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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ﴾

صد الله العلي العظيم

سورة المجادلة الآية ١١

الاهداء

﴿وَآخِرُ دَعْوَاهُمْ أَنِ ٱلْحَمْدُ لِلَّهِ رَبِّ ٱلْعَالَمِينَ﴾

الحمدُ للهِ الذي بفضله تشرق شموسُ العلمِ والمعرفة، وبتوفيقه تُثمر الجهود وتُكلَّل المساعي بالنجاح. الحمدُ لله عدد ما خلق، وعدد ما جرى به القلم، وعدد ما دار به الفكر. الحمدُ لله أولًا وآخرًا، الذي وفّقني لإتمام هذا الجهد المتواضع. الحمدُ لله الذي وهبني نعمة العقل لينير لي الطريق، ووفقني بمشيئته وقدرته لإتمام هذا البحث.

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أهدي هذا النجاح أولًا إلى نفسي الطموحة جدًا؛ فقد ظننتُ يومًا أنني لا أستطيع، ولكن من قال "أنا لها" نالها. اللهم لا تجعله آخر الطريق بل بدايةً لطريقٍ أعظم، فاللهم لك الحمد إذا رضيت ولك الحمد بعد الرضا.

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إلى والدي العزيز، الذي لا ينفصل اسمه عن اسمي، والذي لطالما عاهدته بهذا النجاح، وها أنا اليوم أتممتُ وعدي وأهديته إليك. رحمةُ الله عليك، وأسكنك فسيح جناته.

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وإلى إخوتي وعائلتي جميعًا، الذين مدوني بالقوة والتشجيع، وآمنوا بي في الأوقات الصعبة؛ لولا وجودهم ما كنت ما أنا عليه الآن. عائلتي رفقاء دربي، كنتم موضع اتكاء في كل تعثراتي، وكنتم السند الآمن في متاهات التعب. هذا الإنجاز وإن حمل اسمي، فهو يحمل في طياته أثر دعمكم الذي لا يغيب عني لحظةً واحدة.

وإلى ورثة الأنبياء ومنارة العلم، أساتذتي الأفاضل، وإلى كل من علّمني حرفًا، وإلى الذين لم يبخلوا عليَّ بتوجيه أو معلومة خلال سنوات دراستي، شكراً لرحابة صدوركم وأمانتكم في نقل العلم والمعرفة، وإلى من أشرف على هذا البحث خصوصًا ووجّهني بصبرٍ وحكمة. لكم مني كل الاحترام والتقدير، وجعل الله علمكم في ميزان حسناتكم.

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أرى أن مرحلة البكالوريوس قد شارفت على الانتهاء بعد جهدٍ وصبرٍ طويل، وأسألُ الله تعالى أن يوفقني ويجعلها خيرًا لي.

قال تعالى: ﴿اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ﴾ – العلق

الحمدُ لله الذي وفّقني لإتمام هذا البحث بهذه الصورة، ويدعوني واجبُ العرفان والوفاء إلى أن أتقدم بخالص الشكر والتقدير إلى الدكتورة المشرفة القديرة (رحاب عامر كامل)؛ وذلك لحسن إشرافها على البحث وتقديم يد العون. أسألُ الله تعالى أن يجزيها خير الجزاء.

أتوجه إليها بالشكر لأجل كل نصيحة قدمتها، وكل مساعدة كانت كفيلة بنجاح هذا العمل.

كل الشكر وعظيم الامتنان للمشرفة الفاضلة لما قدمته من دعمٍ كريم وتوجيهات علمية رصينة ومتابعة دقيقة كان لها الأثر البالغ في إنجاح هذا البحث، وكانت مثالًا يُحتذى به في التفاني، وفقها الله.

كما أتقدم بجزيل الشكر والتقدير إلى السادة أعضاء لجنة المناقشة المحترمين بتفضلهم بقبول مناقشة هذا البحث، ولما سيقدمونه من ملاحظات علمية وآراء تُسهم في تطويره والارتقاء بمستواه العلمي.

وأتقدم بأسمى آيات الشكر والاحترام إلى أساتذتي الأجلاء الذين زرعوا فينا شغف البحث والمعرفة، فما كان لهذا الجهد أن يكتمل لولا توجيهاتهم وإرشاداتهم، فجزاهم الله خير الجزاء.

أسألُ الله تعالى أن يجعل هذا العمل نافعًا، وإن أخطأتُ فحسبُ الإنسان أنه اجتهد وسعى.

ختامًا، أسألُ الله تعالى أن يوفقني، وهو نعم المولى ونعم المصير.

