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Vector-Valued Functions in Euclidean Space: An Analytical Study and its Applications in Differential Geometry

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بسم الله الرحمن الرحيم

﴿وَأَنْ لَيْسَ لِلْإِنسَانِ إِلَّا مَا سَعَىٰ وَأَنَّ سَعْيَهُ سَوْفَ يُرَىٰ﴾

(سورة النجم: 39-40)

إهداء

بعد رحلةٍ مليئة بالتعب والسعي، أصل اليوم إلى لحظة طال انتظارها...

لحظة أقطف فيها ثمرة جهدي، وأهديها بكل فخر واعتزاز:

إلى من كان دعاؤهم سرّ نجاحي... أمي وأبي، لكم كل الحب والامتنان.

إلى عائلتي التي كانت دائماً إلى جانبي، شكراً لدعمكم الذي لا ينتهي.

إلى أساتذتي الكرام، الذين زرعوا فينا العلم والأمل، فلكم مني كل التقدير.

إلى أصدقائي، الذين شاركوني تفاصيل هذه الرحلة، فكنتم أجمل ما فيها.

شكر وتقدير

أعبّر عن خالص شكري وامتناني لكل من ساهم في دعمي خلال مسيرتي الدراسية، ولكل من منحني الثقة والتشجيع للاستمرار.

إن هذا الإنجاز ما هو إلا بداية لطموحات أكبر بإذن الله.

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ABSTRACT

This file provides a comprehensive systematic study of vector geometry in Euclidean space (E^3), starting with the definition of vectors and their basic algebraic operations such as addition, dot and cross products, linear dependence, and the determination of bases and components. The content then transitions to the analytical aspect by studying 'Vector-Valued Functions,' explaining the concepts of limits, continuity, and differentiation for these functions, supported by mathematical theorems and practical examples that bridge the gap between vector algebra and the foundations of differential geometry."

INTRODUCTION

. Mathematics serves as the fundamental language for formulating physical and engineering laws. At the heart of this language, "vectors" emerge as an indispensable tool for describing quantities that are only fully defined by both magnitude and direction. The study of vectors is not merely a branch of algebra; it is the cornerstone of quantum mechanics, relativity, and advanced analytical geometry, allowing us to represent complex spaces and simplify calculations in multiple dimensions. This research aims to present a sequential analytical study, starting from the algebraic roots of vectors and leading to the differential analysis of vector-valued functions. To achieve this objective, the research is divided into two main chapters:

Chapter One: Focuses on "Vector Algebra in Euclidean Space," where fundamental concepts are reviewed, including the definition of a vector, addition and subtraction operations, and both dot and cross products. This chapter also addresses the concepts of linear independence and the construction of bases that form the structural framework of vector space.

Chapter Two: Shifts the research from the static aspect of vectors to the dynamic and analytical perspective through the study of "Vector-Valued Functions." This chapter explores how vectors change relative to scalar variables, with a focus on limits, continuity, and the rules of vector differentiation. This chapter serves as the primary gateway to differential geometry, where these functions are used to describe curves and paths in space with high mathematical precision

Chapter one

Vector Algebra in Three-Dimensional Euclidean Space (E^3)

1. Introduction

Vector algebra serves as the fundamental pillar for most modern mathematical and physical applications. A vector is not merely a numerical value but a mathematical entity that combines magnitude and direction, making it the ideal tool for describing the space we inhabit. This chapter focuses on the study of the algebraic structure of vectors in three-dimensional Euclidean space (E^3), beginning with the definition of a vector and its basic properties, such as length (magnitude) and direction. It also covers essential algebraic operations including addition, dot product, and cross product, while reviewing the laws governing these operations. Furthermore, the concepts of linear independence and bases will be explored, which allow us to represent any vector as a linear combination, paving the way for a deeper understanding of geometric and physical transformations.

Definition 1.1[2] : A vector a is the vector (−a) defined by: $\_a = (a_1, a_2, a_3)$

The zero vector: $0 = (0,0,0)$. The length or magnitude of a vector:

$a = (a_1, a_2, a_3)$ is the real number:$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Clearly $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$

Definition 1.2 [1] : Given two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$

in E³, their sum $a + b$ is the vector defined by :

$a + b = (a_1 + b_1, a_2 + b_2, a_3 b_3)$ The difference of two vectors a and b is the vector: $a\_b = a + (-b)$. The vector addition satisfies:

$[A_1]\ a + b = b + a$(Commutative Law)

$[A_2]\ (a + b) + c = a + (b + c)$ (Associative Law)

$[A_3]\ 0 + a = a$for all a

$[A_4]\ a + (-a) = 0$ for all a

Example (1)[3]: Let $a = (1,2,0)$and $b = (0,1,1)$. Then $a + b = (1,-1,1)$

$-a = (-1,2,0)$, b − a = (−1, 3, 1) , $|a| = \sqrt{5}$

Definition1.3 [ ] : The product ka is called multiplication of a vector by a scalar. The multiplication of vectors by scalars satisfies:

$[B_1] K_1(K_2a) = (K_1K_2)a = K_1K_2a$,$(k_1 + k_2)a = k_1a + k_2a$

$[B_2]\ k(a + b) = ka + kb$ (Distributive Laws)

$[B_3]\ 1a = a$

Finally, if $a = (a_1, a_2, a_3)$,then

$$|ka| = \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2} = \sqrt{k^3}\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$$

Thus for all k and a, $|ka| = |k|\,|a|$

Example (2)[3]: In the triangle OAB shown in Fig.(1-1), let a = OA and b = OB and let M be the midpoint of side AB. Then the vector MO can be expressed in terms of a and b as follows:

$$OM = a + AM = a + \frac{1}{2}AB$$

$$= a + \frac{1}{2}(b - a) = a + \frac{1}{2}b - \frac{1}{2}a = \frac{1}{2}a + \frac{1}{2}b$$

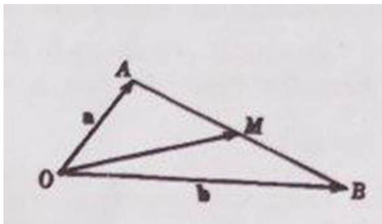


Fig.(1-1)

Definition 1.4.[5]: The vectors u_1, u_2, ... , u_n are said to be **linearly dependent** if there exist scalars k_1, k_2, ... not all zero such that

$k_1u_1 + k_2u_2 + ... + k_nu_n = 0$ The vectors $u_1, u_2, ..., u_n$ are said to be **linearly independent** if they are not linearly dependent. That is, $u_1, u_2, ...,$ u_n are linearly independent if (1.3) implies all $k_1 = k_2 = ... = k_n = 0$.

