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Compactness in Metric Spaces

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(يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَ الَّذِينَ أُوتُوا الْعِلْمَ دَرَجَات)

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صدق الله العظيم

Dedication

This work is dedicated to my beloved parents, whose unconditional love, unwavering support, and constant encouragement were the foundation of my success. It is also dedicated to my family for their patience and understanding throughout my studies, to my brothers, and especially my older brother who helped me with my research, and to everyone who believed in me and inspired me to strive for excellence.

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Abstract

This research discusses the concept of compactness in metric spaces, which is considered one of the fundamental topics in topology and mathematical analysis. Compactness plays an important role in understanding the behavior of sequences and functions within metric spaces. In this study, we review the definition of compactness and its main properties, including sequential compactness and total boundedness. In addition, we explain the relationships between compactness, continuity, and convergence. Several examples are provided to clarify these concepts and demonstrate their applications in different mathematical contexts. The aim of this research is to provide a clear and comprehensive understanding of compactness and its significance in metric spaces.

List of contents

1. **Introduction**

While metrizability is the analyst's favourite topological property, compactness is surely the topologist's favourite topological property. Metric spaces have many nice properties, like being first countable, very separative, and so on, but compact spaces facilitate easy proofs. They allow you to do all the proofs you wished you could do, but never could. The definition of compactness, which we will see shortly, is quite innocuous looking. What compactness does for us is allow us to turn infinite collections of open sets into finite collections of open sets that do essentially the same thing. Compact spaces can be very large, as we will see in the next section, but in a strong sense every compact space acts like a finite space. This behaviour allows us to do a lot of hands-on, constructive proofs in compact spaces. For example, we can often take maxima and minima where in a non-compact space we would have to take suprema and infima. We will be able to intersect "all the open sets" in certain situations and end up with an open set, because finitely many open sets capture all the information in the whole collection. We will specifically prove an important result from analysis called the Heine-Borel theorem that characterizes the compact subsets of Rn. This result is so fundamental to early analysis courses that it is often given as the definition of compactness in that context.

2. Three Equivalent Definitions of Compactness

Our goal in this section is to show that three different definitions of compactness are equivalent to each other.

Definition 2.1.

Let (X,d) be a metric space and $E \subseteq X$. An *open cover* of E is a collection of open sets $(U_\alpha)_{\alpha\in I}$ (indexed by some set I) such that $E \subseteq \bigcup_{\alpha\in I} U_\alpha$. Given an open cover $(U_\alpha)_{\alpha\in I}$, a *subcover* is a subcollection $(U_\alpha)_{\alpha\in J}$ given by some $J \subseteq I$, such that $E \subseteq \bigcup_{\alpha\in J} U_\alpha$. We say that an open cover is *finite* if the index set is finite.

Definition 2.2.

We say that $E \subseteq X$ is *compact* if every open cover has a finite subcover. In other words, if $(U_\alpha)_{\alpha\in E}$ is a collection of open sets with $E \subseteq \bigcup_{\alpha\in I} U_\alpha$, then there exists a finite set $F \subseteq I$ such that $E \subseteq \bigcup_{\alpha\in F} U_\alpha$.

Definition 2.3.

We say that $E \subseteq X$ is *sequentially compact* if for every sequence $(x_n)_{n\in\mathbb{N}})$ in E, there exists a subsequence $(x_{nk})_{n\in\mathbb{N}}$ that converges to some point $x_0 \in E$.

Definition 2.4.

We say that $E \subseteq X$ is *complete* if every Cauchy sequence in E converges to some point in E.

2.1 Definition 2.5.

We say that $E \subseteq X$ is *totally bounded* if for every $r > 0$, there exist a finite set F of points in E such that $E \subseteq \bigcup_{x\in F} B(x,r)$.

Theorem 2.6.

Let (X,d) be a metric space and let E ⊆ X. Then the following are equivalent:

(a) E is compact.

(b) E is sequentially compact.

(c) E is complete and totally bounded.

In the next three subsections, we will show (a) ⇒ (c), (c) ⇒ (b), and (b) ⇒ (a), which will prove the theorem

2.1. Compact implies,Complete and Totally Bounded

Lemma 2.1.1.

Let (X,d) be a metric space and E ⊆ X. The following are equivalent:

(a) E is compact.

(b) Suppose that (Kα)α∈I is a collection of closed subsets of X. Suppose that for every finite F ⊆ I, we have

E ∩ \ Kα ≠ ∅.

