Republic of Iraq
Ministry of Higher Education
And Scientific Research
University of Babylon
College of Education for Pure Sciences
Department of Mathematics

Frobenius's Method as a Tool for Solving Linear Differential Equations

Graduation research submitted to the Council of the Department of Mathematics in the College of Education for Pure Sciences/University of Babylon, as part of the requirements obtaining a bachelor's degree in mathematics

By

Amina Sawadi Kadhem

Supervised By

Assistant Lecturer
Ebaa A. Jaber

1447 A.H 2026 A.D

بِسْمِ اللهِ الرَّحْمنِ الرَّحِيمِ

﴿وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ﴾

صَدقَ الله العَلي العَظيم

التوبة: ١٠٥

Dedication

To my beloved parents, whose unconditional love, endless support, and constant prayers have been the foundation of my success. To my brothers and sisters, who stood by me through every challenge. To my friends and colleagues, who shared this academic journey with me. I dedicate this humble effort to you all.

Acknowledgements

Praise be to Almighty God, pray and peace be upon his Prophet Mohammed and upon his Household, for helping me in completing this work.

In profound gratitude and appreciation of all sincere efforts, and as I conclude the preparation of this work, I must express my deepest thanks and appreciation to my supervisor and professor, **Assist Lecturer. Ebaa A. Jaber**. I am deeply grateful for her guidance in selecting this topic, her endless generosity in sharing valuable knowledge, and her insightful directives and constructive remarks, from which I have immensely benefited throughout this research.

I would also like to extend my sincere thanks and appreciation to my esteemed professors in the **Department of Mathematics, College of Education for Pure Sciences**. It has been a great privilege to learn from them, benefit from their vast expertise, and be affiliated with such a distinguished academic department.

In conclusion, I express my utmost thanks and profound gratitude to everyone who contributed to the realization of this research, as well as to all my colleagues from my hometown and my academic cohort. I pray that Allah blesses them all and grants them the highest ranks in Paradise.

Abstract

This research investigates the application of power series for solving linear second-order ordinary differential equations, with a particular emphasis on their behavior around singular points. While standard series methods are applicable at ordinary points, this study primarily employs the Method of Frobenius to construct valid linearly independent solutions near regular singular points. A significant portion of the work is dedicated to analyzing the asymptotic behavior of these equations for very large values of the independent variable x (the point at infinity). This is effectively achieved by introducing the transformation $w = 1/x$, which maps the infinity point back to the origin, thereby allowing the rigorous application of series techniques. Furthermore, the research comprehensively examines the analytical structure of solutions based on the roots of the indicial equation, categorizing them into three distinct cases: roots not differing by an integer, repeated roots requiring a logarithmic term, and roots differing by an integer. Through detailed mathematical derivations and illustrative examples, the study demonstrates the efficacy of recurrence relations and harmonic numbers in obtaining systematic, exact solutions for complex differential equations where traditional closed-form methods fall short.

Tabel of Contents

Introduction

The study of linear differential equations has been a central theme in mathematics for centuries, driven by the need to model physical phenomena. While standard methods existed for solving equations with constant coefficients, variable-coefficient equations posed significant challenges. The Method of Frobenius is named after the German mathematician Ferdinand Georg Frobenius (1849–1917), who made substantial contributions to the theory of differential equations and group theory. In the late 19th century, Frobenius formalized a systematic approach to finding infinite series solutions for second-order differential equations around regular singular points. His method extended the utility of power series, allowing mathematicians and physicists to solve complex equations—such as Bessel's and Legendre's equations—that appear frequently in mathematical physics.

Chapter One

Partial Differential Equations

1.1 Introduction

Differential equations are important in solving many scientific and technical problems with high efficiency.

There are several different methods and types of (differential equations).

So, it is necessary to identify the properties of the equations to facilitate the solution of scientific problems.

1.2 Definition: Differential Equation (DE)

Is equation that contains a function and one or more of its derivatives.

Example:

- $y'' + y' = 3y$
- $f''(x) + 2f'(x) = 3f(x)$
- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3y$

1.3 Definitions: Partial differential equation (PDE)

Is a Partial differential equation that contains more than one variable.

Example:

- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x$

The order is = 2

1.4 Definition: Order of partial the differential equation

The Order of the differential equation is the order of the highest partial derivative present in the equation.

Example:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 5\frac{dy}{dx} + 7y = 0$$

The degree is = 2

1.5 Definitions: Degree of Partial Differential Equation

The degree of differential equation is the degree of the highest order derivative present in the equation it must be a positive integer.

Example:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 5\frac{dy}{dx} + 7y = 0$$

1.6 Definitions: Homogeneous partial differential equation

A homogeneous partial differential equation is called **homogeneous** if right - hand side is identically zero.