Contents

Abstract

This paper investigates the structure and fundamental properties of fuzzy n-normed linear spaces. We begin by extending the classical concept of n-normed spaces into the fuzzy setting and introduce the notions of convergence and Cauchy sequences in such spaces. Special attention is devoted to the study of completeness, where sufficient conditions for a fuzzy n-normed linear space to be complete are established .

Furthermore, we explore different types of boundedness in fuzzy normed spaces, including bounded, fuzzy bounded, and totally bounded sets, and analyze the relationships among these concepts. Several illustrative examples are provided to clarify the distinctions between these notions. The results presented in this work contribute to a deeper understanding of the interplay between fuzzy structures and functional analysis, and they open the door to further applications in areas such as fuzzy topology and applied mathematics.

Introduction

Fuzzy set theory has become an essential tool in modern mathematical analysis for handling uncertainty and imprecision. In recent years, its integration with functional analysis has led to the development of fuzzy normed linear spaces and their various generalizations, which provide a flexible framework for studying convergence, topology, and structure in abstract spaces [1], [6] .

The concept of n-normed linear spaces, originally introduced to extend classical normed spaces, has been further generalized into fuzzy n-normed linear spaces. These spaces combine the geometric structure of n-norms with the uncertainty modeling of fuzzy theory, making them particularly useful in both theoretical and applied contexts. Recent research has focused on understanding their analytical properties, especially convergence behavior and completeness [2], [3].

In this paper, we investigate the notion of convergence and Cauchy sequences in fuzzy n-normed linear spaces and study the conditions under which such spaces are complete. These concepts play a fundamental role in functional analysis, as completeness ensures the stability of limits and the well-posedness of many problems [3], [5].

Furthermore, different types of boundedness in fuzzy normed spaces have been introduced and studied by various researchers. These include bounded sets, fuzzy bounded sets, and fuzzy totally bounded sets, each reflecting a different aspect of size and control within the space. The relationships between these notions remain an important topic of investigation [5], [6].

The organization of this paper is as follows. In Section 1, we present the basic definitions and preliminary concepts related to n-normed and fuzzy n-normed linear spaces. Section 2 is devoted to the study of completeness and the behavior of Cauchy and convergent sequences. In Section 3, we discuss different notions of boundedness and their properties in fuzzy normed spaces. Finally, Section 4 provides further results and comparisons between these types of boundedness, along with illustrative examples.

Section one

Important definitions :

Definition 1.1:

Let $n \in N$ (natural numbers) and X be a real linear space of dimension $d \geq n$. (Here we allow d to be infinite). A real valued function $|| \bullet, \dots, \bullet ||$ on $X \times X \times X \times X \dots \times X$ (n times) $= X^n$ satisfying the following four properties:

(1) $|| x_1, x_2, \dots, x_n || = 0$ if any only if $x_1, x_2, \dots, x_n$, are linearly dependent

(2) $|| x_1, x_2, \dots, x_n ||$ is invariant under any permutation of $x_1, x_2, \dots, x_n$

(3) $|| x_1, x_2, \dots, cx_n || = |c| \, || x_1, x_2, \dots, x_n ||$, for any real c

(4) $||x_1, x_2, \dots, x_{n-1}, y + z ||, \leq ||x_1, x_2, \dots, x_{n-1}, y || + ||x_1, x_2, \dots, x_{n-1}, z ||$

is called an n-norm on X and the pair (X, $||\bullet, \dots, \bullet||$) is called an n- normed linear space.

Definition 1.2 A sequence $\{x_n\}$ in an n-normed linear space

$(X, || \bullet, \dots, \bullet ||)$ is said to converge to an $x \in X$ (in the n-norm) whenever $\lim_{n,k\to\infty} ||x_1, x_2, \dots, x_{n-1}, x_n - x_k || = 0$.

Definition 1.3 A sequence $\{x_n\}$ in an n-normed linear space $(X, || \bullet, \dots, \bullet ||)$ is called a Cauchy sequence if $\lim\limits_{n,k\to\infty} ||x_1, x_2, \dots, x_{n-1}, x_n - x_k || = 0$

Definition 1.4 An n-normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition 1.5 Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times R$ (R-set of real numbers) is called a fuzzy n-norm on X if and only if:

(N1) For all $t \in R$ with $t \leq 0$, $N(x_1, x_2, \dots, x_n, t) = 0$.

(N2) For all $t \in R$ with $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent.

(N3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$.

(N4) For all $t \in R$ with $t > 0$,

$N(x_1, x_2, \dots, x_n, t) = N(x_1, x_2, \dots, x_n, t/|c|)$ if $c \neq 0$, $c \in F$.