α∈F

Then we have

E ∩ \ Kα ≠ ∅.

α∈I

Proof. By taking the contrapositive, condition (b) is equivalent to the following: If (Kα)α∈I is a collection of closed subsets of X and if E ∩ Tα∈I Kα = ∅, then there exists a finite F ⊆ I such that E ∩ Tα∈F Kα = ∅.

Recall that a set is open if and only if the complement is closed. Note that there is a one-to-one correspondence between collections of open sets {Uα}α∈I and collections of closed sets (Kα)α∈I, given by . Moreover, using DeMorgan's laws,

$$(\cup_{\alpha\in i=I}^{Ua})^c = \cap_{\alpha\in I}^{Ka}$$

It follows that

$E \subseteq U^{Ua}_{\alpha\in I} \Longleftrightarrow E \cap \cap^{K\alpha}_{\alpha\in I} K\alpha = \emptyset$.

Similarly, if F is a finite subset of I, then

$E \subseteq U^{Ua}_{\alpha\in F} \Longleftrightarrow E \cap \cap^{K\alpha}_{\alpha\in F} K\alpha = \emptyset$.

Therefore, if we rewrite condition (b) in terms of the collection of open sets $(U\alpha)\alpha\in I$ rather than the closed sets $(K\alpha)\alpha\in I$, it means that if $(U\alpha)\alpha\in I$ is a collection of open sets in X and if $E \subseteq \alpha\in I\ U\alpha$, then there exists a finite $F \subseteq I$ such that $E \subseteq \alpha\in F\ U\alpha$. This is exactly the definition of compactness.

Lemma 2.1.2.

If $E \subseteq X$ is compact, then E is complete.

Proof. Assume that E is a compact. To prove completeness, suppose that $(x_n)n\in\mathbb{N}$ is a Cauchy sequence in E. Define

$\gamma_n = \sup d(x_m, x_n).\ m\geq n$

Note that $\gamma_n \to 0$ as $n \to \infty$. Indeed, because $(x_n)_{n\in\mathbb{N}}$ is Cauchy, for every $E > 0$, there exists N such that

$$m, n \geq N \implies d(x_m, x_n) < \in.$$

Then if $n \geq N$, we have

$$\sup\ d(x_m, x_n) \leq \in.\ m\geq n$$

This implies that $\gamma_n \to 0$ (since $\gamma_n \geq 0$ obviously).

Now define $K_n = \{x \in X : d(x, x_n) \leq \gamma_n\}$, that is, K_n is the closed ball of radius of γ_n. We want to apply the previous lemma to the collection $(K_n)n\in\mathbb{N}$. Note that K_n is a closed set . Suppose that $F \subseteq N$ and let $m = \max F$. Then for each $n \in F$, we have $m \geq n$ and hence by definition of γ_n, we have $d(x_m, x_n) \leq \gamma_n$. By the previous lemma, since E is compact, we know that $E \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$, let

$x_0 \in E \cap \bigcap_{n\in\mathbb{N}} K_n$Then by definition of K_n, we have $d(x_n, x_0) \leq \gamma_n$ *since* $\gamma_n \to 0$, we have $x_n \to sx_0$. Therefore, every Cauchy sequence in E converges to a point in E as desired.

Lemma 2.1.3.

If E ⊆ X is compact, then E is totally bounded.

Proof. Suppose that E is compact. Let r > 0. Note that $(B(y,r))_{y\in E}$ is an open cover of E. Indeed, we know that an open ball B(x, r) is an open set , and the collection $\{B(x,r)\}_{x\in E}$ covers E because each x is contained in the corresponding ball B(x, r). By compactness, there exists a finite F ⊆ E such that $E \subseteq \bigcup_{x\in F} B(x,r)$. This means precisely that E is totally bounded.

2.2.Complete and Totally Bounded Implies Sequentially Compact

Lemma 2.2.1

Suppose that E ⊆ X is complete and totally bounded. Then E is sequentially compact.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in E, and we will show that there is a subsequence converging to some point $x_0 \in$ E.

We define infinite sets $S_k \subseteq$ N by induction on k as follows. For the base case, let S_0 = N. For the inductive step, suppose that S_{k-1} has been chosen. Because E is a totally bounded, there exists a finite set F_k such that E ⊆ $\bigcup_{y\in F_k} B(y, \frac{1}{k})$. For each n ∈ Sk−1, the pointx_n must be in one of the balls B(y, 1/k). Because S_{k-1} is infinite but there are only finitely many balls, there must be one ball that contains x_n for infinitely many values of n ∈ S_{k-1}. Let us call this ball B(y_k, 1/k) and let S_k = {n ∈ S_{k-1} : x_n ∈ B(y_k, 1/k)}. Note that S_k is infinite by construction. We also have $S_k \subseteq S_{k-1}$.

