Ministry of Higher Education & Scientific Research

University of Babylon

College of Education for pure sciences Mathematics department

Fibonacci Numbers

Submitted by

Rafal Ibrahim Rahi

Supervised by

Asst. Prof. Dr. Ruma Kareem Ajeena

1444H-2023M

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ﴾

صدق الله العلي العظيم

سورة المجادلة: آية/١١

Dedications

وصلت رحلتي الجامعية إلى نهايتها بعد تعب ومشقَّة، وها أنا ذا أختم بحث تخرُّجي بكل همَّة ونشاط، وأمتنُّ لكل من كان له فضل في مسيرتي، وساعدني ولو باليسير، الأبوين، والأهل، والأصدقاء، والأساتذة المُبجَّلين

أُهديكم بحث تخرُّجي

Acknowledgments

History of all great works testify that no great work was ever done without either active or passive support of a person's surrounding and one's close guarters Thus it is not hard to conclude how active assistance from seniors could positively impact the execution of a project Iam highly thankful to our learned faculty Ms. Ruma for his active guidance throughout the completion of the project. Last but not least, I would also like to extend my appreciation to those who could not be mentioned nere but have well played their role to inspire me behind there curtain.

Abstract

In this work, the Rabbits, Fibonacci and Lucas numbers have been studied. some examples of these numbers. have presented. Our work focuses on the Fibonacci numbers. The Fibonacci quadratic equation is also studied. The roots of this equation are determined.

On the other hand, some concepts. of Fibonacci algebra are discussed. The Fibonacci numbers can be expressed by Binet form. Many applications can be proposed based on Fibonacci numbers. One of them, it can use them to design encryption schemes.

List of Contents

Chapter one

1.1 Introduction

Who was Fibonacci?

Leonardo Fibonacci, mathematical innovator of the thirteenth century, was a solitary flame of mathematical genius during the Middle Ages. He was born in Pisa, Italy, and because of that circumstance, he was also known as Leonardo Pisano, or Leonardo of Pisa. While his father was a collector of customs at Bugia on the northern coast of Africa (now Bougie in Algeria), Fibonacci had a Moorish schoolmaster, who introduced him to the Hindu- Arabic numeration system and computational methods**[1]**.

After widespread travel and extensive study of computational systems, Fibonacci wrote, in 1202, **the Liber Abaci**, in which he explained the Hindu- Arabic numerals and how they are used in computation. This famous book was instrumental in displacing the clumsy Roman numeration system and introducing methods of computation similar to those used today. It also included some geometry and algebra. Although he wrote on a variety of mathematical topics, Fibonacci is re-membered particularly for the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

to which his name has been applied. This sequence, even today, is the subject of continuing research, especially by the Fibonacci Association, which pub- lishes The Fibonacci Quarterly **[2]**.

1.2. Rabbits, Fibonacci Numbers, and Lucas Numbers

Fibonacci introduced a problem in the Liber Abaci by a story that may be summarized as follows. Suppose that

1) there is one pair of rabbits in an enclosure on the first day of January

2) this pair will produce another pair of rabbits(always produce one male and one female offspring) on February first and on the first day of every month thereafter
3) each new pair will mature for one month and then produce a new pair on the first day of the third month of its life and on the first day of every month thereafter.
4) the rabbits never die

The problem is to find the number of pairs of rabbits in the enclosure on the first day of the following January after the births have taken place on that day **[2]**.

It will be helpful to make a chart to keep count of the pairs of rabbits. Let A denote an adult pair of rabbits and let **B** denote a "**baby pair**" of rabbits. Thus, on January first, we have only an A; on February first we have that A and a B; and on March first, we have the original A, a new B, and the former B, which has become an A

January 1	A			1	0
February 1	A		B	1	1
March 1	A	B	A	2	1
Date	Pair			Number of A's	Number of B's

To get the next line of symbols, in any line we replace each A by AB and cach B by A. Thus, we have the representation shown in the table **[4]**

- **Rubbits, Fibonacci Numbers, and Lucas Number**

Date	Pairs	Number of A'S	Number of B'S
March 1	**ABA**	**2**	**1**
April 1	**ABAAB**	**3**	**2**
May 1	**ABAABABA**	**5**	**3**
June 1	**ABAABABAABAAB**	**8**	**5**

We now see that the number of A's on July 1 will be the sum of the number of A's on June 1 and the number of B's born on that day (which become A's on July 1). The number of B's on July 1 is the same as the number of A's on June 1 **[6]**.

