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paracompactness of topological spaces through α- open sets

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ
((وَاللَّهُ بِمَا تَعْمَلُونَ خَبِير)) المجادلة اية ١١

صَدَقَ اللهُ العَلِيُّ العَظيم

الإهداء

الى من روحي فداه امامي وسيدي .....

صاحب الزمان ....

إلى من علَّمني كيف أقف بكل ثبات فوق الأرض .....

أبي المحترم ....

إلى نبع المحبة والإيثار والكرم ....

أمي الموقَّرة .....

إلى أقرب الناس إلى نفسي .....

إلى جميع من تلقَّيتُ منهم النصح والدعم .....

أهديكم خلاصة جُهدي العلمي

Abstract:

In this search, the notion of paracompactness topological spaces of class α introduced and several properties of these spaces studied. A comparison between this class and the class of paracompactness topological spaces is presented. In particular, a sufficient condition for $\alpha-$regularity of $\alpha-$topological space is given .

الخلاصه:

في هذا البحث قمنا بدراسه الفضاءات التوبولوجيه الباراكومباكتيه مع بعض من بديهيات الفصل والمعرفه من خلال المجموعه المفتوحه كذلك تم دراسة بعض العلاقات الجديده بين هذه البديهيات من جهه وبينها وبين الفضاءات الباراكومباكتيه التوبولوجيه من جهة اخرى.

1. Introduction

Let A be a subset of a topological space X. Any point $x \in A$ is said to be interior of A, if x belongs to an open set G contained in A, i.e. $x \in G \subseteq A$. The set of interior points of A is denoted by int (A) or A°, which is called the interior of A. Th closure of A is defined as the intersection of all closed sets containing A.The Closure of A is denoted by Cl(A) .

A subset A of a topological space (X, τ) is said to be α-open [10] if A ⊆ int(cl(int(A))). The complement of a α-open set in a space (X, τ) is said to be α-closed [10]. The family of all α-open sets in a topological space (X, τ) is a topology on X finer than τ denoted by τα.

Many athures like [1,2,3,5,6,7,8,9,11,12] use this notion to introduce more general definitions using this concepts. The collection of all $\alpha - open$ set is denoted by αO(X) and the pair (X,αO(X)) is called the $\alpha -$ topological space associated with (X,Ƭ). We remark that (X, αO(X)) is a topological space. The complement of all $\alpha - open$ is called $\alpha - closed$ and the intersection of all $\alpha - closed$ set in X containig A is called $\alpha - clouser$ of A and is denoted by Cl_{α}(A).

CHAPTER ONE

BASIC DEFINITIONS AND PRILIMINARIES

1-1Basic definitions and preliminaries:

In this chapter we will display the basic definitions and the main concepts of our work like the definitions of a topological space and the cover of a topological space and what meaning by compact, paracompct topological spaces and many other related spaces also discussed and showed.

1.1.1. Definition:[10]

A subset A of a topological space (X, τ) is said to be α-open , if A ⊆int(cl(int(A))).

1.1.2. Exammple

As given, X = {a, b, c, d, e}
And τ = {φ, X, {a, b, c}, {d, e}, {c}, {d, e, c}}
So that we have
Open sets:{ φ, X, {a, b, c}, {d, e}, {c}, {d, e, c}
Closes sets: {X, φ, {d, e}, {a, b, c}, (a, b, d, e}, {a, b}
Now as per definition of α -open set, here we have

X, φ,{c},{d,e},{a,c}.{a,b,c}, {c,d,e}, {b, c, d, e}, {c, e, d, a} and {c, e, d, a} are α-open sets with respect to this topology.

1.1.3. Definition:[5]

Let Ѵ be a topological space. A family $\{A_s\}_{s\in S}$ of subset of Ѵ is called a cover of Ѵ if $\bigcup_{s\in S} A_s = Ѵ$. If all the setsA_s are open (closed), we say that the cover $\{A_s\}_{s\in S}$ is open (closed).

1.1.4. Definition: [5]

Let Ѵ be a topological space, a collection F=$\{F_i : i \in I\}$ of subsets of Ѵ is said to be locally finite if for each $v \in Ѵ$, $\exists$ open set U in Ѵ containing v and $U \cap F_\alpha \neq \emptyset$.

