

*Ministry Of Higher Education
And scientific Research
University Of Babylon College
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Applications of complex analysis in physics

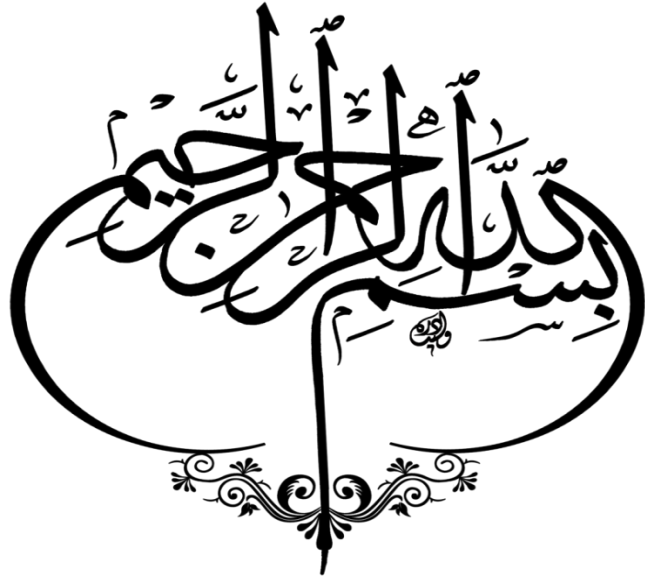
**A PROSED RESEARCH TO THE COUNCIL OF THE COLLEGE OF
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{ مَنْ عَمِلَ صَالِحًا مِّنْ ذَكَرٍ أَوْ أَنْتَى وَهُوَ

مُؤْمِنٌ فَلَنُحْيِيَنَّهٗ حَيَاةً طَيِّبَةً ۗ وَلَنَجْزِيَنَّهُمْ

أَجْرَهُمْ بِأَحْسَنِ مَا كَانُوا يَعْمَلُونَ }

صدق الله العلي العظيم

سورة النحل : الآية ٩٧

الإهداء

إلى من حفتهم القلوب بالمحبة، وأنارت وجوههم دروب
عمري..

إلى والديّ الكريمين، رمز العطاء وبوصلة النجاح..
إلى إخوتي وعائلي وكل من آمن بي حين خفتت
الأضواء..

أهدي هذا الجهد المتواضع."

شكر وامتنان

الحمد لله رب العالمين والصلاة والسلام على اشرف خلق الله سيدنا محمد وعلى اله وأصحابه الطاهرين إلى يوم الدين.

بداية اشكر الله واشكر فضله، الذي يسر لي أمري ومنحني العزم على مواصلة الدراسة، ووقفني إلى الانتهاء من هذا البحث المتواضع وما كان لهذا العمل أن يتم إلا بتوفيق من الله. أتقدم بوافر الشكر وعظيم الامتنان إلى الأستاذ المشرف الدكتور (عقيل كاتب)، الذي كان لتوجيهاته السديد وصبره الجميل الأثر الأكبر في إتمام هذا البحث. كما يطيب لي ان أتقدم بالشكر والتقدير إلى اساتذتي الافاضل في قسم الرياضيات العلوم الصرفة جزاهم الله عني خير الجزاء.

ولا يفوتني أن أشكر كل من ساهم في إنجاز الجانب الميداني للبحث، خاصة أولئك الذين اقتطعوا من وقتهم للمشاركة في العينة، فلهم مني كل التقدير والاحترام. "واقدم شكري وامتناني لكل من شجعني ومدّ لي يد العون لإتمام هذا البحث وفاتني ذكر اسمه.

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ملخص

يُعد التحليل المركب — وهو دراسة الدوال ذات المتغيرات المركبة — أحد أكثر فروع الرياضيات أناقةً وقوةً. وتمتد تطبيقاته إلى ما هو أبعد بكثير من الرياضيات البحتة، لتتغلغل بعمق في مجالي الفيزياء النظرية والفيزياء التطبيقية. وتستكشف هذه الأطروحة الأدوات الأساسية للتحليل المركب، وتُبيّن كيف توفر هذه الأدوات طرائق لا غنى عنها لحل المسائل في مجالات ميكانيكا الكم، والديناميكا الكهربائية، وديناميكا الموائع، والديناميكا الحرارية، ومعالجة الإشارات.