We complete the table for the year:

N	Month	Number of A’s	Number of B’s	Total number of pairs
1	January After births on first	1	0	1
2	February	1	1	2
3	March	2	1	3
4	April	3	2	5
5	May	5	3	8
6	June	8	5	13
7	July	13	8	21
8	August	21	13	34
9	September	34	21	55
10	October	55	34	89
11	November	89	55	144
12	December	144	89	233
13	January	233	144	377

Thus, we see that under the conditions of the problem, the number of pairs of rabbits in the enclosure one year later would **be 377**.

We can draw some conclusions by studying the table. It is clear that the number of A's on the following February 1 is 377. Of these, 376 were originally B's, descendants of the original A. Therefore, if we add all the numbers in the column headed "Number of B's," we have

S= 0+1+1+2+3+5+8+13+21+34+55+89+144=376

From this, we observe that the sum of the first 12 entries in the column headed "Number of A's" is one less than 377, which would be the 14th. entry in that column**[3]**.

- **Rabbits, Fibonacci Numbers, and Lucas Numbers**

This gives the sequence → 1, 1, 2, 3, 5, 8, 13,...

as we wished. For the column headed "Number of B's," we have u_1=0, u_2 = 1, and the same recurrence formula, yielding the sequence

0, 1, 1, 2, 3, 5, 8, 13,....

For the column headed "Total number of pairs," we have u_1 = 1, u_2= 2, and the sequence

1, 2, 3, 5, 8, 13,...

Because of its source in Fibonacci's rabbit problem, the sequence

1, 1, 2, 3, 5, 8, 13,...

is called the **Fibonacci sequence**, and its terms are called **Fibonacci numbers**. We shall denote the *n*th Fibonacci number by F_n; thus **[7]**,

$$F=1, F_2 = 1, F_3=2, F_4=3, F_5=5, F_6 = 8,.....$$

Moreover, we may write these alternative forms:

$$F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2} \qquad n>2$$

We can now give a more formal discussion of the Fibonacci rabbit problem. For all positive integral *n*, we define for the first day of the *n*th month **[7]**:

A_n = number of A's (adult pairs of rabbits)

B_n= number of B's (baby pairs of rabbits)

T_n = total number of pairs of rabbits = $A_n + B_n$

Only the A's on the first day of the *n*th month will produce B's on the first day of the (n + 1)st month. Thus,

$B_{n+1} = A_n \qquad n\geq 1$

we observed that the number of A's on the first day of the (n+2)nd month is the sum of the number of A's on the first day of the (n+1)st month and the number of B's born on that day**[4]**. Thus

$A_{n+2} = A_{n+1} + B_{n+1}$

And since $B_{n+1} = A_n$ we have

$A_{n+2} = A_{n+1} + A_n \qquad n\geq 1$

- **Fibonacci and Lucas Numbers**

We also observe from the table that $A_1 = 1$ and $A_2 = 1$. Thus, the sequence

$A_1, A_2, A_3, \ldots\ldots$

is the Fibonacci sequence, and

$A_n = F_n \qquad n\geq 1$

Since $B_{n+1} = A_n$ for $n \geq 1$, we have

$B_n = A_{n-1} = F_{n-1}$ for $n \geq 2$.