1.1.5. Definition: [5]

Let $\{A_s\}_{s\in S}$ be cover of Ѵ and let $\{B_t\}_{t\in T}$ be another cover we say that $\{B_t\}_{t\in T}$ is a refinement of $\{A_s\}_{s\in S}$ if $\forall\ t \in T, \exists\ s \in S \ni B_t \subseteq A_s$.

1.1.6. Definition: [5]

Let Ѵ be a topological space, then Ѵ is called a Paracompact space if it is hausdorff and every open cover of Ѵ has a locally finite open refinement cover.

1.1.7. Definition: [5]

Let Ѵ be a topological space, then Ѵ is called a compact space if it is hausdorff space with the property that every cover by open sets contains a finite sub cover

1.1.8. Definition: [5]

Let Ѵ be a topological space, then it is called a regular space if and only if for each $v \in Ѵ$ and closed set F in Ѵ with $v \notin F$, there are open sets U, V such that $v \in U, F \subseteq V \text{ and } U \cap V \neq \emptyset$.

1.1.9. Definition: [5]

A topological space V is called T_1 if for any two distinct points v and u of V there exist two disjoint open set U and V such that $u \in U$, $v \notin U$ and $v \in V$, $u \notin V$.

1.1.10. Definition: [5]

Let V be a topological space, then it is called a normal if and only if F_1 and F_2 are two disjoint closed subset of V, then there exists set G , H,∋ $F_1 \subset G$, $F_2 \subset H$ and $G \cap H = \emptyset$.

1.1.1. Definition: [5]

Let V be a topological space, then V is called a LIndelof space if the property that every cover by open sets contains a countable sub cover is hold.

1-2 Some important theorems

In this part we give some important theorem which we shall use and generalized in the second chapter.

1.2.1. Theorem: [4]

Every compact space is paracompact space.

1.2.2. Theorem: [4]

Every open cover of a Lindelof space has locally finite open refinement cover.

1.2.3. Theorem: [4]

Any Lindelof space is pararacompact .

1.2.4. Lemma: [4]

Let V be pararacompact space and A, B a pair of closed subsets of V. If for every $v \in B$ there exists open set U_v, V_v such that $A \subseteq U_v, v \in V_v$ and $U_v \cap V_v = \emptyset$, then there also exists open set U, V such that $A \subset U , B \subset V$ and $U \cap V = \emptyset$.

1.2.5. Theorem: [4]

Every paracompact space is normal.

1.2.6. Lemma: [4]

If V is T_1–space and for every closed set $F \subset$ V and every open $W \subset$ V that contains F threr exists a sequence $W_1, W_2, \ldots$ of open subset of V such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for i=1,2,.., then the space V is normal.

1.2.7. Lemma: [4]

If every open cover of a regular space V has a locally finite refinement (consisting of arbitrary sets), then for every open cover $\{U_s\}_{s \in S}$ of the space V there exists a closed locally finite cover $\{F_s\}_{s \in S}$ of V such that $F_s \subset U_s$ for every $s \in S$.

1.2.8.Lemma: [4]

If an open cover $\mathcal{U}$ of a topological space X has a closed locally finite refinement , then$\mathcal{U}$ has also an open barycentric refinement.

1.2.9.Lemma: [4]

If $\mathcal{A} = \{A_s\}$s $\in$ S of a set X is barycentric refinement of a cover $\mathcal{B} = \{B_t\}$t $\in$ T of X and $\mathcal{B}$ is a barycentric refinement of a cover $\mathcal{C} = \{C_z\} z \in Z$ of the same set , then $\mathcal{A}$ is a star refinement of $\mathcal{C}$.

CHAPTER TWO

α −PARACOMPACTNESS OF TOPOLOGICAL SPACES

2 $\alpha -$ paracompactness of topological spaces

2.1. Definition:

Let V be a topological space. A family $\{A_s\}_{s\in S}$ of subset of V is called a cover of V if $\bigcup_{s\in S} A_s = V$. If all the setsA_s are α −open (α −closed), we say that the cover $\{A_s\}_{s\in S}$ is α −open (α −closed).

2.2. Definition:

Let V be a topological space, a collection F=$\{F_\alpha : \alpha \in I\}$ of subsets of V is said to be α −locally finite if for each $v \in V$, $\exists\ \alpha -$ open set U in V containing v and $U \cap F_\alpha \neq \emptyset$.

