Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure
ciences
Department of Mathematics

THE TYPES OF POINTS ON DIFFERENT SPACE

A Research Paper Submitted to the Council of the College of Education for Pure Sciences/University of Babylon as Partial Fulfillment of the Requirements for the Degree of Bachelor of Science in Mathematics.

By:

Fatima Basem Hamza Attia

Supervised by:

Asst.prof.Dr. .Iftikhar Mudar Talib

2026A.D. 1447A.H

الآية الكريمة

بسم الله الرحمن الرحيم

﴿ وَتَرَى الْجِبَالَ تَحْسَبُهَا جَامِدَةً وَهِيَ تَمُرُّ مَرَّ السَّحَابِ ﴾

صدق الله العلي العظيم

﴿ النمل: ٨٨﴾

الاهداء

الى والديّ العزيزين اللذين كانا دائما النور الذي أضاء طريقي والداعمين الاساسيين في حياتي اهدي هذا لكم هذا البحث بكل الحب والتقدير.

إلى أبي العزيز الذي كان قدوتي ومعلمي الأول شكراً لك على كل الدعم الذي أوصلني إلى يوم تخرجي.

إلى أمي الحبيبة، التي منحتني القوة والحنان والدعاء أهديك هذا الإنجاز.

إلى اساتذتي الكرام الذين غرسوا في نفسي حب البحث والتأمل ,وأمدوني بما يلزم من أدوات الفكر والمنهج، أهدي هذا العمل ليكون امتداداً بما تعلمته على أيديكم، وإلى مشرفتي العزيزة التي وجهتني بحكمة وصبر، فكان لها الفضل الكبير في إتمام هذا الجهد، أهدي هذه الصفحات عربون تقدير وامتنان.

إلى زملائي ورفاق الدرب في طلب العلم، الذين شاركوني النقاشات والتجارب أهديكم هذا العمل ليكون شاهداً على أننا معاً نصنع المعرفة ونبني المستقبل.

الباحثة :فاطمة

الشكر والعرفان

أول الشكر وآخره لله رب العالمين, والصلاة والسلام على سيدنا محمد صلى الله عليه وآل بيته الطيبين الطاهرين.

الحمد لله الذي وفقني وأعانني على اتمام هذا البحث، والصلاة والسلام على معلم البشرية محمد (صل الله عليه وآل وسلم)

أتقدم بخالص الشكر والتقدير الى عمادة كلية التربية للعلوم الصرفة متمثلة برئيس القسم الاستاذ(الدكتور علي العبيدي) وجميع اساتذتي في كلية التربية واخص بالشكر الجزيل الى مشرفتي (الدكتورة افتخار مضر طالب) التي تفضلت بأشرافها على هذا البحث، ولم تبخل عليه بتوجيهاتها القيمة ونصائحها السديدة الذي أنارت لي الطريق.

كما اتقدم بجزيل الشكر لجميع زملائي الذين ساهموا في دعمي، والشكر موصول الى عائلتي التي كانت السند الحقيقي بعد الله سبحانه وتعالى لما بذلوا من جهود استثنائية التي امدتني بالقوة والاستعداد لمواصلة رحلة دراستي.

قائمة المحتويات

INTRODUCTION:

The study of mathematical spaces is considered one of the fundamental pillars in many branches of modern mathematics, as it provides the general framework upon which various mathematical concepts and structures are built, such as continuity,convergence,separation, and other essential properties. Among the most important elements studied within these spaces are points, due to their central role in determining the nature of the space and in understanding its internal structure.

The classification of points in different spaces represents a rich and multifaceted topic, as the types of points vary according to the nature of the space under consideration, whether it is a metric space, a topological space, or more abstract spaces. For instance, there are interior points, boundary points, accumulation points, isolated points, fixed points, periodic points, eventually fixed points, and other classifications that contribute to analyzing the fine structure of a space.

The importance of this topic stems from its contribution to deepening the theoretical understanding of mathematical spaces, as well as its wide applications in various fields such as mathematical analysis, topology, and function theory. Accordingly, the study of types of points is not limited to the theoretical aspect alone, but also extends to support numerous scientific and engineering applications.