(N5) For all $s, t \in R$,

$N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}$.

(N6) $N(x_1, x_2, \dots, x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, \dots, x_n, t) = 1$. Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Definition 1.6 A binary operation *: [0,1] × [0,1]→ [0,1] is called a continuous t-norm if * satisfies the following conditions:

(1) * is commutative and associative

(2) * is continuous

(3) a*1=a for all a $= [0,1]$

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Section Two .

Section two

Complete Fuzzy n-normed linear space

In this section we first redefine the notion of fuzzy n-normed linear space using t-norm.

Definition 2.1 Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy n-norm on X if and only if :

(N1′) $N(x_1, x_2, \dots, x_n, t) > 0$.

(N2′) $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if $x_1, x_2, \dots, x_n$, are linearly dependent.

(N3′) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$.

(N4′) $N(x_1, x_2, \dots, c\, x_n, t) = N(x_1, x_2, \dots, x_n, t/|c|)$ if $c \neq 0$, $c \in F$(field).

(N5′) $N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq N(x_1, x_2, \dots, x_n, s) * N(x_1, x_2, \dots, x'_n, t)$.

(N6′) $N(x_1, x_2, \dots, x_n, t)$ is left continuous and non-decreasing function such that $\lim_{t\to\infty} N(x_1, x_2, \dots, x_n, t) = 1$.

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

To strengthen the above definition, we present the following example.

Example 2.2 Let (X, || •,..., •||) be an n-normed linear space.

Define $a * b = \min\{a, b\}$ and $N(x_1, x_2, \dots, x_n, t) = t/(t + \|x_1, x_2, \dots, x_n\|)$.

Then (X, N) is a f-n-NLS.

Proof :

(N1′) Clearly $N(x_1, x_2, \dots, x_n, t) > 0$

(N2′) $N(x_1, x_2, \dots, x_n, t) = 1$

$\Leftrightarrow t/(t + ||x_1, x_2, \ldots, x_n||) = 1$

$\Leftrightarrow ||x_1, x_2, \ldots, x_n|| = 0$

$\Leftrightarrow x_1, x_2, \ldots, x_n$ are linearly dependent.

(N3′) $N(x_1, x_2, \ldots, x_n, t)$

$= t/(t + ||x_1, x_2, \ldots, x_n||)$

$= t/(t + ||x_1, x_2, \ldots, x_n, x_{n-1}||)$

$= N(x_1, x_2, \ldots, x_n, x_{n-1}, t)$.

It follows similarly for the rest.

(N4′) $N(x_1, x_2, \ldots, x_n, t/|c|)$

$= (t/|c|)/(t/|c| + ||x_1, x_2, \ldots, x_n||)$

$= t/(t + |c|\, ||x_1, x_2, \ldots, x_n||)$

$= t/(t + ||x_1, x_2, \ldots, x_n||)$

$= N(x_1, x_2, \ldots, cx_n, t)$.

(N5′) Without loss of generality assume that $N(x_1, x_2, \ldots, x'_n, t) \leq N(x_1, x_2, \ldots, cx_n, s)$. Then

$t/(t + || x_1, x_2, \ldots, x'_n ||) \leq s/(s + || x_1, x_2, \ldots, x_n||$

$\Rightarrow t\,(s + ||x_1, x_2, \ldots, x_n||) \leq s\,(t + || x_1, x_2, \ldots, x'_n||$

$\Rightarrow t\,\big||x_1, x_2, \ldots, x_n|\big| \leq (s/t)||x_1, x_2, \ldots, x_n||$

Therefore,

$|| x_1, x_2, \ldots, x_n|| + || x_1, x_2, \ldots, x'_n||$

$\leq (s/t)\, || x_1, x_2, \ldots, x'_n|| + || x_1, x_2, \ldots, x'_n||$

$\leq (s/t + 1)|| x_1, x_2, \ldots, x'_n||$

$= ((s+t)/t) \| x_1, x_2, \ldots, x'_n \|$

But,

$\|x_1, x_2, \ldots, x_n + x'_n \|$

$\leq \|x_1, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x'_n\|$

$\leq ((s+t)/t) \| x_1, x_2, \ldots, x'_n \|$

$\Rightarrow (\|x_1, x_2, \ldots, x_n + x'_n)/(s+t) \leq (\|x_1, x_2, \ldots, x'_n)\|/t$

$\Rightarrow 1 + (\| x_1, x_2, \ldots, x_n + x'_n\|)/(s+t) \leq 1 + (\| x_1, x_2, \ldots, x'_n)/t$

$((s+t) + \|x_1, x_2, \ldots, x_n + x'_n\|)/(s+t) \leq \left(t + \left\|x_1, x_2, \ldots, x'_n\right\|\right)$

$\Rightarrow (s+t)/(\|x_1, x_2, \ldots, x_n + x'_n\| + s + t) \geq t/(t + \| x_1, x_2, \ldots, x'_n)$

$\Rightarrow N(x_1, x_2, \ldots, x_n + x'_n, s+t) \geq \min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x'_n, t)\}.$

(N6') Clearly $N(x_1, x_2, \ldots, x_n, t)$ is a left continuous function.