If we now let n= 1 in this last formula, we have

$B_1 = F_0$

if we let n = 1 in the formula $F_{n+1} = F_n + F_{n-1}$, we have

$F_2 = F_1 + F_0$

OR

$F_0=F_2-F_1=1-1=0$

which checks with $B_1 = 0$ in the table. Thus, we have now defined F_n for

$n = 0$

Finally, the total number of pairs on the first day of the *n*th month is

$T_n = A_n + B\square = F_n + F_{n-1} = F_{n+1}$

We can now establish the following result

The sum of the first n Fibonacci numbers is one less than the (n+2)nd Fibonacci number. Symbolically:

$F_1+ F\square+\ldots..+F_n = F_{n+2} - 1, \qquad n\geq 1.$

We remember that $F_{n+2} =A_{n+2}$ and that A_{n+2} is the number of A's (adult pairs of rabbits) in the enclosure on the first day of the (n + 2)nd month.

The number of extra A's is $A_{n+2}-1$.

Now, one month after being born, each B became an A. If we add the number of B's from the first day of the first month to the first day of the (n+1)st month, the sum is the number of A's other than the original pair that we have on the first day of the (n+2)nd month. Thus,

$B_1 +B_2+B_3+\ldots\ldots.+B_{n+1} =A_{n+2} -1$

- **Rabbits, Fibonacci Numbers, and Lucas Numbers**

But, remembering that $B_1 = 0$, $B_n= F_{n-1}$, and $A_{n+2}= F_{n+2}$, we have

$F_1+F\square+\ldots\ldots..+ F_n =F_{n+2}-1 , \qquad n\geq 1$

as we wished to show. This formula is an example of a Fibonacci number identity.

1.3. The Golden Section and the Fibonacci Quadratic Equation

Suppose that we are given a line segment AB, and that we are to find a point C on it (between A and B) such that the length of the greater part is the mean proportional between the length of the whole segment and the length of the lesser part

$$\frac{AB}{AC}=\frac{AC}{CB}$$

where AB ≠0, AC ≠0, and CB ≠ 0 **[5]**.

We first find a positive numerical value for the ratio $\frac{AB}{AC}$For convenience, let

$X=\frac{AB}{AC}$ (x > 0)

Then

$$X=\frac{AB}{AC}=\frac{AC+C}{AC}=1+\frac{CB}{AC}=1+\frac{1}{\frac{AC}{CB}}=1+\frac{1}{\frac{AB}{AC}}=1+\frac{1}{X}$$

From

$$X=1+\frac{1}{X}$$

we obtain, by multiplying both members of the equation by x,

$$x^2= x + 1$$

or

$$x^2-x-1= 0.$$

(F)

The roots of this quadratic equation are

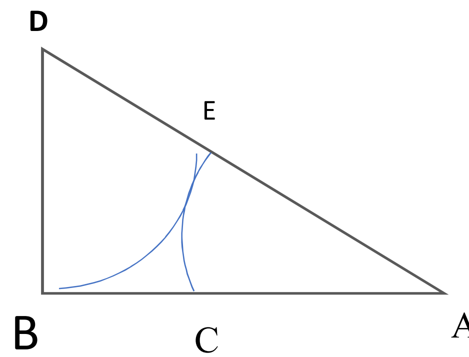
$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

(α is the Greek letter alpha, and $\beta$is the Greek letter beta.)

- **Fibonacci and Lucas Numbers**

by computation that $\alpha > 0$ and $\beta < 0$; α = 1.618 and β =-1.618 Thus, we take the positive root, α , as the value of the desired ratio: $\frac{AB}{AC} = \frac{1+\sqrt{5}}{2}$

We can now use this numerical value to devise a method for locating C on $\overline{AB}$. Draw $\overline{AD}$ perpendicular to $\overline{AB}$ at $\bar{B}$, but half its length. Draw $\overline{AD}$. Make $\overline{DE}$ the same length as $\overline{BD}$, and $\overline{AC}$ the same length as $\overline{AE}$ **[6]** Then



AB = 2BD, ED = BD

and, by the Pythagorean theorem,

AD = $\sqrt{5}$ BD;

hence

AC = AE = AD-ED =($\sqrt{5}$- 1)BD

$$\frac{AB}{AC} = \frac{2BD}{(\sqrt{5}-1)BD} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2}$$

