Education Ministry of Higher Education
University of Babylon
College of Education and Pure Sciences
Mathematics department

Modes of Convergence

A proposed research to the council of the college of Education for pure Sciences/University of Babylon As part of the requirements for Bachelor's degree

By

Ayat Kazem Talib

Supervised by
Asst . prof . Dr. Janan Hamza

H 1444 A 2023

بسم الله الرحمن الرحيم

وَقُلْ اعْمَلُوا فَسَيَرَى اللَّهُ عَملَكُمْ وَرَسُولُهُ
وَالْمُؤْمِنُونَ وَسَتُرَدُّونَ إِلَى عَالِمِ الْغَيْبِ وَالشَّهَادَةِ
فَيُنَبِّئُكُمْ بِمَا كُنتُمْ تَعْمَلُونَ (١٠٥)

صدق الله العظيم
سورة التوبة (١٠٥)

الأهداء.....

إلى...
أول من نظر كلمة "اقرأ... سيد المرسلين الذي أنار برسالته ظلام الجاهلية
"محمد (**اللهم صل على محمد وال محمد**)"

إلى...
الذي كلما ذكر أسمهُ فاضت الدموع
"وطني الحبيب"

إلى...
من أضاف إلى سنين الصبر صبراً ليراني كما يحب
"والدي العزيز"

إلى...
من حملتني وهناً على وهن وسهرت الليالي على راحتي
"والدتي العزيزة"

إلى...
الأزهار التي معهم أحلى وأجمل الأيام التي لن أنساها
"زملائي وزميلاتي"

إلى...
كل من علمني حرفاً ليكون لي سلاحاً بوجه الظلام
"أساتذتي الأفاضل"

إليكم جميعاً أهدي ما وفقنا به ربنا حباً واعتزازاً...

شكر وتقدير ...

الحمدلله والشكر له على تيسير وإكمال هذا البحث، والصلاة والسلام على رسولنا الكريم محمد (**اللهم صل على محمد وال محمد**) الذي نور لنا الطريق القويم وبعد...

يشرفنا أن نقدم جميع شكرنا وتقديرنا الى أستاذتنا المشرفة الدكتورة (**جنان حمزة**) الذي تكرمت بالأشراف على هذا البحث، وكان له الدور الفاعل في العون والدعم والاسناد والصبر الطويل في تذليل الكثير من صعوبات البحث والعناية بمتابعة وفحص وتدقيق كل التفاصيل، التي تخص البحث، فجزاها الله كل خير وأبقاه ذخراً للبحث العلمي.

وأتوجه بالشكر الجزيل والاحترام والمحبة، إلى أساتذة قسم الرياضيات

الذين أخذوا بيدي في طريق الخير والعلم والمعرفة فجزاهم الله خير الجزاء...

ومن الله التوفيق...

الباحثة

Contents

Introduction

We offer in this research subject on modes of convergence that represents by the convergence in distribution, convergence in probability, Convergence almust shurly and convergence in rth mean such that There is only one sense in which a sequence of real numbers $(a_n)_{n \in N}$ is said to converge to a limit. Namely, $a_n \to a$ if for every $\epsilon > 0$ there exists a positive integer N such that the sequence after N is always within ϵ of the supposed limit a. In contrast, the notion of convergence becomes somewhat more subtle when discussing convergence of functions. In this note we briefly describe a few modes of convergence and explain their relationship

In this research we introduce some theorems. about this like weak law of large number, strong law of large number and the central limit theorem .We express some examples about this subject.

1.Convergence in Distribution

Let F and F_n be the distribution functions of X and X_n, respectively the sequence of random variables $\{X_n\}$ is said to converge in distribution to random variable X as $n \to \infty$ if

$\lim_{n\to\infty} F_n(z) = F(z)$........(1)

For all $z \in R$ and z is a continuity points of F. We write $X_n \overset{d}{\to} X$ or $F_n \overset{d}{\to} F$

Example(1):

Let Y_n denote the nth order statistic of a random sample $X_1, X_2, \dots \dots X_n$ from a distribution having

Probability density function:

$$f(x) = \frac{1}{\theta}, \qquad 0 < x < \theta, \qquad 0 < \theta < \infty$$

=0 elsewhere.