Suppose that $t_2 > t_1 > 0$ with $t_1, t_2 = [0, \infty)$ then,

$t_2/(t_2 + \|x_1, x_2, \ldots, x_n\|) -/t_1 \left(t_1 + \left\|x_1, x_2, \ldots, x_n\right\|\right)$

$= \|x_1, x_2, \ldots, x_n\| (t_2 - t_1)/((t_2 + \|x_1, x_2, \ldots, x_n\|)(t_1 + \|x_1, x_2, \ldots, x_n\|)) \geq 0,$

for all $(x_1, x_2, \ldots, x_n) \in X^n$

$\Rightarrow t_2/(t_2 + \| x_1, x_2, \ldots, x_n\| \geq / t_1/ (t_1 + \|x_1, x_2, \ldots, x_n\|)$

$\Rightarrow N(x_1, x_2, \ldots, x_n, t_2) \geq N(x_1, x_2, \ldots, x_n, t_1).$

Thus $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in [0, \infty)$.

Also,

$\lim_{t\to\infty} N(x_1, x_2, \ldots, x_n, t)$

$\lim_{t\to\infty} t/(t + \|x_1, x_2, \ldots, x_n\|)$

$\lim_{t\to\infty} t/t(1 + 1/t\|x_1, x_2, \ldots, x_n\|)$

$= 1$.

Thus, (X, N) is a f-n-NLS.

Definition 2.3 A sequence $\{x_n\}$ in a f-n-NLS (X, N) is said to converge to x if given

$r > 0, t > 0$, $0 < r < 1$, there exists an integer noe N such that $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) > 1 - r$ for all $n \geq n_0$.

Theorem 2.4 In a f-n-NLS (X, N) a sequence {x} converges to x if and only if $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.

Proof.

Fix $t > 0$. Suppose $\{x_n\}$ converges to x.

Then for a given r, $0 < r < 1$, there exists an integer $x_0 \in N$ such that $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) > 1 - r$.

Thus $1 - N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) < r$ and hence $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.

Conversely, if for each $t > 0$, $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) \to 1$ as

$n \to \infty$, then for every r, $0 < r < 1$, there exists an integer no such that $1 - N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) < r$ for all $n \geq n_0$.

Thus $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) > 1 - r$ for all $n \geq n_0$.

Hence {x} converges to x in (X, N).

Definition 2.5. A sequence {x} in a f-n-NLS (X, N) is said to be Cauchy sequence if given $\varepsilon > 0$ with $0 < \varepsilon < 1, t > 0$, there exists an integer no N such that $N(x_1, x_2, \dots, x_{n-1}, x_n - x_k, t) > 1 - \varepsilon$ for all n,k $\geq n_0$.

Theorem 2.6. In a f-n-NLS (X, N) every convergent sequence is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a convergent sequence in (X, N). Suppose $\{x_n\}$ converges to x.

Let $t > 0$ and $\varepsilon = (0,1)$. Choose $r = (0,1)$ such that

$(1 - r) * (1 - r) > l - \varepsilon$.

Since {x} converges to x, we have an integer no such that $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t/2) > 1 - r$.

Now, $N (x_1, x_2, \dots, x_{n-1}, x_n - x_k, t)$

$= N (x_1, x_2, \dots, x_{n-1}, x_n - x + x - x_k, t)$

$= N (x_1, x_2, \dots, x_{n-1}, x_n - x, t/2) * N (x_1, x_2, \dots, x_{n-1}, x - x_k, t/2)$

$\geq (1 - r) * (1 - r)$ for all $n, k > n_0$

$> 1 - \varepsilon$ for all $n, k > n_0$.

Therefore $\{x_n\}$ is a Cauchy sequence in (X, N).

Definition 2.7. A f-n-NLS is said to be complete if every Cauchy sequence in it is convergent.

The following example shows that there may exist Cauchy sequence in a f-n-NLS which is not convergent.