This computation verifies that the construction does indeed locate C on $\overline{AB}$ such that $\frac{AB}{AC} = \frac{1+\sqrt{5}}{2}$

Since α is a root of equation (F), we have

$$\alpha^2 = \alpha + 1.$$

Multiplying both members of this equation by α^n (n can be any integer) yields

(A) $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$

If we let $u_n = \alpha^n \geq 1$ then $u_1 = \alpha$ and $u_2 = \alpha^2$, and we have the sequence

$$\alpha, \alpha^2 = \alpha + 1, \alpha^3 = \alpha^2 + \alpha, \dots \dots \dots \dots,$$

which satisfies the recursive formula (R) Similarly, we have

(B) $\beta^{n+2} = \beta^{n+1} + \beta^n$

and the sequence

$$\beta, \beta^2 = \beta + 1, \beta^3 = \beta^2 + \beta, \dots \dots$$

- **The Golden Section and the Fibonacci Quadratic Equation**

You can easily verify that

$$\alpha + \beta = 1 \; and \; \alpha - \beta = \sqrt{5}$$

If we now subtract the members of equation (β) from the members of equation (A) and divide each member of the resulting equation by$\alpha - \beta (= \sqrt{5 \neq 0})$, we find

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

If we now let u□ $= \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

And

$U_1 = \frac{\alpha - \beta}{\alpha - \beta} = 1$

$$u_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} = \frac{(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta} = \frac{(\sqrt{5})(1)}{\sqrt{5}} = 1$$

Thus, this sequence u_n, is precisely the Fibonacci sequence and so

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n = 1,2,3, \dots \dots \dots.$$

This is called the **Binet form** for the Fibonacci numbers after the French mathematician Jacques-Phillipe-Marie Binet (1786-1856).

Because of the relationship of the roots, α and β, of the equation (F),

$$x^2 - x - 1 = 0$$

to the Fibonacci numbers, we shall call equation (F) the **Fibonacci quadratic** equation. We shall call the positive root of (F).

$$\alpha = \frac{1+\sqrt{5}}{2}$$

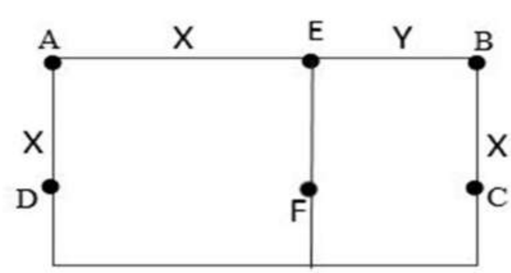
the **Golden Section**. [This is often represented by $\emptyset$ (Greek letter phi) or by some other symbol, but we shall continue to use α ,The point C in Figures 1 and 2, dividing $\overline{AB}$ such that

$$\frac{AB}{AC} = \alpha = \frac{1+\sqrt{5}}{2}$$

is said to divide $\overline{AB}$ in the **Golden Section.**

- **Fibonacci and Lucas Numbers**

Suppose that the rectangle ABCD in this Figure is such that if the square AEFD is removed from the rectangle, the lengths of the sides of the remaining rectangle, BCFE, have the same ratio as the lengths of the sides of the rectangle ABCD **[7]**.



That is,

$\frac{BC}{EB} = \frac{AB}{DA}$

Then if DA= AE =BC= x and EB = y,

we have

$$\frac{X}{Y} = \frac{X+Y}{X}$$

Multiplying both members of

$$\frac{X}{Y} = 1 + \frac{X}{Y}$$

by $\frac{x}{y}$ we find

$$\left(\frac{X}{Y}\right)^2 = \frac{x}{y} + 1$$

which is in the form of equation (F), the variable now being

Chapter Two

2.1. Some Fibonacci Algebra

As we mentioned earlier

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the root of (F)

And so $\alpha^2 = \alpha + 1$ *and* $\beta^2 = \beta + 1$ Also ,$\alpha + \beta = 5$

Moreover,

(A) $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$

And

(B) $\beta^{n+2} = \beta^{n+1} + \beta^n$

and by using these equations, we found that the Fibonacci numbers can be expressed in the **so-called Binet form**:

(C) $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}$,n=1,2,3,…….