نستهل عملنا بإرساء الأسس النظرية، والتي تشمل جبر الأعداد المركبة، والدوال التحليلية، ومعادلات كوشي-ريمان، والتكامل الكفافي، ومبرهنة كوشي التكاملية، ومتسلسلات لوران، ومبرهنة البواقي. ومن ثم، ننتقل لتطبيق هذه الأدوات بصورة منهجية على المسائل الفيزيائية؛ حيث نوضح كيف تُستخدم التكاملات الكفافية في حساب قيم التكاملات الفيزيائية الحقيقية التي تظهر في نظرية الحقل الكمومي، وكيف تُحوّل الإسقاطات المطابقة مسائل القيم الحدية الصعبة في الكهروستاتيكا إلى مسائل قابلة للحل، وكيف تصف الكمونات المركبة جريان الموائع ثنائي الأبعاد غير الدوراني، وأخيراً كيف يعتمد تحويل فوربيه ولا بلاس اعتماداً جوهرياً على التحليل المركب في صياغتهما الرياضية الدقيقة.

وعلى مدار هذه الأطروحة، ينصب التركيز على إبراز الوحدة العميقة القائمة بين البنية الرياضية والبصيرة الفيزيائية، مما يوضح السبب الذي يجعل من التحليل المركب حجر زاوية راسخاً ضمن الحزمة الرياضية التي يعتمد عليها الفيزيائي. (١)

1: Introduction

1.1 Motivation and Overview

The set of complex numbers C extends the real number system by introducing the imaginary unit i , defined by the relation $i^2 = -1$. At first glance, this algebraic extension might seem like a formal curiosity. Yet the decision to work over C rather than R transforms the landscape of analysis profoundly. Functions that are differentiable in the complex sense — called analytic or holomorphic functions — turn out to have remarkable rigidity properties that have no real analogue: infinite differentiability, representation by convergent power series, and a global behaviour that is completely determined by local data.

These properties make complex analysis an extraordinarily efficient tool in physics. Many of the most important integrals in quantum mechanics, statistical mechanics, and electrodynamics are real integrals that are most naturally evaluated by embedding them in the complex plane and applying the residue theorem. Boundary-value problems in two-dimensional electrostatics and fluid mechanics that would be forbidding in Cartesian coordinates often yield to the technique of conformal mapping. The Fourier and Laplace transforms, which underpin the analysis of linear systems throughout physics, are most cleanly understood as operations on functions of a complex variable. This thesis is intended to provide a self-contained, mathematically precise account of the principal methods of complex analysis together with their major physical applications. It is written at the level of an advanced undergraduate student in mathematics who has completed standard courses in calculus, linear algebra, and ordinary differential equations, and who has had some exposure to introductory physics.(1)

1.2 Historical Background

The history of complex numbers stretches back to the sixteenth century, when Gerolamo Cardano and Rafael Bombelli found it useful to admit square roots of negative numbers when solving cubic equations, even when all the roots were ultimately real. Rene Descartes coined the dismissive term 'imaginary' in 1637, and the phrase 'complex number' was introduced by Carl Friedrich Gauss. Gauss was also the first to give a rigorous geometric interpretation, identifying complex numbers with points in the plane, a representation independently proposed by Caspar Wessel and Jean-Robert Argand.

The systematic theory of functions of a complex variable was developed in the nineteenth century, primarily by Augustin-Louis Cauchy, Bernhard Riemann, and Karl Weierstrass. Cauchy formulated the integral theorem that bears his name around 1825 and introduced the calculus of residues. Riemann's 1851 doctoral dissertation introduced the concept of a Riemann surface and proved the conformal mapping theorem (the Riemann Mapping Theorem). Weierstrass provided a rigorous power-series foundation for the subject.

The applications to physics came quickly. In the 1850s, Riemann and, independently, Helmholtz applied complex potentials to two-dimensional fluid mechanics. Lord Rayleigh, Kirchhoff, and others used conformal mapping in electrostatics and hydrodynamics. In the twentieth century, complex analysis became indispensable in quantum mechanics and quantum field theory: the path-integral formulation due to Feynman relies on analytic continuation, and the systematic evaluation of loop integrals in perturbation theory is a vast extension of the residue calculus.(2)

1.3 Scope of the Thesis

The thesis is organised as follows. Chapter 2 develops the theoretical foundations of complex analysis, covering complex numbers and the Argand plane, limits and continuity, analytic functions and the Cauchy-Riemann equations, elementary complex functions, complex integration and Cauchy's theorem, Taylor and Laurent series, and the residue theorem. Chapter 3 treats the application of contour integration to the evaluation of definite real integrals, including Fourier-type integrals and integrals involving branch cuts.