Example 2. 8. Let $(X, || \bullet, \dots, \bullet ||)$ be an n-normed linear space and define $a * b = \min \{a, b\}$ for all $a, b \in [0,1]$ and $N (x_1, x_2, \dots, x_n, t) = t/(t + || x_1, x_2, \dots, x_n||)$. Then (X, N) is shown to be a f-n-NLS.

Let $\{x_n\}$ be a sequence in f-n-NLS, then

(a) $\{x_n\}$is a Cauchy sequence in $(X, || \bullet, \dots, \bullet ||)$ if and only if $\{x_n\}$ is a Cauchy sequence in (X, N).

(b) $\{x_n\}$is a convergent sequence in $(X, ||\bullet,\dots,\bullet||)$ if and only if {x} is a convergent sequence in (X, N).

Proof.

(a) {x}is a Cauchy sequence in (X, ‖ •,..., •‖)

$\Leftrightarrow \lim_{n,k\to\infty} \|x_1, x_2, \dots, x_{n-1}, x_{n-1}, x_n - x_k\| = 0$

$\Leftrightarrow \lim_{n,k\to\infty} (x_1, x_2, \dots, x_{n-1}, x_n - x_k, t) = 0$

$= \lim_{n,k\to\infty} t/(t + \|x_1, x_2, \dots, x_{n-1}, x_n - x_k\|) = 1$

$\Leftrightarrow$ N $(x_1, x_2, \dots, x_{n-1}, x_n - x_k, t) \to 1$ as $n \to \infty$

$\Leftrightarrow$ N $(x_1, x_2, \dots, x_{n-1}, x_n - x_k, t) > 1 - r$, for all $n, k \geq 0$.

$\{x_n\}$ is a Cauchy sequence in (X, N).

(b) $\{x_n\}$ is a convergent sequence in (X, ‖ •,..., •‖)

$\Leftrightarrow \lim_{n\to\infty} \|x_1, x_2, \dots, x_{n-1}, x_{n-1}, x_n - x\| = 0$

$\Leftrightarrow \lim_{n\to\infty}$ N $(x_1, x_2, \dots, x_{n-1}, x_n - x_k, t)$

$= \lim_{n\to\infty}$ N $(x_1, x_2, \dots, x_{n-1}, x_n - x_k, t)$

$N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$

$\Leftrightarrow N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) > 1 - r$, for all $n \geq n_0$.

$\{x_n\}$ is a convergent sequence in (X, N).

Thus if there exists an n-normed linear space (X, ‖ •,..., •‖) which is not complete, then the fuzzy n-norm induced by such a crisp n-norm ‖•,...,•‖ on an incomplete n-normed linear space X is an incomplete fuzzy n- normed linear space.

Theorem 2.9 A f-n-NLS (X, N) in which every Cauchy sequence has a convergent subsequence is complete.

Proof.

Let $\{x_n\}$ be a Cauchy sequence in (X, N) and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $\{x_n\}$ converges to x. Let $t > 0$ and $\varepsilon = (0,1)$. Choose $r \in (0,1)$ such that $(1-r) * (1-r) > 1-\varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer noe N such that $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t/2) > 1 - r$ for all $n, k \geq n_0$. Since $\{x_{n_k}\}$ converges to x, there is a positive integer $i_k > n_0$ such that $N(x_1, x_2, \dots, x_{n-1}, x_{i_k} - x, t/2) > 1 - r$.

Now,

$N(x_1, x_2, \dots, x_{n-1}, x_n - x, t/2)$

$= N(x_1, x_2, \dots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, t/2 + t/2)$

$\geq N\left(x_1, x_2, \dots, x_{n-1}, x_n - x_{i_k}, \frac{t}{2}\right) * N\left(x_1, x_2, \dots, x_{n-1}, x_{i_k} - x, \frac{t}{2}\right)$

$> (1-r) * (1-r)$

$> 1 - \varepsilon$.

Therefore $\{x_n\}$ converges to x in (X, N) and hence it is complete.

Section Three

Different type of boundedness in fuzzy normed spaces

Definition 3 . 1 . A binary operation

$$*: [0,1] \times [0,1] \rightarrow [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall)\ a, b \in [0,1]$;

2. $a * 1 = a, (\forall) a \in [0,1]$;

3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0,1]$;

4. *If* $a \leq c$ *and* $b \leq d$, *with* $a, b, c, d \in [0,1]$, *then* $a * b \leq c * d$.

