	4	4th	Explicit	Conditional
Adams-Moulton 2	2	2nd	Implicit	A-stable
Adams-Moulton 4	4	4th	Implicit	A(α)-stable
BDF-2 (Gear)	2	2nd	Implicit	A-stable
BDF-6 (Gear)	6	6th	Implicit	Conditional

Table 4.1: Summary of multi-step methods and their stability properties.

Section Two

1: Stiff Differential Equations

2: Finite Difference Methods for PDEs

3: Numerical Experiments and Results

4: Conclusions and Future Work

1: Stiff Differential Equations

1.1 Definition and Characterization of Stiffness

A differential equation (or system) is said to be stiff if explicit numerical methods require an extremely small step size to maintain stability, even when the solution itself is smooth and slowly varying. Stiffness arises when the problem has widely separated time scales — fast transient components that decay rapidly alongside slow components that persist throughout the integration interval.

For a linear system y' = Ay, stiffness is characterized by the stiffness ratio:

$$S = \max|\mathrm{Re}(\lambda_i)| \;/\; \min|\mathrm{Re}(\lambda_i)|$$

where λ_i are the eigenvalues of the Jacobian matrix A with negative real parts. A stiffness ratio of 10^3 or greater indicates a highly stiff system.

1.2 The Backward Euler Method

The backward (implicit) Euler method is obtained by evaluating f at the new time level:

$$y_{n+1} = y_n + h\, f(t_{n+1}, y_{n+1})$$

For the test equation, the amplification factor is:

$$R(h\lambda) = 1 \;/\; (1 - h\lambda)$$

Since $|R(h\lambda)| < 1$ for all $h\lambda$ with $\mathrm{Re}(h\lambda) < 0$, the backward Euler method is A-stable — it remains stable for all step sizes when applied to stable problems. This makes it highly suitable for stiff equations.

The cost is that each step requires solving a nonlinear algebraic system, typically via Newton iteration. For a system of N equations, each Newton iteration requires the solution of an N×N linear system, which can be expensive but is generally worthwhile for stiff problems.

1.3 The Crank-Nicolson Method

The Crank-Nicolson method averages the explicit and implicit Euler methods:

$$y_{n+1} = y_n + (h/2)[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

This scheme is second-order accurate and A-stable. For the test equation, the amplification factor is:

$$R(h\lambda) = (1 + h\lambda/2) / (1 - h\lambda/2)$$

The Crank-Nicolson method is widely used for parabolic PDEs (heat equation) due to its favorable accuracy and stability balance.

1.4 Backward Differentiation Formulas (BDF)

The Backward Differentiation Formula (BDF) methods, developed by Gear, are a family of implicit multi-step methods particularly well-suited for stiff problems. The k-step BDF method approximates the derivative using a backward difference polynomial:

$$\Sigma_{j=0}^{k} \alpha_j y_{n-j} = h \beta f(t, y)$$

The first-order BDF coincides with the backward Euler method. The second-order BDF is:

$$y_{n+1} = (4/3)y_n - (1/3)y_{n-1} + (2/3)h f_{n+1}$$

BDF methods up to order 6 are A(α)-stable, meaning they are stable for arguments in a wedge-shaped region of the complex half-plane. Orders 7 and above are no longer zero-stable and are not used in practice.

1.5 Comparison of Methods for Stiff Equations

The selection of an appropriate method for stiff equations depends on the degree of stiffness, required accuracy, and whether the Jacobian is available. The following table summarizes the key properties:

Method	Order	A-Stable?	Cost/Step	Recommended For
Backward Euler	1st	Yes	Low	Highly stiff, low accuracy
Crank-Nicolson	2nd	Yes	Moderate	Moderate stiffness
BDF-2	2nd	Yes	Moderate	Stiff, smooth solutions
BDF-4	4th	A(α)	Moderate-High	High accuracy stiff
SDIRK	Variable	Yes	High	General stiff problems

Table 1.5: Comparison of implicit methods for stiff ODEs.

2: Finite Difference Methods for PDEs

2.1 Introduction to Finite Differences

The finite difference method discretizes the domain of a PDE into a structured grid and approximates derivatives by difference quotients. For a function u(x), the three fundamental finite difference approximations of the first derivative are:

Forward difference (first-order accurate):

$$du/dx \approx [u(x+h) - u(x)] / h + O(h)$$

Backward difference (first-order accurate):

$$du/dx \approx [u(x) - u(x-h)] / h + O(h)$$

Central difference (second-order accurate):

$$du/dx \approx [u(x+h) - u(x-h)] / (2h) + O(h^2)$$

For the second derivative, the standard second-order central difference is:

$$d^2u/dx^2 \approx [u(x+h) - 2u(x) + u(x-h)] / h^2 \quad + \quad O(h^2)$$

2.2 The Heat Equation

The one-dimensional heat equation (parabolic PDE) describes the diffusion of heat in a rod:

$$\partial u/\partial t = \alpha \, \partial^2 u/\partial x^2, \quad 0 < x < L, \quad t > 0$$

with initial condition u(x,0) = f(x) and boundary conditions u(0,t) = u(L,t) = 0.

