جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم رياضيات

Block matrix

بحث مقدم الى قسم رياضيات - كلية التربية - للعلوم الصرفة - جامعة بابل – و هو جزء من متطلبات نيل شهادة البكالوريوس في علوم الرياضيات

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بسم الله الرحمن الرحيم

وَلَقَدْ آتَيْنَا دَاوُودَ وَسُلَيْمَانَ عِلْمًا ۖ وَقَالَا الْحَمْدُ لِلَّهِ

الَّذِي فَضَّلَنَا عَلَىٰ كَثِيرٍ مِّنْ عِبَادِهِ الْمُؤْمِنِينَ

صَدَقَ اللَّهُ الْعَلِيُّ الْعَظِيمُ

الاهداء

اهدي تخرجي إلى معلم البشرية أجمعين الهادي الأمين ﷺ إلى من تعجز الكلمات عن إيفائه حقه إلى والدي العزيز أطال الله في عمره ورزقه الصحة والعافية وأحسن عمله إلى صاحبة القلب الصابر الحنون إلى من أنار لي دعائها حياتي والدتي العزيزة أطال الله في عمرها وأحسن عملها

شكر وتقدير

الحمد لله، عَدَدَ خلـقه ، ، وَزِنَـةَ عَـرْشِـه ، وَمِـدادَ كَلِمـاتِـه، الذي رفع شأن العلم وشرفه بقوله في اول كلماته التي انزلها من السماء (اقرأ باسم ربك الذي خلق) والصلاة والسلام على سيد الأنام محمد و على آله وصحبه الكرام وبعد ...

يشرفني في هذا المقام الكريم ان أقف إجلالا و احتراما امام القناديل العلمية التي أضاءت لي دروب العلم و بالأستاذة (د.رومى كريم) نتقدم أليها بجزيل الشكر وخالص المعرفة ، بدء التقدير لقبولها الاشراف على هذا البحث ،الذي ما كان لترى النور لولا جهودها -في ما تحقق فيه من إيجابيات.

كما اعبر عن شكري إلى كل أساتذتي في قسم الرياضيات لمساعدتي طيلة مدة السنوات التي قضيتها في الجامعة لمواقفهم الجميلة مع الطلاب.

Contents

Abstract

In this work , the Block matrices have been studied as an important topic in many applications. Some Fundemental facts of the block matrices are studied such as Block matrix multiplication, Addition of block matrices, the block matrix inversion

and others. On the other hand, Some theorems of the block matrices are used in applications, for example the special reduction formula, Generalization of Brauer theorem on stochastic matrices and other applications.

Chapter one
Block matrix

1.1 Introduction

In mathematics, a block matrix or a partitioned matrix is a matrix that is interpreted as having been broken into sections called blocks or submatrices. Intuitively, a matrix interpreted as a block matrix can be visualized as the original matrix with a collection of horizontal and vertical lines, which break it up, or partition it, into a collection of smaller matrices.Any matrix may be interpreted as a block matrix in one or more ways, with each interpretation defined by how its rows and columns are partitioned.

1.2 Important Facts to Block Matrices

A matrix is a rectangular array of numbers treated as a single object. A block matrix is a matrix whose elements are themselves matrices, which are called submatrices. By allowing a matrix to be viewed at different levels of abstraction, the block matrix viewpoint enables elegant proofs of results and facilitates the development and understanding of numerical algorithms.

A block matrix is defined in terms of a partitioning, which breaks a matrix into contiguous pieces. The most common and important case is for an $n \times n$ matrix to be partitioned as a block 2×2
matrix (two block rows and two block columns).
For n = 4, partitioning into 2×2 blocks gives

$$A = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} = A(1:2, 1:2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and similarly for the other blocks.[2]

Definition 1.2.1(Block Matrix)

A block matrix is a matrix that is defined using smaller matrices, called blocks. For example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad (1)$$

where A, B, C, and D are themselves matrices, is a block matrix. In the specific example

$$A=\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \tag{2}$$

$$B=\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \tag{3}$$

$$C=\begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \tag{4}$$

$$D=\begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}; \tag{5}$$

therefore, it is the matrix

$$\begin{bmatrix} 0 & 2 & 3 & 3 & 3 \\ 2 & 0 & 3 & 3 & 3 \\ 4 & 4 & 5 & 0 & 5 \\ 4 & 4 & 0 & 5 & 0 \\ 4 & 4 & 5 & 0 & 5 \end{bmatrix}. \tag{6}$$

Block matrices can be created using ArrayFlatten.

