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Department of mathematics

# Analytic Solution for Integral Equation by Using Integral Transform

**Graduation Research**

**Research submitted to University of Babylon/College of Education for pure Sciences /Department of Mathematics as Partial Fulfillment of the Requirements for the Degree of B.Sc. in Mathematical Science**

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بسم الله الرحمن الرحيم

{ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ }

صدق الله العلي العظيم

[المجادلة :١١]

إِهْدَاءُ

إلى من كلله الله بالهيبة والوقار ... إلى من علمني العطاء دون انتظار .... إلى من احمل أسمة بكل افتخار والدي العزيز

إلى صاحبة القلب الطيب والنوايا الصادقة إلى من رافقتني منذ حملنا حقائب صغيره معكم سرت الدرب خطوة خطوة أمي الحبيبة

إلى من هم اقرب إلية من روحي إلى من شاركوني حضن أمي ومنهم استمد عزتي وإصراري . اخواني

## الشُكر والتقدير

الحمد لله الذي علم بالقلم، علم الانسان مالم يعلم، والصلاة والسلام على النبي الامين الذي ارسله الله بالهوى ودين الحق ورحمة للعالمين، وعلى الة وصحبة المنتجبين، ولا حول ولا قوة الابالله العلي العظيم وبعد.... قال الرسول (صلى الله عليه واله وسلم)

(ومن صنع اليكم معروفا فكافئوه، فان لم تجدوه ما تكافئونه فادعوا له حتى ترو انكم قد كافئتموه) صدق رسول الله

تعجز الكلمات، وتتوارى الحروف، ويخجل القلم ان يقف هذا الموقف، فقد تخونه العبارات ، وتتشتت الجمل، ويضيع المعنى، ولا يصل الهدف، ولكني اجتهدت لا رد اليكم بعضا مما اخذت

لذا يسرني ان اتقدم بكلمة شكر وامتنان ووفاء اضعها في صدر هذا البحث، فانني اسجلها بكل اعتزاز وتقدير للدكتورة **(د. هدى عامر هادي)** الذي تحملت عبء الاشراف على هذا البحث فقد بذلت الجهد وقدم التوجيه السديد والراي الناصح، وهذا بعض الوفاء والتقدير والامتنان نظير ما قدمتة لي، فجزاه الله عني خير الجزاء.

# Contents

# Abstract

In this research, the analytical solution of the integral equation was studied using integral transformation. This research includes two chapters.

The first chapter: We studied the integral transformation in general and touched on some definitions and proofs related to the analytical solution of the Integral equation using the integral transformation. The second chapter: We touched on the basic idea of the newly proposed method of Hebel integration to clarify the basic idea of the HPTM for solving a differential equation. We touched on a group of mathematical examples.

# Section One

# 1. Introduction

Integral transformations are important for solving real problems. Appropriate selection of integral transformations helps convert differential equations as well as integral equations into an algebraic equation that can be easily solved.

During the last two decades, many integral transformations have been introduced into the Laplace transform class, such as the Soumdo, Al-Zaki, Al-Natury, Aboudah, Bureza, Muhannad, G-Transform, Al-Sawy, and Kamal transformations. Integral transformations are valuable for the simplification they bring about, most often when dealing with differential equations subject to specific boundary conditions. The solution achieved is of course a transformation of the solution of the original differential equatorial transform.

Let $f(t)$ be a integrable function defined for $t \geqslant 0, p(s) \neq 0$ and $q(s)$ are positive real functions, we define the general integral transform $\mathscr{T}(s)$ of $f(t)$ by the formula

$$T\{f(t);s\}= \mathscr{T}(s)= p(s)\int_0^{\infty} f(t)e^{-q(s)t}\, dt, \qquad (1)$$