Now we choose the indices n_k for our subsequence inductively. For the base case, let n_0 = 1. For the inductive step, once n_{k-1} has been chosen, we may select $n_k \in S_K$ such that $n_k > n_{k-1}$ (because S_k is infinite). If j, k ≥ K, then we have $S_j \subseteq S_k$ and $S_k \subseteq S_K$, and hence nj, $n_k \in$ SK, which implies that x_{nj} , x_{nk} $\in$ B(y_k,1/K). But if x_{nj} , $x_{nk} \in$ B(y_k, 1/K), then

$$d(x_{nj} , x_{nk}) \leq d(x_{nj},y_k) + d(y_k, x_{nk}) \leq \frac{1}{K}+ \frac{1}{K}= \frac{2}{K} .$$

Overall, we have for natural numbers j, k, and K that

$$j, k \geq K \Rightarrow d(x_nj , x_nk) \leq \frac{2}{K}.$$

Given E > 0, we may choose K such that 2/K < E. Then

$$j, k \geq K \Rightarrow d(x_{nj}, x_{nk}) \leq \frac{2}{K} < \in.$$

This means that $(x_n)_{n\in N}$ is a Cauchy sequence. Therefore, by completeness of E, $(x_n)_{n\in N}$ converges to some $x_0 \in$ E. Therefore, $(x_n)_{n\in N}$ has a convergent subsequence as desired.

2.3. Sequentially Compact Implies Compact

Lemma 2.3.1

Suppose that E ⊆ X is sequentially compact. Then E is totally bounded.

Proof. We proceed by contrapositive. Suppose that E is not totally bounded. Then there exists an r > 0 such that no finite collection of balls of radius r will cover E. Now we construct a sequence $(x_n)_{n\in N}$ by induction. Note that E must

be nonempty, so we can choose x1 ∈ E. For the induction step, suppose that x_1, . . . , x_{n-1} have been chosen. Then the balls B(x_1, r), . . . , B(x_{n-1}, r) do not cover E, and therefore, we may choose some x_n in E \ n−1 B(xj, r).

By construction, if m > n, then x_m I∈ B(x_n, r) and hence d(x_m, x_n) ≥ r. It follows that if $(x_{nk}\ )$k∈ ℕ is a subsequence of $(x_n)_{n\in N}$, then d(x_{nj} , x_{nk}) ≥ r

. Therefore, any subsequence of $(x_n)_{n \in N}$ cannot be Cauchy and hence it cannot converge. Thus, $(x_n)_{n \in N}$ has no convergent subsequence, so E is not sequentially compact.

Lemma 2.3.2

Suppose that E ⊆ X is sequentially compact, and let (Uα)α∈I is an open cover of E. Then there exists some r > 0, such that for every x ∈ E, the ball B(x, r) is contained in one of the sets Uα.

Proof. We proceed by contradiction. Assume that X is sequentially compact but that the conclusion fails. Then for every r > 0, there exists some x ∈ E such that B(x, r) is not contained in one of the sets Uα. In particular, for each n ∈ N, there exists x_n ∈ E such that B(x_n, 1/n) is not contained in one of the sets Uα.

Because E is sequentially compact, the sequence (x_n)n∈N has a subsequence (x_{nk})k∈ ℕ which converges to some x_0 ∈ E. Because (Uα)α∈I is an open cover, there exists some index α such that x_0 ∈ Uα. Because Uα is open, there exists some r > 0 such that B(x0, r) ⊆ Uα. Since $x_{nk} \to x_0$ and $1/n_k \to 0$, we have

$$lim_{k\to\infty}\left(d_{(x_{nk},x_0)+\frac{1}{nk}}\right)=0$$

In particular, there exists some k such that

$$d(x_{nk},x_0)+\frac{1}{nk}\ <r.$$

Now if y ∈ B(xnk , 1/nk), then we have

$$d(y,x_0)\leq d(y,x_{nk})+d(x_{nk}\ ,x_0\ )\leq\frac{1}{nk}+d(x_{nk},x_0)\ \ <r.$$

and hence y ∈ B(x_0, r); this implies that

$$B(x_{nk}\ ,\ 1/n_k)\ \subseteq B(x_0,\ r)\ \subseteq U_{\alpha}.$$

However, by our choice of x_n, we know that B(x_{nk} , $1/n_k$) cannot be contained in any set Uα. This is a contradiction, so the proof is complete.