If right - hand side is not identically zero, then the equation is called **non - homogeneous.**

Example:

1. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$
2. $\frac{\partial^2 v}{\partial x^2} - 4\frac{\partial v}{\partial y} + 3v = 0$

Homogeneous

1. $\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = x^2$
2. $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \sin(x)$

Non- Homogeneous

1.7 Definitions: Partial differential equations with constant coefficients

A partial differential equation is said to have constant coefficients if all the coefficients of the unknown function and its partial derivatives are real constants.

Example:

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} - 2u = 0$$

1.8 Definitions: Partial differential equations with variable coefficients

A partial differential equation is said to have variable coefficients if at least one of the coefficients of the unknown function or its partial derivatives is a function of the independent variables.

Example:

$$x^2\frac{\partial^2 u}{\partial x^2}+y\frac{\partial u}{\partial y}=0$$

Chapter Two

Series Solution of Second Order Differential Equations

(Frobenius Method)

2.1 Introduction

This chapter addresses the solution of linear second-order differential equations using power series, with a particular focus on the Method of Frobenius at regular singular points.

2.2 Power Series and Their Limitations

Before discussing Frobenius's method, it is essential to review the standard Power Series Method. For a linear second-order differential equation of the form:

$$\mathrm{P}(x)y'' + \mathrm{Q}(x)y' + \mathrm{R}(x)\mathrm{y} = 0 \qquad . . . (2.1)$$

If the point x_0 is an ordinary point (where $\mathrm{P}(x_0) \neq 0$), the solution can be successfully represented as a Taylor series:

$$\mathrm{y}(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad . . . (2.2)$$

This method is powerful and sufficient for many problems. However, this standard power series approach fails when dealing with singular points (where $\mathrm{P}(x_0) = 0$). Near a singular point, the coefficients of the differential equation may become unbounded, and a simple power series solution may not exist or may fail to capture the general solution. According to Mahaffy (2019), solutions near a singular point often exhibit behavior different from standard power series, potentially involving terms like $\ln(x)$ or negative powers x^{-n} . Consequently, a more generalized approach was required to handle these singularities, leading to the adoption of the Method of Frobenius. [5]

2.٣ Classification of Points

We consider the general homogeneous second-order linear differential equation (2.1)

where $\mathrm{P}(x)$, $\mathrm{Q}(x)$, and $\mathrm{R}(x)$ are functions of the independent variable x.

2.3.1 Ordinary Points [5]

A point x_0 is defined as an ordinary point of Equation (2.1) if the coefficient of the highest derivative $P(x_0) \neq 0$. Mathematically, this ensures that the functions:

$$p(x) = \frac{Q(x)}{P(x)} \quad \text{and} \quad q(x) = \frac{R(x)}{P(x)}$$

are analytic at x_0. Analyticity implies that these functions can be represented by a Taylor series convergent in x_0.

Theorem (Existence of Power Series Solution):

If x_0 is an ordinary point, then every solution of Equation (2.1) can be represented by a power series of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Example:

Find the solution of the differential equation:

$$y'' + y = 0$$

Solution:

$$y' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} ,$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)\ a_n\, x^{n-2}$$

Substituting into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1)\ a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Shifting the index of the first summation by letting $(k = n-2)$ so $(n = k+2)$, we obtain:

$$\sum_{k=0}^{\infty}(k+2)(k+1)\ a_{k+2}\, x^{k} + \sum_{k=0}^{\infty} a_k\, x^k = 0$$

Combining the series gives the recurrence relation:

$$(k+2)(k+1)\, a_{k+2} + a_k = 0 \rightarrow a_{k+2} = \frac{-a_k}{(k+2)(k+1)}$$

By iterating this relation, we find two linearly independent solutions (corresponding to even and odd powers), which represent the Maclaurin series for $\cos(x)$ and $\sin(x)$:

$$y(x) = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

This confirms that standard power series methods are effective at ordinary points.[3]

2.3.2 Singular Points

If x_0 is not an ordinary point (i.e., $P(x_0) = 0$), it is classified as a singular point. At such points, the standard power series method often fails because the solution may not be analytic (e.g., it may become unbounded or involve fractional powers). To determine the appropriate method of solution, singular points are classified into two types based on the behavior of the coefficient functions near x_0.[5]

2.٣.2.1 (Regular and Irregular Singular Points):

A singular point x_0 is said to be a regular singular point if both of the following limits exist and are finite:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

If either of these limits does not exist or is infinite, the point is classified as an irregular singular point.