Note that a set of vectors which includes the zero vector is dependent; for we can always write

$$10 + 0u_1 + ... + 0u_n = 0$$

Example(3)[5]: The vectors $a = (1, -1, 0)$, $b = (0, 2, -1), c = (2, 0, -1)$
are linearly dependent, since $2a + b - c = 0$.

Theorem 1.5 [4]: If a vector is expressed as a linear function of independent vectors, then it is expressed so uniquely. That is, if $u_1, u_2, ...,$ u_n are independent, and if

$u = k_1u_1 + k_2u_2 + ... + k_nu_n = k'_1u_1 + k'_2u_2 + \dots + k'_nu_n$

$then\ k_1 = k'_1, k_2 = k'_2, ... , k_n = k'_n$.

Definition 1.6.[5]: The three vectors $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ are independent. For

$k_1e_1 + k_2e_2 + k_3e_3 = (k_1, k_2, k_3)$ and so if

$k_1e_1 + k_2e_2 + k_3e_3 = 0$, then $k_1 = k_2 = k_3 = 0$. Also any vector $a = (a_1, a_2, a_3)$ can be written as $a = a_1e_1 + a_2e_2 + a_3e_3$ as a linear combination of e_1, e_2 and e_3, and by Theorem 1.1 this representation is unique. In general we call a set of vectors B a basis for $E^3\ if$

(i) Every vector in E^3 can be written as a linear combination of the vectors in B,

(ii) B is a linearly independent set of vectors.

Theorem 1.2[5]: Any three linearly independent vectors form a basis in E^3. $Conversely, every\ basis\ in\ E^3$ consists of three linearly independent vectors.

Proof:

Let u_1, u_2, u_3 be a basis in space and let $a = a_1u_1 + a_2u_2 + a_3u_3$.The scalars a_1, a_2, a_3, for short a_i, i = 1, 2, 3, are called the components of a with respect to the basis u_1, u_2, u_3. It follows from Theorem 1.1 that the components of a vector with respect to a given basis are unique. However, note that the components of a vector depend upon the basis chosen and in general the components will change if there is a change in basis. An exception is the vector 0 whose components are always 0, 0, 0. In general we shall denote the components of vectors a, b, x, y, u, with respect to some prescribed basis by

a_i, b_i, x_i, y_i, u_i.

Example(3) [1] : Let u_1, u_2, u_3 be a basis and let

$a = 2u_1 - u_2, b = u_2 - 2u_3, c = 3u_1 + u_3$.

We will show that a, b, c are linearly independent and hance also from abasis. for, suppose

$k_1a + k_2b + k_3c = (2k_1 + 3k_3)u_1 + (-k_1 + k_2)u_2 + (-2k_2 + k_3)u_3 = 0$.

Since the u_i are independent, it follows that

$$2k_1 + 3k_3 = 0,, -k_1 + k_2 = 0, -2k_2 + k_3 = 0.$$

This is a system of three homogeneous linear equations in k_1, k_2, k_3.

Since the determinant of the coefficients det $= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} = 8 \neq 0$

the only solution is $k_1 = k_2 = k_3 = 0$.Hence the vectors a, b, c are independent. Observe that the components of a, b, c appear as the columns in the above determinant.

As suggested in the above example, we have in general.

Definition 1.7.[1]: The dot or scalar product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is the real number $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$ In particular, for a = b we have the formula $a \cdot a = |a|^2$

The scalar multiplication satisfies

$[C_1]$ $a \cdot b = b \cdot a$(Symmetric Law)

$[C_2]$ $(ka) \cdot b = k(a \cdot b)$(k = scalar)

$[C_3]$$a \cdot (b + c) = a \cdot b + a \cdot c$(Distributive Law)

$[C_4]$ Scalar multiplication is positive definite; that is,

$(i) a \cdot a \geq 0\ for\ all\ a$

$(ii) a \cdot a = 0 if and only if a = 0$

Clearly, from the definition, $a \cdot 0 = 0\ for\ all\ a$. Also, if $a \cdot b = 0$ for all a, then

$b \cdot b = 0$ and hence from $C_4(ii)$,$b = 0$.

Example (4) [2] : Let $a = (-2, 1, 0)$ and $b = (2,1,1)$. Then $a \cdot b = -3$ and $a \cdot a = 5 = |a|^2$.

Definition 1.8.[2]: Two vectors a and b are said to be orthogonal, written $a \perp b$ if $a \cdot b = 0$. In other words a and b are orthogonal if and only if either $a = 0, b = 0$, or $\theta = \measuredangle(a, b) = \frac{\pi}{2}$.

Example (5)[4]: Let a and b be linearly independent and let $c = a - P_b(a)$.Then c is a nonzero vector orthogonal to b.

For suppose c = 0; then from equation (1.6), $0 = 1a - P_b(a) = 1a - kb$, where $k = \frac{(a \cdot b)}{|b|^2}$ which is impossible since a and b are independent.

Hence $c \neq 0$. Finally, $c \cdot b = (a - \frac{((a \cdot b)\, b)}{|b|^2}) \cdot b$

$$= a \cdot b - (\frac{(a \cdot b)(b \cdot b))}{|b|^2})$$

$$= (a \cdot b) - (a \cdot b) = 0.$$

Thus $c \perp b$.

Definition 1.9.[4]: Let e_1, e_2, e_3 be three mutually orthogonal unit vectors as shown in Fig. 1-2. These vectors are independent; for if $k_1e_1 + k_2e_2 + k_3e_3 = 0$, then $0 = e_i \cdot 0 = e_i \cdot (k_1e_1 + k_2e_2 + k_3e_3) = e_i \cdot k_ie_i = k_i$ or $k_i = 0$ for each i.

Therefore they form a basis called an orthonormal basis.

