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Semi Open Sets On Minimal and Maximal Topological Spaces

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بسم الله الرحمن الرحيم
{ نَرْفَعُ دَرَجَاتٍ مَّن نَّشَاءُ وَفَوْقَ كُلِّ ذِي عِلْمٍ عَلِيمٌ }
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الاهداء

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شهداء العراق..

الى من افنى شبابه من اجلي.. والدي

الى من وضع الله سبحانه وتعالى الجنة تحت اقدامها... امي

الى القلوب المعلقه بي طوال حياتي .... اخوتي

الى الذين اجتهدو لايصال العلم لي .... اساتذتي الاعزاء

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Abstract

The aim of this work is to expand the study of concepts of Minimal and maxmal regular open sets and investigate some of their fundamental properties

Preface

Nakaoka, F. and Oda, N. [4], [5] introduced and studied the notions of minimal open sets and maximal open sets in topological spaces. As a simulation of these studies, minimal semi-open sets and maximal semi-open sets have been introduced and studied.

In 1937, Stone [6] gave a new class of open sets called regular open sets which is used to define the semi-regularization of a topological space. The authors define the notion of minimal regular open sets and maximal regular open sets.

The aim of this work is to expand the study of concepts of minimal and maximal regular open sets and investigate some of their fundamental properties.

The work consists of two chapters. In chapter one, we give some basic concepts of topological spaces that will be used during this work.

Chapter two is divided into two sections. In the first section, we study the concepts of minimal and maximal open sets and give the definition of generalized minimal closed sets. In the second section, we deals with the concepts of minimal and maximal semi-open sets and discuss some of their fundamental properties.

Chapter one

Basic Topological Concepts

Basic Topological Concepts

In this chapter, we give some basic concepts of topological spaces that will be used during this work. We study a class of open sets called regular open sets and give some of related properties. Finally, we give definitions and some basic properties of generalized open sets as semi-open, clopen, regular closed sets.

Definition 1.1 [7]

Let X be a nonempty set. A topology T on X is a collection of subsets of X satisfying the following:

(a) X and ϕ belong to T.
(b) Any finite intersection of elements of T is an element of T.
(c) An arbitrary union of elements of T is an element of T.

Definition 1.2 [1]

If X is a topological space and $E \subseteq X$, we say E is *closed* if $X \setminus E$ is open.

Remark 1.3

A subset A of X may be open, closed, both (clopen) and neither.

Definition 1.4 [7]

Let X be a topological space and $A \subseteq X$. The ***closure*** of A, denoted by $Cl(A)$, is the closed set defined as:
$Cl(A) = \cap \{K \subseteq X : K \text{ is closed and } A \subseteq K\}$

Or $Cl(A)$ can be defined as the smallest closed subset containing A

Theorem 1.5 [7]

Let A and B be subsets of X. Then,
(1) $A \subseteq Cl(A)$.
(2) If $A \subseteq B$, *then* $Cl(A) \subseteq Cl(B)$.
(3) $Cl(\phi) = \phi$.
(4) $Cl(Cl(A)) = Cl(A)$.
(5) $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
(6) $Cl(A \cap B)$ _ $Cl(A) \cap Cl(B)$.
(7) A is closed if and only if $Cl(A) = A$.

Definition 1.6 [7]
Let A be a subset of a space X. The ***interior*** of A in X, denoted as Int(A), is the open set defined as:
$Int(A) = \cup \{G \subseteq X : G \text{ is open and } G \subseteq A\}$:

Or $Int(A)$ can be defined as the largest open subset contained in A.

Theorem 1.7 [7]
Let A and B be subsets of X. Then,
(1) $Int(A) \subseteq A$.
(2) If $A \subseteq B$, *then* $Int(A) \subseteq Int(B)$.
(3) $Int(X) = X$.
(4) $Int(Int(A)) = Int(A)$.
(5) $Int(A \cap B) = Int(A) \cap Int(B)$.
(6) $Int(A) \cup Int(B) \subseteq Int(A \cup B)$.
(7) A is open if and only if $Int(A) = A$.

Remark 1.8
The complement of a subset A of a space X, denoted by A^c, is the set $X \setminus A$.

Remark 1.9 [7]
Let X be a toplogical space and $A \subseteq X$, then $(Cl(A))^c = Int(A^c)$.

Definition 1.10 [7]
Let x be an element of a space X. A ***neighborhood*** of x is a set U which contains an open set V containing x. The collection ρ_x of all neighborhoods of x is the neighborhood system of x.