Show that the sequence of nth order statistics $\{Y_n, n = 1,2,3, \dots\}$ convergence in distribution to a random variable that has degenerate distribution at the point $x = \theta$.

<u>Solution:</u>

The pdf of Y_n is:

$$gy_n(y_n) = g_n(y_n) = n\big(F(y_n)\big)^{n-1} \ f(y_n)$$

$$= n\left[\int_{-\infty}^{y} f(t)\ dt\right]^{n-1} \frac{1}{\theta} = n\ \left[\int_0^y \frac{1}{\theta} dt\right]^{n-1} \frac{1}{\theta}$$

$$= n\left[\frac{t}{\theta}\Big|_0^y\right]^{n-1}\frac{1}{\theta} = n\left[\frac{y}{\theta}\right]^{n-1}\frac{1}{\theta}$$

$$= n\frac{y^{n-1}}{\theta^{n-1}}\ \frac{1}{\theta} = n\ \frac{y^{n-1}}{\theta^{n}}$$

The p.d.f. of Y_n is

$$g_n(y) = \frac{ny^{n-1}}{\theta^2},\ \ 0 < y\ \ < \theta,$$

$$= 0 \text{ elsewhere},$$

The distribution function of Y_n is:

$$F_n(y) = \int_0^y g_n(t)dt = \int_0^y n\frac{t^{n-1}}{\theta^n}\ dt = n\ \frac{t^n}{n\ \theta^n}\Big|_0^y = \frac{y^n}{\theta^n},\quad 0 < y < \theta$$

we get,

$$F_n(y) = 0,\qquad y < 0$$

$$F_n(y) = \left(\frac{y}{\theta}\right)^n\quad,\quad 0\ \leq y < \theta$$

$$= 1,\quad \theta \leq y < \infty$$

Then:

$$\lim_{n\to\infty} F_n(y) = 0,\quad -\infty < y < \theta$$

$$= 1,\quad \theta \leq y < \infty$$

$$F(y) = 0,\qquad -\infty < y < \theta,$$

$$= 1,\quad \theta \leq y < \infty$$

is a distribution function moreover,

$$\lim_{n \to \infty} F_n(y) = F(y)$$

The above distribution is degenerate distribution at the point $x = \theta$.

Example(2):

Let Y_n denote the nth order statistic of a random sample from the uniform distribution having pdf

$$f(x) = \frac{1}{\theta}, \quad 0 < x < \theta,$$

$$0 < \theta < \infty,$$

=0 elsewhere.

Let $Z_n = n(\theta - Y_n)$. Show that the sequence of nth order statistic $\{Z_n, n = 1,2,3, \dots\}$ convergence in distribution to a random variable that has an exponential distribution with mean θ.

Solution:

$$gy_n(y_n) = g_n(y_n) = n\big(F(y_n)\big)^{n-1} f(y_n)$$

$$= n\left[\int_{-\infty}^{y} f(t)\ dt\right]^{n-1} \frac{1}{\theta} = n\left[\int_{0}^{y} \frac{1}{\theta} dt\right]^{n-1} \frac{1}{\theta}$$

$$= n\left[\frac{t}{\theta}\Big|_{0}^{y}\right]^{n-1} \frac{1}{\theta} = n\left[\frac{y}{\theta}\right]^{n-1} \frac{1}{\theta}$$

$$= n\frac{y^{n-1}}{\theta^{n-1}}\ \frac{1}{\theta} = n\ \frac{y^{n-1}}{\theta^{n}}$$