## 1.1 Integral Transform

**Theorem 1** Let f(t) is differentiable and p(s) and q(s) are positive real functions. Then

(I) $T\{f'(t);s\} = q(s)T(s) - p(s)f(0).$

(II) $T\{f''(t);s\} = q^2(s)T\{f(t);s\} - q(s)p(s)f(0) - p(s)f'(0).$

(III) $T\left\{f^{(n)}(t);s\right\} = q^n(s)T\{f(t);s\} - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)f^{(k)}(0).$

**Proof.** (I). In view of (1) we have

| Function | New integral transform |
|---|---|
| $f(t) = T^{-1}\{\mathscr{T}(s)\}$ | $\mathscr{T}(s) = T\{f(t);s\}$ |
| $1$ | $\frac{p(s)}{q(s)}$ |
| $t$ | $\frac{p(s)}{q(s)^2}$ |
| $t^\alpha$ | $\frac{\Gamma[\alpha+1]p(s)}{q(s)^{\alpha+1}}, \quad \alpha > 0$ |
| $\sin t$ | $\frac{p(s)}{q(s)^2+1}$ |
| $\sin(at)$ | $\frac{ap(s)}{a^2+q(s)^2}$, if $q(s) > \lvert \Im(a) \rvert$ |
| $\cos t$ | $\frac{q(s)p(s)}{q(s)^2+1}$ |
| $e^t$ | $\frac{p(s)}{q(s)-1}, q(s) > 1,$ |
| $tH(t-1)$ | $\frac{e^{-q(s)}(q(s)+1)p(s)}{q(s)^2}$ |
| $f'(t)$ | $q(s)\mathscr{T}(s) - p(s)f(0)$ |

**Table 1. Table of new integral transform.**

*H. Jafari*

$$T\{f'(t);s\} = p(s)\int_0^\infty f'(t)e^{-q(s)t}dt$$
$$= p(s)\left[e^{-q(s)t}f(t)|_0^\infty + q(s)\int_0^\infty f(t)e^{-q(s)t}dt\right]$$
$$= q(s)T\{f(t);s\} - p(s)f(0).$$

To proof (II), we assume $h(t) = f'(t)$ so $f''(t) = h'(t)$ now

$$T\{h'(t);s\} = p(s)\int_0^\infty h'(t)e^{-q(s)t}dt = q(s)T\{h(t);s\} - p(s)h(0)$$
$$= q(s)T\{f'(t);s\} - p(s)f'(0)$$
$$= q(s)[q(s)T\{f(t);s\} - p(s)f(0)] - p(s)f'(0)$$
$$= q^2(s)T\{f(t);s\} - q(s)p(s)f(0) - p(s)f'(0).$$

**Theorem 2** let $p(s)$, $q(s)$ and $f(t)$ are differentiable ($q'(s) \neq 0$), then:

## 1.2 Fourier Transform

$$T\left\{t^2 f^{(n)}(t)\right\} = \frac{p(s)}{q'(s)}\frac{d}{ds}\left[\frac{1}{q'(s)}\left(\frac{d}{ds}\left(\frac{1}{p(s)}T\left\{f^{(n)}(t)\right\}\right)\right)\right].$$

**Proof**

$$T\left\{t\, f^{(n)}(t)\right\} = p(s)\int_0^\infty t\;\; f^{(n)}(t)\;\; e^{-q(s)t}dt = -\frac{p(s)}{q'(s)}\frac{d}{ds}\left(\frac{1}{p(s)}T\left\{f^{(n)}(t)\right\}\right).$$

By derivation of above equation respect to $s$ we have

$$-\frac{d}{ds}\left[\frac{1}{q'(s)}\left(\frac{d}{ds}\left(\frac{1}{p(s)}T\left\{f^{(n)}(t)\right\}\right)\right)\right] = -q'(s)\int_0^\infty t^2 f^{(n)}(t)e^{-q(s)t}dt = -q'(s)\frac{T\left\{t^2 f^{(n)}(t)\right\}}{p(s)},$$