2.3. Definition:

Let $\{A_s\}_{s\in S}$ be cover of V and let $\{B_t\}_{t\in T}$ be another cover we say that $\{B_t\}_{t\in T}$ is a refinement of $\{A_s\}_{s\in S}$ if $\forall\ t \in T$, $\exists\ s \in S \ni B_t \subseteq A_s$.

2.4. Definition:

Let V be a topological space, then V is called a Paracompact space of class α if it is Hausdorff and every open cover of V has a locally finite α −open refinement.

2.5. Definition:

Let V be a topological space, then V is called a compact space of class α if it is hausdorff space with the property that every cover by $\alpha -$ open sets contains a finite sub cover.

2.6. Definition:

Let V be a topological space, then it is called a regular space of class αif and only if for each $v \in$ V and $\alpha -$closed set F in V with $v \notin F$, there are $\alpha -$open sets U, V such that $v \in U , F \subseteq V \ and \ U \cap V \neq \emptyset$.

2.7. Theorem: Every compact space of class α is paracompact space of class α

Proof:

Let V is a compact space of class α →V is hausdorff and every $\alpha -$open cover of V has a finite sub cover , let V = {v_1 , v_2 , …., v_n } and $\{G_\lambda\}$ be $\alpha -$ open cover of V→V$\subseteq \cup G_\lambda$ → V $\subseteq \bigcup_{i=1}^{n} G_\lambda$, since by remark(every $\alpha -$ open cover is open cover) →$\{G_\lambda\}$ is open cover .

2.8. Definition:

Let V be a topological space, then it is called a Lindelof space of class α if V is regular and every $\alpha -$ open cover of V has a countable subcover.

2.9. Theorem:

Every open cover of a Lindelof space of class α has locally finite open refinement cover.

Proof:

It is clear by definition of a Lindelof space of class α and the definition of locally finite open refinement cover

2.10.Theorem . Any Lindelof space of class α is pararacompact of class α

Proof:

Let Ѵ a Lindelof space of class α, then it Ѵ is regular and every $\alpha -$ open cover of Ѵ has a countable sub cover, since Ѵ is regular that it is hausdorff , Let U be $\alpha -$ open cover of Ѵ, that has $\alpha -$locally finite open refinement cover which mean that Ѵ is pararacompact of class α.

2.11. Lemma:

Let Ѵ be pararacompact of class α and A, B a pair of $\alpha -$closed subsets of Ѵ. If for every $v \in B$ there exists $\alpha -$open set U_v, V_v such that $A \subseteq U_v, v \in V_v$ and $U_v \cap V_v = \emptyset$, then there also exists $\alpha -$open set U, V such that $A \subset U , B \subset V$ and $U \cap V = \emptyset$

Proof:

The family $\{V_v : v \in B\} \cup \{Ѵ \setminus B\}$ is an δ-open cover of the space Ѵ, so that it has a locally finite $\alpha -$open refinement $\{W_s\}_{s \in S}$. Letting $S_O = \{ s \in S : W_s \cap B \neq \emptyset \}$ we have

$A \cap \overline{W_s} = \emptyset$ for every $s \in S_O$ and $B \subset \bigcup_{s \in S_o} W_s$.

By theorem (foe every locally finite family $\{A_s\}_{s\in S}$ we have the equality $\overline{\bigcup_{s\in S} A_s} = \bigcup_{s\in S} \overline{A_s}$) the set U= X\ $\bigcup_{s\in S_o} \overline{W_s}$ is α −open ; one readily sees that U and V= $\bigcup_{s\in S_o} W_s$ have all the required properties.

2.12. Definition:

A topological space (Ѵ, α O(Ѵ)) is called $\alpha - T_1$if for any two distinct points v and u of Ѵ there exist two disjoint− α − open set U and V such that $u \in U , v \notin U \ and \ v \in V , u \notin V$.

2.13. Definition:

Let Ѵ be a topological space, then it is called a normal of class α if and only if F_1 and F_2 are two disjoint closed subset of Ѵ, then there exists α − open set G , H,∋ $F_1 \subset G , F_2 \subset H \ and \ G \cap H = \emptyset$.