Definition 1.2.2 (Block Matrix Multiplication)

It is possible to use a block partitioned matrix product that involves only algebra on submatrices of the factors.

The partitioning of the factors is not arbitrary, however, and requires "conformable partitions" between two matrices A and B such that all submatrix products that will be used are defined.
Given an (m ×p) matrix A with q row partitions and S column partitions

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1s} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \cdots & \mathbf{A}_{qs} \end{bmatrix}$$

and a (p×n) matrix B with S row partitions and r column partitions

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1r} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{s1} & \mathbf{B}_{s2} & \cdots & \mathbf{B}_{sr} \end{bmatrix},$$

that are compatible with the partitions of A ،the matrix product C=AB
can be performed blockwise, yielding C as an (m× n) matrix with q row partitions and r column partitions. The matrices in the resulting matrix C are calculated by multiplying:

$$\mathbf{C}_{qr} = \sum_{i=1}^{s} \mathbf{A}_{qi} \mathbf{B}_{ir}.$$

Or, using the Einstein notation that implicitly sums over repeated indices:

$$\mathbf{C}_{qr} = \mathbf{A}_{qi}\mathbf{B}_{ir}.$$

[2]

Example 1.2.1

$$M_1 = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 7 & 8 \\ 10 & 12 \end{bmatrix}$$

$$M_1 M_2 = \begin{bmatrix} (2)(7) + (1)(10) & (2)(8) + (1)(12) \\ (4)(7) + (5)(10) & (4)(8) + (5)(12) \end{bmatrix}$$

$$= \begin{bmatrix} 14 + 10 & 16 + 12 \\ 28 + 50 & 32 + 60 \end{bmatrix} = \begin{bmatrix} 24 & 28 \\ 78 & 92 \end{bmatrix}$$

Theorem 1.2.1 : that the dot product formula can be applied to block matrices follows.

Proof:·

The proof, although tedious, allows us to better understand under what condition all the blocks can be multiplied. The partitions need to be such that a vertical partition of M□ leaves S□ columns to the left and S□ to the right if and only if an horizontal partition of M□ leaves S□ rows in the upper part of the matrix and S□ in the lower part.There are no constraints on the horizontal partitions of M□ and the vertical partitions of M_2 · ■

Definition 1.2.3 (Addition of block matrices)

If two block matrices M□ and M□ have the same dimension and are partitioned in the same way, we obtain their sum by adding the corresponding blocks.

Example 1.2.2 If

$$M_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

we can compute their sum as

$$M_1 + M_2 = \begin{bmatrix} A_1 + A_2 & B_1 + B_2 \\ C_1 + C_2 & D_1 + D_2 \end{bmatrix}$$

All the couples of summands need to have the same dimension. For instance, in the example above, if A_1 is J⍰K (J rows and K columns), then A_2 must be J⍰K·
This property of block matrices is a direct consequence of the definition of matrix addition.Two matrices having the same dimension can be added together by adding their corresponding entries.

Definition 1.2.4 (Block Matrix Inversion)

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix},$$

where A and D are square blocks of arbitrary size, and B and C are conformable with them for partitioning. Furthermore, A and the Schur complement of A in P: $P/A = D - CA^{-1}B$ must be invertible.
If A and D are both invertible, then:

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}.$$

Here, D and the Schur complement of D in P: $P/D = A - BD^{-1}C$ must be invertible.
If A and D are both invertible, then:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I} \end{bmatrix}.$$

The formula for the determinant of a 2×2 matrix above continues to hold, under appropriate further assumptions, for a matrix composed of four submatrices A,B,C,D. The easiest such formula, which can be proven using either the Leibniz formula or a factorization involving the Schur complement, is

$$\det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det(A)\det(D) = \det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

If A is invertible (and similarly if D is invertible), one has

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det\left(D - CA^{-1}B\right).$$

If D is a 1×1-matrix, this simplifies to $\det(A)(D - CA^{-1}B)$.

If the blocks are square matrices of the *same* size further formulas hold. For example, if C and D commute (i.e., $CD = DC$), then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

This formula has been generalized to matrices composed of more than 2×2 blocks, again under appropriate commutativity conditions among the individual blocks.