Example:

$$4xy'' + 2y' + y = 0$$

Here, $P(x) = \frac{1}{2x}$, $Q(x) = \frac{1}{4x}$,

We apply the limits test to classify this singularity:

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{2}{4x} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2} \quad \text{(Finite)}$$

$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{1}{4x} = \lim_{x\to 0}\frac{x}{4} = 0 \quad \text{(Finite)}$$

Since both limits are finite, a regular singular point.

2.٤ The Method of Frobenius

The Method of Frobenius is applied to solve the differential equation near a regular singular point. Without loss of generality, we assume the regular singular point is at $x_0 = 0$.

This implies that $xQ(x)/P(x) = p(x)$ and $x^2R(x)/P(x) = q(x)$ are analytic at $x = 0$, allowing the equation to be written in the form:

$$x^2y'' + xp(x)y' + q(x)y = 0 \quad ...(2.3)$$

where $p(x)$ and $q(x)$ have convergent power series expansions.[5]

2.٤.1 The Frobenius Series Solution

We seek a solution of the form:

$$y(x) = x^{r\sum_{n=0}^{\infty} a_n x^n} = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (a_0 \neq 0) \quad ...(2.4)$$

where r is an exponent to be determined.

2.٤.2 The Indicial Equation

By substituting the series solution into Equation (2.2) and examining the coefficient of the lowest power of x (where $n = 0$), we obtain the quadratic Indicial Equation:

$$r(r-1) + p_0 r + q_0 = 0 \quad ...(2.5)$$

Here, p_0 and q_0 are the leading terms of the power series for $\mathrm{p}(x)$ and $\mathrm{q}(x)$, equivalent to the limits:

$$p_0 = \lim_{x \to 0} \frac{xQ(x)}{P(x)} \quad \text{and} \quad q_0 = \lim_{x \to 0} \frac{x^2 R(x)}{P(x)}$$

2.٥ Classification of Solutions (The Three Cases)

The roots of the indicial equation, denoted by r_1 and r_2 (where $\mathrm{Re}(r_1) \geq \mathrm{Re}(r_2)$), determine the form of the general solution. The Method of Frobenius distinguishes three cases.[5]

Case 1: This case occurs when the roots of the indicial equation $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$) are distinct and their difference is not an integer ($\boldsymbol{r_1 - r_2 \neq N}$). In this scenario, the Method of Frobenius yields two linearly independent solutions in the form of standard power series.

Example:

Find the solution to the differential equation:

$$2xy'' + (x+1)y' + 3y = 0$$

Solution:

First, we analyze the point $x = 0$.

We evaluate the limits:

$$\lim_{x \to 0} \frac{(x+1)x}{2x} = \frac{1}{2}, \quad \lim_{x \to 0} \frac{3x^2}{2x} = 0$$

Since both limits are finite, the point $x = 0$ is a **regular singular point**.

We assume the solution is of the form:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting these into the given differential equation, we obtain:

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n x^{n+r} = 0$$

Equating the coefficient of the lowest power of x (which is x^{n+r-1}) to zero yields the Indicial Equation:

$c_0[2r(r-1)+r] = c_0[2r^2-2r+r] = c_0(2r^2-r) = 0$

Since $c_0 \neq 0$, we have:

$r(2r-1) = 0$

The roots are:

$$r_1 = \frac{1}{2}, \quad r_2 = 0$$

The difference $r_1 - r_2 = 1/2$ is not an integer.

The Recurrence Relation:

Equating the coefficient of x^{n+r-1} to zero gives the recurrence relation:

$$c_n(n+r)(2n+2r-1) + c_{n-1}(n+r+2) = 0$$

$$c_n = \frac{-(n+r+2)}{(n+r)(2n+2r-1)} c_{n-1}, \quad n \geq 1$$

$r = 0$

$$\therefore C_n = \frac{-(n+2)}{n(2n-1)} C_{n-1}, \quad n \geq 1$$

$$\therefore C_1 = \frac{-3}{1(1)} C_0 = -3C_0$$

$$C_2 = \frac{-4}{2(3)} C_1 = \frac{(-4)(-3)}{(2)(3)} C_0 = 2C_0$$

$$C_3 = \frac{-5}{(3)(5)} C_2 = \frac{(-5)(2)}{(3)(5)} C_0 = \frac{-2}{3} C_0$$

$r = 1/2$

$$C_n = \frac{-(n+5/2)}{(n+1/2)} C_{n-1}$$

$$\text{or } C_n = \frac{-(2n+5)}{2n(2n+1)} C_{n-1}, \quad n \geq 1$$

$$C_1 = \frac{-7}{2(3)} C_0 = \frac{-7}{6} C_0$$

$$C_2 = \frac{-9}{(4)(5)} C_1 = \frac{9(7)}{(4)(5)(6)} C_0 = \frac{21}{40} C_0$$

$$C_3 = \frac{-11}{(6)(7)} C_2 = \frac{(11)(21)}{(6)(7)(40)} C_0 = \frac{-11}{80} C_0$$