Lemma 2.3.3.

If E ⊆ X is sequentially compact, then E is compact.

Proof. Assume that E is sequentially compact. To prove compactness, suppose that (Uα)α∈I is an open cover of E. By Lemma 2.3.2., there exists an r > 0 such that for every x ∈ E, the ball B(x, r) is contained in one of the sets Uα. By Lemma 2.3.1., E is totally bounded, and hence there exists a finite set F ⊆ E such that E ⊆ $\cup_{x \in F} B(x, r)$For each x ∈ F , there exists index α_x ∈ I such that

B(x,r) ⊆ $\cup_{\alpha x}$, by our choice of r. Therefore, E⊆ $\cup_{x \in F} \cup_{\alpha x}$, and hence $(\cup_{\alpha x})_{x \in F}$ is our desired finite subcover.

2.4. Corollaries; the Heine-Borel Theorem

In the special case X = R^n, the general Theorem 2.6 reduces to the Heine-Borel theorem, which says that a subset of R^n is compact if and only if it is closed and bounded. The point is that the conditions of completeness and total boundedness can be expressed in a simpler way if E is a subset of R^n. We now explain how to deduce this special case, as well as making other general observations.

Lemma 2.4.1.

Let X be a metric space and E ⊆ X. If E is complete, then E is closed. The converse holds if X is complete.

Proof. For the first claim, suppose that E is complete. To show that E is closed, suppose that $(x_n)_{n \in N}$ is a sequence in E that converges in X to some point x0 ∈ X, and we will show x0 ∈ E. Since $(x_n)_{n \in N}$ is convergent in X, it is a Cauchy sequence. By assumption $(x_n)_{n \in N}$ must converge to some point x in E. The limit of a sequence in X is unique and hence $x = x_0$. Thus, $x_0 = x \in E$

as desired. For the second claim, suppose that X is complete and E is closed. To show that E is complete, suppose that $(x_n)_{n \in N}$ is a Cauchy sequence in E and we will show that x_n is convergent in E. Then (xn)n∈N is a Cauchy sequence in X. Therefore, it converges to some $x_0 \in X$. Because E is closed and $x_n \to x_0$, we know $x_0 \in E$. Therefore, $x_n \to x_0$ in E.

Lemma 2.4.2.

Let X be a metric space and E ⊆ X. If E is totally bounded, then E is bounded. The converse holds if X = R^d.

Proof. Since E is totally bounded, there exist finitely many balls of radius 1 that cover E; call them B(x_1, 1), . . . , B(x_n, 1). Pick some point $x_0 \in E$ and let

R = max(d(x_0, x_j)) + 1. Then for each x ∈ E, we have x ∈ B(x_j, 1) for some j and by the triangle inequality d(x, x_0) < R. Therefore, E ⊆ B(x_0, R), so E is bounded.

For the second claim, suppose that E is a bounded subset of R^d. Let r > 0. Because E is bounded, there exists some M > 0 such that E ⊆ $[-M, M]^d$. For each n ∈ N, we may subdivide $[-M, M]^d$ in a grid-like fashion into (2n)d (d-dimensional) closed cubes of side length M/n. Each such cube is contained

in an open ball of radius d1/2(M/n) about the center point of the cube. Thus, if we choose n large enough, that Md1/2/n < r, then we have covered E by finitely many balls of radius of r.

Theorem 2.4.3

A subset of R^d is compact if and only if it is closed and bounded.

Proof. Let E ⊆ R^d. We know that E is compact if and only if it is complete and totally bounded. Because R^n is complete, we know that E is complete if and only if it is closed. Moreover, by the previous lemma, E is totally bounded if and only if it is bounded. Hence, E is compact if and only if it is closed and bounded. Remark. For a general metric space, a compact set must be closed and bounded, but the converse is not true. The next results allow us to test when E is compact.

Lemma 2.4.4

Let X be a metric space and E ⊆ X. If E is totally bounded, then E is totally bounded.

Proof. Let r > 0. Since E is totally bounded, E can be covered by finitely many balls (B(x, r/2)) x∈F . Let B(x, r/2) be the closed ball of radius r/2, and recall that this is a closed set. Since a finite union of closed sets is closed, we know B(x, r/2) is closed. Now since E ⊆ B(x, r/2), which is closed, we know that E ⊆ B(x, r/2). Clearly, B(x, r/2) ⊆ B(x, r) and hence E ⊆ x∈F B(x, r). Thus, E is totally bounded.