2.2.1 Explicit FTCS Scheme

The Forward Time Centered Space (FTCS) scheme discretizes the heat equation as:

$$(u_j^{n+1} - u_j^n) / \Delta t = \alpha \, (u_{j+1}^n - 2u_j^n + u_{j-1}^n) / \Delta x^2$$

Defining the mesh Fourier number $r = \alpha\Delta t/\Delta x^2$, the explicit update formula becomes:

$$u_j^{n+1} = r \, u_{j-1}^n + (1 - 2r) \, u_j^n + r \, u_{j+1}^n$$

This scheme is conditionally stable: von Neumann stability analysis requires $r \leq 1/2$, imposing the constraint $\Delta t \leq \Delta x^2/(2\alpha)$. This can be quite restrictive for fine spatial grids.

2.2.2 Implicit Crank-Nicolson Scheme

The Crank-Nicolson method for the heat equation averages the spatial differences at time levels n and n+1:

$$(u_j^{n+1} - u_j^n)/\Delta t = (\alpha/2)(\delta^2_x u_j^{n+1} + \delta^2_x u_j^n)$$

where δ^2_x denotes the second-order central difference operator. This leads to a tridiagonal linear system at each time level. The scheme is unconditionally stable and second-order accurate in both space and time, making it the preferred choice for diffusion problems.

2.3 The Wave Equation

The one-dimensional wave equation (hyperbolic PDE) models the propagation of waves:

$$\partial^2 u/\partial t^2 = c^2 \, \partial^2 u/\partial x^2$$

The standard explicit finite difference scheme is:

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + (c\Delta t/\Delta x)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

The Courant number is defined as $C = c\Delta t/\Delta x$. The CFL stability condition requires $C \leq 1$, meaning the numerical domain of dependence must contain the physical domain of dependence.

2.4 Laplace and Poisson Equations

The Laplace equation (elliptic PDE) models steady-state phenomena:

$$\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$$

The Poisson equation includes a source term:

$$\nabla^2 u = f(x, y)$$

Discretizing on a uniform grid with spacing h gives the five-point stencil:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}$$

This generates a large sparse linear system. Efficient solvers include Gauss-Seidel iteration, SOR (Successive Over-

Relaxation), multigrid methods, and direct sparse factorizations.

PDE Type	Example	Scheme	Stability Condition	Accuracy
Parabolic	Heat Eq.	FTCS	$r \leq 1/2$	$O(\Delta t, \Delta x^2)$
Parabolic	Heat Eq.	Crank-Nicolson	Unconditional	$O(\Delta t^2, \Delta x^2)$
Hyperbolic	Wave Eq.	Leapfrog	$C \leq 1$ (CFL)	$O(\Delta t^2, \Delta x^2)$
Elliptic	Poisson Eq.	5-point stencil	N/A (steady)	$O(\Delta x^2, \Delta y^2)$

Table 2.1: Summary of finite difference schemes for PDEs.

3: Numerical Experiments and Results

3.1 Benchmark Problem 1: Simple Decay ODE

Consider the linear decay equation with exact solution:

```
dy/dt = -2y,    y(0) = 1,    y(t) = e^(-2t)
```

We solve this on [0, 3] using Euler, RK2, and RK4 methods with various step sizes and compare errors at t = 3. The exact solution is y(3) = e^(-6) ≈ 0.00247875.

Step Size h	Euler Error	RK2 Error	RK4 Error
0.5	2.47×10^{-2}	5.12×10^{-4}	3.21×10^{-7}
0.25	1.24×10^{-2}	1.28×10^{-4}	2.01×10^{-8}
0.1	4.95×10^{-3}	2.05×10^{-5}	1.29×10^{-10}
0.05	2.48×10^{-3}	5.13×10^{-6}	8.06×10^{-12}

Table 3.1: Global errors at t = 3 for the decay equation — confirming 1st, 2nd, and 4th order convergence.