General Solution

Assuming $c_0 = 1$, the general solution is

$$y = A_1 \sum_{n=0}^{\infty} C_n x^{n+0} + A_2 \sum_{n=0}^{\infty} C_n x^{n+\frac{1}{2}}$$

$$\therefore y = A_1 \left[1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots\right] + A_2 \left[1 - \frac{7}{6}x + \frac{21}{40}x^2 - \frac{11}{80}x^3 + \cdots\right] x^{1/2}$$

Case 2: This case arises when the indicial equation yields two identical roots ($r_1 = r_2$). While the first solution is a standard Frobenius series, the second linearly independent solution must inevitably include a logarithmic term ($\ln x$) to satisfy the differential equation.

Example:

Solve the differential equation:

$$x^2 y'' + 3xy' + (1-2x)y = 0$$

Solution:

The point $x = 0$ is a regular singular point because the limits are finite:

$$\lim_{x\to 0}\frac{3x}{x^2}=3, \quad \lim_{x\to 0}\frac{(1-2x)x^2}{x^2}=1$$

We assume the solution is:

$$y=\sum_{n=0}^{\infty} C_n x^{n+r}$$

$$\therefore y'=\sum_{n=0}^{\infty} C_n(n+r)x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} C_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting into the differential equation, we obtain:

$$\therefore \sum_{n=0}^{\infty} C_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty} 3C_n(n+r)x^{n+r}$$

$$+\sum_{n=0}^{\infty} C_n x^{n+r}-\sum_{n=0}^{\infty} 2C_n x^{n+r+1}=0$$

Indicial Equation

Equating the coefficient of the lowest power x^{n+r}

$$\therefore C_n(n+r)(n+r-1)+3C_n(n+r)+C_n-2C_{n-1}=0$$

$$\therefore C_n[(n+r)(n+r+2)+1]=2C_{n-1}$$

$$\therefore C_n[(n+r+1)^2]=2C_{n-1} \quad \dots(2.6)$$

For $n=0$

$$C_0(r+1)^2=2C_{-1}=0$$

(Since $C_{-1}=0$)

$$\therefore r=-1,-1$$

From (2.6), the recurrence relation is

$$C_n = \frac{2}{(n+r+1)^2} C_{n-1}, \quad n \geq 1$$

We determine the coefficients $C_1, C_2, C_3, \ldots$ in terms of r

$$\therefore C_1 = \frac{2}{(r+2)^2} C_0$$

$$C_2 = \frac{2}{(r+3)^2} C_1 = \frac{2^2}{[(r+2)(r+3)]^2} C_0$$

$$C_n = \frac{2^n}{[(r+2)(r+3)(r+4)\ldots(r+n+1)]^2} C_0$$

Assuming $C_0 = 1$ (arbitrary), the first solution is:

$$x > 0, \quad y_1(x,r) = x^r + \sum_{n=1}^{\infty} C_n(r) x^{n+r}$$

$$\text{where } C_n(r) = \frac{2^n}{[(r+2)(r+3)\ldots(r+n+1)]^2}, \quad n \geq 1$$

$$y_1(x,r) = x^r \left[1 + \frac{2}{(r+2)^2} + \frac{2^2}{(r+2)^2(r+3)^2} x^2 + \frac{2^3}{(r+2)^2(r+3)^2(r+4)^2}\right]$$

$$y_1(x,-1) = \left[1 + 2x + x^2 + \frac{2}{9}x^3 + \cdots\right]$$

Also

$$\frac{\partial y_1(x,r)}{\partial r} = \left[1 + \frac{2}{(r+2)^2} x + \frac{2^2}{(r+2)^2(r+3)} x^2 + \cdots\right]$$
$$+ x^r \left[\frac{-4}{(r+2)^3} x - 4\left[\frac{2^2}{(r+2)^2(r+3)} + \frac{2^2}{(r+2)^2(r+3)^3}\right] x^2 + \cdots\right]$$

$$y_1(x,-1) = \frac{\partial y_1(x,r)}{\partial \mathrm{r}} |_{r=-1}$$

$$= x^{-1}\ln(x)\left[1 + 2x + x^2 + \frac{2}{9}x^3 + \cdots\right] + x^{-1}\left[-4x - 4\left(\frac{2}{4} + \frac{2}{8}\right)x^2 + \cdots\right]$$

The general solution is

$$\therefore y = (A + B \ln x)x^{-1}\left[1 + 2x + x^2 + \frac{2}{9}x^3 + \cdots\right]$$
$$-Bx^{-1}[4x + 3x^2 + \cdots]$$

Case 3: This case is identified when the difference between the distinct roots is a positive integer ($r_1 - r_2 = N$). The solution corresponding to the larger root is a standard series, whereas the second solution (associated with the smaller root) typically requires a logarithmic term and involves modified coefficients to avoid singularities.