Lemma 2.4.5.

Suppose that X is a complete metric space. If E ⊆ X is totally bounded, then E is compact.

Proof. By the previous lemma, E is totally bounded. Also, since X is complete and E is closed, we know E is complete. Thus, E is complete and totally bounded, hence compact

3 . Consequences of Compactness

In this section, we prove various well-known and useful consequences of compactness. We present two proofs of each result, one using open covers and one using sequences. By examining these parallel proofs, we hope the reader will get a better intuitive grasp on why compactness and sequential compactness are equivalent.

3.1 Closed Subsets of Compact Sets

Proposition 3.1.1

Suppose that X is a compact metric space and $E \subseteq X$ is closed. Then E is compact. Covering proof. Suppose that $(U_a)_{)\alpha\in I}$ is an open cover of E. Since E is closed $E^c = X \setminus E$ is open. Thus, $(U_a)_{\alpha\in I} \cup \{E^c\}$ is an open cover of X, since $X = E \cup E^c \subseteq$ αЄI $\cup_{\alpha\in I}$ ∪ Ec. Since X is compact, there exists a finite subcover of X. This finite subcover will certainly cover E. If this finite subcover includes the E^c, we may delete E^C from it and the remaining sets will still cover E. Thus, $(U_a)_{\alpha\in I}$ contains a finite subcover of E. Sequential proof. Suppose $(x_n)_{n\in N}$ is a sequence in E. Then $(x_n)_{n\in N}$ is also a sequence in X, and by compactness of X, there is a subsequence $(x_{nk})_{k\in N}$ that converges to some $x_0 \in E$. Because E is closed, we know that $x_0 \in$ E. Therefore, $(x_n)_{n\in N}$ has a subsequence that is convergent in E. Because $(x_n)_{n\in N}$ was an arbitrary sequence in E, we know that E is compact.

3.2. Images Under Continuous Functions

Proposition 3.2.1.

If X and Y are metric spaces, $f : X \to Y$ is continuous, and $E \subseteq X$ is compact, then f(E) is compact.

Covering proof. Suppose that $(U_a)_{\alpha\in I}$ is an open cover of $f\ (E)$. Because f is continuous, we know that $f^{-1}(U_a)$ is open in X. Moreover, $(U_a)_{\alpha\in E}$ covers E, because $if\ x \in E$, then $f\ (x)$ is in one of the U_a's, and hence x ∈ $f^{-1}(U_a)$. Because E is compact, there exists a finite set $F \subseteq I$ such that $E \subseteq \alpha \in F$ $f^{-1}(U_a)$. Then we claim that (U_a)α∈F cover f (E). Indeed, if y ∈ f (E), then y = f (x) for some x ∈ E. By assumption x is contained in f−1(U_a) for some α ∈ F . But that means that f (x) ∈ Uα, that is, y ∈ U_a. Therefore, (Uα)α∈F is the desired finite subcover. Sequential proof. Suppose that (y_n)n∈ℕ is a sequence in f (E). Then by def-inition of f (E), we have y_n = f (x_n) for some x_n ∈ E. Because E is com-pact, the sequence $(x_n)_{n\in N}$ has a subsequence $(x_{nk})_{k\in N}$ that converges to some point x_0 ∈ E. By continuity of f , we have f (x_{nk}) → f (x_0), which means that $y_{nk} \to f\ (x_0) \in$ f (E). Thus, $(y_n)_{n\in N}$ has a convergent subsequence as desired.

Proposition 3.2.2.

Let X be a metric space and $f : X \to R$ be continuous. Then f achieves a maximum and a minimum on X and hence is a bounded function.

Proof from the previous result. Because f is continuous and X is compact, the previous proposition shows that $f\ (X)$ is a compact subset of R. So $f(X)$ is closed and bounded. Since $f\ (X)$ is bounded, it has a finite supremum and infimum. Because $f\ (X)$ is closed, the supremum and infimum must be in the set $f\ (X)$, and hence they are achieved by the function f . Direct sequential proof. By basic properties of the supremum, there exists a sequence $(x_n)_{n\in N}$ such that $f\ (x_n) \to sup_{x\in X}\, f\ (x)$. By compactness, the sequence $(x_n)_{n\in N}$ has a subsequence $(x_{nk}$)k∈ℕ that converges to some x_0 ∈ X. Then $f\ (x_{nk}) \to f$ (x_0). Hence, $f\ (x_0) = sup_{x\in X}\, f\ (x)$, so f achieves a maximum. Similarly, f achieves a minimum.