Now suppose that we add the members of equation (B) to the members of equation (A), giving

$(\alpha^{n+2} + \beta^{n+2}) = (\alpha^{n+1} + \beta^{n+1}) + (\alpha^n + \beta^n)$

If we let $u_n = \alpha^n + \beta^n$,then we have

$$u_n + 2 = u_{n+1} + u_n$$

And

$u_1 = \alpha + \beta = 1$

$u_2 = \alpha^2 + \beta^2 = \alpha + 1 + \beta + 1 = (\alpha + \beta) + 2 = 1 + 2 = 3$

Thus, this sequence u, is the sequence of Lucas numbers defined in Section 2 **[8]**, and so we have a Binet form for the Lucas numbers:

(D)

$L_n = \alpha^n + \beta^n, \quad n = 1,2,3, ...$

Now look at the following comparison of the Fibonacci numbers and the Lucas numbers:

$F_1\ F_2\ F_3\ F_4\ F_5\ F_6\ F_7\ F_8\ \ F_9\ \ F_{10}$

1	1	2	3	5	8	13	21	34	55

1	3	4	7	11	18	29	47	76	123
L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}

Notice that

$F_1 + F_3 = L_2\ {,} F_2 + F_4 = L_3, and\ so\ on$

It can be proved ,that in general

$L_n = F_{n-1} + F_{n+1}$

From which ,since $F_{n+1} = F_n + F_{n-1}$,it follows that

(E) $L_n = F_n + 2F_{n-1}$

You can verify this latter statement for specific examples; that is, you can show that $L_6 = F_6 + 2F_5$ and so on .

We now have F_n, and L_n, expressed in terms of α^n and β^n. We can also find α^n and β^nin terms of F_n, andL_n. If we note that $\alpha - \beta = \sqrt{5}$, then,

from the Binet forms, we have

$$\sqrt{5}\, F_n = \alpha^n - \beta^n$$

$$L_n = \alpha^n + \beta^n$$

Adding, we find

$$2\alpha^n = L_n + \sqrt{5} F_n$$

Or

$$\alpha^n = \frac{L_n + \sqrt{5} F_n}{2}$$

Subtracting ,we find

$$\beta^n = \frac{L_n + \sqrt{5} F_n}{2}$$

Recall that in Section 2 we had occasion to define $F_0\ as F_2 - F_1$ Similarly, we can define $L_0 as L_2 - L_1 = 3 - 1 = 2$ (Notice that this agrees with the definition by the Binet form, since $\alpha^0 + \beta^0$=1+1=2) Since$L_1 = \alpha + \beta = 1$, we can now write expression (E) for L_n, (above) as **[9]**

(E)$L_n = L_1 F_n + L_0 F_{n-1}$

2.2 .Fibonacci and Lucas Numbers

The method of defining $F_0$and L_0 suggests that we can also define F-1, L-1, and so on, by applying the formulas**[4]**

$$F_{n-1} = F_{n+1} - F_n$$

and

$$L_{n-1} = L_{n+1} - L_n$$

repeatedly. Thus, we have:

F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4
-3	**2**	**-1**	**1**	**0**	**1**	**1**	**2**	**3**

7	**-4**	**3**	**-1**	**2**	**1**	**3**	**4**	**7**
L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4

We can derive a formula for F_{-n}, n> 0, by assuming that the Binet form also holds for negative values of the exponents (compare derivation)

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{(\frac{1}{\alpha})^n - (\frac{1}{\beta})^n}{\alpha - \beta}$$