Chapter 4 introduces conformal mapping and applies it to two-dimensional electrostatics and potential theory. Chapter 5 develops the complex-potential formalism for two-dimensional inviscid, incompressible fluid flow. Chapter 6 surveys applications in quantum mechanics, including the role of analytic continuation in scattering theory, the use of Green's functions in the complex plane, and selected applications to quantum field theory. Chapter 7 discusses the Fourier and Laplace transforms from the perspective of complex analysis. Chapter 8 contains the conclusion and an outline of further topics.(1)

2: Theoretical Foundations of Complex Analysis

2.1 The Complex Number System

2.1.1 Algebraic Definition

The set of complex numbers C is defined as the set of ordered pairs (a, b) of real numbers equipped with the operations of addition and multiplication:

$$(a, b) + (c, d) = (a+c, b+d)$$

$$(a, b) * (c, d) = (ac - bd, ad + bc)$$

Under these operations C forms a field. Identifying the real number a with the complex number $(a, 0)$ embeds R as a subfield of C . The element $(0, 1)$ is denoted i and satisfies $i^2 = (0,1)*(0,1) = (-1, 0) = -1$. Every complex number (a, b) can therefore be written as $z = a + ib$, where $a = \text{Re}(z)$ is the real part and $b = \text{Im}(z)$ is the imaginary part.

The complex conjugate of $z = a + ib$ is defined as $z^* = a - ib$ (also written \bar{z}). The modulus (or absolute value) of z is $|z| = \sqrt{a^2 + b^2}$, which equals $\sqrt{z * z^*}$. The modulus satisfies the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ for all z_1, z_2 in C , which endows C with the structure of a metric space via the distance function $d(z_1, z_2) = |z_1 - z_2|$.(4)

2.1.2 Polar Form and Euler's Formula

A nonzero complex number $z = a + ib$ can be written in polar form $z = r * e^{i*\theta}$, where $r = |z| > 0$ is the modulus and $\theta = \text{Arg}(z)$ is the argument (or phase). Here Euler's formula asserts:

$$e^{i*\theta} = \cos(\theta) + i*\sin(\theta)$$

This fundamental identity, which can be established rigorously via power series, links exponential and trigonometric functions and will be central throughout the thesis. The principal value of the argument, $\text{Arg}(z)$, is taken to lie in $(-\pi, \pi]$. Multiplication of complex numbers corresponds to multiplication of moduli and addition of arguments:

$$|z_1 z_2| = |z_1| |z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

De Moivre's theorem, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$, follows immediately and provides an elegant approach to trigonometric identities.(5)

2.2 Analytic Functions and the Cauchy-Riemann Equations

Let $f : U \rightarrow \mathbb{C}$ be a function defined on an open subset U of \mathbb{C} . Writing $z = x + iy$ and $f(z) = u(x, y) + i v(x, y)$, where u and v are real-valued functions, we define the complex derivative of f at z_0 in U by:

$$f'(z_0) = \lim_{z \rightarrow z_0} [f(z) - f(z_0)] / (z - z_0)$$

provided this limit exists and is independent of the direction in which z approaches z_0 in the complex plane. If $f'(z_0)$ exists, f is said to be complex-differentiable at z_0 . If f is complex-differentiable at every point of U , it is called holomorphic (or analytic) on U .

A necessary condition for f to be complex-differentiable at z_0 is that the partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations:

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx}$$

Conversely, if u and v have continuous first-order partial derivatives on U that satisfy the Cauchy-Riemann equations, then f is holomorphic on U . One immediate consequence is that both u and v are harmonic, i.e. they satisfy Laplace's equation:

$$\begin{aligned} \nabla^2 u &= \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \\ \nabla^2 v &= 0 \end{aligned}$$

The function u is called the harmonic conjugate of v (and vice versa). This connection between holomorphic functions and harmonic functions is the foundation of the application of complex analysis to electrostatics and fluid mechanics.(5)

2.3 Elementary Complex Functions

The exponential function e^z is defined for all z in \mathbb{C} by the convergent power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