Let $z = n(\theta - y) \to y = \theta - \frac{z}{n} \to dz = -ndy$

The pdf of Z_n by using the transformation technique to get the following

$$h_n(z) = g_n\left(y = \theta - \frac{z}{n}\right)\left|\frac{dy}{dz}\right| = n\ \frac{\left(\theta - \frac{z}{n}\right)^{n-1}}{\theta^n}\ \left|-\frac{1}{n}\right| = \frac{\left(\theta - \frac{z}{n}\right)^{n-1}}{\theta^n}$$

and the distribution function of Z_n is

$$G_n(z) = \int_0^z h_n(t)dt = \int_0^z \frac{(\theta - \frac{t}{n})^{n-1}}{\theta^n}\ dt = -n\ \frac{(\theta - \frac{t}{n})^n}{n\theta^n}\Bigg|_0^z$$

$$= -\left(1 - \frac{t}{n\ \theta}\right)^n\Bigg|_0^z = 1 - \left(1 - \frac{z}{n\ \theta}\right)^n, \qquad 0 \le z \le n\theta$$

$$\lim_{n\to\infty} G_n(z) = \lim_{n\to\infty}\left[1 - \left(1 - \frac{z}{n\ \theta}\right)^n\right] = 1 - \lim_{n\to\infty}\left(1 - \frac{z}{n\ \theta}\right)^n$$

$$= 1 - \lim_{n\to\infty}\left(1 - \frac{\frac{z}{\theta}}{n}\right)^n = 1 - e^{-\frac{z}{\theta}} = G(z)$$

$$\therefore \lim_{n\to\infty} G_n(z) = G(z) = 1 - e^{-\frac{z}{\theta}}$$

The above distribution is an exponential distribution with mean θ.

2. Convergence in Probability

The sequence of random variables $X_1, \ldots . X_n$ converges in probability to constant c, denoted

$$X_n \overset{p}{\to} c$$

If

$\lim_{n\to\infty} P[|X_n - c| < \epsilon] = 1$........(2)

or, equivalently,

$\lim_{n\to\infty} P[|X_n - c| \geq \epsilon] = 0$…….(3)

That is, if the limiting distribution of $X_1, \ldots . X_n$ is degenerate at c.

Theorem(1)(Weak Law of Large Numbers)

Suppose that $X_1, \ldots . X_n$ is a sequence of i.i.d. random variables with expectation μ and finite variance σ^2. Let Y_n be defined by

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

then, for all $\epsilon > 0$,

$$\lim_{n\to\infty} P[|Y_n - \mu| < \epsilon] = 1$$

that is, $Y_n \overset{p}{\rightarrow} \mu$, and thus the mean X1……Xn convergence in probability of μ

Proof. Using the properties of expectation, it can be shown that Y_n has expectation μ and variance $\frac{\sigma^2}{n}$, and hence by the chebychev inrquality,

$$P[|Y_n - \mu| \geq \epsilon] \leq \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \qquad \text{as } n \rightarrow \infty$$

for all $\epsilon > 0$. Hence

$$P[|Y_n - \mu| < \epsilon] \rightarrow 1 \qquad \text{as } n \rightarrow \infty$$

and $Y_n \overset{p}{\rightarrow} \mu$.

Example(3):

Let $Y_n \sim b(n,p)$. Show that $\frac{Y_n}{n}$ convergence in probability to p (that is $\frac{Y_n}{n} \xrightarrow{p} p$).