3.3. Continuity Implies Uniform Continuity

Proposition 3.3.1

Suppose that X and Y are metric spaces, X is compact, and

$f : X \to Y$ is continuous. Then f is uniformly continuous.

Covering Proof. To prove uniform continuity, choose some E > 0. Then for each ξ ∈ X, there exists δξ > 0 such that

$$d(x,\xi) < \delta \implies d(f(x), f(\xi)) < \in/2.$$

Note that the balls (BX (ξ, δξ/2))ξ∈X are an open cover of X. By compact-ness, there exists a finite set F ⊆ X such that X = ξ∈F BX (ξ, δξ). Let δ = min ξ∈F δξ/2.

Suppose that x, xt ∈ X with d(x, x_t) < δ. Then x must be in BX (ξ, δξ/2) for some ξ ∈ F . Hence,

$$d(x,\xi) < \delta\xi/2 < \delta\xi$$

and

$$d(xt,\xi) \leq d(xt,x) + d(x,\xi) < \delta + \delta\xi/2 \leq \delta\xi.$$

Therefore, by our choice of δξ, we have d(f (x), f (ξ)) < E/2 and d(f (x'), f (ξ)) < E/2. So by the triangle inequality, d(f (x_t), f (x)) < E.

Overall, we have obtained a δ such that d(x, x_t) < δ implies d(f (x), f (x')) < €. Thus, f is uniformly continuous.

Sequential proof. Suppose for the sake of contradiction that f is not uniformly continuous. That means that there exists an € > 0 such that for every δ > 0, there exist x and x'- such that d(x, x') < δ but d(f (x), f (x')) ≥ €. In particular, for each n ∈ N, we may take δ = 1/n, and thus there exist x_n and x_n ‘ such that

d(x_n, x'_n) < 1/n but d(f (x_n), f (x'_n)) ≥ €.

By compactness, the sequence $(x_n)n∈ℕ$ has a subsequence $(x_{nk})k∈ ℕ$ that converges to a point x_0 ∈ X. Note that

$$d(x_{nk} , x_0) \leq d(x'_{nk} , x_{nk}) + d(x_{nk}, x_0) \leq \frac{1}{n_k} + d(x_{nk} , x_0) \rightarrow 0.$$

Therefore, $(x'_{nk})k∈N$ also converges to x_0. Because f is continuous we have

$$f(x_{nk}) \rightarrow f(x_0), \quad f(x'_{nk}) \rightarrow f(x_0).$$

Since the distance function is continuous, it follows that

$$d(f(x_{nk}), f(x'_{nk})) \rightarrow d(f(x_0), f(x_0)) = 0.$$

But this contradicts the fact that $d(f(x_{nk}), f(x'_{nk})) \geq$ € for all k.

3.4. Uniform Convergence of Monotone Sequences

Proposition 3.4.1

Let X be a compact metric space. Suppose that f_n: X → [0, +∞) is continuous, $f_{n+1} \leq f_n$, and $f_n \rightarrow 0$ pointwise. Then $f_n \rightarrow 0$ uni-formly. Covering proof. Fix € > 0. Let U_n= {x ∈ X : $f_n(x)$ < €}. Then $U_n= f_n^{-1}((-\infty,€))$ and hence Un is open. Note that $(U_n)_{n∈N}$ is Note that $(U_n)_{n∈N}$ is an open cover of X; indeed, if $x \in X$, then $f_n(x) \rightarrow 0$ and hence $f_n(x)$ < E for sufficiently large n, which means that $x \in U_n$.By compactness $(U_n)_{n∈N}$ has a finite subcover, so there exists $F \subseteq N$ finite with $X = n \in F$ Un. Now becauuse

$$f_n + 1 \leq f_n, we\ have\ Un + 1 \subseteq Un. Thus, if$$

$f_N(x)$ < E for all x. Thus, $f_n \rightarrow 0$ uniformly.

Sequential proof. Suppose for contradiction that f_n does not converge uniformly to zero. Then there exists an $\epsilon > 0$ such that for every N , there exists n ≥ N and x ∈ X such that f_n (x) ≥ E. Now given N , we know there exists n ≥ N and x ∈ X such that $f_n(x) \geq \epsilon$, but we also have $f_N(x) \geq f_n(x)$. Thus, for every N , there exists x_N such that $f_N(x_N) \geq \epsilon$.