For $A = D$ and $B = C$, the following formula holds (even if A and B do not commute)

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A - B)\det(A + B).$$

Example 1.2.3

Example 1.2.3

$$M = \begin{pmatrix} 1 & -3 & 2 & 4 \\ 0 & 5 & 0 & 2 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & -6 & 3 \end{pmatrix}$$

$$\det M = \det(AD - BC)$$

we see that $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det M = \det\left(\begin{pmatrix} 1 & -3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 16 & -8 \\ -3 & 15 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 14 & -2 \\ -30 & 13 \end{pmatrix}$$

$$= (14)(13) - (-12)(-30) = -178$$

Definition 1.2.5 (Block Diagonal Matrices)

A block diagonal matrix is a block matrix that is a square matrix such that the main-diagonal blocks are square matrices and all off-diagonal blocks are zero matrices. That is, a block diagonal matrix A has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix}$$

where A_k is a square matrix for all k = 1, ..., n.
In other words, matrix A is the direct sum of A_1, ..., A_n. It can also be indicated as $A_1 \oplus A_2 \oplus \ldots \oplus A_n$ or diag(A1, A2, ..., An) (the latter being the same formalism used for a diagonal matrix). Any square matrix can trivially be considered a block diagonal matrix with only one block.
For the determinant and trace, the following properties hold

$$\det \mathbf{A} = \det \mathbf{A}_1 \times \cdots \times \det \mathbf{A}_n,$$
$$\operatorname{tr} \mathbf{A} = \operatorname{tr} \mathbf{A}_1 + \cdots + \operatorname{tr} \mathbf{A}_n.$$

A block diagonal matrix is invertible if and only if each of its main-diagonal blocks are invertible, and in this case its inverse is another block diagonal matrix given by

$$\begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n^{-1} \end{bmatrix}.$$

The eigenvalues and eigenvectors of A are simply those of the A□ combined.

Definition 1.2.6(Block tridiagonal matrices)

A block tridiagonal matrix is another special block matrix, which is just like the block diagonal matrix a square matrix, having square matrices (blocks) in the lower diagonal, main diagonal and upper diagonal, with all other blocks being zero matrices. It is essentially a tridiagonal matrix but has submatrices in places of scalars. A block tridiagonal matrix A has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{C}_1 & & & \cdots & & 0 \\ \mathbf{A}_2 & \mathbf{B}_2 & \mathbf{C}_2 & & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & \mathbf{A}_k & \mathbf{B}_k & \mathbf{C}_k & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ & & & & \mathbf{A}_{n-1} & \mathbf{B}_{n-1} & \mathbf{C}_{n-1} \\ 0 & & \cdots & & & \mathbf{A}_n & \mathbf{B}_n \end{bmatrix}$$

where A□, B□ and C□ are square sub-matrices of the lower, main and upper diagonal respectively.
Block tridiagonal matrices are often encountered in numerical solutions of engineering problems (e.g., computational fluid dynamics). Optimized numerical methods for LU factorization are available and hence efficient solution algorithms for equation systems with a block tridiagonal matrix as coefficient matrix.
The Thomas algorithm, used for efficient solution of equation systems involving a tridiagonal matrix can also be applied using matrix operations to block tridiagonal matrices .

Definition 1.2.7 (Direct sum)

For any arbitrary matrices A (of size m × n) and B (of size p × q), we have the direct sum of A and B, denoted by A⊕B and defined as

$$\mathbf{A} \oplus \mathbf{B} = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{p1} & \cdots & b_{pq} \end{bmatrix}.$$

For instance,

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This operation generalizes naturally to arbitrary dimensioned arrays (provided that A and B have the same number of dimensions).

Note that any element in the direct sum of two vector spaces of matrices could be represented as a direct sum of two matrices.

Chapter two

Applications of Block Matrix

Applications of Theorem on Partitioned Matrices

2.1Introduction

In a previous paper a general reduction formula was given for certain Partitioned matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{bmatrix}$$

of order N , where the submatrices $A_{i,j}$ have dimensions $n_i \times n_j$, $i,j=1,2\ldots\ldots,t$ and

$$\sum_{i=1}^{t} n_i = N.$$

One of the special cases of this general theorem

is of particular interest in practical applications to the problem of ftnding the eigenvalues of a matrix and, in fact, has been applied successfully to such
problems. This special case is given below as theorem 1, since its proof is simpler than that for the general theorem and exhibits the transformation matrices needed for the reduction formula. This result is used insection 3 to give generalizations of some theorems by A. Brauer 2 on stochastic matrices. Other applications are given in later sections. As in the previous paper, for a given partitioning of a matrix A we shall call the submatrices, Ai the blocks of A and we sball write A = (Aij). The clements of the blocks will be denoted by, $a_{hk}^{(ij)}$ i.e.

$$A_{ij} = (a_{hk}^{(ij)}).$$

Unless otherwise stated, the matrices will be arbitrary complex matrices. Also, since we will be dealing throughout with matrices A = (Aij) of form (1) , we will assume,unless otherwise indicated, that the statements and formulas given are true for i,j= 1,2, ... , t. [3].