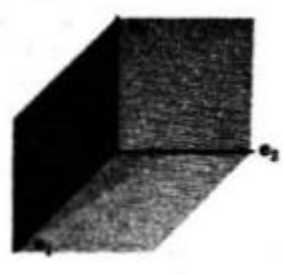
We observe that $e_1, e_2, e_3$is an orthonormal basis if and only if

$e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1$(Unit vectors)

$e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = 0$(Mutually orthogonal)

or, in short, $e_i \cdot e_j = \delta_{ij} = \{ 1, if\ j = i$

$\{ 0, if\ j \neq i\ (i, j = 1, 2, 3)$The quantity $\delta_{ij}$is called the Kronecker symbol and will be used repeatedly.



Fig(1-2)

Theorem 1.3[4] : Let e_1, e_2, e_3 be an orthonormal basis and let

$a = a_1 e_1 + a_2 e_2 + a_3 e_3$, $b = b_1 e_1 + b_2 e_2 + b_3 e_3$

(i) $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i$

(ii) $|a| = \sqrt{a \cdot a} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{i=1}^{3} a_i^2}$

(iii) $a_i = a \cdot e_i, (i = 1, 2, 3)$.

Definition 1.10[5]: Let (e_1, e_2, e_3) and (g_1, g_2, g_3) be ordered orthonormal bases and imagine that the triad (g_1, g_2, g_3) is rotated to make g_1 *and* g_2 *coincide with* e_1 *and* e_2 respectively.

Then $g_3$will either coincide with e_3, in which case we say that (g_1, g_2, g_3)*has the same orientation as* (e_1, e_2, e_3), *or* g_3*will point in the direction opposite to* e_3,

in which case the bases are said to have opposite orientation.

To formulate this concept of orientation in a precise manner, not only for orthonormal bases but for arbitrary bases, we proceed as follows:

Let (u_1, u_2, u_3) *and* (v_1, v_2, v_3) be ordered bases and

let $v = \sum_{i=1}^{3} = 1 a_{ij} u_i$.

Then (v_1, v_2, v_3) *has the same as* (u_1, u_2, u_3) if $det(a_{ij}) > 0$.

This defines an equivalence relation of all ordered bases in E^3. This relation partitions the bases into exactly two equivalence classes. Ordered bases in the same class have the same orientation and ordered bases in different classes have opposite orientation.

Definition 1.11[5]: Let (e_1, e_2, e_3) be a right-handed orthonormal basis and

$Let\ a = a_1e_1 + a_2e_2 + a_3e_3\ and\ b = b_1e_1 + b_2e_2 + b_3e_3$.The cross or vector product of a and b, denoted by a × b, is the vecto

$a \times b = (a_2\, b_3 - a_3\, b_2\,)e_1 + (a_3\, b_1 - a_1\, b_3\,)e_2 + (a_1b_2 - a_2b_1)e_3$As an aid in computing the above, we observe that it can be obtained as the expansion of the determinant

$$a \times b = det\begin{pmatrix} e_1 & a_1 & b_1 \\ e_2 & a_2 & b_2 \\ e_3 & a_3 & b_3 \end{pmatrix}$$

$$= e_1\, det\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} - e_2\, det\begin{pmatrix} a_1 & b_1 \\ a_3 & a_3 \end{pmatrix}$$

$$+ e^3 det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$$= (a_2\, b_3 - a_3\, b_3\,)\ e_1 + (a_3\ b_1 - a_1\ b_3\,)e_2$$

$$+ (a_1\, b_2 - a_2\, b_1\,)e_3$$

Example(6)[5]:

$Let\ a = e_1 - e_2, b = e_2 + 2e_3, c = -2e_1 - e_3.$

Then

$$a \times b = det\begin{pmatrix} e_1 & 1 & 0 \\ e_2 & -1 & 1 \\ e_3 & 0 & 2 \end{pmatrix} = -2e_1 - 2e_2 + e_3.$$

Theorem 1.4[1] :

$(i)|a \times b| = |a||b|\, sin\, \theta, where\ \theta = \measuredangle(a, b).$

$(ii)a \cdot (a \times b) \perp a\ and (a \times b) \perp b$

$b.\ If\ a \times b \neq$

$0, then\ (a, b, a \times b) is\ a\ right -$

$handed\ linearly\ independent\ triplet.$

$Since\ |a||b|sin\ \theta\ =\ 0 if\ and\ only\ if$

$|a| =\ 0, or\ |b| =\ 0, or\ \theta\ =\ 0, or\ \theta$

$=\ \pi, we\ have\ from\ (i)\ above\ a\ form\ of\ the\ Schwarz\ inequality$

$(|a\ \cdot\ b|\ =\ |a||b| if\ and\ only\ if\ a\ and\ b\ are\ linearly\ dependent).$

Example(7)[2]:

For an orthonormal basis (g_1, g_2, g_3)shown in, it follows from Theorem 1.5 that

$g_1 \times g_1 = 0$	$g_2 \times g_1 = -g_3$	$g_3 \times g_1 = g_2$
$g_1 \times g_2 = -g_3$	$g_2 \times g_2 = 0$	$g_3 \times g_2 = -g_1$
$g_1 \times g_3 = -g_2$	$g_2 \times g_3 = -g_1$	$g_3 \times g_3 = 0$

In Problem 1.29 we prove that the vector product satisfies

$[E_1]\ a\ \times\ b\ =\ -(b\ \times\ a)\ (Anticommutative\ Law)$

$[E_2\ ]a\ \times\ (b\ +\ c) =\ a\ \times\ b\ +\ a \times\ c$

(Distributive Law)

$[E_3]\ (ka)\ \times\ b\ =\ k(a\ \times\ b)\ (k\ =\ scalar)$

$[E_4]a\ \times\ a\ =\ 0$

Note that the vector product is not only not commutative but also not associative; that is, in general

$a \times\ (b\ \times\ c) \neq\ (a\ \times\ b) \times\ c. For\ as\ shown\ in\ Example$ 1.20,

$g_1\ \times\ g_2\ =\ -g_3$, where as

$(g_1\ \times\ g_1)\ \times\ g_2\ =\ 0\ \times\ g_2\ =\ 0$

Definition 1.12[3]: The product $a\ \cdot\ b\ \times\ c$is called the mixed or triple scalar product. Parentheses are not required, for this can only mean