This research aims to achieve several scientific objectives, including:

1. To provide a systematic and precise presentation of the concept of points in different mathematical spaces.

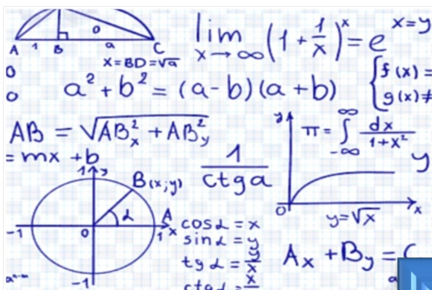
2. To classify the types of points in various spaces, such as metric and topological spaces.

3. To study the fundamental properties of each type of point and analyze the relationships between them.

4. To highlight the essential differences in the classification of points depending on the nature of the space.

5. To support the study with illustrative examples and theoretical results that enhance a deeper mathematical understanding of the topic.

In light of these objectives, this research seeks to present a rigorous and well-structured treatment of the topic “Types of Points in Different Spaces,” thereby contributing to the enrichment of the mathematical literature in this field and opening avenues for more specialized future studies.



FIXED POINTS AND STABILITY IN ONE-DIMENSIONAL DYNAMICS

Inthis section , we introduce on the spaces of dimension noe the elementries which we need then wonq our work .

1.1 Definition :

Let f be a function and let x_0 and $f(x_0)$ be in the domain of f.

Then $f(x_0)$ =the first iterate of x_0 for

$f(f(x_0)) =$ the second iterate of x_0 for f.

$f^n(x) = f(f(\dots(f(x)\dots) = the\ n - th$ iterate of x_0 for f [1]

1.2 Definition :

The orbit $O(x_0)$ of apoint $x_0\ \epsilon\mathcal{R}$ is defined to be the set of points

$O(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \dots\}$

Where

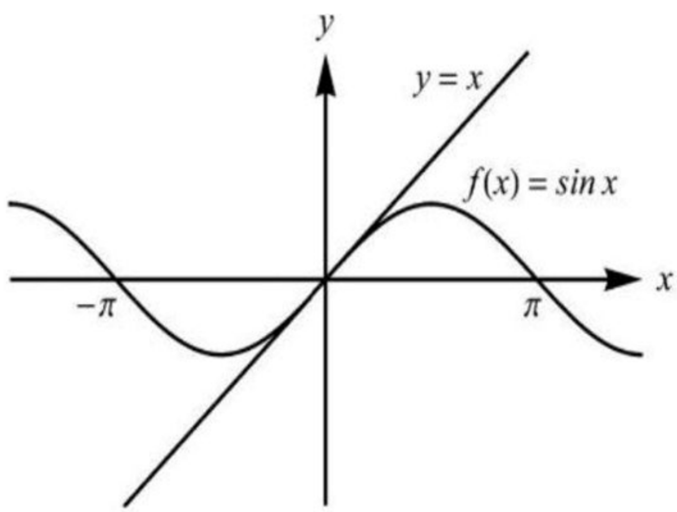
$$f^2 = fof,$$

$$f^3 = fofof$$

$$f^n = f o f o \ldots o\ f$$

1.3 Definition :

Let p be in the domain of f. Then p is a **fixed point** of f if $f(p) = p$.



1.4 From Figure we might conjecture that the orgin is the only point at which the graph of sin$x$ and the line y=x touch each other . [2]

We will prove that this is true . In the proof we will use the mean value theorem, which says that if f is continuose on [a,b] and differentiable on (a,b) such that

$f'(c) = = \frac{f(b)-f(a)}{b-a}$

, or equivalently, $f(b)-f(a)=f'(c)(b-a)$.

1.5 Example :

Let $f(x)=\sin x$ .We prove that x=0 is the unique fixed point of f.

Solve.