It satisfies $e^{(z_1+z_2)} = e^{z_1} * e^{z_2}$ and reduces to the real exponential when z is real. The complex trigonometric and hyperbolic functions are defined in terms of the exponential:

$$\sin(z) = (e^{(iz)} - e^{(-iz)}) / (2i), \quad \cos(z) = (e^{(iz)} + e^{(-iz)}) / 2$$

$$\sinh(z) = (e^z - e^{(-z)}) / 2, \quad \cosh(z) = (e^z + e^{(-z)}) / 2$$

The complex logarithm is the multi-valued inverse of the exponential. For $z \neq 0$, we define $\text{Log}(z) = \ln|z| + i*\text{Arg}(z)$ as the principal branch, where $\text{Arg}(z)$ in $(-\pi, \pi]$. Other branches differ by integer multiples of $2*\pi*i$. The logarithm introduces the concept of branch cuts, which will be important when evaluating certain real integrals by contour methods.

Complex power functions $z^\alpha = e^{(\alpha*\text{Log}(z))}$ and the complex roots of unity are also fundamental. The n -th roots of unity are the n complex numbers $e^{(2*\pi*i*k/n)}$ for $k = 0, 1, \dots, n-1$, which form the vertices of a regular n -gon inscribed in the unit circle.(6)

2.4 Contour Integration and Cauchy's Theorem

A contour (or path) in C is a piecewise continuously differentiable map $\gamma : [a, b] \rightarrow C$. The contour integral of a continuous function f along γ is defined by:

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) * \gamma'(t) dt$$

Cauchy's Integral Theorem states that if f is holomorphic on a simply connected domain D , then for any closed contour γ lying in D :

$$\oint_{\gamma} f(z) dz = 0$$

This result has the remarkable consequence that the integral of an analytic function between two points is path-independent within a simply connected domain. From Cauchy's theorem one derives the Cauchy Integral Formula:

$$f(z_0) = (1 / 2\pi i) * \oint_{\gamma} f(z) / (z - z_0) dz$$

for any z_0 interior to a simple closed contour γ oriented counterclockwise. This formula expresses the value of an analytic function at any interior point in terms of its values on the boundary, a result with no analogue in real analysis. Differentiating under the integral sign yields the generalized Cauchy integral formula:

$$f^{(n)}(z_0) = (n! / 2\pi i) * \oint_{\gamma} f(z) / (z - z_0)^{(n+1)} dz$$

which shows that a holomorphic function is infinitely differentiable and its derivatives at any point are determined by the boundary values.(7)

2.5 Laurent Series and the Residue Theorem

If f has an isolated singularity at z_0 , it can be expanded in a Laurent series in a punctured disk $0 < |z - z_0| < R$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

The coefficient a_{-1} is called the residue of f at z_0 , written $\text{Res}(f, z_0)$. If the Laurent series contains finitely many negative-power terms, the singularity is a pole; otherwise it is an essential singularity. For a pole of order m , the residue can be computed as:

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} * \lim_{z \rightarrow z_0} \frac{d^{(m-1)}}{dz^{(m-1)}} [(z-z_0)^m * f(z)]$$

The Residue Theorem generalises Cauchy's theorem to functions with isolated singularities:

$$\text{contour_integral_gamma } f(z) dz = 2\pi i * \sum_k \text{Res}(f, z_k)$$

where the sum is over all singularities z_k enclosed by γ . This theorem is the master tool for evaluating definite integrals in both mathematics and physics, as the following chapters demonstrate extensively.(7)

3: Evaluation of Real Integrals by Contour Integration

3.1 Strategy and Methodology

The power of complex analysis in evaluating real integrals lies in the following strategy: given a definite integral over the real line (or part of it) that is difficult or impossible to evaluate by elementary methods, one extends the integrand to the complex plane, selects a closed contour that includes the real segment of interest, applies the residue theorem to evaluate the total contour integral, and then shows that the contributions from the non-real parts of the contour vanish (or can be related to the original integral) in an appropriate limit.