Remark 3.2 Three basic examples of continuous t-norms are $\wedge, ., *$ L, which are defined by $a \wedge b = \min\{a, b\}$ (the minimum t-norm) , $a.b = ab$ (usual multiplication in $[0,1]$) and $a * b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm). Our basic reference for fuzzy metric spaces and related structures is , while for t-norms, is .

Definition 3.3 . A t-norm * is strictly monotonic if

$$(\forall) x \in (0,1), y < z \Rightarrow x * y < x * z.$$

A t-norm is strict if it is continuous and strictly monotonic.

Remark 3.4 *We note that the usual multiplication is a strict t-norm but the minimum t-norm is continuous but not strictly monotonic. This remark leads us to the following more general definition.*

Definition 3.5 . A t-norm is called almost strictly monotonic if

$$(\forall) x, y \in (0,1) = x * y > 0.$$

A t-norm is called almost strict if it is continuous and almost strictly monotonic.

Remark 3.6 *The usual multiplication and the minimum t-norm* $\wedge$ *are almost strict.*

Definition 3.7 .*The triple* (X, M, *) *is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in* $\mathrm{X} \times \mathrm{X} \times [0, \infty)$ *satisfying the following conditions:*

(M1) $\mathrm{M}(\mathrm{x}, \mathrm{y}, 0) = 0, (\mathrm{V})\mathrm{x}, \mathrm{y} = \mathrm{X};$

(M2) $(\forall)x, y \in \mathrm{X}, x = y$ *if and only if* $\mathrm{M}(x, y, t) = 1$ *for all* $t > 0$;

(M3) $\mathrm{M}(x, y, t) = \mathrm{M}(y, x, t), (\forall)x, y \in X, (\forall)t > 0;$

(M4) $\mathrm{M}(x, z, t + s) \geq \mathrm{M}(x, y, t) * \mathrm{M}(y, z, s), (\forall)\ x, y, z \in X, (\forall)t, s > 0;$

(M5) $(\forall)x, y \in \mathrm{X}, \mathrm{M}(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t \to \infty} \mathrm{M}(x, y, t) = 1.$

Definition 3.8 . *Let* $(X, M, *)$ *be a fuzzy metric space. A subset A of* X *is said to be F-bounded if*

$(\exists)\alpha \in (0,1), (\exists)t > 0$ *such that* $M(x, y, t) > 1 - \alpha, (\forall)x, y \in A$.

Definition 3.9 . *Let* X *be a vector space over a field* K *(where* K *is* $\mathbb{R}$ *or* $\mathbb{C}$*) and* $*$ *be a continuous t-norm. A fuzzy set* N *in* $\mathrm{X} \times [0, \infty)$ *is called a fuzzy norm on* X *if it satisfies:*

(N1) $\mathrm{N}(\mathrm{x}, 0) = 0, (\forall)\mathrm{x} \in \mathrm{X};$

(N2) $[\mathrm{N}(\mathrm{x}, \mathrm{t}) = 1, (\forall)\mathrm{t} > 0]$ if and only if $\mathrm{x} = 0;$

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)\ x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathrm{K}^*;$

(N4) $\mathrm{N}(\mathrm{x} + \mathrm{y}, \mathrm{t} + \mathrm{s}) \geq \mathrm{N}(\mathrm{x}, \mathrm{t}) * \mathrm{N}(\mathrm{y}, \mathrm{s}), (\forall)\mathrm{x}, \mathrm{y} \in \mathrm{X}, (\forall)\mathrm{t}, \mathrm{s} \geq 0;$

(N5) $(\forall)\ x \in \mathrm{X}, \mathrm{N}(x, .)$ *is left continuous and* $\lim_{t \to \infty} N(x, t) = 1.$

The triple (X, N,*) *will be called fuzzy normed linear space (briefly FNL space).*

Example 3.10 . *Let* $(X, \|\cdot\|)$ *be a normed linear space. Let* $N: X \times [0,\infty) \to [0,1]$ *defined by*

$$N(x,t) = \begin{cases} \dfrac{t}{t+||x||} & if \quad t > 0 \\ 0 & if \quad t = 0 \end{cases}$$

Then $(X, N, \wedge)$ *is a FNL space.*

Theorem 3.11 . Let $(X, N, *)$ be a FNL space.