To begin, we observe that $f(x)\neq x$ if $|x|>1$,

since $|\sin x|\leq 1$ for all x . Next, if $0<x\leq 1$, then the mean value theorem implies the existence of z between 0 and x such that

sin x=sin x-sin 0=f'(z)(x-0) =cos z

Since 0<cos z<1 for such z,

it follows that 0<sin x =x cos z<x,

true for all x>0.

The fact that $f(-x)=-f(x)$ implies that x<sine x<0 for all x<0 . Finally, sin 0=0, so we conclude that 0 is the unique fixed point of f.

1.6 Example:

To find the fixed point of $f(x)=1+x-\frac{1}{2}x$^2 the fixed point of $f(x)$ is at (x,x) .

Solve.

$f(x)=x$

$$=1+x-\frac{1}{2}x^2$$

$$1+x-\frac{1}{2}x^2=X$$

$$1-\frac{1}{2}x^2=0$$

$$2-x^2=0$$

$$x^2=2$$

$x=\mp\sqrt{2}$ fixed point :(($\sqrt{2},\sqrt{2}$) and ($-\sqrt{2},-\sqrt{2}$).

1.5 Definition:

The point p is attracting fixed point of *f* provided that there is an interval$(p-\varepsilon,p+\varepsilon)$ containing p such that if x is in the domain of f and in $(p-\varepsilon,p+\varepsilon)$,then $f^n(x)\to p$ as $n\to\infty$.[3]

1.6 Definition :

The point p is a repelling fixed point of f provided that there is an interval $(p-\varepsilon,p+\varepsilon)$ containing p ,Such that there if x is in the domain of f and in $(p-\varepsilon,p+\varepsilon)$ but $x\neq p$ then $|f(x)-p|>|x-p|$. [4]

1.7 Theorem

Suppose that f' is differentiable at a fixed point p .

a.If $|f'(p)|<1$, then p is attracting .

b.If $|f'(p)|>1$, then p is repelling.

c.If $|f'(p)|=1$, then p can be attracting ,repelling or neither.

Proof:

To begin our proof of (a),

we notice that since $|f'(p)|<1$,the definition of derivative implies there is a positive constant A<1 and an open interval

J=(p-ε,p+ε) such that if x is in J and x≠p,

then$\left|\frac{f(x)-f(p)}{x-p}\right| \leq A$

Therefor |*f*(x)-*f*(p)|≤A |X-P| ,for all x in *J*

For each such x ,this means that

$$|f(x)-p|=|f(x)-f(p)|\leq A|x-p| \qquad (1)$$

So that *f*(x) is in J because 0<A<1 and x is in J .

$$|f^n(x) \rightarrow p| \leq A^n|x-p| \; for \; all \; n \geq 1 \quad (2)$$

By(1),the inequality holds for n=1 ,Next we assume that (2) holds for agiven n>1.

Then *f*^n (x)is in J since *0 < A^n < A < 1* .Therefor by(1) with *f*^n (x) sub stituted for x ,and then by(2),we find that

|f^(n+1) (x)-p|=|f (f^n (x)-p|≤A| f^n (x)-p|≤A(A^n |x-p|)=A^(n+1) |x-p|.

By the law of induction we deduce that (2) holds for all integers

n≥1. Since $A^n \to 0$ as increases with out

bunded,it follows that $f^n \to p$ for every x in J .

Thus (a) is proved

To begin our proof of (b) ,

we notice that since $|f'(p)|>1$,the definition of derivative implies

that there is apositive constant A≥1 and an open interval

$J=(p-\varepsilon,p+\varepsilon)$ such that if x ∈ J and x≠p, then

$$\left|\frac{f(x)-f(p)}{x-p}\right| \geq A > 1$$

$$|f(x)-f(p)|>|x-p|$$

p is a repelling fixed point .Thus (b) is proved.

1.8 Example:

To find the repelling fixed point of $f(x)=x^2$ $x \in R$ $f:R \to R$

Solve .

$$x \in \mathcal{R} \;\; f: \mathcal{R} \to \mathcal{R}$$

$$f(x)=x$$

$$x^2=x$$

$$x(x-1)=0$$

$$x=0 \text{ and } x=1,$$

$$f'(x)=2x \quad |f'(0)|=0<1$$

x=0 is attracting fixed point

If $|f'(1)|=2>1$, $x=1$ is repeling fixed point

1.9 Example :

Let f:R→R be a fuction such that

f(x)=$\frac{-1}{2}x$ x∈R .To find the attracting fixed point .