This procedure requires justification at each step, in particular the bounding of integrals over auxiliary arcs. Jordan's lemma and its variants provide the key estimates: if $f(z) \rightarrow 0$ uniformly on a semicircular arc of radius R as $R \rightarrow \infty$, then the integral of $f(z)e^{iaz}$ over the arc tends to zero for $a > 0$. (8)

3.2 Rational Functions and the Semicircle Contour

Consider integrals of the form $\int_{-\infty}^{\infty} P(x)/Q(x) dx$, where P and Q are polynomials with $\deg(Q) \geq \deg(P) + 2$ and Q has no real zeros. We close the real axis with a large semicircle in the upper half-plane. By Jordan's lemma (or a direct estimate), the integral over the semicircle vanishes as $R \rightarrow \infty$. The residue theorem then gives:

$$\int_{-\infty}^{\infty} P(x)/Q(x) dx = 2\pi i * \sum_{\text{Im}(z_k) > 0} \text{Res}(P/Q, z_k)$$

As a concrete example, consider the integral $\int_{-\infty}^{\infty} dx/(1+x^2)$. The integrand $f(z) = 1/(1+z^2) = 1/[(z+i)(z-i)]$ has simple poles at $z = \pm i$. Only $z = i$ lies in the upper half-plane, with residue $1/(2i)$. The residue theorem gives $2\pi i * (1/2i) = \pi$, confirming the elementary result.

This integral appears physically in the Lorentzian line shape of a quantum resonance: if an energy level has a width Γ (decay rate), the spectral density is proportional to $\Gamma/[(E-E_0)^2 + (\Gamma/2)^2]$, and the normalization integral is evaluated exactly by the residue method.(8)

3.3 Fourier Integrals and Jordan's Lemma

Integrals of the form $\int_{-\infty}^{\infty} f(x) e^{(iax)} dx$, which arise as Fourier transforms, are evaluated by closing the real axis in the upper half-plane when $a > 0$ (and in the lower half-plane when $a < 0$). Jordan's lemma guarantees that the semicircular arc contributes nothing in the limit $R \rightarrow \infty$, provided $f(z) \rightarrow 0$ uniformly. The result is:

$$\int_{-\infty}^{\infty} f(x) e^{(iax)} dx = 2\pi i * \sum_{\text{Im}(z_k) > 0} \text{Res}(f(z) e^{(iaz)}, z_k)$$

A physically important example is the causal propagator in quantum mechanics. The retarded Green's function for a harmonic oscillator involves an integral of the form $\int_{-\infty}^{\infty} e^{-iEt} / (E^2 - E_0^2 + i\epsilon) dE$, where $\epsilon \rightarrow 0+$ is a small positive quantity that shifts the poles off the real axis. Evaluating this integral using residues gives the time-domain response, establishing causality (the response vanishes for $t < 0$). (4)

3.4 Integrals Involving Branch Cuts

When the integrand involves a multivalued function such as a fractional power or logarithm, one must introduce a branch cut and choose a contour that avoids it. The standard example is $\int_0^{\infty} x^{\alpha-1}/(1+x) dx = \pi/\sin(\pi\alpha)$ for $0 < \alpha < 1$, the Euler beta function.

One takes the branch cut along the positive real axis, introduces a keyhole contour consisting of two rays just above and below the cut connected by circles of radii $\epsilon \rightarrow 0$ and $R \rightarrow \infty$. Computing the residue at $z = -1$ (the only pole) and noting the phase difference $e^{2\pi i\alpha}$ between the two rays, one obtains the Euler reflection formula. Such integrals arise in the theory of the Gamma function and in the computation of quantum mechanical path integrals(9).

4: Conformal Mapping and Electrostatics

4.1 Definition and Properties of Conformal Maps

A holomorphic function $f : U \rightarrow V$ with $f'(z) \neq 0$ is a conformal map: it preserves angles between smooth curves at every point. This follows because the argument of $f'(z)$ gives the local rotation angle, and $|f'(z)|$ gives the local scale factor, both of which are the same for all directions at z . Conformal maps also preserve harmonicity: if ϕ is harmonic in V , then $\phi(f(z))$ is harmonic in U (by the chain rule for Laplace's equation).

The Riemann Mapping Theorem guarantees that any simply connected domain U (other than \mathbb{C} itself) can be mapped conformally onto the open unit disk. In practice, one constructs such maps by composing known elementary functions. The most important family is the Möbius transformations:

$$w = (az + b) / (cz + d), \quad ad - bc \neq 0$$

which map circles and lines to circles and lines, and include rotations, translations, dilations, and inversions as special cases. The Joukowski transformation $w = z + 1/z$ maps the exterior of a circle to the exterior of a line segment and is fundamental in aerodynamics(10)

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4.2 Application to Two-Dimensional Electrostatics

In two-dimensional electrostatics (or, equivalently, for configurations with translational symmetry in the z -direction), the electrostatic potential $\phi(x, y)$ satisfies Laplace's equation $\nabla^2 \phi = 0$ in charge-free regions, with appropriate boundary conditions on

conductors ($\phi = \text{constant}$) or at infinity. The electric field is $E = -\text{grad}(\phi)$.