2.2 Special Reduction Formula

For the sake of completeness we include a lemma from the previous paper which is needed in the proofs of theorems 1 and 3. In this lemma we consider three cases where there are zeros in convenient places in the blocks of a partitioned matrix, A. In each of these cases A is reducible, and the proof consists merely in defining the permutation matrix which puts A into the reduced form,

$$\tilde{A}=\begin{pmatrix} C & O \\ D & T \end{pmatrix}, \qquad (2)$$

where C and T are square matrices and O is a matrix composed entirely of zeros. [4]

Lemma2.2.1: Given a partitioned matrix A of order N with $n_i\times n_j$ blocks Aij:
1. If ni=n (i= 1,2, ... , t) and the blocks Aij are lower triangular with elements $\lambda_{ij}^{(h)}$, h=1,2, ... n, on the diagonal, A is similar to a matrix Ã, with blocks $\tilde{A}_{hh}=(\lambda_{ij}^{(h)})$ h= 1,2, ...,n, on the diagonal, and zero blocks above the diagonal

$$A_{ij}=\begin{pmatrix} C_{ij} & O \\ D_{ij} & T_{ij} \end{pmatrix}, \qquad (3)$$

.

where all matrices T_{ij} are square, of order r, and the matrices C_{ij} are $(n_i-r)\times(n_j-r)$, then A has the form (2), where C= (C_{ij}) and T = (T_{ij}).
3. If A_{ij} has the form (3) where the matrices T_{ij} are lower triangular, we will say A_{ij} is partially triangular. Then if A has blocks A_{ij} which are all partially trianoular with submatrices, T_{ij} of order r , having
elements $t_{ij}^{(h)}$), h= 1,2, ... ,r, on the diagonal, tr roots of A are roots of the r matrices, $(t_{ij}^{(h)})$.

The proofs given below would follow in a corresponding manner if all blocks were transposed.

Proof :

1. The rows and columns of A should be arranged in the order

    1, n+ 1, 2n+ 1,..., (t- l)n+1; …(4)

    2, n+ 2, 2n+ 2,...,(t- l)n+ 2;

    n, 2n, 3n, . . . , tn.

Then the new matrix Ã will have the matrices A□□= (λ_{ij}□^{h}□) on the diagonal and $\tilde{A}_{h\ k}$ = 0 for i< j, so its roots are the roots of the n matrices ,Ã□□ .

2. Let $\sum_{i=1}^{k} n_i = N_k$. Then $N_1 = n_1$, $N_t = N$

r

If we arrange the rows and columns of A in the following order:

1,2,...,N□-r,

N_1 +1,N_1 +2,...,N_2 -r, …(5)

N_{t-1} +1,N_{t-1} +2,...,N_t -r,

N□-r+1,N□-r+2,...,N□,

N□-r+1,N□-r+2,...,N□,

we have a new matrix A in which the matrices C_{ij} are together in the upper left corner, and the matrices T_{ij} are together in the lower right corner. So A will have the form (2) where C= (Cij) and T= (T_{ij}).

3. Case 3 follows immediately now by first applying the permutation in 2 to the rows and columns of A and then applying the permutation in 1 to the rows and columns containing T.

Theorem2.2. 1: Suppose the blocks A_{ij} of the partitioned matrix given in (1) satisjy the equation

$$A_{ij}X_j=X_iB_{ij} \quad ....(6)$$

where B_{ij} is a square matrix of order

$0< r \leq n_i$

with strict inequality for at least one value of i, and X_i is an $n_i \times r$ matrix with a nonsingular matrix of order r, X_j□□□, in the first r rows. Let the last n_i-r rows of X_i be X□□□□, and let

$$A_{ij}=\begin{pmatrix} A_{11}^{(ij)} & A_{12}^{(ij)} \\ A_{21}^{(ij)} & A_{22}^{(ij)} \end{pmatrix} \qquad (7)$$

where A□□□□^{j}□ is square, of order r. Then A is similar to the matrix,

$$R=\begin{pmatrix} B & * \\ O & C \end{pmatrix} \tag{8}$$

where B is a partitioned matrix of order tr with blocks B_{ij}, as defined in (6), and C has blocks,

$$C_{ij}=(A_{22}^{(ij)}-X_2^{(i)}(X_1^{(i)})^{-1}A_{12}^{(ij)}) \tag{9}$$

with dimensions $(n_i-r)\times(n_j-r)$. (If either n_i or $n_j=r$, the corresponding block C_{ij} does not appear. By hypothesis not all n_i are equal to r, or else we would be left with the matrix $B=(B_{ij})$ which would be similar to A.)