Solution:

To prove $\frac{Y_n}{n}$ convergence in probability to p

We must prove $\lim_{n\to\infty} P[\left|\frac{Y_n}{n} - p\right| \geq \varepsilon] = 0$

$$P\left(\left|\frac{Y_n}{n} - p\right| \geq \varepsilon\right) = P(|Y_n - np| \geq n\,\varepsilon)$$

For any $\varepsilon > 0$ let $n\varepsilon = K\sqrt{np(1-p} \rightarrow$

$$K = \frac{n\varepsilon}{\sqrt{np(1-p)}} = \frac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}$$

where var (Y_n)=$np(1-p)$

$$P\left(\left|\frac{Y_n}{n} - p\right| \geq \varepsilon\right) = P(|Y_n - np| \geq n\,\varepsilon)$$

$= P\left(|Y_n - np| \geq K\sqrt{np(1-p)}\right) \leq \frac{1}{\left(\frac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\right)^2}$ by chebyshev's

theorem,

$$\therefore P\left(\left|\frac{Y_n}{n}-p\right| \geq \varepsilon\right) \leq \frac{1}{\left(\frac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\right)^2}$$

Take the limit of two sides:

$$\lim_{n\to\infty} P\left(\left|\frac{Y_n}{n}-p\right| \geq \varepsilon\right) \leq \lim_{n\to\infty}\left(\frac{1}{\left(\frac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\right)^2}\right) = 0$$

$$\therefore \lim_{n\to\infty} P\left(\left|\frac{Y_n}{n}-p\right| \geq \varepsilon\right) = 0$$

$$\therefore \frac{Y_n}{n} \overset{p}{\to} p$$

Example(4):

Let $Y_n \sim b(n,p)$. Show that $1-\frac{Y_n}{n}$ convergence in probability to **1–p** (that is $1-\frac{Y_n}{n} \overset{p}{\to} 1-p$).

Solution:

$$P\left(\left|\left(1-\frac{Y_n}{n}\right)-(1-p)\right| < \varepsilon\right) = P\left(\left|1-\frac{Y_n}{n}-1+P\right| < \varepsilon\right)$$

$$= P\left(\left|-\frac{Y_n}{n}+P\right| < \varepsilon\right) = P\left(\left|(-1)\left(\frac{Y_n}{n}-p\right)\right| < \varepsilon\right)$$

$$=P\left(|-1|\left|\left(\frac{Y_n}{n}-p\right)\right| < \varepsilon\right) = P\left(\left|\left(\frac{Y_n}{n}-p\right)\right| < \varepsilon\right)$$

$$= P(|Y_n-np| < n\varepsilon)$$

By last example, we get

$$\lim_{n\to\infty} P\left(\left|\left(1-\frac{Y_n}{n}\right)-(1-p)\right|<\varepsilon\right)$$

$$=\lim_{n\to\infty} P\left(\left|\left(\frac{Y_n}{n}-p\right)\right|<\varepsilon\right)=1$$

$$\therefore \lim_{n\to\infty} P\left(\left|\left(1-\frac{Y_n}{n}\right)-(1-p)\right|<\varepsilon\right)=1$$

$$\therefore 1-\frac{Y_n}{n} \overset{p}{\rightarrow} 1-p$$

3.Convergence Almost Surely (Convergence with Probability one)

The sequence of random variables $X_1, \ldots . X_n$ converges almost surely to random variable x, denoted

$$X_n \overset{a.s}{\rightarrow} X$$

If

$P\left[\lim_{n\to\infty} |X_n - X| < \epsilon\right] = 1$,……..(4)

That is, if $A = |\omega: X_n(\omega) \rightarrow X(\omega)|, then\ P(A) = 1.$

Theorem(2) (Strong Law of Large Numbers)

Suppose that $X_1, \ldots . X_n$ is a sequence of i.i.d. random variables with expectation μ and (finite) variance σ^2 let Y_n be defined by

$$Y_n = \frac{1}{\mu}\sum_{i=1}^{n} X_i$$

Then, for all $\epsilon > 0$

$$P\left[\lim_{n\to\infty} |Y_n - \mu| < \epsilon\right] = 1,$$

4.Convergence in Rth Mean

The sequence of random variables $X_1, \ldots . X_n$ converges in rth mean to random variable X, denoted

$$X_n \xrightarrow{r} X$$

If

$$\lim_{n\to\infty} E[| X_n - X |^r] \ldots\ldots..(5)$$

For example, if

$\lim_{n\to\infty} E|(X_n - X)^2| = 0 \ldots\ldots..(6)$

Then we write

$$X_n \xrightarrow{r=2} X$$

In this case, we say that $|X_n|$ converges to X in mean-square or in quadratic mean.