Since the real part of any holomorphic function is harmonic, we can seek a complex potential $\Omega(z) = \phi(x,y) + i\psi(x,y)$, where ϕ is the electrostatic potential and ψ is the stream function (the harmonic conjugate of ϕ). The equipotential lines $\phi = \text{const}$ and the field lines $\psi = \text{const}$ form two orthogonal families of curves, as befits an analytic function.

The key application of conformal mapping is the following: if $\Omega(z)$ is the complex potential for a given configuration, and $w = f(z)$ is a conformal map, then $\Omega(f^{-1}(w))$ is the complex potential for the transformed configuration. This allows one to map a complicated geometry onto a simple one (such as parallel plates or concentric circles), solve Laplace's equation there, and transform back.

Example: Find the potential between two coaxial cylinders of radii a and b ($a < b$) at potentials V_a and V_b . The complex potential is simply $\Omega(z) = A \ln(z) + B$, with A and B determined by the boundary conditions. The logarithm maps concentric circles to vertical lines. The potential is $\phi(r) = V_a + (V_b - V_a) \ln(r/a) / \ln(b/a)$, independent of the angle, as expected by symmetry. The capacitance per unit length is $C = 2\pi\epsilon_0 / \ln(b/a)$. (10)

4.3 The Schwarz-Christoffel Transformation

The Schwarz-Christoffel formula gives an explicit conformal map from the upper half-plane to the interior of a polygon with specified vertices z_1, \dots, z_n and interior angles $\alpha_k \pi$:

$$\frac{dz}{dw} = C \cdot \prod_{k=1}^n (w - x_k)^{\alpha_k - 1}$$

where $x_1 < x_2 < \dots < x_n$ are the real preimages of the vertices. This transformation is used in electromagnetic problems involving conducting wedges, slots, and other polygonal geometries. For instance, the fringing field at the edge of a parallel-plate capacitor can be computed exactly using a Schwarz-Christoffel map.(10)

5: Complex Potential in Fluid Mechanics

5.1 Irrotational Incompressible Flow

Consider a two-dimensional, inviscid, incompressible fluid flow. The velocity field $v = (u, v)$ satisfies $\text{div}(v) = 0$ (incompressibility) and $\text{curl}(v) = 0$ (irrotationality). These conditions are precisely the Cauchy-Riemann equations if we define the velocity potential ϕ by $v = \text{grad}(\phi)$, so that ϕ satisfies $\nabla^2 \phi = 0$. The stream function ψ (constant along streamlines) is the harmonic conjugate of ϕ .

The complex potential is $w(z) = \phi + i\psi$, and the complex velocity is:

$$\frac{dw}{dz} = u - iv$$

where $u = d(\phi)/dx$ and $v = d(\phi)/dy$ are the x- and y-components of velocity. The magnitude of the complex velocity gives the flow speed: $|\frac{dw}{dz}|^2 = u^2 + v^2$. Stagnation points, where the velocity vanishes, correspond to zeros of $\frac{dw}{dz}$.(11)

5.2 Standard Flows and Superposition

Since Laplace's equation is linear, complex potentials can be superposed. The standard elementary flows and their complex potentials are:

- Uniform flow at angle α : $w(z) = U e^{-i\alpha} z$
- Line source/sink of strength m at origin: $w(z) = (m/2\pi) \ln(z)$
- Line vortex of circulation Γ at origin: $w(z) = - (i\Gamma/2\pi) \ln(z)$
- Doublet of strength μ at origin: $w(z) = \mu/(2\pi z)$

By superposing these, one builds up complex flows. For example, uniform flow plus a doublet gives the flow past a circular cylinder: $w(z) = U^*(z + a^2/z)$, where a is the cylinder radius. The streamlines are symmetric front-to-back, consistent with D'Alembert's paradox (zero drag in inviscid flow).