Thus



$$T\left\{t^2 f^{(n)}(t)\right\} = \frac{p(s)}{q'(s)}\frac{d}{ds}\left[\frac{1}{q'(s)}\left(\frac{d}{ds}\left(\frac{1}{p(s)}T\left\{f^{(n)}(t)\right\}\right)\right)\right].$$

One of the most useful of the infinite number of possible transforms

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt.$$

Two modifications of this form, developed in Section 15.3, are the Fourier cosine and Fourier sine transforms:

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos \omega t \, dt,$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin \omega t \, dt.$$

is the Fourier transform, given by

In physics, engineering and mathematics, the Fourier transform (FT) is an integral transform that takes as input a function and outputs another function that describes the extent to which various frequencies are present in the original function. The output of the transform is a complex-valued function of frequency. The term Fourier transform refers to both this complex-valued function and the mathematical operation. When a distinction needs to be made the Fourier transform is sometimes called the frequency domain representation of the original function. The Fourier transform is analogous to decomposing the sound of a musical chord into the intensities of its constituent pitches.

The Fourier transform is based on the kernel $e^{iwt}$ and its real and imaginary parts taken separately, $cos\omega t$ and $sin\omega t$. Because these kernels are the functions used to describe waves, Fourier Transforms appear frequently in studies of waves and the extraction of information. From waves, particularly when phase information is involved.

The Fourier transform of a Gaussian function $e^{-a^2t^2}$,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2t^2} e^{i\omega t}\, dt,$$

can be done analytically by completing the square in the exponent,

$$-a^2t^2 + i\omega t = -a^2\left(t - \frac{i\omega}{2a^2}\right)^2 - \frac{\omega^2}{4a^2},$$

which we check by evaluating the square. Substituting this identity we obtain

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a^2} \int_{-\infty}^{\infty} e^{-a^2t^2}\, dt,$$

upon shifting the integration variable $t \to t + \frac{i\omega}{2a^2}$. This is justified by an application of Cauchy's theorem to the rectangle with vertices $-T$, $T$, $T + \frac{i\omega}{2a^2}$, $-T + \frac{i\omega}{2a^2}$ for $T \to \infty$, noting that the integrand has no singularities in this region and that the integrals over the sides from $\pm T$ to $\pm T + \frac{i\omega}{2a^2}$ become negligible for $T \to \infty$. Finally we rescale the integration variable as $\xi = at$ in the integral (

$$\int_{-\infty}^{\infty} e^{-a^2t^2}\, dt = \frac{1}{a} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \frac{\sqrt{\pi}}{a}.$$

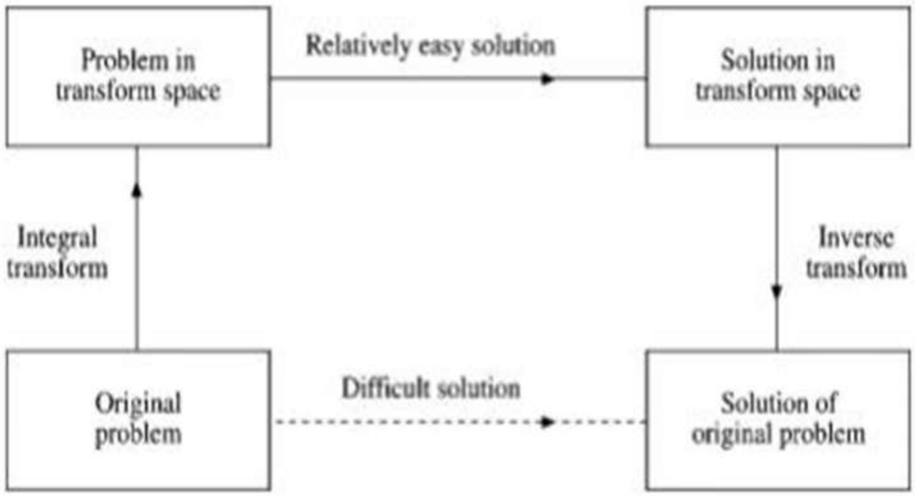
Substituting these results we find

$$g(\omega) = \frac{1}{a\sqrt{2}} \exp\left(-\frac{\omega^2}{4a^2}\right),$$

again a Gaussian, but in $\omega$-space. The bigger $a$ is, that is, the narrower the original Gaussian $e^{-a^2t^2}$ is, the wider is its Fourier transform $\sim e^{-\omega^2/4a^2}$. ■