Example:

Solve the differential equation

$$xy'' - 3y' + xy = 0$$

Solution:

Checking singularity at $x = 0$

$$\lim_{x\to 0} x\left(\frac{-3x}{x}\right) = -3, \quad \lim_{x\to 0}\frac{x^3}{x} = 0$$

It is a regular singular point.

Indicial Equation:

Substituting the series leads to the indicial equation

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} C_n (n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1)x^{n+r-2}$$

Substituting into the differential equation:

$$\sum_{n=0}^{\infty} C_n (n+r)(n+r-1)x^{n+r-1}$$
$$-\sum_{n=0}^{\infty} 3C_n (n+r)x^{n+r-1} + \sum_{n=0}^{\infty} C_n x^{n+r+1} = 0$$

Equating the coefficient of x^{n+r-1} to zero, we obtain

$$C_n(n+r)(n+r-4) + C_{n-2} = 0 \quad . \; . \; . \; (2.7)$$

For $n = 0$, the equation becomes:

$$C_0 r(r-4) + C_{-2} = 0$$

Since $C_{-2} = 0$ and $C_0 \neq 0$ (arbitrary)

∴ The indicial equation is

$$r(r-4) = 0 \quad \Rightarrow \quad r_1 = 0, \quad r_2 = 4$$

Note that $r_2 - r_1 = 4$ (a positive integer).

The recurrence relation from (2.7) is

$$C_n = \frac{-1}{(n+r)(n+r-4)} C_{n-2}, \quad n \geq 2$$

It is clear that for $n = 1, C_1 = 0$. Thus, $C_{2n+1} = 0$ (all odd coefficients vanish).

$$C_2 = -\frac{1}{(r+2)(r-2)} C_0$$

$$C_4 = -\frac{1}{r(r+4)} C_2 = \frac{-1^2}{(r-2)r(r+2)(r+4)} C_0,$$

$$C_6 = -\frac{1}{(r+6)(r+2)} C_4$$

$$= \frac{-1}{(r-2)r(r+2)^2(r+4)(r+6)} C_0$$

Thus, the solution in terms of r is

$$\mathrm{y}(\mathrm{x}, \mathrm{r}, C_0) = C_0 x^r \left[1 - \frac{1}{(r-2)(r+2)} x^2 + \frac{1}{(r-2)r(r+2)(r+4)} x^4 - \frac{1}{(r-2)r(r+2)^2(r+4)(r+6)} x^6 + \ldots\right]$$

We observe that for $r = 4$, all coefficients are defined.

However, for $r = 0$ the coefficients starting from x^4 become indeterminate. Therefore, we set:

$$C_0 = b_0 r \Leftarrow C_0 = b_0(r-0)$$

We obtain

$$\bar{y} = b_0 \mathrm{r} x^r \left[r - \frac{r}{(r-2)(r+2)} x^2 + \frac{1}{(r-2)(r+2)(r+4)} x^4 - \frac{1}{(r-2)(r+2)^2(r+4)(r+6)} x^6 + \ldots \right]$$

First Solution y_1

$$y_1 = \bar{y}|_{r=0}$$

$$= b_0 \left[-\frac{1}{16} x^4 + \frac{1}{192} x^6 - \cdots \right]$$

$$\frac{\partial \bar{y}}{\partial r} = \bar{y} \ln(x)$$

$$+ b_0 x^r \left[1 - \left(\frac{1}{(r-2)(r+2)} - \frac{r}{(r-2)^2(r+2)} - \frac{r}{(r-2)(r+2)^2} \right) x^2 - \left(\frac{1}{(r-2)^2(r+2)(r+4)} + \frac{1}{(r-2)(r+2)^2(r+4)} + \frac{1}{(r-2)(r+2)(r+4)} \right) x^4 + \cdots \right]$$

The second solution is found by differentiating with respect to r:

$$y_2 = \frac{\partial \bar{y}}{\partial r}|_{r=0}$$

$$= y_1 \ln(x)$$

$$+ b_0 \left[1 - \left(\frac{-1}{4} \right) x^2 - \left(\frac{1}{32} + \frac{-1}{32} + \frac{-1}{64} \right) x^4 + \cdots \right]$$

$$= y_1 \ln(x) + b_0 \left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots\right)$$