By compactness, the sequence $(x_n)_{n\in N}$ must have a subsequence $(x_{nk})_{k\in N}$ that converges to some $x_n \in X$. Now since limn→∞ $f_n(x_0) = 0$, there exists N such that $f_n(x_0)) < \epsilon$ for n ≥ N and in particular, $f_N(x_0) < E$. Because f_N is continuous, we have

$$lim\, f_N(x_{nk}) = f_N(x_0) < \epsilon.$$

In particular, for sufficiently large k, we have $f_N(x_{nk}) < \epsilon$. But if k is sufficiently large, we also have nk ≥ N and hence

$$f_{nk}(x_{nk}) \leq f_N(x_{nk}) < \epsilon$$

which contradicts our choice of $(x_n)_{n\in N}$.

4. The Arzela-Ascoli Theorem

4.1. The Space of Continuous Functions

Let X be a compact metric space. Let C(X; R) denote the space of continuous functions X → R. For f ∈ C(X; R), denote

$$||f||_\infty\, sup_{x\in X}|f(x)|.$$

Because X is compact, a continuous function must achieve a maximum and a minimum and therefore $||f||_\infty$ is always finite. One can check that $||.||_\infty$ satisfies the following axioms:

$$1. ||f||_\infty = 0 \text{ if and only if } f = 0 .$$

$$2. ||cf||_\infty = |c|\, ||f||_\infty \text{ for } c \in R .$$

$$3. ||f + g||_\infty \leq ||f||_\infty + ||g||_\infty .$$

In other words, (C(X; R), $||.||_\infty$) is a normed vector space. It follows from these axioms that

$$d_\infty(f,g) := ||f - g||_\infty$$

defines a metric on C(X; R).

Lemma 4.1.1.

C(X; R) is complete with respect to the metric d∞.

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For each x ∈ X and for each m, n ∈ N, we have

$$|f_m(x) - f_n(x) \leq d_\infty(f,g).$$

It follows that for each x ∈ X, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Because R is complete, this sequence converges. We define $f(x) = lim_{n\to\infty} f_n(x)$.

By construction $f_n$converges pointwise to f, but we claim that in fact f_n converges uniformly to f. Suppose that ϵ > 0. Then there exists N such that

$$m,n \geq N \;\; \rightarrow\Longrightarrow d_\infty(f_m, f_n) \leq \epsilon$$

In particular, if m, n ≥ N , then for each x ∈ X, we have

$$|f_m(x) - f_n(x)| \leq \epsilon.$$

We know that $f_m(x) \to f(x) as\ m \to \infty$and hence for n ≥ N and x ∈ X,

$$|f(x) - \; f_n(x) \leq \epsilon$$

Since this holds for all x, we may take the supremum over X and hence

$$n \geq N \;\; \Longrightarrow ||f_n - f||_\infty \leq \epsilon$$

Since ϵ was arbitrary, we have shown that $f_n \to f$uniformly.Next, we show that $f$is continuous. Suppose that x_0∈ X and ϵ > 0. By uniform convergence, there exists N such that n ≥ N implies that$||f_n - f||_\infty \leq \epsilon/3$. In particular,$||f_N - f||_\infty \leq \epsilon/3$. Because $f_N$is continuous, there exists δ > 0 such that

$$|x - x_0| < \delta \Longrightarrow |f_N(x) - f_N(x_0)| < \epsilon/3$$

Therefore, if $|x - x_0|< \delta$, we have

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Therefore, f is continuous. We have shown that d∞(f_n, f) → 0 and hence our Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ converges in (C(X; R), d∞).

4.2. Proof of Arzela-Ascoli

Let X be a compact metric space. When is a set $\varepsilon \subseteq$ C(X; R) compact? Recall that a set is compact if and only if it is complete and totally bounded. We also know that C(X; R) is complete, and henceε is complete if and only if it is closed in d_∞. Thus, we are left with the question of when a set $\varepsilon\subseteq$ C(X; R) is totally bounded. The next theorem will answer this question

Definition 4.2.1.

Let $\varepsilon\subseteq$ C(X; R). Then we say ε is pointwise bounded if for every x ∈ X, the set $\{f(x): f \in \varepsilon \}$ is a bounded subset of R, or equivalently,

$$sup_{f\epsilon F}|f(x)| < +\infty$$

Definition 4.2.2

Let $\varepsilon \subseteq$ C(X; R). Then we say that ε is equicontinuous if for every x_0 ∈ X and ε > 0, there exists δ > 0 such that

$$d(x, x_0) < \delta \Longrightarrow |f(x) - f(x_0)| for\ all\ x\ \in and\ f \in\ \varepsilon$$

Remark. The point of equicontinuity is that the same δ will work for all the functions in the set ε .