## Example 1.2.1 FOURIER TRASFORM OF GAUSSIAN

## FFIGURE 1. Schematic integral transforms




**Example 1.2.2** **The Fourier Transforms for a Function of two Variables**

$$F(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int f(x, y) e^{i(ux+vy)}\, dx\, dy,$$

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int F(u, v) e^{-i(ux+vy)}\, du\, dv.$$

Using $f(x, y) = f([x^2 + y^2]^{1/2})$, show that the zero-order Hankel transforms

$$F(\rho) = \int_0^{\infty} r f(r) J_0(\rho r)\, dr,$$

$$f(r) = \int_0^{\infty} \rho F(\rho) J_0(\rho r)\, d\rho,$$

are a special case of the Fourier transforms.

## 1.3 Analytic Solution

Noticing that quartics admits analytical solutions, Mortari [4] found a closed-form solution for Wahba's Problem. We will show by an example that this method can be numerically unstable. Our proposed Algorithm will use the characteristic polynomial of the K-matrix presented in [4] which is given as follows.

P(x) = x

4 + ax3 + bx2 + cx + d = 0, (6)

Where a = 0, b = −2(tr[B])2 + tr[adj(S)] – z

Tz, S = B + BT, adj(S) the adjugated matrix of S,

C = −tr[adj(K)], and d = det(K) are all known parameters. It is well-known that a polynomial of degree 4admits analytic.

## 1.4 Analytic Function

Analytic functions are a particular subset of the class of all complex-valued functions of a complex variable. The real and imaginary parts of complex numbers are constantly required. We use the standard notation that if

$$z = x + iy, \qquad i^2 = -1,$$

is a complex number, with $x$, $y$ real numbers, then

$$x = \mathrm{Re}(z), \qquad y = \mathrm{Im}(z).$$

We shall refer to the set of all complex numbers as the complex plane, and denote it by $\mathbb{C}$.

An alternative polar form is often used, using the modulus $|z|$ and argument $\theta = \arg(z)$:

$$|z| = \sqrt{x^2 + y^2}, \qquad x = |z| \cos\theta, \qquad y = |z| \sin\theta.$$

# Section Tow

## 2. Basic idea of newly proposed method for Abel's integral equation

**2.1** To illustrate the basic idea of the HPTM for solution of singular integral equation of Abel type, we consider the following tables integral equation

To illustrate the basic idea of the HPTM for solution of singular integral equation of Abel type, we consider the following Abel's integral equation of second kind as

$$y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt, \quad 0 \leqslant x \leqslant 1. \tag{2.1}$$

Operating the Laplace transform on both sides in Eq. (2.1), we get

$$L[y(x)] = L[f(x)] + L\left\{\int_0^x \frac{y(t)}{\sqrt{x-t}}\,dt\right\}. \tag{2.2}$$

By using the convolution property of the Laplace transform, Eq. (2.2) takes the form

$$L[y(x)] = L[f(x)] + \sqrt{\frac{\pi}{s}}L[y(x)]. \tag{2.3}$$

Operating the inverse Laplace transform on both sides in Eq. (2.3), we get

This is coupling of the Laplace transform and homotopy perturbation method. Now, equating the coefficient of corresponding power of $p$ on both sides, the following approximations are obtained as:

$$p^0 : \psi_0(x) = f(x), \quad p^n : \psi_n(x) = L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\psi_{n-1}(x))\right\},$$
$$n = 1, 2, 3, \ldots \tag{2.7}$$