Thus the roots of A are the roots of the smaller matrices B and C.

PROOF: Let P_i be a matrix of order n_i:

$$P_i=\begin{pmatrix} X_1^{(i)} & O \\ X_2^{(i)} & I_{n_i-r} \end{pmatrix}; \tag{10}$$

then

$$P_i^{-1}=\begin{pmatrix} (X_1^{(i)})^{-1} & O \\ -X_2^{(i)}(X_1^{(i)})^{-1} & I_{n_i-r} \end{pmatrix}, \tag{11}$$

where I_k represents an identity matrix of order k.

Since by (7) and (10)

$$A_{ij}P_j=\begin{pmatrix} [A_{11}^{(ij)}X_1^{(j)}+A_{12}^{(ij)}X_2^{(j)}] & A_{12}^{(ij)} \\ [A_{21}^{(ij)}X_1^{(j)}+A_{22}^{(ij)}X_2^{(j)}] & A_{22}^{ij} \end{pmatrix},$$

then by (6),

$$A_{ij}P_j=\begin{pmatrix} [X_1^{(i)}\cdot B_{ij}] & A_{12}^{(ij)} \\ [X_2^{(i)}\cdot B_{ij}] & A_{22}^{(ij)} \end{pmatrix},$$

and, using (11) and (9),

$$\tilde{A}_{ij}=P_i^{-1}A_{ij}P_j=\begin{pmatrix} B_{ij} & (X_1^{(i)})^{-1}A_{12}^{(ij)} \\ O & C_{ij} \end{pmatrix}. \tag{12}$$

So, if we let P be the direct sum of the matrices P_i,

$$P=\Sigma\cdot P_i=P_1\dot{+}P_2\dot{+}\ \dots\ \dot{+}P_t, \tag{13}$$

and let

$$\tilde{A}=P^{-1}AP$$

then $\tilde{A}$ has the blocks $\tilde{A}_{ij}$ given in (12), and the simultaneous permutation of rows and columns of A given by case 2 of the lemma above, will put it into form (8)

2.3 Generalizations of Brauer Theorems on Stochastic Matrices

A. Brauer proved a number of interesting theorems about generalized stochastic matrices, i.e., matrices $A = (a_{ij})$ of order n snch

$$\sum_{j=1}^{n} a_{ij}=s, \qquad (i=1,2,\ldots n). \qquad (14)$$

Such matrices have as one root, s, and the vector corresponding to this root is

$Un= (1,1,...,1)$ ...(14a)

The reduction formula of theorem 1 is applied in this section to matrices which can be partitioned into stochastic blocks, to give generalizations of Brauer's results.

If a matrix A of order N can be partitioned in to rectangular ($n_i \times n_j$) blocks, in each of which the row-sums are all equal , i.e.,

$$A_{ij}=(a_{hk}^{(ij)}), \qquad (h=1,2,\ldots,n_i, \quad k=1,2,\ldots,n_j)$$

with

$$\sum_{k=1}^{n_j} a_{hk}^{(ij)}=s_{ij}, \qquad (h=1,2,\ldots,n_i), \tag{15}$$

A is a block-stochastic matrix.

2.4 Other Applications

In this section we will mention several other types of partitioned matrices,
$A = (A_{ij})$, such that

$$PAP^{-1}=\begin{pmatrix} B & * \\ O & C \end{pmatrix}, \tag{20}$$

where P is the direct sum of transformation matrices
P_i, and $B = (B_{ij})$, $C = (C_{ij})$, are partitioned matrices
which will be defined in each case. We also give a case where the transformation PAP^{-1} produces a real matrix from a complex one. Since the proof in each case is the same and consists merely in performing the indicated matrix multiplications, it will not be given in detail.

2.4.1. Block-Circulant Matrices

This is the case where each block A_{ij} is a circulant
matrix of order n. The result (21) is given by Williamson.
The roots of A are roots of the n matrices of order t

$$(\lambda_{ij}^{h})=\left(\sum_{k=1}^{n} a_{hk}^{(ij)}\epsilon_h^k\right) \qquad (h=1, 2, \ldots, n) \qquad (21)$$

where ϵ_h is one of the nth roots of unity.

[5]

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