Solve.

$f(x)=x$

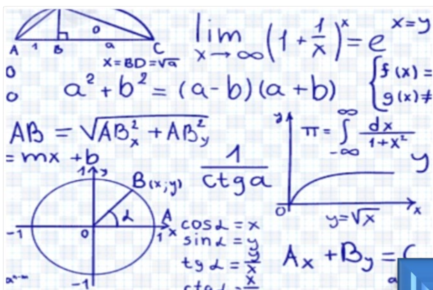
$$\frac{-1}{2}x = x$$

$$x=0$$

The derivative of f is $f'(x)=\frac{-1}{2}x$

$$|f(0)|=\frac{-1}{2}x<1$$

Since the abslute value of the derivative at the fixed point x=0 is an attracting fixed point x_0∈R



Linear Dynamics and Stability in Two-Dimensional Spaces

In this section , me study the fixed point one the spaces of hiqh dimension .

2.1 Definition:

The function $L:R^2 \rightarrow R^2$ is linear if $L(bv+cw)=bL(v)+cL(w)$, for all v and w in R^2 , and real numbers b and c .A linear function is also called a linear map[.4]

2.2 Theorem

A 2×2 matrix A has an inverse if and only if det A≠0.

2.3 Theorem

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For appropriate real numbur λ,μ,β,and Υ, the matrix A is similar to one of the normal forms, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ if A has two distinct real eigenvalues λ,μ

$\begin{pmatrix} \lambda & \beta \\ 0 & \lambda \end{pmatrix}$ if A has one real eigenvalue λ

$\begin{pmatrix} \beta & -\Upsilon \\ \Upsilon & \beta \end{pmatrix}$ if A has two complex eigenvalues ,

$\beta+\imath\Upsilon$ and $\beta-\imath\Upsilon$.

2.4 Theorem

Let A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 matrix . Suppose that

$V_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} x_n i + y_n j$ for n=1,2,…, and

that $v_n \to 0$. Then $Av_n \to 0$.

2.5 Theorem

Let L: $R^2 \to R^2$ have the property that A_L has distinct real eigenvalues λ and μ, with $|\lambda|<1$ and $|\mu|<1$.

Then for every v in R^2 ,L^n (v)→0 .Therefor 0is an attracting fixed point of L,and R^2 is the basin of attraction of 0.

Proof:

Let v be arbitrary in $\mathcal{R}^2. Since\ L^n(v) = (A_l)^n v$, by (), we need only show that $(A_L)^n v \to 0$. Now let $B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. By theorem2.3 $A_L \approx B$,so that there is an invertible matrix E such that $A_L \approx B_E$ then in **2.2** implies that $(A_L)^n \approx (B_E)^n$,so that $(A_L)^n = E^{-1}B^nE$,

and hence

$(A_L)^n v = E^{-1}B^nEv = E^{-1}(B^nEv)$

The proof will be complete if we show that $E^{-1}(B^n\ Ev) \to 0$.

From we know that

$B^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix}$. Next let $Ev = \binom{x}{y}$. Then

$$B^nEv = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix} \binom{x}{y} = \binom{\lambda^n x}{\mu^n y}$$

since $|\lambda| < 1$ and $|\mu| < 1$ by hypothesis ,we find that $\|B^n Ev\| = \left\|\begin{pmatrix}\lambda^n x\\ \mu^n y\end{pmatrix}\right\| = \sqrt{\lambda^{2n}x^2 + \mu^{2n}y^2} \to 0$

as n increases with out bound .Therefore (8) and

by Theorem 2.4 with v_n replaced by $B^n Ev$, tell us that $L^n(v) = (A_L)^n v = E^{-1}(B^n Ev) \to 0$

Since v is an arbitrary vector in $\mathcal{R}^2$, it follows that not only is 0 an attracting fixed point of L, but also the basin of attracting of 0 is $\mathcal{R}^2$ itself . This completes the proof.