Hence the solution of the Eq. (2.1) is given as

$$y(x) = \lim_{p \to 1} \psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \tag{2.8}$$

It is to be noted that the rate of convergence of the series (2.8) depends upon the initial choices $\psi_0(x)$ as illustrated by the given numerical examples. It is worth to note that the major advantage of homotopy perturbation transform method is that the perturbation equation can be freely constructed in many ways (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected.

of second kind as

## 2.2 Illustrative Examples

**example1.We consider the following Abel integral equation of the second kind as**

$$y(x) = x + \frac{4}{3}x^{3/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt, \quad 0 \leqslant x \leqslant 1, \qquad (3.1)$$

By applying the aforesaid homotopy perturbation transform.

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = x + \frac{4}{3}x^{3/2} - p\left\{L^{-1}\left(\sqrt{\frac{\pi}{s}}L\left(\sum_{n=0}^{\infty} p^n \psi_{n-1}(x)\right)\right)\right\}. \qquad (3.2)$$

Now equating the coefficients of corresponding power of p on Both sides in Eq. (3.2), the following iterates $wn(x\ ,n\ =\ 0,1,2,33,...$ are given as

$$p^0 : \psi_0(x) = x + \frac{4}{3}x^{3/2},$$

$$p^1 : \psi_1(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\psi_0(x))\right\} = -\frac{4x^{3/2}}{3} - \frac{\pi x^2}{2},$$

$$p^2 : \psi_2(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\psi_1(x))\right\} = \frac{\pi x^2}{2} + \frac{8\pi x^{5/2}}{15},$$

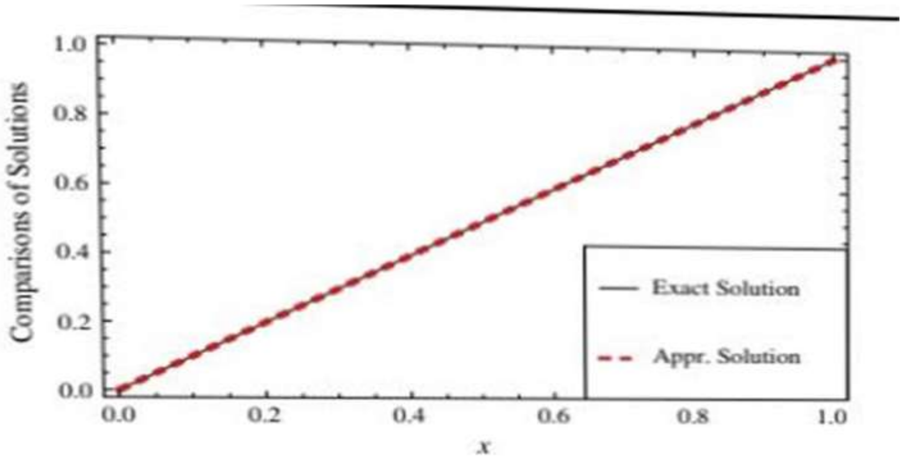
$$p^3 : \psi_3(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\psi_2(x))\right\} = -\frac{8\pi x^{5/2}}{15} - \frac{\pi^2 x^3}{6},\ldots$$

$$p^{25} : \psi_{25}(x) = -L^{-1}\left\{\sqrt{\frac{\pi}{s}}L(\psi_{24}(x))\right\}$$

$$= -\frac{16384\pi^{12}x^{27/2}}{213458046676875} - \frac{\pi^{13}x^{14}}{87178291200}.$$