Definition 1.11 [7]

A ***neighborhood base*** of x in a topological space X is a sub collection β_x taken from the neighborhood system ρ_x having the property that each $U \in \rho_x$ contains some $V \in \beta_x$. The elements of a neighborhood base at x are called ***basic neighborhoods*** of x.

Definition 1.12 [6]
Let A be a subset of a topological space X. Then A is called a **regular open** set if $A = Int(Cl(A))$. A set A is called *regular closed* if A^c is regular open; that is, $A = Cl(Int(A))$.

The collection of all regular open (resp. regular closed) sets in a topological space X is denoted by $RO(X)$ (resp. $RC(X)$).

Theorem 1.13 [7]
The intersection of any regular open sets is regular open set.

Remark 1.14
If A and B are both regular open sets in a topological space X, then $A \cup B$ need not be regular open set as shown in the following example:

Example 1.15
Let X = R with standard topology. Let A = (0; 1) and B = (1; 2),
then A and B are regular open sets, but A ∪ B = (0; 1) ∪ (1; 2) is not regular open set.

Remark 1.16
Let X be a space and $A \subseteq X$. Then:

(1) If A is regular open set, then $Cl(A)$ is regular closed set.
(2) If A is regular closed set, then $Int(A)$ is regular open set.

Definition 1.17
The family of all clopen sets of a space X is denoted by $CO(X)$.

Theorem 1.18 [3]
$CO(X) = RO(X) \cap RC(X)$.

Remark 1.19

If A is regular open set, then $(Cl(A))^{c}$ is also regular open set.

In [6], it was shown that the regular open sets of a space (X, T) is a base for a topology T_s on X coarser than T. The space (X, T_s) was called the *semi-regularization* space of (X, T). The space (X, T_s) is *semi-regular* if the regular open sets of (X, T) is a base for T; that is, $T = T_s$. For a space (X, T), the regular open sets of (X, T) equal the regular open sets of (X, T_s). Hence, the semi-regularization process generates at most one new topology, thus $(T_s)_s = T_s$.

Theorem 1.20 [3]
Let (X, T_s) be the semi-regularization space of a topological space (X, T), then CO(X, T)= CO(X, T_s)

Chapter two

Minimal and Maximal Semi Open Sets

2.1. Minimal and Maximal Open Sets

Definition 2.1.1

Let X be a topological space. A proper nonempty open subset U of X is said to be:
(i) A ***minimal open*** set [4] if any open set which is contained in U is ϕ or U, and a ***minimal closed*** set [4] if any closed set which is contained in U is ϕ or U.

(ii) A maximal open set [5] if any open set which contains U is X or U, and a maximal closed set [5] if any closed set which contains U is X or U.

The collection of all minimal open (resp. maximal open, minimal closed, maximal closed) sets in a topological space X is denoted by $MiO(X)$ (resp. $MaO(X)$, $MiC(X)$, $MaC(X)$).

Example 2.1.2

Let $X = \{1,2,3,4\}$ with a topology $T = \{\phi, X, \{1\}, \{1,2\}, \{3,4\}, \{1,3,4\}\}$.

Then, the set {3,4} is minimal open and the set {1,3,4} is maximal open. Also the set {2,3,4} is maximal closed and the set {2} is minimal closed.

Theorem 2.1.3

Let X be a topological space and $U \subseteq X$. Then, U is minimal open [4] (resp. minimal closed [5]) set if and only *if* $X\backslash U$ is maximal closed (resp.maximal open) set.

Proof:

Let U be a minimal open set in X, then $X\backslash U$ is closed set. Let V be a closed set such that $X\backslash U \subseteq V$, then $X\backslash V \subseteq U$. But $X\backslash V$ is open set contained in the minimal open set U, so $X\backslash V = \phi$ or $X\backslash V = U$. This implies that $V = X$ or $V = X\backslash U$. Therefore $X\backslash U$ is maximal closed set. Similarly, if $X\backslash U$ is maximal closed set, then U is minimal open set.

Corollary 2.1.4

Let X be a topological space with a; b ∈ X. Then we have the following:

(1) if {a} is an open set in X, then {a} is a minimal open set and so X\{a} is a maximal closed.

(2) if {b} is a closed set, then {b} is a minimal closed set and so X\{b} is a maximal open.

Lemma 2.1.5 [4]

Let (X, T) be a topological space.
(1) If U is a minimal open set and W is an open set such that $U \cap W \neq \emptyset$, then $U \subseteq W$.
(2) If U and V are minimal open sets such that $U \cap V \neq \emptyset$, then U = V .