2.6 Corollary :

Let $L: \mathcal{R}^2 \to \mathcal{R}^2$ be a linear function with a single eigenvalue λ ,with $|\lambda| < 1$, or two complex eigenvalues $\beta + \iota\lambda$ and $\beta - \iota\lambda$, where β and λ are real numbers and $\beta^2 + \lambda^2 < 1$. Then $L^n(n) \to 0$ for each v in $\mathcal{R}^2$.Therefor 0 is an attracting fixed point whose basin of attraction is $\mathcal{R}^2$.

Proof:

By Theorem 2.5 proves the result if λ and μ are real and distinct. The cases in which L has one real eigenvalue, or complex eigenvalues since in all cases the eigenvalues satisfy the condition for stability ,it follows that

$\lim_{n\to\infty} L^n(v) = 0$ for every $v \in \mathcal{R}^2$.

This confirms that 0 is an attracting fixed point and its basin of attraction is entir space $\mathcal{R}^2$.

2.6 Example :

Let $L = \binom{x}{y} = \binom{x/2}{y/4}$. Show that 0 is an attracting fixed point whose basin of attraction is $\mathcal{R}^2$. Also show that for any number $r \neq 0$, the parabola $y = rx^2$ is invariant under L.

Solve .

We find that

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 1/4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

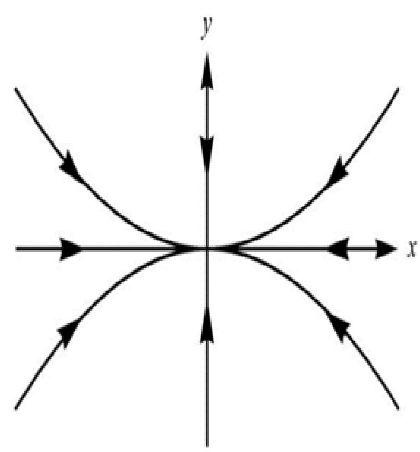
So that the eigenvalues of L are $1/2$ *and* $1/4$ Theorem 2.5 impliesthat

0 is an attracting fixed point whose basin of attracting is $\mathcal{R}^2$.

If $y = rx^2$ with $r \neq 0$ then

$$L\begin{pmatrix}x\\y\end{pmatrix} = L\begin{pmatrix}x\\rx^2\end{pmatrix} = \begin{pmatrix}x/2\\rx^2/4\end{pmatrix}\begin{pmatrix}x/2\\r(x/2)^2\end{pmatrix}.$$

The latter point lies on the parabola $y = rx^2$, so it is invarint under L



1. Introduction to the Figure: As shown in **Figure 3.1**, which is called a **portrait** of L, we describe the dynamics of a linear function. Generally, when a linear function L has two real eigenvalues whose absolute values are less than 1, the portrait of L follows the general form where trajectories converge toward the origin.[5]

2. Mathematical Proof (The Core): To support this claim, let

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ax\\dy\end{pmatrix} \text{, so that}$$

$$A_L\begin{pmatrix}a & 0\\0 & d\end{pmatrix}\text{. Assuming } a > 0 \text{ and } d > 0\text{,}$$

we determine a value of α for which the graph C of $y = rx^{\alpha}$ is invariant under L. If $\begin{pmatrix}x\\y\end{pmatrix}$ lies on C, then:

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ax\\drx^{\alpha}\end{pmatrix}$$

This point lies on C only if $drx^{\alpha} = r(ax)^{\alpha}$, which leads to: $\alpha = \frac{\ln d}{\ln a}$

As a result, curves of the form $y = rx^{\alpha}$ are invariant under L.