Finally, we approximate the analytical solution$y(x)$by the truncated series as

**FFIGURE 3 The Comparison between the exact solution (straight line)**




**And the approximate solution (dotted line) of the Abel integral equation.**

**Example 2.** **In this example, we consider the following Abel integral equation of the second kind as follows** [6]:

$$y(x) = \sum_{i=0}^{\infty} \psi_i(x) = \sum_{i=0}^{n} \psi_i(x) + O(x^{1+n/2}) \to x \text{ as } n \to \infty. \quad (3.3)$$

$$y(x) = 2\sqrt{x} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \leqslant x \leqslant 1, \tag{3.4}$$

which has $y(x) = 1 - e^{\pi x} erfc(\sqrt{\pi x})$ as the exact solution, where the complimentary error function $erfc$is defined as $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du.$

Now applying the aforesaid homotopy perturbation transform method, we get

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = 2\sqrt{x} - p\left\{L^{-1}\left(\sqrt{\frac{\pi}{s}} L\left(\sum_{n=0}^{\infty} p^n \psi_n(x)\right)\right)\right\}. \tag{3.5}$$

The various $\psi_n(x)$, $n = 0, 1, 2, 3, \ldots$ are given as

$$\psi_0(x) = 2\sqrt{x}, \quad \psi_1(x) = -\pi x, \quad \psi_2(x) = \frac{4\pi x^{3/2}}{3},$$
$$\psi_3(x) = -\frac{\pi^2 x^2}{2}, \quad \psi_4(x) = \frac{8\pi^2 x^{5/2}}{15}, \ldots$$

Hence the solution of the given problem (3.4) is give as

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} \psi_n(x) \\ &= 2\sqrt{x} - \pi x + \frac{4\pi x^{3/2}}{3} - \frac{\pi^2 x^2}{2} + \frac{8\pi^2 x^{5/2}}{15} - \frac{\pi^3 x^3}{6} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (\pi x)^{n/2}}{\Gamma(1 + n/2)} = 1 - E_{1/2}(-\sqrt{\pi x}) \\ &= 1 - e^{\pi x} erfc(\sqrt{\pi x}). \end{aligned} \tag{3.6}$$

This is the exact solution of the Abel integral Eq. (3.4) and $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ is the Mittag-Leffler function [25] of one parameter.

**Example 3.** **Consider the following Volterra integral equation.**

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,dt,$$

By applying this new transform on both sides of this equation

And using Convolution Theorem we get

$$\mathscr{Y}(s) = T\{t\} + \frac{1}{p(s)}\mathscr{F}(s)\cdot\mathscr{Y}(s) \longrightarrow \mathscr{Y}(s) = \frac{p(s)}{q(s)^2}$$
$$+ \frac{1}{p(s)}\frac{p(s)}{q(s)^2+1}\mathscr{Y}(s),$$
$$\mathscr{Y}(s) = \frac{q(s)^2+1}{q(s)^2}\frac{p(s)}{q(s)^2} = \frac{p(s)q(s)^2}{q(s)^4} + \frac{p(s)}{q(s)^4} \longrightarrow y(t) = t$$
$$+ \frac{1}{6}t^3.$$

**Example 4.** **Consider the following Volterra integral equation**

$$y(t) = 1 - \int_0^x (t-\tau)y(t)dt.$$

By applying this new transform on both sides of this equation

$$\mathscr{Y}(s) = T\{1\} - \frac{1}{p(s)}\mathscr{F}(s)\cdot\mathscr{Y}(s) \longrightarrow \mathscr{Y}(s)$$
$$= \frac{p(s)}{q(s)} - \frac{1}{p(s)}\frac{p(s)}{q(s)^2}\mathscr{Y}(s),$$
$$\mathscr{Y}(s) = \frac{q(s)^2}{q(s)^2+1}\frac{p(s)}{q(s)} = \frac{p(s)q(s)}{q(s)^2+1} \longrightarrow y(t) = \cos t.$$

And using Convolution Theorem we get

# Conclusion

At the conclusion of this research, our scientific study the analytical solution of the integral equation using integral transformation. The theory of integral equation is very useful tool to deal with problems in applied mathematics and engineering applications. Therefore, it was important to touched on the basic idea of the newly proposed method of Hebel integration to clarify the basic idea of the HPTM for solving a differential equation.

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