$a \cdot (b \times c), the\ scalar\ product\ of\ the\ vector\ a\ and\ the\ vector\ b \times c.$

This product is also conveniently given in terms of a determinant. For let

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3, b = b_1 e_1 + b_2 e_2 + b_3 e_3, c = c^1 e^1 + c^2 e^2 + c^3 e^3.$$

Then

$$a \cdot (b \times c) = (a_1 e_1 + a_2 e_2 + a_3 e_3) \cdot det \begin{pmatrix} e_1 & b_1 & c_1 \\ e_2 & b_2 & c_2 \\ e_3 & b_3 & c_3 \end{pmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

$$det = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

It follows from properties of the determinant that

$$a.b \times c = c.a \times b = b.c \times a = \_(b.a \times c) = \_(c.b \times a) = \_(a.c \times b)$$

In particular,it follows that$a \cdot b \times c = a \times b \cdot c$. Thus we can drop the dot and cross in the notation of the triple scalar product and use instead $[abc] = a \cdot (b \times c) = (a \times b) \cdot c$.as an immediate consequence of Theorem 1.3 and equation (1.8) we have

Theorem 1.5[2]:

$[abc] = 0\ if\ and\ only\ if\ a, b, c$are linearly dependent.

a number of useful identities relate vector and products of vectors. a basic identity, which is derived in Problem 1.35, is

Example(8)[3]:

$S1\ /\ _{100}(5)\ on\ E^3$is the set of numbers x satisfying

$|x\ -\ 5| < \frac{1}{100}$

Or

$5\ - \frac{1}{100} <\ x\ <\ 5\ + \frac{1}{100}. Note\ that\ S_1\ /\ _{100}(5)$ is the open interval of length $\frac{1}{50}$ $centered\ about\ 5.$

Definition 1.13[4]: The assignment of vector $f(t)$to each real number$t$ $of\ a\ set\ of\ real\ a\ \ numbers\ S\ defines\ a\ vector\ function\ f\ of\ the\ single\ var$

as in the case of scalar functions of a real variable, the set S is called the domain of definition of

$f;\ and\ the\ set\ of\ assigned\ vectors, denoted\ by\ f(S), is\ called\ the\ image\ of$

Example (9)[5]:

Let a, b, c be fixed vectors in space. The equation

$F(t)\ =\ a\ -\ 2tb\ +\ t^2c,\quad -2\ \leq\ t$

$\leq\ 2 defines\ a\ vector\ function\ of\ t\ with\ domain\ -2$

$\leq\ t\ \leq\ 2.$

A table of some assigned vectors is

t	-2	-1	0	1	2
$f(t)$	$a + 4b + 4c$	$a + 2b + c$	a	$a - 2b + c$	$a - 4b + 4c$

Definition 1.14[5]:A function $f(t)$is said to be bounded on the interval I if there exists a scalar $M > 0$such that $|f(t)| \leq M$for t in I.Observe in Fig.1-3 that if$x = f(t), then\ f(t)$ is bounded on I if there exists a sphere of radius M about the origin such that the point xis in the sphere for t in I.

Fig.1-3

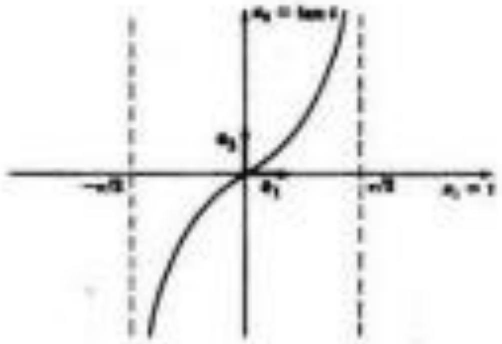


Fig.1-4

Example(10)[6]:

The curve traced by

$X = te_1 + (tan\ t)e_2 on\ -\frac{\pi}{2} < t < \frac{\pi}{2}$Is shown in Fig. 1-

Observe that $|x|$becomes arbitrarily large for t close to

$\frac{\pi}{2}. Thus\ x\ is\ not\ bounded\ on\ -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Note, however, that x is bounded on the interval

$$-\frac{\pi}{2} + \varepsilon < t < \frac{\pi}{2} - \varepsilon for\ any\ \varepsilon > 0. For\ these\ t,$$

$|x| = |te_1 + (\tan e_2| \leq |te_1| + |(tan\ t)e_2| \leq |t| + |\tan \leq M,$

$Where$

$M = \frac{\pi}{2} - \varepsilon + \tan\left(\frac{\pi}{2} - \varepsilon\right).$

$a\ function\ f(t) is\ said\ to\ be\ bounded\ at\ t = t^0$if there exists $\varepsilon > 0$

$such\ that\ f(t) is\ bounded\ for\ t\ in\ S\varepsilon(t^0);\ or, equivalently, f(t)$

$is\ bounded\ at\ t_0$ if there exists an $M > 0\ and\ an\ \varepsilon > 0\ such\ that$

$|f(t)| \leq M\ for\ |t - t_0| < \varepsilon$.Clearly,if

$f(t)\ is\ defined\ and\ bounded\ on\ an\ interval\ I, then\ it\ is\ bounded\ at\ each\ t_0$

However, the converse is not true, as shown by the example above,

where $f(t)\ is\ bounded\ at\ each\ t_0$ in

$-\frac{\pi}{2} < t < \frac{\pi}{2}$ but not on the whole interval

Chapter two

Lines and planes, Nighborhood, Limts , properties oflimite Continuity, Differentiation

2.1 Lines and planes

Definition2.1[1]: Let a and u be vectors in E^3 $with\ M \neq 0 \in a = (a_1, a_2, a_3), u = (u_1, u_2, u_3)$ By the Straight line through aparallel to u, we can be represented by

$X = ku + a, \_\infty < k < \infty \ldots 1 or$

In Component from, $X = (x_1, x_2, x_3)$

$X_1 = ku_1 + a_1, X_2 = ku_2 + a_2, X_3 =$

$ku_3 + a_3 \ldots 2\ The\ equation$ 1 or 2 are called the

Parametric equations of the line. And the point x generates the line

As the parameter K varies over the real line

Example(11): $Let\ a = e_1 + 2e_2\ \ and\ u = e_1 - e_3$ be two vectors

$then\ the\ parametric\ equation\ of\ the\ line\ through\ a\ parallel\ to\ 4\ is\ x = ku + a$

$$X: k(e_1 - e_3) + (e_1 + 2e_2) = ke_1 - ke_3 + e_1 + e_2$$

$$= (k + 1)e_1 + 2e2 \_ke_3 or\ x_1 = k + 1, x_2 = 2, x_3 = k$$

Example (12)[2]:

$Let\ a = (1, -3{,}4), b = (-5, 1, 7)\ are\ distinct\ point, on\ aline. If\ a$ be $apoint\ in\ the\ line\ and\ b_a\ be\ avector\ parallel\ to\ the\ ling.$.The parametric equation of the line is

$$Or\ X = k(b\_a) + a$$

$X_1 = k(b_1 - a_1) + a_1, X_2 = k(b_2 - a_2) + a_2, X_3 = k(b_3 - a_3) \quad +a_3 \rightarrow$

$X_1 = 1\_6k, X_2 = 3 + 4k, X_3 = 4 + 3k$

Definition2.1.2[3]: The plane through apant a parallel to two independent Vectors $u\ and\ \ v\ we\ mean\ the\ set\ of\ x$in E^3 which can be represented by

$X = hu + kv + a, \_\infty < k < \infty \ldots 1$

Or equating Components,

$X_1 \quad = \ hu_1 + \ kv_2 + \ a_1$

$X_2 = hu_2 + \ kv_2 \ \ + a_2$

X_3

$= \ hu_3 \ \ + \ kv_3$

$+ \, a_3 \ldots \ 2\, The\ equations\ (1) and\ (2) are\ called\ the\ parametric\ equations\ of\ t$

$plan$

Definition2.1.3[3]: A vector will be said to be parallel to the plane if it is Iinerly dependent upon $U\ and\ V$

Definition2.1.4[4]: A vector will be said to be normal to the plane if it is orthogonal to both $u\ and\ v$

Definition2.1.5[5]: If n is anonzero vector normal to the plane $X =$ $hu + kv \ + a$

Then $x\ lies\ on\ the\ plane\ iff\ (x\_a).n = 0$

Example(13)[5]: The parametric equation of the plane through $a =$ $e_2\ parallel\ to$

$U = e_1 \ \ and\ V = -e_1 + e_2 \ \ is$

$X = hu + ku + a$

$X = \ (h - k)e_1 + e_2 + ke_3$

Or

$X_1 = (h - k)\ , X_2 = 1, X_3 = k$

2.2 Neighborhood

Local properties of functions are Conveniently described in terms of the Concept of a spherical neighborhood.

Definition2.2.1[5]: The $\in -opensphere\ or\ (\in -spherical$

$neighborhood\ of avectora\ denoted\ by\ S_{\in}(a)\ is\ the\ set\ of\ all\ points\ whose\ di$

$\in (j-e)|x-a| <\in$

Example(14)[5]: The geometric representation of this neighborhood varies based on the spatial $\in (a) is\ an\ open\ interval\ of\ length\ 2\in$

$with\ the\ point\ a\ at\ its\ center. It\ Can\ be\ expressed\ as(\ a_{\in}, a{+}\in)$

dimension (E^n)

1-In one-dimensional space E':

$Theneighborhoodde\quad S_{\in} is\ an\ open\ interval\ of\ length\ 2\ \in$

$with\ the\ point\ a\ at\ its\ center -\ It\ can\ be\ expressed\ as(\ a\_\ \in, a{+}\in)$

2- In two-dimensional space (E^3)! The

neighborhood $S_{\in}(a)$ $is\ the\ interior\ of\ a$ circle (often called an open disk) with raduis and Center a.

3-In three-dimensional space (E^3): $The\ neighborhood\ S_{\in}(a)$ is the interior of a sphere (offen called an often bell) with vaduis $\in$ and Center a

Definition2.2.2[4]: The $\in$-deleted spherical neighborhood of a point (or vector) a denoteel by $S'_{\in}(a), is\ defined\ as\ the\ set\ of\ all\ pornits\ x$that belong to the spherical

neighborhood $S_{\in}(a) excluding\ the\ Center\ a\ Hself.$

$(j-e) this\ following\ nequality\ 0 < |x\_a| <\in$

Notes:

1- The Condition $|x-a|\ <\in$ ensures the point lies within the sphere

2- The aaddition of theCondition $o < |x-a|$

$ensures\ the\ exclusion\ of\ the\ Center\ a, Since\ the\ distance\ is\ Zero\ iff$

$$x = a$$

Example(15)[4]: Let $\in = \frac{1}{10}$ be spherical neighborhood of the the vector $a = e_1 + 2e_2 + 3e_3$

Let the vector $x = x\_1\, e\_1 + x_2\, e_2 + x_3 e_3$ $satisfying\ |x\_a| =$ $[x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2]$ ½ $< 1/10$

Or $(x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2 < \frac{1}{100}$

Definition2.2.3[1]:

$A\ Vector\ Functione\ \&\ of\ asingle\ real\ Variable\ t\ is\ a\ vule\ that\ assigns\ a\ uni$

The set

$S\ is\ called\ the\ domain\ of\ definition\ of\ f;\ and\ the\ set\ of\ assigned$ $vectors\ denoted\ by\ f(S)$, is called

The Image of f

Example(16)[2]:

If the vector function defined as $f(t) = a - 2tb + t^2 c,$ and the Constant vectors are given by $a = e_1 + e_3$

$b = e_2 - e_3\ and\ c = e_1 - e_2$ Then by substituting and grouping terms We can express the vector function in terms of its Components as follows: $f(t) = (1 + t^2)e_1 + (2 - 2t)e_2 + (2t - t^2)e_3$ Consequently, the three scalar functions representing these components are:

$$F_1(t) = 1 + t^2, f_2\ (t) = 2 + 2t^2, f_3\ (t) = 2 + t^2$$

$\boldsymbol{Definition 2.2.4}$[**4**]: $Let\ x = f(t)$,then as varies the point x will trace out a curve. The equation $x = f(t)$ or, Componentwise, The three scalar equations $x_1 = f_1(t), x_2 = f_2\,(t), x_3 = f(t)$ will be called a parametric representation