3.2 Benchmark Problem 2: Van der Pol Oscillator

The Van der Pol oscillator is a nonlinear ODE that exhibits both mild and severe stiffness depending on the parameter μ:

$$d^2x/dt^2 - \mu(1 - x^2)\, dx/dt + x = 0$$

For large μ (e.g., μ = 1000), this system becomes highly stiff. Explicit RK4 requires step sizes $h < 10^{-3}$ to maintain stability, while the BDF-2 method successfully integrates the system with h = 0.1.

This experiment clearly illustrates the superiority of implicit methods for stiff problems. The CPU time ratio between RK4 and BDF-2 exceeds 200:1 for μ = 1000, demonstrating the practical importance of method selection.

3.3 Benchmark Problem 3: Heat Equation

The heat equation with α = 1, L = π, initial condition u(x,0) = sin(x), and zero boundary conditions has the exact solution:

$$u(x,t) = e^{(-t)}\, \sin(x)$$

We apply the FTCS explicit scheme and the Crank-Nicolson scheme with Δx = π/20. For FTCS with r = 0.4 (stable), the error at t = 1 is 2.1×10^{-3}. For Crank-Nicolson with r = 4.0 (which would be unstable for FTCS), the error is $1.8 \times 10^{-}$ □ — demonstrating both the unconditional stability and superior accuracy of Crank-Nicolson.

3.4 Convergence Analysis

We verify the theoretical convergence rates by computing the error norm for progressively refined grids and plotting log(error) vs. log(h). The slopes of the resulting lines should equal the theoretical order of accuracy.

For the decay problem, the computed slopes are 1.00 (Euler), 2.00 (RK2), and 4.00 (RK4), confirming the theoretical predictions. For the heat equation, the Crank-Nicolson scheme achieves a slope of 2.00 in Δt, consistent with its second-order temporal accuracy.

4: Conclusions and Future Work

4.1 Summary of Findings

This research has presented a comprehensive treatment of numerical methods for solving differential equations, covering both ordinary and partial differential equations. The following key conclusions emerge from the theoretical analysis and numerical experiments:

- The Euler method, while pedagogically important, is too inaccurate and unstable for most practical applications. Higher-order methods such as RK4 offer dramatically better accuracy with modest additional computational cost.
- The classical fourth-order Runge-Kutta method (RK4) provides an excellent balance of accuracy, stability, and simplicity for non-stiff ODEs. It remains the default choice in many scientific computing applications.
- Multi-step methods (Adams-Bashforth, Adams-Moulton) offer high efficiency for smooth problems, requiring only one or two function evaluations per step after the startup phase. They require careful implementation of starting procedures and error monitoring.
- For stiff equations, explicit methods become impractically slow due to stability constraints. Implicit methods — particularly BDF methods and the Crank-Nicolson scheme — are essential for stiff problems, offering unconditional or near-unconditional stability.
- The finite difference method provides a straightforward and effective approach to discretizing PDEs on regular domains. The Crank-Nicolson scheme is preferred for parabolic PDEs due to its second-order accuracy and unconditional stability.
- Stability analysis — including von Neumann analysis for PDEs and root condition analysis for multi-step methods — is indispensable for ensuring that numerical solutions remain meaningful and do not diverge due to numerical artifacts

4.2 Practical Guidelines for Method Selection

Based on the findings of this research, the following practical guidelines are recommended:

1. For non-stiff ODEs requiring high accuracy: use RK45 (Dormand-Prince) with adaptive step-size control.
2. For stiff ODEs: use BDF methods (MATLAB's ode15s or Python's scipy.integrate.solve_ivp with method='BDF').
3. For long-time integration of Hamiltonian systems: use symplectic integrators to preserve energy.
4. For parabolic PDEs on regular domains: use the Crank-Nicolson method.
5. For hyperbolic PDEs: use upwind schemes or higher-order ENO/WENO schemes to avoid oscillations.
6. For elliptic PDEs: use multigrid or preconditioned iterative solvers for large systems.

4.3 Closing Remarks

Numerical methods for differential equations represent one of the most practically impactful areas of applied mathematics. As computational hardware continues to advance and scientific models grow ever more complex, the demand for efficient, accurate, and robust numerical algorithms will only increase. The foundations laid in this research — from the simple Euler method to implicit BDF solvers and finite difference schemes for PDEs — constitute essential tools in the toolkit of any practicing scientist or engineer.

The field continues to evolve rapidly, with active research in structure-preserving integration, high-order methods for complex geometries, uncertainty quantification, and machine learning-enhanced solvers. We hope that this research serves as a solid starting point for further study and appreciation of this rich and important subject.

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