Theorem(3):

For $r_1 > r_2 \geq 1,$

$$X_n \xrightarrow{r=r1} X \quad \Rightarrow \quad X_n \xrightarrow{r=r2} X$$

Proof.

$$E[|X_n - X|^{r2}]^{1/r2} \leq E[|X_n - X|^{r1}]^{1/r1}$$

so that

$$E[|X_n - X|^{r2}] \leq E[|X_n - X|^{r1}]^{r2/r1} \rightarrow 0$$

as $n \rightarrow \infty$, as $r_2 < r_1$, thus

$$E[|X_n - X|^{r2}] \rightarrow 0$$

and $X_n \xrightarrow{r=r2} X$

Note: the converse does not hold in general.

Example(5):

Let $X_n \sim Uniform\left(0, \frac{1}{n}\right). show\ that$

$$X_n \xrightarrow{r} 0, for\ any\ r \geq 1.$$

Solution:

The PDF of X_n is given by

$$fX_n(x) = \begin{cases} n & 0 \leq x \leq \dfrac{1}{n} \\ 0 & otherwise \end{cases}$$

We have

$$E(|X_n - 0|^r) = \int_0^{\frac{1}{n}} x^r n\ \ dx$$

$$= \frac{1}{(r+1)n^r} \rightarrow 0, \qquad for\ \ all\ \ r \geq 1.$$

5.Relating the Modes of Convergence

Theorem4:

For sequence of random variables $X_1, \ldots . X_n$, following relationships hold

$$X_n \xrightarrow{a.s} X \searrow$$

$$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$$

$$X_n \xrightarrow{r} X \nearrow$$

So almost sure convergence and convergence in rth mean for some r both imply convergence in probability, which in turn implies convergence in distribution to random variable X.

No other relationships hold in general.

That is ,we can prove that

(a) $x_n \xrightarrow{a.s} x \Longrightarrow x_n \xrightarrow{p} x$

(b) $x_n \xrightarrow{r} x \Longrightarrow x_n \xrightarrow{p} x$

(c) $x_n \xrightarrow{p} x \Longrightarrow x_n \xrightarrow{d} x$

Proof.

$(a)\ X_n \xrightarrow{a.s} X \ \Rightarrow X_n \xrightarrow{p} X$, suppose $X_n \xrightarrow{a.s} X,$ and let $\epsilon > 0$ then

$$P[|X_n - X| < \epsilon] \ \geq P\ [|X_m - X| < \epsilon, \forall m \ \geq n]$$

as, considering the original sample space,

$(\omega\colon |X_m(\omega) - X(\omega)| < \epsilon, \forall m \ \geq n) \subset (\omega\colon |X_n(\omega) - X(\omega)| < \epsilon)$

But, as $X_n \xrightarrow{a.s} X, P\left[\left|X_m - X\right| < \epsilon, \forall m \ \geq n\right] \to 1,$

as $n \to \infty$, so after taking limits in equation (1),we have

$$\lim_{n \to \infty} P[|X_n - X| < \epsilon\,] \geq \lim_{n \to \infty} P[|X_m - X| < \epsilon, \forall m \geq n\,] = 1$$

and so:

$$\lim_{n \to \infty} P[|X_n - X| < \epsilon] = 1 \qquad \therefore X_n \xrightarrow{p} X$$

(b) $X_n \xrightarrow{r} X \Longrightarrow X_n \xrightarrow{p} X$. suppose $X_n \xrightarrow{r} X,$ and let **ϵ >0** then, using an argument similar to chebychev's Lemma.