2.14. Theorem:

Every paracompact space of class α is normal of class α.

Proof:

Substituting one –point sets for A in the above lemma, we set that every paracompact space of class α is α −regular; using this fact and applying the lemma again we obtain the theorem.

2.15. Lemma:

If V isα $-$T $_1$ –space and for every closed set $F \subset$ V and every open $W \subset$ V that contains F threr exists a sequence W_1,W_2,...of α $-$open subset of V such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for i=1,2,.., then the space V is normal of class α .

2.16. Lemma:

If every α $-$open cover of a α $-$regular space V has a locally finite refinement (consisting of arbitrary sets), then for every α $-$open cover $\{U_s\}_{s\in S}$ of the space V there exists a α $-$closed locally finite cover $\{F_s\}_{s\in S}$ of V such that $F_s \subset U_s$ for every $s \in S$.

Proof:

By $\alpha -$ regularity of V there exists an $\alpha -$ open cover $\mathcal{W}$ of the space V such that is $\{\overline{W}: W \in \mathcal{W}\}$ refinement of$\{U_s\}_{s\in S}$. Take a locally finite refinement $\{A_t\}_{t\in T}$ of the cover $\mathcal{W}$, for every $t \in T$ choose an s(t) $\in$ S such that $\overline{A_t} \subset U_{s(t)}$, and let $F_s = \bigcup_{s(t)=s} \overline{A}_t$. From theorem (for every locally finite family $\{A_s\}_{s\in S}$ we have the equality $\overline{\bigcup_{s\in S} A_s} = \bigcup_{s\in S} \overline{A_s}$)) and (If $\{A_s\}_{s\in S}$ is a locally finite (discrete) family then the family that $\{\overline{A}_s\}_{s\in S}$ also is a locally finite (discrete)) it follows readily that $\{F_s\}_{s\in S}$ is $\alpha -$ closed locally finite cover of V and the definition of the F_s's implies that $F_s \subset U_s$ for every $s \in S$.

2.18. Remark:

Let us note that if the cover $\{A_t\}_{t\in T}$ in the last proof is $\alpha -$ open, then the sets $V_s = \bigcup_{s(t)=s} A_t$ are open and $\overline{V}_s = F_s$. Hence, for every α $-$open cover $\{U_s\}_{s\in S}$ of paracompact space of class α there exists a locally finite α $-$open cove $\{V_s\}_{s\in S}$ r such that $\overline{V}_s \subset U_s$ for every $s \in S$.

4.22. Lemma:

Every open σ $-$locally finite cover $\mathcal{V}$ of a topological space X has a locally finite refinement.

Proof:

Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ where $\mathcal{V}_i = \{V_s\}$ s $\in S_i$ is locally finite family of open sets and $S_i \cap S_j$ whenever $\neq j$. For every $s_o \in S_i$ let

$$A_{so} = V_{so} \setminus \bigcup_{k<i} \bigcup_{s \in S_k} V_s$$

The family $\mathcal{A} = \{A_s\}$ s $\in$ S , where $= \bigcup_{i=1}^{\infty} S_i$, covers X and is a refinement of $\mathcal{V}$. We shall show $\mathcal{A}$ that is locally finite. Consider a point $\in X$, denote by the smallest natural number such that $x \in \bigcup_{s \in S_k} V_s$, and take an $s_o \in S_k$ satisfying $x \in V_{so}$; clearly V_{so} is a neighbourhood of x disjoint from all sets A_s with $s \in \bigcup_{i>k} S_i$. Since the families $\mathcal{V}_i$ are locally finite, for every $i \leq k$ there exists a neighbourhood U_i of x which meets only finitely many members of $\mathcal{V}_i$. The neighbourhood $U_1 \cap U_2 \cap \ldots \cap U_k \cap V_{so}$ of the point x meets only finitly many members of $\mathcal{A}$.

4.23. Theorem:

For every σ −regular space X the following condition are equivalent:

(i) The space X is paracompact of class α .

ii) Every α −open cover of the space X has an open σ −locally finite refinement

(iii) Every α −open cover of the space X has a locally finite refinement.

(iv) Every α −open cover of the space X has a closed locally finite refinement

Proof:

It is clear by definitions of paracompact of class α, α −open cover and open σ −locally finite refinement.

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