Adding a vortex of circulation Γ yields $w(z) = U^*(z + a^2/z) - (i*\Gamma/2*\pi)*\ln(z)$, which models the flow around a spinning cylinder (the Magnus effect). The Kutta-Joukowski theorem states that the lift per unit length on the cylinder is $L = \rho*U*\Gamma$, where ρ is the fluid density. This result, derived by integrating the pressure using Bernoulli's equation and Blasius's theorem (itself a residue calculation), is the basis of aerodynamic lift.(5)

5.3 Blasius's Theorem and the Kutta-Joukowski Lift Formula

Blasius's theorem relates the aerodynamic force on a rigid body to a contour integral of the complex potential. If the body occupies a region with boundary C , the force (F_x, F_y) and moment M about the origin are:

$$F_x - i*F_y = (i*\rho/2) * \text{contour_integral_C} (dw/dz)^2 dz$$

This is evaluated by expanding $(dw/dz)^2$ in a Laurent series and applying the residue theorem. For a body with circulation Γ in a uniform stream U , the residue calculation yields $F_x = 0$ and $F_y = \rho*U*\Gamma$ (the Kutta-Joukowski lift formula). Blasius's theorem thus reduces the computation of hydrodynamic forces to a purely algebraic operation on residues.(11)

6: Applications in Quantum Mechanics

6.1 Green's Functions in the Complex Plane

The Green's function $G(x, x'; E)$ for a quantum mechanical Hamiltonian H is defined as the resolvent operator $(E - H)^{-1}$ evaluated between position eigenstates. In momentum space, for a free particle of mass m , the Green's function is:

$$G_0(\mathbf{k}, \mathbf{k}'; E) = 1 / (E - \hbar^2 k^2 / 2m)$$

The position-space Green's function is obtained by a Fourier transform that requires integration over real k with a pole on the real axis. The prescription for handling this pole — whether to shift it slightly into the upper or lower half-plane — determines the boundary conditions. Shifting the pole as $E \rightarrow E + i\epsilon$ ($\epsilon > 0$) gives the retarded Green's function, which vanishes for $t < 0$, encoding causality. Evaluating the resulting integral by the residue theorem yields:

$$G_0^+(x, x'; E) = -i m / \hbar^2 * e^{(ik|x-x'|)} / k, \quad k = \sqrt{2mE} / \hbar$$

The retarded Green's function is the fundamental tool in scattering theory (the Lippmann-Schwinger equation) and in the many-body perturbation theory used to describe interacting electron systems.(12)

6.2 Analytic Continuation and the S-Matrix

In quantum scattering theory, the S-matrix element $S(k)$ encodes the amplitude for scattering from a given potential. For a spherically symmetric potential, $S(k)$ is a meromorphic function of the complex wave number k . Poles of $S(k)$ in the upper half k -plane ($\text{Im}(k) > 0$) correspond to bound states at energies $E = \hbar^2 k^2/2m < 0$. Poles in the lower half-plane correspond to resonances (or quasi-bound states) with complex energies $E = E_r - i\Gamma/2$, where Γ is the decay width.

The physical cross-section $\sigma(E)$ for scattering near a resonance has the Breit-Wigner form:

$$\sigma(E) \sim \frac{\Gamma^2}{(E - E_r)^2 + (\Gamma/2)^2}$$

which is a Lorentzian peak of width Γ centred at E_r . The evaluation of the cross-section integral over energy to obtain the total reaction rate is a straightforward residue calculation of exactly the type discussed in Chapter 3.(12)

6.3 Path Integrals and Analytic Continuation to Imaginary Time

Feynman's path integral formulation of quantum mechanics expresses the transition amplitude as a sum over all paths $x(t)$ weighted by $\exp(iS[x]/\hbar)$, where S is the classical action. For a

particle in a potential $V(x)$, the Minkowski-space path integral is often ill-defined due to the oscillatory phase factor. The standard remedy is Wick rotation: the analytic continuation $t \rightarrow -i\tau$, which transforms the Minkowski action S into the Euclidean action S_E by:

$$i S_{\text{Minkowski}} \rightarrow -S_{\text{Euclidean}} = - \int_0^\beta [m/2 (\dot{x})^2 + V(x)] d\tau$$

The Euclidean path integral $\exp(-S_E/\hbar)$ is now a real, convergent expression. Setting $\beta = \hbar/(k_B T)$ identifies the Euclidean path integral with the partition function of statistical mechanics at temperature T . This deep connection between quantum mechanics and statistical mechanics, mediated by the analytic continuation of the time variable, is exploited throughout quantum field theory and condensed matter physics.(13)