$$E[|X_n - X|^r] \geq E[|X_n - X|^r I_{\{|X_n - X| > \epsilon\}}] \geq \epsilon^r P[|X_n - X| > \epsilon]$$

Taking limits as $n \to \infty$, as $X_n \xrightarrow{r} X$, $E[|X_n - X|]^r \to 0$

as $n \to \infty$, so therefore, also, as $n \to \infty$,

$$P[|X_n - X| > \epsilon] \to 0 \qquad \therefore \qquad X_n \xrightarrow{p} X$$

$(c) X_n \xrightarrow{p} X \Longrightarrow X_n \xrightarrow{d} X$, suppose $X_n \xrightarrow{p} X,$ and let $\epsilon > 0$, denote, in the usual way,

$F_{X_n}(x) = P[X_n \leq x]$ and $Fx(x) = [X \leq x]$

Then, by the theorem of total probability, we have two inequalities

$$F_{X_n}(x) = P[X_n \leq x] = P[X_n \leq x, X \leq x + \epsilon] +$$

$$P[X_n \leq x, X > x + \epsilon] \leq F_X(x + \epsilon) + P[\{|X_n - X] > \epsilon]$$

$$F_X(x - \epsilon) = P[X \leq x - \epsilon] = P[X \leq x - \epsilon, X_n \leq x] +$$

$$P[X \leq x - \epsilon, X_n > x] \leq F_X(x) + P\{[X_n - X] > \epsilon].$$

as $A \underline{c} B \Rightarrow P(A) \leq P(B)$ yields

$$P[X_n \leq x, X \leq x + \epsilon] \leq F_X(x + \epsilon) \text{ and}$$

$$P[X \leq x - \epsilon X_n \leq x] \leq F_{X_n}(x).$$

Thus

$$F_x(x-\epsilon) - P[|X_n - X| > \epsilon] \le F_{X_n}(x)$$

$$\le F_X(x+\epsilon) + P[|X_n - X| > \epsilon]$$

and taking limits as $n \to \infty$ (with care; we cannot yet write

$$\lim_{n\to\infty} F_{X_n}(x)$$

as we do not know that this limit exists) recalling that $X_n \xrightarrow{p} X$,

$$F_x(x-\epsilon) \le \lim_{n\to\infty} inf\ F_{X_n}(x) \le \lim_{n\to\infty} sup\ F_{X_n}(x) \le F_X(x+\epsilon)$$

Then if F_X is continuous at x, $F_X(x-\epsilon) \to F_X(x) and$

$F_X(x+\epsilon) \to F_X(x)\ as\ \epsilon \to 0$, and hence

$$F_X(x) \le \lim_{n\to\infty} inf\ F_{X_n}(x) \le \lim_{n\to\infty} sup\ F_{X_n}(x) \le F_X(x)$$

and thus $F_{X_n}(x) \to F_X(x)$ as $n \to \infty$.

Theorem (5):Central Limit Theorem

Let $X_1,\dots,X_n$ be iid random variables with $E[X_k] = \mu$ and $Var(X_k) = \sigma^2 < \infty$, Then

$$\sqrt{n}(Y_n - \mu) \xrightarrow{d} N(0,\sigma^2) \quad \text{......... (7)}$$

where $Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k$.

Proof.

It is sufficient to prove that

$$\sqrt{n}\left(\frac{Y_n-\mu}{\sigma}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