General Solution:

$$y = Ay_1 + By_2$$

$$y = (A + B\ln(x))\left(\frac{-x^4}{16} + \frac{x^6}{192} - \cdots\right) + B\left(1 + \frac{x^2}{4} + \frac{x^4}{64} + \cdots\right)$$

where A and B are arbitrary constant

Chapter Three

Finding the Solution of the Differential Equation in Power Series for Very Large Values of x

3.1 Introduction

In this chapter study the solving the differential equation using power series when x is very large, especially if the point $x = 0$ is an irregular singular point. Therefore, we use the transformation $w = \frac{1}{x}$. If $w = 0$ is an ordinary point or a regular singular point, we can find a solution around it, which yields a valid solution for large values of x.

Example: Find the solution of the differential equation valid for very large values of x.

$$xy'' + (3x - 1)y' + y = 0 \quad ...(3.1)$$

Solution:

Since $x = 0$ is a singular point

$$\lim_{x \to 0} \frac{(3x - 1)}{x} = -\infty$$

That is, $x = 0$ is an irregular singular point.

To find solutions in terms of very large x, we set

$$w = \frac{1}{x} \Rightarrow x = \frac{1}{w}$$

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = \frac{-1}{x^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dw}\left[-w^2\frac{dy}{dw}\right]\frac{dw}{dx} = \frac{-1}{x^2}\left[-w^2\frac{d^2y}{dw^2} - 2w\frac{dy}{dw}\right]$$

$$= w^4\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw}$$

Substituting into the given differential equation

$$\therefore \frac{1}{w^2}\left[w^4\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw}\right] + \left[\frac{3}{w} - 1\right]\left(-w^2\frac{dy}{dw}\right) + y = 0$$

Or

$$w^2\frac{d^2y}{dw^2}-w(1-w)\frac{dy}{dw}+y=0 \quad \dots(3.2)$$

Thus, $w=0$ is a regular singular point for equation (3.2).

Consequently, the point at infinity is a regular singular point for equation (3.1).

We assume that $y=\sum_{n=0}^{\infty} C_n w^{n+r}$ is a solution for equation (3.2).

Substituting into (3.2), we obtain

$$\sum C_n(n+r)(n+r-1)w^{n+r}-\sum C_n(n+r)w^{n+r}$$

$$+\sum C_n(n+r)w^{n+r+1}+\sum C_n w^{n+r}=0$$

Equating the sum of coefficients of w^{n+r} (the lowest power in the equation) to zero

$$C_n(n+r)(n+r-1)-C_n(n+r)+C_n+C_{n-1}(n+r-1)=0$$

$$C_n(n+r-1)^2+C_{n-1}(n+r-1)=0 \quad \dots(3.3)$$

When $n=0$

$$\therefore C_0(r-1)^2+C_{-1}(r-1)=0$$

Where $C_{-1}=0$ and $C_0\neq 0$ is arbitrary.

$$(r-1)^2=0$$

∴Thus, the indicial equation is

$$\therefore r=1,1$$

From equation (3.3), the recurrence relation is obtained

$$C_n=\frac{1}{n+r-1}C_{n-1}, n\geq 1$$

$$\therefore C_1=\frac{1}{r}C_0$$

$$C_2=\frac{-1}{r+1}C_1=\frac{(-1)^2}{r(r+1)}C_0$$

$$C_3 = \frac{-1}{r+2} C_2 = \frac{(-1)^3}{r(r+1)(r+3)} C_0, \ldots$$

By choosing $C_0 = 1$ ،we obtain

$$y_1(w,r) = w^r \left[1 - \frac{1}{r} w + \frac{(-1)^2}{r(r+1)} w^2 + \frac{(-1)^3}{r(r+1)(r+3)} w^3 + \cdots \right]$$

$$y_1(w,1) = w \left[1 - \frac{(-1)^2 w^2}{2!} + \frac{(-1)^3 w^3}{3!} + \cdots \right]$$

$$= w \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} w^n$$

To obtain the second solution, we find

$$\frac{\partial y_1(w,r)}{\partial r} = y_1(w,r) \ln w + w^r \left[\frac{1}{r^2} w - \frac{1}{r(r+1)} \left(\frac{1}{r} + \frac{1}{r+1} \right) w^2 + \frac{1}{r(r+1)(r+2)} \left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2} \right) w^3 \ldots \right]$$

$$\frac{\partial y_1(w,r)}{\partial r} |_{r=1} = y_1 \ln w + w \left[\frac{w^2}{2!} \left(1 + \frac{1}{2} \right) + \frac{w^3}{3!} \left(1 + \frac{1}{2} + \frac{1}{3} \right) + \cdots \right]$$