7: Fourier and Laplace Transforms

7.1 The Fourier Transform as a Complex Integral

The Fourier transform of a function $f(t)$ is defined as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{(-i\omega t)} dt$$

and the inverse Fourier transform reconstructs f from F :

$$f(t) = (1/2\pi) * \int_{-\infty}^{\infty} F(\omega) e^{(i\omega t)} d(\omega)$$

For a causal function ($f(t) = 0$ for $t < 0$), the Fourier transform $F(\omega)$ is analytic in the upper half of the complex ω -plane ($\text{Im}(\omega) > 0$). This analyticity encodes causality through the Kramers-Kronig relations: the real and imaginary parts of $F(\omega)$ on the real axis are related by Hilbert transforms, which are themselves contour integrals evaluated by the residue theorem.

The Kramers-Kronig relations have profound physical significance: they express the connection between dispersion (frequency dependence of the real part of a response function) and absorption (imaginary part), and they follow purely from causality plus the analyticity it implies.(13)

7.2 The Laplace Transform and the Bromwich Integral

The (one-sided) Laplace transform of a causal function $f(t)$ is:

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

which converges for $\text{Re}(s) > c$, where c is the abscissa of convergence. The inverse Laplace transform is given by the Bromwich integral:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) e^{st} ds$$

where the vertical contour $\text{Re}(s) = \gamma$ lies to the right of all singularities of $F(s)$. This is a contour integral that is evaluated by closing to the left (for $t > 0$) and applying the residue theorem. Each pole of $F(s)$ at s_k contributes a term proportional to $e^{s_k t}$ to $f(t)$, so the poles of the Laplace transform control the temporal behaviour of the system.

The Laplace transform is the standard tool for solving linear ODEs and PDEs with initial conditions. In circuit theory, the poles of the impedance $Z(s)$ in the complex s -plane determine the natural frequencies of the circuit. In control theory, the stability of a linear system is determined by whether all poles lie in the left half-plane $\text{Re}(s) < 0$. The residue theorem thus provides the direct link between the frequency-domain analysis of linear systems and their time-domain behaviour.(14)

7.3 Dispersion Relations in Physics

A dispersion relation connects the real and imaginary parts of a response function by means of a principal-value integral derived from the Cauchy integral formula. For a response function $\chi(\omega)$ that is analytic in the upper half ω -plane:

$$\text{Re}[\chi(\omega)] = (1/\pi) \text{P.V.} \int_{-\infty}^{\infty} \text{Im}[\chi(\omega')] / (\omega' - \omega) d(\omega')$$

where P.V. denotes the principal value. These Kramers-Kronig relations connect the dispersive and absorptive parts of the dielectric susceptibility, the magnetic permeability, the refractive index, and any other causal linear response function. They have been verified experimentally across many orders of magnitude in frequency, providing a powerful consistency check on measured optical spectra.(14)

8: Conclusion

This thesis has surveyed the main applications of complex analysis to physics, progressing from the algebraic and analytic foundations to concrete physical problems in electrostatics, fluid mechanics, quantum mechanics, and linear systems theory. The unifying theme is the extraordinary power that analyticity — the requirement of complex differentiability — confers: functions satisfying this condition are globally constrained by their local behaviour, and this global structure can be probed efficiently by contour integrals.

The residue theorem, Cauchy's integral formula, conformal mapping, and the theory of analytic continuation are not mere mathematical tools grafted onto physics: they reflect deep physical principles. Causality is encoded in analyticity. Conservation laws constrain the singularity structure of S -matrices. Thermodynamic partition functions are analytic continuations of quantum amplitudes. These connections suggest that the complex plane is, in some fundamental sense, the natural arena for formulating physical law.

Several important topics lie beyond the scope of this thesis but deserve mention for further study. These include the Riemann Hypothesis and its connections to the distribution of primes and to quantum chaos; the role of complex analysis in the theory of Riemann surfaces and string theory; the Penrose transform and twistor theory, which reformulate relativistic field equations in terms of complex geometry; and the theory of several complex variables, which underlies modern algebraic geometry and has deep connections to quantum field theory.

We hope this thesis has communicated both the mathematical elegance and the physical richness of complex analysis, and has provided the reader with a solid foundation for further study.(15)

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