Let $Z_n = \sqrt{n}(\frac{Y_n-\mu}{\sigma})$. The moment generating function of Z_n is

$$MZn(s) \overset{\text{def}}{=} E[e^{sZn}] = E\,[e^{\,s\sqrt{n}\,(\frac{y_{n}-\mu}{\sigma})}\,]= \prod_{K=1}$$

$$E[\frac{s}{e\sigma/\sqrt{n}}\ \ (Xk-\mu)\ ]$$

By Taylor approximation, we have

$$\mathbb{E}\left[e^{\frac{s}{\sigma\sqrt{n}}(X_k-\mu)}\right] = \mathbb{E}\left[1+\frac{s}{\sigma\sqrt{n}}(X_k-\mu)+\frac{s^2}{\sigma^2 n}(X_k-\mu)^2+O(\frac{1}{\sigma^3\sqrt{n^3}}(X_k-\mu)^3)\right]$$

$$= (1+0+\frac{s^2}{2n}).$$

Therefore,

$$M_{Z_n}(s) = \left(1+0+\frac{s^2}{2n}\right)^n \xrightarrow{(a)} e^{\frac{s^2}{2}},$$

as $n \to \infty$. To prove (a), we let $y_n = (1+\frac{s^2}{2n})^n$. Then, $\log y_n = n\log(1+\frac{s^2}{2n}$), and by Taylor

approximation we have

$$\log(1+x_0) \approx x_0 - \frac{x_0^2}{2}.$$

Therefore,

$$\log y_n = n\log(1+\frac{s^2}{2n}) = n(\frac{s^2}{2n}-\frac{s^4}{4n^2}) = \frac{s^2}{2}-\frac{s^4}{4n} \xrightarrow{n\to\infty} \frac{s^2}{2}.$$

As a corollary of the Central Limit Theorem, we also derive the following proposition.

Example(6):

Let $\bar{X}$ denote the mean of a random sample of size 75 from the distribution that has the pdf:

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & elsewhere. \end{cases}$$

Find P $(0.45 < \bar{X} < 0.55)$

Solution:

$$\Rightarrow \mu = E(x) = \int_0^1 x\, f(x)\, dx = \int_0^1 x dx = \frac{1}{2}.$$

$$E(x^2) = \int_0^1 x^2 f(x) dx = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\Rightarrow \sigma_x^2 = E(x^2) - \big(E(x)\big)^2 = \frac{1}{3} - \frac{1}{4}$$

$$= \frac{4-3}{12} = \frac{1}{12}$$

$$\therefore n = 75 \Rightarrow \sqrt{n} = \sqrt{75}$$

$$p(0.45 < \bar{X} < 0.55) = P\,(\frac{\sqrt{n}(0.45-\mu)}{\sigma} <$$

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < \frac{\sqrt{n}(0.55-\mu)}{\sigma})$$

$$= P(\frac{\sqrt{75}(0.45-\frac{1}{2}}{\sqrt{\frac{1}{12}}} < \frac{\sqrt{75}(\bar{X}-\frac{1}{2}}{\sqrt{\frac{1}{12}}} < \frac{\sqrt{75}(0.55-\frac{1}{2}}{\sqrt{\frac{1}{12}}} <)$$

$$= P(-1.5 < z < 1.5) = P(z < 1.5) - p\,(z < -1.5)$$

$$P(z < 1.5) - (1 - P(z < 1.5))$$

From below table normal distribution, we have

Example(7) :

Let $X_1, X_2, \dots\dots, X_n$ be independent and identically random sample from $b(1,p)$ where $p = 0.5$ such that $\bar{X}$ denote a mean of a random sample of size 100 from this distribution. Find $P(47.5 < Y < 52.5)$ where $Y = \sum_{i=1}^{n} X_i$ i.e, find the approximate value of $P(47.5 < Y < 52.5)$ where $Y = \sum_{i=1}^{n} X_i$ when one uses the central limit theorem.