Letting $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then

$$y_2 = y_1 \ln w + w_2 \left[H_1 - \frac{w}{2!} H_2 + \frac{w^2}{3!} H_3 - \cdots \right]$$

$$= y_1 \ln w + w^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} w^{n-1}}{n!} H_n$$

Thus, the solution to equation (3.2) is

$$y = (A + B \ln w) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} w^{n+1} - B \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} H_n w^{n+1}$$

Where A, B are arbitrary constants and $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

$$y = \left[A + B\ln\left(\frac{1}{x}\right)\sum_{n=1}^{\infty}\frac{(-1)^n x^{-n-1}}{n!} B\sum_{n=1}^{\infty}\frac{(-1)^n H_n x^{-n-1}}{n!}\right]$$

Example:

Solve the equation $4x^3y'' + 6x^2y' + y = 0$ for very large values of x.

Solution:

We see that $x = 0$ is a singular point.

$$\lim_{x\to 0}\frac{6x^2}{4x^3}x = \frac{6}{4} = \frac{3}{2}$$

$$\lim_{x\to 0}\frac{1(x-0)^2}{4x^3} = \lim_{x\to 0}\frac{1}{4x} = \infty$$

That is, $x = 0$ is an irregular singular point.

Thus, $w = \frac{1}{x} andhence\ x = \frac{1}{w}$.

$$\frac{dy}{dx} = \frac{dy}{dw}.\frac{dw}{dx} = \frac{-1}{x^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dw}\left(-w^2\frac{dy}{dw}\right)\frac{dw}{dx} = \frac{-1}{x^2}\left(-w^2\frac{d^2y}{dx^2} - 2w\frac{dy}{dw}\right)$$

$$= w^4\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw}$$

$$\therefore \frac{4}{w^3}\left(w^4\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw}\right) + \frac{6}{w^2}\left(-2w\frac{dy}{dw}\right) + y = 0$$

$$4w\frac{d^2y}{dw^2} + 2\frac{dy}{dw} + y = 0$$

We observe that $w = 0$ is a regular singular point.

We assume the solution is

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}, \quad C_n \neq 0$$

$$y' = \sum C_n(n+r)w^{n+r}$$

$$y'' = \sum C_n(n+r)(n+r-1)w^{n+r}$$

$$\therefore 4\sum C_n(n+r)(n+r-1)w^{n+r-1} + 2\sum C_n(n+r)w^{n+r-1} + \sum C_n w^{n+r} = 0$$

$$\therefore C_n[4(n+r)(n+r-1) + 2(n+r)] + C_{n-1} = 0$$

$$C_n[(n+r)(4n+4r-2)] + C_{n-1} = 0 \quad \ldots (3.4)$$

Where $C_{-1} = 0$ and $C_0 \neq 0$,

By setting $n = 0$

$$C_0(r)(4r-2) = 0 \Rightarrow r = 0, r = \frac{1}{2}$$

From the recurrence relation (3.4), we obtain

$$C_n = \frac{-1}{(n+r)(4n+4r-2)} C_{n-1}, n \geq 1$$

Substituting into the recurrence relation

$\boldsymbol{r = 0}$

$$C_n = \frac{-1}{(n)(4n-2)} C_{n-1} = \frac{-1}{2n(2n-1)} C_{n-1}$$

$$C_1 = \frac{-1}{2.1} C_0, \quad C_2 = \frac{-1}{3.4} C_1 = \frac{(-1)^2}{4!} C_0$$

$$C_3 = \frac{-1}{6.5} C_2 = \frac{(-1)^2}{6.54!} C_0, \ldots$$

And the solution is

$$y_1(w,0) = C_0\left[1 - \frac{w}{2!} + \frac{w^2}{4!} - \frac{w^3}{6!} + \cdots\right]$$

$\boldsymbol{r = \frac{1}{2}}$

$$C_n = \frac{-1}{\left(n+\frac{1}{2}\right)\left(4n + 4.\frac{1}{2} - 2\right)} C_{n-1}, \quad n \geq 1$$

$$= \frac{-1}{(2n+1)(2n)} C_{n-1}$$

$$C_1 = \frac{-1}{2 \cdot 3} C_0, \quad C_2 = \frac{-1}{4 \cdot 5} C_1 = \frac{(-1)^2}{5!} C_0$$

$$C_3 = \frac{-1}{6 \cdot 7} C_2 = \frac{(-1)^2}{7!} C_0$$

The second solution is

$$y_2 = C_0 w^{\frac{1}{2}} \left[1 - \frac{w}{3!} + \frac{w^2}{5!} - \frac{w^3}{7!} + \cdots \right]$$

Taking $C_0 = 1$, the solution to the original equation is

$$y = Ay_1 + By_2$$

$$= A \left[1 - \frac{x^{-1}}{2!} + \frac{x^{-2}}{4!} - \frac{x^{-3}}{6!} + \cdots \right] + Bx^{-\frac{1}{2}} \left[+1 - \frac{x^{-1}}{3!} + \frac{x^{-2}}{5!} - \frac{x^{-3}}{7!} + \cdots \right]$$

Where A, B are two arbitrary constants.