3. Generalization (The Conclusion): "Now suppose that the two eigenvalues of L satisfy $|\lambda| < 1$ and $|\mu| < 1$, but the associated matrix is not necessarily diagonal. This is where similar matrices play a decisive role. The reason is that if M corresponds to a diagonal matrix such that $L \approx M_p$, and if C is a set that is invariant under M , then

$(PLP^{-1})(C) = M(C) \subseteq C$,so that $L(P^{-1}(C)) \subseteq P^{-1}(C)$

This means that $P^{-1}(C)$ is the adjusted curve ,and is invariant under L .

2.7 theorem

If $A \approx B$, then A and B have identical eigenvalues.

2.8 Example:

Let $A_L = \begin{pmatrix} 1/2 & 1/8 \\ 1/2 & 1/2 \end{pmatrix}$. Analyze the dynamics of L .

Solve.

We observe that the eigenvalues of A_L are ${}^1/_4$ and ${}^3/_4$,

since $\det(A_L - \lambda I) = \lambda^2 - \lambda + \frac{3}{16} = 0$ if $\lambda = 1/4$ or $3/4$.

You can calculate that corresponding eigenvectors for

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Next ,

if $A_M = \begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}$, Then by Theorem 2.3 and 2.7 , $A_L \approx A_M$, and also A_L and A_M have the same eigenvalues. Using(9), we calculate that with respect to M, iterates of points converge to 0 along

curves C of the form $y = rx^{\alpha}$, where $x > 0$ and

$$\alpha = \frac{\ln(3/4)}{\ln(1/4)} = \frac{\ln 3 - \ln 4}{\ln 1 - \ln 4} = 1 - \frac{\ln 3}{\ln 4} \approx .2075$$

If P is a linear function such that $L \approx M_p$, then from the remarks preceding the example ,$P^{-1}(C)$ is invariant under L .

Abstract

In discrete dynamical systems, the classification of points provides a fundamental framework for understanding the global behavior of a system. A point is considered a fixed point if it remains unchanged under the mapping, representing an equilibrium state whose stability depends on the magnitude of the derivative. When the absolute value of the derivative is less than one, the point is attracting, whereas values greater than one indicate instability. Beyond fixed points, periodic points play a central role, as they return to their initial position after a finite number of iterations. The presence and density of such periodic orbits reveal an underlying structure even in systems that appear irregular.

As system parameters vary, a transition from regular to complex behavior often occurs through a sequence of bifurcations, most notably period-doubling, which leads progressively to chaotic dynamics. In this regime, points exhibit sensitive dependence on initial conditions,

meaning that arbitrarily close starting values can generate significantly different trajectories over time. Despite this apparent randomness, chaotic systems retain an intrinsic order characterized by the coexistence of infinitely many periodic points and intricate geometric structures.

The concept of attractors further refines this understanding, as each point in the space evolves toward a specific long-term behavior determined by its basin of attraction. These basins partition the space into regions with distinct dynamical outcomes, explaining how identical systems can produce varied responses depending on initial conditions. In higher-dimensional spaces, the dynamics become more complex, giving rise to structures such as strange attractors, where trajectories follow non-repeating yet bounded patterns.

Moreover, the geometric nature of these systems is often fractal, indicating that the distribution of points is neither random nor smooth but follows self-similar patterns with non-integer dimensions. This connection between dynamical behavior and fractal geometry highlights that chaos is not a lack of order, but rather a sophisticated form of organization governed by precise mathematical laws.

Referencs

1- Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.

2- Strogatz, S. H. (2015). Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (2nd ed.). Westview Press.

3- Devaney, R. L. (2018). A First Course in Chaotic Dynamical Systems: Theory and Experiment (2nd ed.). CRC Press.

4- Strogatz, S. H. (2015). Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (2nd ed.). Westview Press.

5- Lay, D. C., Lay, S. R., & McDonald, J. J. (2016). Linear Algebra and Its Applications (5th ed.). Pearson.

6- Devaney, R. L. (2018). A First Course in Chaotic Dynamical Systems: Theory and Experiment (2nd ed.). CRC Press.

7- Munkres, J. R. (2000). Topology (2nd ed.). Prentice Hall.

8- Discrete Chaos, Second Edition: With Applications in Science and Engineering

9- ENCOUNTERS، with CHAOS AND FRACTALS ،Second Edition.