1. *We define* $M: X \times X \times [0,\infty) \to [0,1]$ *by* $M(x,y,t) = N(x-y,t)$. *Then* M *is a fuzzy metric on* X .
2. *For* $x \in X$, $\alpha \in (0,1)$, $t > 0$ *we define the open ball*

$$B(x,\alpha,t) := \{y \in X: N(x - y,t) > 1-\alpha\}.$$

Then

$$\mathcal{T}_N = \{T \subset X : x \in T \text{ iff } (\exists) t > 0, (\exists)\, \alpha \in (0,1): B(x,\alpha,t) \subset T\}$$

is a topology on X *and* $(X, \mathcal{T}_N)$ *is a metrizable topological vector space.*

Recall that considering $(X_1, N_1, *), (X_2, N_2, *)$ two FNL spaces, the application

$$N: X_1 \times X_2 \times [0,\infty) \to [0,1]$$

$$N((x_1,x_2),t) = N_1(x_1,t) * N_2(x_2,t), (\forall)(x_1,x_2) \in X_1 \times X_2, (\forall) t > 0$$

is a fuzzy norm on the Carthesian product $X_1 \times X_2$, named the fuzzy product norm.

We denote by pr_i the projection function from $X_1 \times X_2$ onto X1, defined by $p_{r_i}(x_1,x_2) = x_i$; for $i \in \{1,2\}$.

The next result deals with the Carthesian product of fuzzy normed linear spaces.

Theorem 3.12 . *Let* $(X_1, N_1,*), (X_2, N_2,*)$ *be FNL spaces with the topologies* $\mathcal{T}_{N_1}$ *and* $\mathcal{T}_{N_2}$*, respectively. If N is the fuzzy product norm, then* $\mathcal{T}_N$ *is the product topology on* $X_1 \times X_2$.

Definition 3.13 . Let $(X, N,*)$ be a FNL space and (x_n) be a sequence in X. The sequence (x_n) is said to be convergent if $(\exists)x \in X$ such that $\lim N(x_n - x, t) = 1$, $(\forall)t > 0$. In this case, x is called the limit of the sequence (xn) and we denote $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 3.14 . Let $(X, N,*)$ be a FNL space. A subset B of X is called the closure of the subset A of X if for any $x \in B$, $(\exists)(x_n) \subset A$ such that $x_n \to x$. We denote the set B by A.

A subset A of X *is called closed if* $A = \bar{A}$.

Remark 3.15 *As any FNL space is a fuzzy metric space, the notion of F-bounded set can be used in the context of FNL spaces. More precisely, a subset A of a FNL space X will be called F-bounded if*

$$(\exists)\alpha \in (0,1), (\exists)t > 0 \text{ such that } N(x - y, t) > 1 - \alpha, (\forall)x, y \in A.$$

We will denote by FB(X) the family of all F-bounded subset of X.

Definition 3.16 . A subset A of a FNL space X is said to be bounded if

$$(\exists)\alpha \in (0,1), (\exists)t > 0 \text{ such that } N(x, t) > 1 - \alpha, (\forall)x \in A.$$

We will denote by B(X) the family of all bounded subset of X.

Definition 3.17 . A subset A of a FNL space X is called fuzzy bounded if

$$(\forall)\alpha \in (0,1), (\exists)t_\alpha > 0 \text{ such that } N(x, t_\alpha) > 1 - \alpha, (\forall)x \in A.$$

We will denote by fB(X) the family of all fuzzy bounded subset of X.

Definition 3.18 . A subset A of a FNL space X is called fuzzy totally bounded if

$$(\forall)\alpha \in (0,1), (\exists)\{\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{n}\} \subset X: A \subset \bigcup_{i=1}^{n} (x_i + B(0, \alpha, \alpha))$$

We will denote by ftB(X) the family of all fuzzy totally bounded subsets of X.

Section four

Fuzzy Bounded Sets

One might think by looking at the concepts of boundedness presented above, that there is still one concept missing, namely the one in which the boundedness of a set A is defined as follows:

$$(\forall)\alpha \in (0,1), (\exists)t_\alpha > 0 \; such\; that\; N(x-y, t_\alpha) > 1-\alpha, (\forall)x, y \in A.$$

In fact it is not missing because it coincides with one above, as the following theorem shows. 411

Theorem 4.1 Let (X, N,∗) be a FNL space. A subset A of X is fuzzy bounded if and only if

$$(\forall)\alpha \in (0,1), (\exists)t_\alpha > 0 \; such\; that\; N(x-y, t_\alpha) > 1-\alpha, (\forall)x, y \in A.$$

Proof. " $\Longrightarrow$ "Let

$\alpha \in (0,1)$. Then there exists $\beta \in (0,1)$ such that $(1-\beta) * (1-p) > 1-\alpha$. Since A is fuzzy bounded, for $\beta \in (0,1)$ there exists $t_\beta > 0$ such that $N(x, t_\beta) > 1-\ t_\beta$, $(\forall)x \in A$. Let $x, y \in A$ and $t_\alpha = 2t_\beta$. We have that

$$N(x-y, t_\alpha) \geq\ N(x, t_\beta) * N(y, t_\beta) \geq (1-\beta) \times\ (1-\beta) > 1-\alpha.$$

" $\Longleftarrow$ " Let $\alpha \in (0,1)$. Using Lemma 3.6 we obtain that there exist $\gamma, \delta \in (0,1)$ such that $1-\frac{\alpha}{2} = (1\ -\gamma) * \delta$.