Proof:

(1) Let W be an open set such that $U \cap W \neq \emptyset$. Since U is minimal open set and $U \cap W$ is open set with $U \cap W \subseteq U$, we have $U \cap W = U$. Therefore $U \subseteq W$.

(2) If $U \cap V \neq \emptyset$,, then by (1), $U \subseteq V$ and $V \subseteq U$. Therefore $U = V$.

Lemma 2.1.6 [5]

Let (X; T) be a topological space.
(1) If U is a maximal open set and W is an open set such that $U \cup W \neq X$, then $W \subseteq U$.

(2) If U and V are maximal open sets such that $U \cup V \neq X$, then U = V .

Proof:
(1) Let W be an open set such that $U \cup W \neq X$. Since U is maximal open set and $U \cup W$ is open set with $U \subseteq U \cup W$, then $U \cup W = U$. Therefore $W \subseteq U$.

(2) If $U \cup V \neq X$, then by (1), $U \subseteq V$ and $V \subseteq U$. Therefore $U = V$.

Theorem 2.1.7 [4]
Let U be a minimal open set. If x is an element of U, then $U \subseteq W$, for any open neighborhood W of x.

Proof: Let W be an open neighborhood of x such that U not in W. Then $U \cap W$ is an open set such that $U \cap W \subseteq U$ (:= $U \cap W$ _ U and $U \cap W \neq U$) and $U \cap W \neq \phi$. This contradicts our assumption that U is a minimal open set.

2.2 Minimal and Maximal Semi Open Sets

Definition 2.2.1.
Let (X, T) be a topological space, then:

(1) A proper nonempty semi-open set U of X is said to be a minimal semi-open set if any semi-open set which contained in U is ϕ or U. A proper nonempty semi-closed set F of X is said to be a minimal semi-closed set if any semi-closed set which contained in F is ϕ or F.

(2) A proper nonempty semi-open set M of X is said to be a maximal semi-open set if any semi-open set which contains M is X or M. A proper nonempty semi-closed set if any semi-closed set which contains E is X or E.

The collection of all minimal semi-open sets in X is denoted by $miSO(X, T)$ and the collection of all maximal semi-open sets in X is denoted by $MaSO(X, T)$.

Example 2.2.2.

Let X = {a,b,c,d} with the topology T = {ϕ,X, {a}, {b}, {d}, {a,b}, {a,d}, {b,d}, {a,b,d}, {}a,b,c}. Then SO(X; T) = {ϕ,X, {a}, {b}, {d}, {a,b}, {a,d}, {b,d}, {a,b,d}, {a,b,c},{a,c}, {b,c}; {b,c,d}; {a,c,d}, then the set {1} is minimal semi-open, the set {a,c,d} is maximal semi-open set.

Theorem 2.2.3.

Let U be a proper nonempty subset of X. Then U is a minimal semi-open if and only if X\U is a maximal semi-closed

Proof:

Let U be a minimal semi-open set in X, then $X \setminus U$ is semi-closed. Let V be a semi-closed set such that $X \setminus U \subseteq V$, then $X \setminus V \subseteq U$. But $X \setminus V$ is semi-open set contained in a minimal semi-open set U, so $X \setminus V = \phi$ or $X \setminus V = U$. This implies that $V = X$ or $V = X \setminus U$. Therefore $X \setminus U$ is maximal semi-closed set. Similarly, if $X \setminus U$ is maximal semi-closed, then U is minimal semi-open.

Remark 2.2.4.

The collection of minimal semi-open sets and minimal open sets are, in general, independent.

Example 2.2.5

Let X = {1,2,3} with a topology T = {ϕ;X; {1}}. SO(X; T) ={ϕ;X; {1}; {1,2}; {1,3}}. Then the set {1} is maximal open, but not maximal semi-open and the set {1,2} is maximal semi-open, but not maximal open.

Theorem 2.2.6.

For any topological space X, the following statements hold:
(1) Let A be a maximal semi-open set and B a semi- set. If A∪B ≠ X, then B ⊆ A.
2) Let A and B be two maximal semi-sets. If A∪B ≠ X, then A = B

Proof:

(1) Let A be a maximal semi-open set and B a semi-open set. Assume A∪B ≠ X, then $A \subseteq$ A∪B, but A∪B is a semi-open and A is a maximal semi-

open, so A = A∪B; that is, $B \subseteq A$.

(2) Let A and B be a maximal semi-open. If A∪B ≠ X, then from (1) $A \subseteq B$ and $B \subseteq A$, so $A = B$.

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