Of the curve, and the variable t will be the parameter

Example(17)[4]: The equation $X = a(Cost)e_1 + a\,(sint)e_2$its parametric representation of the curve

$X_1 = \ a\ Cost$

$X_2 = a\ sint$

Where $a > \ 0\ and\ \ 0 \leq t \leq 2t$ and the cirel is traceal in a Counter cluckwise direction

Definition 2.5.5[3]: A function $f(t)$is said to be bounded on the interval I if there existsa

scale$r\ M > o\ such\ that\ |f(t)| \leq M\ In\ geometrical\ terms, if\ X =$

$f(l), t$hen

$f(t)\ is\ bounded\ on\ I\ iff\ there\ exists\ asphere\ of\ radius\ M$ about the origin such

Example(18)[3]: The Curve Traced by. $X = \ te_1, + (tan\ t)e_2, on \frac{\pi}{2} < t < \frac{\pi}{2}$ such that $f_1(t) = t\ \ f_2(t) = \ Tan\ t$

Then $|x| = |te_1, +(\ tant)e_2|$

$\leq |t||e_1\ | + |tant||e_2\ \ | \leq |t| + (tan\ t)| \leq M$ Where $M = -\frac{\pi}{2}\_ \in$

$+ Tan\left(\frac{\pi}{2} \in\right)$

2.3 LIMTS

Definition 2.3.1[3] : A vector function $f(t)$ has a limit L as t approaches to,written $limt \to t_0\, f(t) = L\ or\ f(t) \to L\ as\ t \to t_0, if\ for\ every\ \in>0, one\ can\ find\ a\ S > 0, depending\ on\ \in,$ $such\ that\ the\ vectors\ f(t)\ are\ in\ S_{\in}(L)\ for\ t\ in\ S_s(t_0).$ Observe in Fig.1-5

Fig (1.5)

Example (19)[3]:

Let $f(t) = a = constant. Then\ for\ any\ t0, limt \to t_0 f(t) = a$

$For\ f(t) = a$ is in every $S \in (a)\ for\ all\ Ss(t_0\ )all\ t_0$

Now, we recall thatla scalar function $g(t) \to 0\ as\ t \to t_0\ if\ for\ every\ \in > 0$there exists $a\ 3 > 0\ such\ that\ |g\ (t)| <\in\ for\ t\ in\ S'(t_0). If\ we\ let\ g(t) = |f(t) - L|, then\ |g\ (t)| = |f(t) - L| <\in\ if\ and\ only\ if\ f(t) is\ in\ S \in (L).$ Thus we have the important

Lim $t \to t_0\ f(t) = L$

Or$f(t) \to L$ $as\ t \to t0\ if\ for\ every\ \in > 0, one\ can\ find\ a\ S > 0,$
$depending\ on\ \in, such\ that\ the\ vectors\ f(t)\ are\ in\ S \in$
$(L)\ for\ t\ Ss'(t_0)$

Theorem2.3.2[2]:$f(t) \to L\ as\ t \to t_0\ iff\ |f(t) - L| \to 0\ as\ t \to t_0$

Example(20)[2]:

$Lim\ t \to 1\ (t^2 e_1 - (t+1)e_2) = e_1 - 2e_2$ $, since\ lim\ lf(t) -$
$L|$ =$lim\ t \to 1\ |(t^2 - 1)e_1\_(t-1)e_2| = lim\ t \to 1$ $[(t^2\_1)^2+1(t\ \_\ 1)^2]^{1/2}$
= 0 Finally, suppose $f(t) \to L\ as\ t \to t0\ Then\ for\ an\ arbitrary\ \in <$
$0, exists\ there\ S > 0$ suchthat $|f(t) - L| < \in$
$for\ t\ in\ Ss'\ (t_0)\ Hence\ for\ t\ in\ Ss(t_0), |f(t)| = |f(t) - L + L| \leq$
$|f(t) - L| + |L| \leq M$
Where $M = Max(\in, |f(t_0) - L|) + |L|$thus we hav

Theorem 2.3.3[4]:

$If\ f(t) has\ a\ limit\ as\ t \to t_0, then\ f(t) is\ bounded\ at\ t_0.$

Proof:

$suppose\ f(t)\ has\ a\ limit\ then\ f(t) \to$ L$as\ t \to$
$t_0\ So\ for\ an\ arbitrary\ \in > 0, \exists\ S > 0 \in |f(t)\_L| \in$
$For\ t\ in\ S's\ (t_0)$
$For\ t\ S_s'\ (t_0)$We have
$|f(t)| = |f(t)\_L + L| \leq |f(t)\_L| + |L| \leq M$

$Then\ f(t) is\ bounded\ at\ t_0$

2.4 PROPERTIES OF LIMITS

Suppose $lim\ fi(t) = Li, i = 1,2,3\ then\ lim\ [f_1\ (t)e_1 + f_2\ (t)e_2 + f_3\ (t)e_3) = L_1 e_1 + L_2\ e_2 + L_3 e_3$

For,let $f(t) = f_1(t)e_1 + f_2(t)e_2 + f_3\ (t)e_3\ \ and\ L = L_1 e_1 + L_2 e_2 + L_3\ e_3;\ then$ $\lim(t) - L \mid = lim\ (f_1(t)e_1 + f_2(t)e_2 + f_3(t)e_3) - (L_1 e_1 + L_2 e_2 + L_3 e_3) = \lim t \to t_0[(f_1(t) - L_1)^2 + (f_2(t) - L_2)^2 + (f_3(t) - L_3)^2]$ ½ $= 0$

Theorem 2.4.1[3]:

The function $f(t) = f_1\ (t)e_1 + f_2(t)e_2 + f_3\ (t)e_3\ \ has\ a\ limit\ as\ t$ to if and only if $f(t), I = 1,2,3\ have\ limits\ as\ t \to t_0, in\ which\ case = limt \to t_0\ f(t) = \left(limt \to t_0 f_1(t)\right)e_1 + \left(limt \to t_0 f_2(t)\right)e_2 + (limt \to t_0 f_3(t))e_3$

Example (21)[3]:

$$Limt \to 0\ (sin\ t)e_1 + (cost)e_2 + te_3) = (limt \to 0\ sint) + (limt \to 0\ cost)e_2 + (limt\ t \to 0)e_3 = e_2$$

Example(22)[3]:

$Let\ f(t) = t^2 e_1 + te_2 . then$

$Lim\ h \to 0 \frac{f(2+h)\_f(2)}{h} = lim\ h \to 0 = lim\ \ h \to 0((2+h)^2$

$E_1 + (2+h)e_2)\_(4e\_1 + 2e_2)/h$

$Lim\ h \to 0[\frac{((2+h)^2_4)e_1}{h} + \frac{he2}{h}] = 4e_1 + e_2$

Theorem 2.4.2[5] :

If $f(t) \to L\ as\ t \to t_0, then\ |f(t)| \to |L| as\ t \to t_0$

Proof:

$Let\ = f_1(t)e_1 + f_2(t)e_2 + f_3(t)e_3\ \ and\ Let\ L = L_1e_1 + L_2e_2 +$
$L_3e_3\ then\ Iim\ t \to t_0|f(t)| = limt \to t_0\ |f_1(t)e_1 + f_2(t)e_2 + f_3(t)e_3|$

$$= lim\ t \to t_0\sqrt{\left(f_1(t)\right)^2 +\ \ (f_2(t)^2) + \left(f_3(t)\right)^2}$$

$$+\sqrt{\lim t \to t_0 f_1(t)^2 + f_2(t)^2 +\ f_3(t)^2} = \sqrt{L_1^2 + L_2^2 + L^2{}_3}$$

$$= |L|$$

Theorem2.4.3[5]: If$\lim t \to t_0\ \ f(t) = L, \lim t_0\ \ g(t) =\ M\ and$
$limt \to t_0\ h(t)\ \ =\ \ N$,then

$[H_1] li\,m\,t \to t_0\left(f(t) +\ g(t)\right) = li\,m\,t \to t_0\,f(t) + limt \to t_0\,g(t)$
$=\ L\ +\ M$

$[H_2] limt \to t_0\left(h(t)g(t)\right) =\ lim\ t \to t_0\ h(t) lim\ t \to t_0 g(t)\ =\ NM$

$[H_3]\ If\ N \neq 0, then\ limt)/h(t))\ \ =\ \ limt \to t_0\ \ \frac{f(t)}{\lim t_0\ \ h(t)}\ \ \ = \frac{l}{n}$

$[H_4] lim\ t \to t_0\left(f(t). g(t)\right) = li\,m\,t_0\,f(t). lim\ t \to t_0\ g(t)\ =\ L. M$

$[H_5] lim\left(f(t)x(t)\right) =\ lim\ t \to t_0 f(t)x\ limt \to t_0\ g(t)\ =\ L \times M.$

$[H_6]\ If\ lim\ t \to t_0 f(t) = f(t_0\ )and\ lim\theta \to\ \ h\theta = t_0\ \ lim\theta \to \theta_0\ h\theta$
$= f t_0$

Example(23)[5]:

$Let\ limt \rightarrow t_0\ f(t) = \ L, limt \rightarrow t_0\ g(t) = M$

$limt \rightarrow t_0\ \ h(t) = \ N. Then\ lim\ t \rightarrow t_0\ \ f(t)g(t)h(t) = lim\ t$

$\rightarrow t_0\ \ \big(f(t)g(t)h(t)\big) = \ lim\ t \rightarrow t_0\ f(t). lim\ t$

$\rightarrow t_0\ \ \big(g(t)h(t)\big)limt$

$\rightarrow t_0 f(t). \text{limt} \rightarrow t_0 g(t) \times \lim t \rightarrow t_0 h(t))$

$= limt \rightarrow t_0 f(t)\,.\lim t \rightarrow t_0\ g(t) \times li\,m\,t \rightarrow t_0\ h(t)$

$= [LMN]$

2.5 CONTINUITY

Definition2.16[]: Avector function$f(t)defined\ at\ t_0$ is continuous at to if

forevery$\in > \ 0\ there\ exists\ a\ 3 > \ 0, depending\ on\ \in$

$, such\ that\ f(t) is\ in\ S_\in\big(f(t_0)\big)$foralltin

$S_s(t_0)(t_0) or, equivalently, f(t) is\ continuous\ at\ t_0\ if\ lim\ \ t \rightarrow$

$t_0\ \ f(t)\ \ = \ \ f(t_0)$

$The\ functionf(t)\ is\ said\ to\ be\ continuous\ on\ I\ if\ it\ is\ continuous\ at\ all\ t\ \ =$

$t_0\ \ in\ I\ f(t)\ is\ continuous\ if\ and\ only\ if\ its\ components\ fi(t)\ I\ \ =$

$1, 2, 3$ are continuous. The sum, product, and scalar and vector products of continuous functions are continuous and that a continuous function of a continuous function is continuous.

$lim\ t \rightarrow t_0\ (\ f(t)\ \ \_f(t_0)) = 0\ \ or, if\ we\ let\ h = t\_t_0,$

$Lim\ h\ \ \rightarrow\ \ 0\ \big(f(t_0 +\ h) -\ f(t_0)\big) = \ 0$

Example (24)[2]:

$$Let\ f(t) = a + bt + c\,t^2\ with\ a, b, c = constants. Then\ lim\ t \rightarrow t_0\ f(t) = lim\ t \rightarrow t_0\ (a + bt + ct^2)$$

$$= a + bt\_0 + c\,t^20 = f(t_0)$$

$Hence\ f(t)\ is\ continuous\ for\ all\ t.$

2.6 DIFFERENT IATION

Definition2.6.1[3]: The limit $f'(t_0) = \lim t_0 \frac{(f(t) - f(t_0))}{(t - t_0)}$if it $exists, defines\ the\ derivativeof f(t) att = t_0\ \ If\ f'\ (t_0\ )exists,$ $we\ say\ f(t)$is differentiable at to. Observe that if we substitute $t = t_0 + \Delta t$ in the above,the derivative at to is also given as

$$F'(t_0) = lim\ \Delta \rightarrow t_0 \frac{f(t_0 + \Delta t)\ \ \_f(t_0)}{\Delta t}$$

Example(25)[3]:

Let$f(t) = a + bt + ct^2 with\ a, b, c\ \ constants. Then\ f'(t_0\ ) =$ $\lim t_0 \frac{f(t_0 + \Delta t) - (f t_0\ )}{\Delta t} = limt \rightarrow t_0 \frac{([a + b(t_0 + \Delta t) + c\ (t_0 + \Delta t)^3])\_(a + bt_0 + ct^20))}{(\Delta\ t)} =$ $\lim t \rightarrow 0 \frac{((b\Delta t + 2ct0\ + c(\Delta t)^2))}{(\Delta\ t)} = \lim t \rightarrow 0\ (b + 2ct0 + c\Delta t) = b + 2ct_0$

Thus f(t) is differentiable at t0 with derivative $f'(t_0) = b + 2ct_0$

Now, if $f(t) = f_1\ (t)\ e_1 + f_2\ (t)\ e_2\ + f_3(t)e_3$, then it follows

Theorem2.6.2[5].A$function\ (f(t) = f_1(t)e_1 + f_2(t)e_2 + f_3(t)e_3$

$is\ differentiable\ at\ to\ if\ and\ only\ if\ each\ component\ fi(t), i =$

$1,2,3,$ is differentiable at to

Proof:

$Let\ f(t) = f_1(t)e_1\ +\ f_2(t)e_2\ +\ f_3(t)e_3\ by\ th.(2.3)\ we\ have$

$f'(t_0)limt \to t_0\ \to 0\frac{f(t)\_f(t_0)}{t\_t_0}$

$$= lim\left(\frac{(f_1\ (t) - f_1(t_0)\ )}{t\_t_0}e_1\right) + \frac{(f_2\ (t) - f_2(t_0)\ )}{t\_t_0}e_2$$

$$+\frac{(f_3\ (t) - f_3(t_0)\ )}{t\_t_0}e_3 =$$

$$\boldsymbol{Lim\ t \to t_0}\ \left(\frac{(f_1\ (t) - f_1(t_0)\ )}{t\_t_0}e_1\right)\ \boldsymbol{e_1} + [\boldsymbol{limt}$$

$$\to \boldsymbol{t_0}\ \left(\frac{(f_2\ (t) - f_2(t_0)\ )}{t\_t_0}\right)\boldsymbol{e_2}\ \ + [\boldsymbol{limt}$$

$$\to \boldsymbol{t_0}\ \left(\frac{(f_3\ (t) - f_3(t_0)\ )}{t\_t_0}\right)\boldsymbol{e_3}$$

$$=\ \ \boldsymbol{f_1'(t_0) + f_2'(t_0)e_2 + f_3'(t_0)e_3 Thus\ we\ have}$$

Example(26)[5]:

If $u\ =\ (t^2 +\ 2t)e_1 + (sin\ t) + ete_3, then$

$$u' =\ \frac{du}{dt} = \frac{d}{dt}(t^3 + 2t)e_1\ \frac{d}{dt}(sin\ t)e_2$$

$$+\ \frac{d}{dt}(et)\ \ e_3$$

$$=\ (3t^2 + 2)e_1 + (cos\ t)e_2 +\ ete_3$$

$$u'' = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\ (3t^2 + 2)e_1 + \frac{d}{(dt\,)}(\cos(t)e_2 + \frac{d}{dt}(et)e_3 =$$

$$6te_1(\sin t)e_2 + ete_3$$

$$u''' = \frac{d}{dt}\ \frac{d^2u}{dt^2}\ = \frac{d}{dt}\ (6t)e_1 - \frac{d}{dt}(\ sint)e_2\ - \frac{d}{dt}(e^2)e_3 =$$

$$6e_{1\_}(\cos t)e_2\ +\ e^te3$$

$\boldsymbol{Theorem\ 2.6.3}[\ \mathbf{5}].If\ f(t) is\ differentiable\ at\ t_0, then\ f(t) is\ continuous\ at\ t_0$

DIFFERENTlATION FORMULAS 2.6.4[5]:

If u, v,h are differentiable functions of t on I, then

$[J1]u + v\ is\ differentiable\ on\ I\ and\ \frac{d}{dt}\ (u\ +\ v) = \frac{du}{dt} + \frac{dv}{dt}$

$[J2]hu\ is\ differentiable\ on\ I\ and\ \frac{d}{dt}(hu) =\ h\ \frac{du}{dt}\ + \frac{dh}{dt}\ u$

$[J3]u.v\ is\ differentiable\ on\ I\ and\ \frac{d}{dt}(u.v) =\ u.\frac{dv}{dt} + \frac{du}{dt}.v$

$[J4]u \times v\ is\ differentiable\ on\ I\ and\ \frac{d}{dt}(u \times v) =\ u\ \times \frac{dv}{dt} + \frac{du}{dt} \times v$

Finally we have the chain rule:

$[J5]\ If\ u\ =\ f(t) is\ differentiable\ on\ I, and\ t\ =\ h(\theta) is\ dif\ ferentiable\ on\ It, where\ the\ image\ h\ (I_0\ ) contained\ in\ Itthen\ u\ =\ g(\theta) = f(h(\theta) is\ differentible\ on\ \ I_0\ \ and$

$$\frac{du}{d\theta}\ = \frac{du}{dt}\ \frac{dt}{d\theta}$$

Example(27)[5]:

$u = (a(co\,s\,t)e_1 - a(si\,n\,t)e_2 , \theta = (1 + t^2)½, t > 0.$

Then

$\frac{du}{d\theta} = \frac{du}{dt}\frac{dt}{d\theta} = ((- a \frac{a\,sin(t)e_1 - a(cos\,t)\,e_2}{[t\,(1 + t^2)½]}$ -(a / t) (1 + t^2)½ $((sin(t)e_1 +$

$(cos\,t)e_2$

Where we used the fact that for scalar functions $\theta = h(t) such\ that \frac{d\theta}{dt} \neq$

$0\ we\ have \frac{dt}{d\theta}\ (t) = \frac{1}{\frac{d\theta}{dt}}$.

Therom 2.6.5[6]:

If $u\ is\ a\ unit\ vector\ function, then \frac{du}{dt}$ in orthogonal to uThis theorem is an important result which will be used often.

Proof:

$Let\ u\ is\ aunıt\ vector\ function\ then\ |u| = 1 \rightarrow\ u.u =$

$1, and\ differentiating\ we\ obtain \frac{d}{dt}(u.u) = u.\frac{du}{dt} + u.\frac{du}{dt} = 0 + 0 = 0$

Source

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