Let $x_0 \in$ A be fixed. As $\lim_{t\to\infty} N(x_0, t) = 1$, we have that there exits $t_1 > 0$ such that $N(x_0, t_1) > \delta$. From our hypothesis, for $\gamma \in (0,1)$ there exists $t_1 > 0$ such that $N(x-x_0, t_2) > 1-\ \gamma, (\forall)x \in A. Let\ t = t_1 + t_2$. Then, for all $x \in A$, we have

$$N(x,t) \geq N(x\ -\ x_0, t_2) * N(x_0, t_1) \geq (1-\gamma) * \delta = 1-\frac{\delta}{2} > 1-\alpha$$

Remark 4.2 One can observe that a subset A of a topological linear space X is called bounded if for each neighbourhood V of 0x, there exists a positive number k such that A ⊆ kV.

Theorem 4.3 Let $(\mathrm{X}, \mathrm{N}, *)$ be a FNL space. A subset A of X is fuzzy bounded if and only if A is bounded in topology TN.

Proof. " $\Longrightarrow$ "Let V be a neighbourhood of 0_x. Then there exist $\alpha \in (0,1), t > 0$ such that $B(0, \alpha, t) \subseteq V$. Since A is fuzzy bounded, for $\alpha \in (0,1), (\exists) t_\alpha > 0$ such that $N(x, t_\alpha) > 1 - \alpha, (\forall) x \in A$. Let $k = \frac{t_\alpha}{t}$ We have that $N(x, tk) = N(x, t_\alpha) > 1 - \alpha, (\forall) x \in A$. Thus

$$A \subset B(0, \alpha, tk) = kB(0, \alpha, t) \subseteq kV.$$

" $\Longleftarrow$ "Let $\alpha \in (0,1)$. Since $B(0, \alpha, 1)$ is a neighbourhood of 0_x , there exists $k > 0$ such that $A \subseteq kB\ (0, \alpha, 1) = B(0, \alpha, k)$. Thus $N(x, k) > 1 - \alpha, (\forall) x \in A$. Hence A is fuzzy bounded.

Remark 4.4 Previous result was mentioned by Sadeqi and Kia the context of FNL spaces of type $(X, N, \wedge)$

Corollary 4.5 . Let $(\mathrm{X}, \mathrm{N}, *)$ be a FNL space. Then:

1. If A, B are fuzzy bounded, then AUB and $A + B$ are fuzzy bounded;
2. If A is fuzzy bounded, then $\bar{A}$ is fuzzy bounded.

Corollary 4.6 Let $(X_i, N_i, *), i = \{1,2\}$, be two FNL spaces. Then $A \in fB(\mathrm{X}_1 \times \mathrm{X}_2)$ if and only if $pr_i(A) \in fB(\mathrm{X}_i), i \in \{1,2\}$.

Proposition 4.7 Let $(\mathrm{X}, \mathrm{N}, *)$ be a FNL space and $\{An\}_{n=1}^{\infty}$ be fuzzy bounded subsets of X. Then there exist $\{t_n\} = 1, t_n > 0, (\forall) n \in N^*$ such that $\bigcup_{n=1}^{\infty} t_n A_n$ tnAn is a fuzzy bounded subset of X.

Proof. Let ρ be the metric of X. Let $\alpha \in (0,1)$ and $t > 0$. Since $B(0, \alpha, t)$ is a neighbourhood of 0_x,

there exists $\lambda > 0$ such that

$$S(\lambda) = \{x \in X: \rho(x,0) < \lambda\} \subset B(0,\alpha,t).$$

As $\{An\}_{n=1}^{\infty}$ are fuzzy bounded subsets of X , there exists $s_n > 0$ such that $A_n \subseteq s_n S(1)$. Let $t_n = \frac{1}{s_n}$. then

$$\bigcup_{n=1}^{\infty} t_n A_n \subseteq S(1) = \lambda^{-1} S(\lambda) \subseteq \lambda^{-1} B(0,\alpha,t) = B\left(0,\alpha,\frac{t}{\lambda}\right)$$

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