Theorem 4.2.3

Let X be a compact metric space, and let $\varepsilon \subseteq$ C(X; R). Then ε is totally bounded in d_∞ if and only if it is equicontinuous and pointwise bounded .

Proof. (⇒) Suppose $\varepsilon$is totally bounded. Then it must be bounded with respect to d∞. Therefore, there exists some $f_0 \in \varepsilon$ and R > 0 such that E ⊆

Bd∞ (f_0, R). Let M = $||f_0||_\infty$+ R. Then we have $||f||_\infty \leq$ M for all $f \in$ E . In particular, $|f\ (x)| \leq$ M for all $f \in \varepsilon$ and x ∈ X, and hence the set

$\{f\ (x) : f\ \in\ E\} \subseteq\ R$ is bounded for each $x\ \in\ X$. (This proof in fact shows $\varepsilon$is uniformly bounded since the M does not depend on x.) Next, let us show that ε is equicontinuous. Let $x_0\ \in\ X$ and $\varepsilon\ >\ 0$. By assumption, ε can be covered by finitely many balls of radius E/3; call them Bd∞ (f_1, ε/3), . . . , Bd∞ (f_n, ε/3). For each fk, there is a δ_k such that

$$d(x, x_0) < \delta_k implies\ |f_k(x) - f_k(x_0)| < \frac{\epsilon}{3}\ forall\ x\ \in X\ .$$

Let δ = min(δ_1, . . . , δn). Every $f \in \varepsilon$ is contained in some ball $B_{d\infty}$ (f_k, ϵ/3), which implies $\left||f_k - f|\right|$< E/3. If d(x, x_0) < δ ≤ δ_k, then we have

$$|f(x) - f(x_0)| \leq f(x) - f_k(x)| + |f_k(x) - f_k(x_0)| + |f_k(x_0) - f(x_0)|$$
$$\leq ||f - f_k||_\infty + |f_k(x) - f_k(x_0)| + ||f - f_k||_\infty < \epsilon$$

Since the same δ works for all f , we have equicontinuity.

4.3. Variants and Corollaries

Corollary 4.3.1.

Suppose that X is a compact metric space and $\varepsilon \subseteq$ C(X; R). Then ε is compact if and only if it is closed (in d_∞), equicontinuous, and point-wise bounded. Proof. As remarked earlier, C(X; R) is complete. Hence, ε is complete if and only if it is closed. Theorem 3.4 shows that ε is totally bounded if and only if it is equicontinuous and pointwise bounded.

Corollary 4.3.2.

Suppose that X is a compact metric space and $\varepsilon \subseteq$ C(X; R). If E is equicontinuous and pointwise bounded, then ε is compact.

Proof. By Theorem 4.2.3, we know that ε is totally bounded. Since C(X; R) is complete, Lemma 2.4.5 shows that ε is compact.

Corollary 4.3.3.

Let X be a compact metric space. If (f_n)n∈N is a sequence in C(X; R) which is equicontinuous and pointwise bounded, then there exists a uniformly convergent subsequence.

Proof. Let $\varepsilon = \{f_n$: n ∈ N} ⊆ C(X; R). The sequence being equicontinuous and pointwise bounded means exactly that the set ε is equicontinuous and pointwise bounded. By the previous Corollary, E is compact in d∞. Since (f_n)n∈N is a sequence in ε, it must have a convergent subsequence with respect to d_∞.

Corollary 4.3.4.

Let X be a compact metric space. If (f_n)n∈N is a sequence in C(X; R) that converges uniformly, then $(f_n)_{n\in\mathbb{N}}$ is equicontinuous and pointwise bounded.

Proof. Let f be the limit of the sequence fn. Let $\varepsilon = \{f_n$: n ∈ N} ∪ {f}. Then ε is totally bounded in d∞. Indeed, given r > 0, there exists N such that n ≥ N implies $d_\infty(f_n, f) < r$. Therefore

$$\varepsilon \subseteq B_{d\infty}(f, r) \cup \cup_{j=1}^{N-1} B_{d\infty}(f_j, r).$$

Thus, Ɛ is totally bounded. Hence, by Theorem 4.2.3, Ɛ is equicontinuous and pointwise bounded.

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