Solution:

$$since, E(x) = np = (1)p = p = 0.5$$

$$var\ (x) = npq = (1)pq = pq$$

$$= p(1-p) = 0.5(1-0.5) = 0.25$$

$$n = 100 \ \rightarrow \ \sqrt{n} = 25$$

We get

$$E(y) = E\left(\sum_{i=1}^{n} x_i\right) = \left(\sum_{i=1}^{n} E(x_i)\right)$$

$$= \sum_{i=1}^{n} p = np = (100)(0.5) = 50$$

$$var(y) = var\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} var\ (x_i)$$

$$= \sum_{i=1}^{n} pq = npq = (100)(0.25) = 25$$

$$\therefore z = \frac{\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} x_i\right)}{\sqrt{var\left(\sum_{i=1}^{n} x_i\right)}} = \frac{Y - np}{\sqrt{npq}} \sim N(0.1)$$

$$P(47.5 \ < Y < 52.5) = P\left(\frac{47.5 - np}{\sqrt{npq}} < \frac{Y - np}{\sqrt{npq}} < \frac{52.5 - np}{\sqrt{npq}}\right)$$

$$= P\left(\frac{47.5 - 50}{5} < \frac{Y - 50}{5} < \frac{52.5 - 50}{5}\right)$$

$$P(-0.5 < z < 0.5) = P(z < 0.5) - P(z < -0.5)$$

$$P(z < 0.5) - \left(1 - P(z < 0.5)\right)$$

From the table normal distribution, we have

$$P(47.5 < Y < 52.5) = P(z < 0.5) - \left(1 - P(z < 0.5)\right)$$

$$= 0.691 - (1 - 0.691) = 0.691 - 0.309 = 0.382$$

Example(8):

Let $X \sim N(6.1)$ and $Y \sim N(7.1)$. find $P(X > Y)$.

Solution:

Since $P(X > Y) = P(X - Y > 0) = 1 - P\ (X - Y \leq 0)$

$$\therefore X - Y \sim N(6 - 7.1 + 1) \Longrightarrow X - Y \sim N(-1.2)$$

$$\therefore P(X > Y) = P(X - Y > 0) = 1 - P(X - Y \leq 0)$$

$$= 1 - P\left(\frac{(X - Y) - (-1)}{\sqrt{2}} \leq \frac{0 - (-1)}{\sqrt{2}}\right)$$

$$= 1 - P\left(z \leq \frac{1}{\sqrt{2}}\right) = 0.24$$

Example(9):

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample of size n=25 from a population that has a mean $\mu = 71.43$ and variance $\sigma^2 = 56.25.$ let $\bar{X}$ be the sample mean. What is the probability that the sample mean is between 68.91 and 71.97?

Solution:

Since, $E(\bar{X}) = E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \mu = 71.43$

$$v(\bar{X}) = v\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = \frac{\sigma^2}{n} = \frac{56.25}{25} = 2.25$$

$$P(68.91 \leq \bar{X} \leq 71.97)$$

$$= P\left(\frac{68.91 - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{71.97 - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= P\left(\frac{68.91 - 71.43}{\sqrt{2.25}} < \frac{\bar{X} - 71.43}{\sqrt{2.25}} < \frac{71.97 - 71.43}{\sqrt{2.25}}\right)$$

$$= P(-0.68 < Z < 0.36) = P(z < 0.36) - P(z < -0.68)$$

$$P(z < 0.36) - \big(1 - P(z < 0.68)\big) = 0.5941$$

References

[1]Bexheti, B. , Ibraimi, a. , sadiki, F. and pollozhani, F. L., (the Relations Between models of convergence for sequences of Random Variables), Department of Natural Sciences and Mathematic, University of tetovo, NMK

[2]Imai, k. , (pol 571: Convergence of Random Variables), Department of politics, princeton University, 2006.

[3] Jordan, M. I., (stat210B:Theoretical Statistics), Lecture Date, January, 2007.

[4]Salim, R.E., (Analysis of Random signals: Lecture Chapter 6:Convergence of Random sequences), ECE 511,2016.

[5] Whitt, (Stochastic Models 1:Lecture Notes Modes of convergence), IEoR 6711, 2013.

[6]https://engineering.purdue.edu/ChanGroup/ECE645Notes/StudentLecture04.pdf

[7]https://www.math.mcgill.ca/dstephens/OldCourses/556-2006/Math556-ModesOfConvergence.pdf