Since $\alpha\beta = -1$, we have

$$\frac{1}{\alpha} = -\beta \text{ and } \frac{1}{\beta} = -\alpha$$

Therefore

$$F_{-n} = \frac{(-\beta)^n - (-\alpha)^n}{\alpha - \beta} = \frac{(-1)^n(\beta^n - \alpha^n)}{\alpha - \beta} = (-1)^{n+1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

and so

$$F_{-n} = (-1)^{n+1}F_n$$

Similarly, you can show that for n> 0,

$$L_{-n} = (-1)^n L_n$$

Now suppose that we compute the first 14 successive ratios $\frac{F_{n+1}}{F_n}$ and $\frac{L_{n+1}}{L_n}$ The values of the successive ratios as shown at the top of the next page suggest that in both cases the value of the ratio becomes closer and closer to α as we take larger and larger values of n. However, we shall not undertake to prove this here. We can also observe that the first Fibonacci ratio is less than α, the second is greater than α, and so on, while the first Lucas ratio is greater thanα, the second is less thanα, and so on. Moreover **[10].**

$$\frac{F_2}{F_1} < \alpha < \frac{L_2}{L_1}, \qquad \frac{F_3}{F_2} > \alpha > \frac{L_3}{L_2}$$

And so on

$\frac{F_{n+1}}{F_n}$	$\frac{L_{n+1}}{L_n}$
$\frac{1}{1} = 1.0000$	$\frac{3}{1} = 3.0000$
$\frac{2}{1} = 2.0000$	$\frac{4}{3} = 1.3333$
$\frac{3}{2} = 1.5000$	$\frac{7}{4} = 1.7500$
$\frac{5}{3} = 1.6667$	$\frac{11}{7} = 1.5714$
$\frac{8}{5} = 1.6000$	$\frac{18}{11} = 1.6363$
$\frac{13}{8} = 1.6250$	$\frac{29}{18} = 1.6111$
$\frac{21}{31} = 1.6154$	$\frac{47}{29} = 1.6207$
$\frac{34}{21} = 1.6190$	$\frac{76}{47} = 1.6170$
$\frac{55}{34} = 1.6176$	$\frac{123}{76} = 1.6184$
$\frac{89}{55} = 1.6182$	$\frac{199}{123} = 1.6179$
$\frac{144}{89} = 1.6180$	$\frac{322}{199} = 1.6181$
$\frac{233}{144} = 1.6181$	$\frac{521}{322} = 1.6180$
$\frac{377}{233} = 1.6180$	$\frac{843}{521} = 1.6180$
$\frac{610}{377} 1.6180$	$\frac{1364}{843} = 1.6180$

$$\boldsymbol{\alpha = 1.61803398875\ldots.}$$

References

1. Avant L, The So-called Fibonacci Numbers in Ancient and Medieval India, Raj Narain college India, 1985, P.45.
2. S. Vajda, Fibonacci & Lucas Numbers, and the Golden Section, Theory and Applications, Ellis Horwood Ltd., Chichester, 1989, P.78.
3. D. Kalman. "Generalized Fibonacci Numbers by Matrix Methods." The Fibonacci
4. Quarterly 20, no. 1 (1982):73-76.
5. Melham, R. S., A Fibonacci Identity in the spirit of Simson and Gelin- Cesaro, Fibonacci Quarterly, 2003, P.129.
6. Fair grieve, S., Gould, H. W., Product Difference Fibonacci Identities of Simson, Gelin- Cesaro, Tagiuri and Generalizations، Fibonacci Quarterly, 2005, P.301.
7. J.A. Bondy and U.S.R. Murty. Graph Theory with Applications, North–Holland, Amsterdam،1969. P132.
8. R.E. Merrifield and H.E. Simmons. Topological Methods in Chemistry, T. Wiley, New York, 1989, P.121.
9. V.E. Hoggat, Fibonacci and Lucas Numbers, Palo Alto, Houghton-Mifflin, CA, 1969, P.76.
10. H. Civciv, A note on the determinants of pentadiagonal matrices with the generalized Fibonacci and Lucas numbers, Appl. Math. Comput., in press. doi:10.1016 /j.amc.2008.06.007.