Example:

$$(1 - x^2)y'' - 2xy' + 6y = 0 \quad . \; . \; . \; (3.5)$$

In power series about $x = \infty$.

Solution:

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

By setting

$$x = \frac{1}{t}$$

$$\therefore t = \frac{1}{x}, \quad \frac{dt}{dx} = \frac{-1}{x^2}$$

$$\frac{dy}{dx} = -t^2 \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = t^2 \left(2t \frac{dy}{dt} + t^2 \frac{d^2y}{dt^2} \right)$$

Substituting into (3.5), we obtain

$$t^2(t^2-1)\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}+6y=0$$

We see that $t=0$ is a regular singular point (check this?), and based on that we assume the solution of equation (5) is of the form

$$y=\sum_{n=0}^{\infty} C_n t^{n+r}, \quad C_0 \neq 0$$

Substituting into the equation and equating the coefficient of the lowest power of t to zero, we obtain the recurrence relation

$$(n+r-3)(n+r+2)C_n-(n+r-2)(n+r-1)C_{n-2}=0$$

And by setting $n=0$, we obtain the indicial equation

$$C_0(r-3)(r+2)=0$$

$$\therefore r = 3, \quad r = -2$$

The roots are distinct and the difference between them is an integer.

From the recurrence relation we find

$$C_n=\frac{(n+r-2)(n+r-1)}{(n+r-3)(n+r+2)}C_{n-2}, \quad n\geq 2$$

From this we see that

$$C_1=C_3=C_5=\cdots=0$$

$$C_2=\frac{r(r+1)}{(r-1)(r+4)}C_0$$

$$C_4=\frac{r(r+1)(r+2)(r+3)}{(r-1)(r+1)(r+4)(r+6)}C_0$$

$$=\frac{r(r+2)(r+3)}{(r-1)(r+4)(r+6)}C_0$$

Thus, the solution is of the form

$$y = t^r C_0 \left[1 + \frac{r(r+1)}{(r-1)(r+4)} t^2 + \frac{r(r+2)(r+3)}{(r-1)(r+4)(r+6)} t^4 + \cdots \right]$$

By setting $r = 3$, we obtain the first solution

$$y_1 = t^3 \left[1 + \frac{3 \cdot 4}{2 \cdot 7} t^2 + \frac{3 \cdot 5 \cdot 6}{2 \cdot 7 \cdot 9} t^4 + \cdots \right]$$

By setting $r = -2$, we obtain the second solution

$$y_2 = t^{-2} \left[1 - \frac{t^2}{3}\right]$$

And the general solution for equation is

$$y = Ay_1 + By_2$$

$$At^3 \left[1 + \frac{3 \cdot 4}{2 \cdot 7} t^2 + \frac{3 \cdot 5 \cdot 6}{2 \cdot 7 \cdot 9} t^4\right] + \frac{B}{t^2}\left[1 - \frac{t^2}{3}\right]$$

That is

$$y = \frac{A}{x^3}\left[1 + \frac{3 \cdot 4}{2 \cdot 7x^2} + \frac{3 \cdot 5 \cdot 6}{2 \cdot 7 \cdot 9x^4}\right] + Bx^2 \left[1 - \frac{1}{3x^2}\right]$$

Where A, B are arbitrary constants.

References

[1]. Hawkins, T. (2013). The mathematics of Frobenius in context. Sources and Studies in the History of Mathematics and Physical Sciences, Springer, New York.

[2]. Simmons, G. F. (2017). *Differential equations with applications and historical notes* (3rd ed.). CRC Press.

[3]. Zill, D. G. (2018). A first course in differential equations with modeling applications (11th ed.). Cengage Learning.

[4]. Bonchi, F., Gadducci, F., Kissinger, A., Sobocinski, P., & Zanasi, F. (2018, July). Rewriting with frobenius. In Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science (pp. 165-174).

[5]. Mahaffy, J. M. (2019). *Lecture notes: Method of Frobenius*. Department of Mathematics and Statistics, San Diego State University.