جمهورية العراق
وزارة التعليم العالي البحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

Some Special Stochastic Processes

بحث تقدمت به الطالبة (ندى سعد عبد الأمير) الى مجلس جامعة بابل كلية التربية للعلوم الصرفة قسم الرياضيات لنيل شهادة البكالوريوس في الرياضيات

اشراف
م. م. فاطمة علي عبد الحسين

٢٠٢5م ١٤٤٦هـ

بسم الله الرحمن الرحيم

((وَقُــلِ اعْمَلُــوا فَسَــيَرَى اللَّهُ عَمَلَكُــمْ وَرَسُــولُهُ وَالْمُؤْمِنُــونَ ۖ
وَسَتُرَدُّونَ إِلَىٰ عَالِمِ الْغَيْبِ وَالشَّهَادَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ))

صدق الله العلي العظيم

إهداء

إلى من لولاهما لما وطئت قدماي هذا الطريق، إلى منارة حياتي وسرّ وجودي.. إلى والدي العزيز، صاحب القلب الكبير واليد التي لم تنفك عن مساندتي، إليك يا من علمتني أن الطموح لا سقف له، وأن الصبر مفتاح كل نجاح.

إلى نبع الحنان الذي لا ينضب، إلى دعوات الفجر التي كانت تحرسني في غربتي وبين دفات كتبي.. إلى والدتي الغالية، التي اقتطعت من راحتها لتهبني مستقبلاً، إليكِ يا من أرى الدنيا من خلال رضاكِ، أهدي ثمرة هذا الجهد المتواضع.

إلى إخوتي وعائلتي، سندي وعزوتي في الشدائد، من شاركوني لحظات القلق وبثوا في نفسي الأمل كلما خبت شعلتي.

إلى أساتذتي الأفاضل، ورثة الأنبياء وحملة مشاعل التنوير، الذين غرسوا في عقولنا بذور المعرفة، فأنبتت فكراً وعملاً. ولكم مني أسمى آيات الشكر والتقدير

وعلى ما قدمتموه من توجيه وعلم.

والى (م. م. فاطمة علي عبد الحسين)

أتقدم لكِ بوافر الشكر والتقدير على إشرافكِ المتميز ودعمكِ اللامحدود طوال فترة إعداد هذا البحث. شكرًا لكونكِ قدوة يحتذى بها في الأمانة العلمية والعطاء الأكاديمي لقد كانت نعم العون والسند بفضل سعة صدرها وحرصها الدائم على الدقة العلمية، فجزاها الله عني وعن العلم خير الجزاء."

إلى كل زميل ورفيق درب، تقاسمنا معاً عناء الدراسة وحلاوة المعرفة، وإلى كل من آمن بقدراتي بكلمة طيبة أو تشجيع صادق..

الى منارة العلم والعلماء الى الصرح الشامخ جامعة بابل.

أهديكم جميعاً هذا البحث، ليكون بصمة فخر في مسيرة بدأت بكم، وتستمر بدعواتكم وحبكم.

Contents

Introduction

Interest in stochastic processes began in the 17th century when Pascal and Fermat laid the foundations of probability theory to analyze games of chance, but the major shift occurred in 1827 when botanist Robert Brown observed the random motion of pollen particles (Brownian motion). This was followed in 1906 by Russian mathematician Andrei Markov, who introduced "Markov chains" based on the property that the future depends only on the present. Around the same time, in 1905, Albert Einstein provided the first mathematical model for Brownian motion. In the 1920s, Norbert Wiener rigorously formalized this model, which became known as the "Wiener process," while in the 1930s, Andrey Kolmogorov established the modern axiomatic foundations of probability and linked stochastic processes to differential equations. The 1940s and 1950s saw the development of "martingale theory" by Paul Lévy and Joseph Doob to model fair games. A true revolution came when Kiyoshi Itô invented stochastic calculus in the 1940s, allowing integration with respect to random processes. Finally, the most famous practical application emerged in the 1970s when Black, Scholes, and Merton used these ideas to price financial options. Thus, stochastic processes evolved from a simple observation of drifting pollen to a fundamental mathematical tool behind modern finance, artificial intelligence, and quantum physics.

Chapter One

1-Definition and General Properties

Let T be an index set, (Ω, F, P) be a probability space, and let (S, β) be a measurable space.

Definition(١.١)

A function $X: \Omega \times T \longrightarrow S$ is called S-valued stochastic process, if for each $t \in T$, the function $\omega \longrightarrow X(t, \omega)$ is an S-valued random variable, i.e. $\{\omega : X(t, \omega) \in A\} \in A$ for every $A \in \beta$.

Remarks

The set T will always be a subset of the extended of real numbers R, and T is often called the parameter space or the index set of the process.

$\mathbb{T} = \{1,2,\ldots\}$, $T\{0, \pm, 1, \pm 2, \ldots\}$, $T = \{t : t \geq 0\} = [0, \infty)$

$$T = \{t : -\infty < t < \infty\} = (-\infty, \infty) = R,\ T = [-\infty, \infty] = \overline{R}$$

• For a fixed sample $\omega \in \Omega$, the function $t \rightarrow X(t, \infty)$ from T into S is the (the sample) pat (or, trajectory or realization) of the S-valued stochastic process X associated with ω .

The totality of all sample paths is called an ensemble.

• Let S be a topological space .An S-valued stochastic process X is said to be a

1. continuous (resp. a right continuous, resp. a left continuous) when its paths are continuous (resp. a right continuous, resp. a left continuous)

2. cad lag when its paths are right continuous with left limits.

3.cad lag when its paths are left continuous with right limits.

Definition (1.2)

Consider a stochastic process $\{X_1\}_{1 \in T}$ For a fixed timet_1 , $X_{t1}=X_1$ is a random variable, and its distribution function $F_X(x_1;t_1)$ is defined as

$$F_X(x_1;t_1) = P\{X_{t1} \leq x_1\}$$

$F_X(x_1;t_1)$ is known as the first ordered distribution of X_1 .similarly, given t_1, and X_{t2}=X_2 represent two random variables. Their joint distribution is known as the second order distribution of X, and is given by

$$F_x(x_1, x_2; t_1, t_2)=P\{X_{t1} \leq x_1, X_{t2} \leq x_2\}$$

In general, we define the nth order distribution of X_t by

$$F_x(x_1, x_2,\ldots\ldots x_n; t_1, t_2, \ldots\ldots t_n)=P\{X_{t1} \leq x_1, X_{t2} \leq x_2 ,\ldots\ldots, X_{tn} \leq x_n\}$$

• If X_t is a discrete time process, then X_t is specified by a collection of p.d.f:

$$F_x(x_1, x_2,\ldots\ldots x_n; t_1, t_2, \ldots\ldots t_n)=P\{X_{t1}=x_1, X_{t2}=x_2,\ldots, X_{tn}=x_n\}$$

If X, is a continuous time process, then X is specified by a collection of p.d.f:

$F_x(x_1, x_2, \ldots\ldots x_n; t_1, t_2, \ldots\ldots t_n) = \frac{\partial^n Fx(x_1, x_2, \ldots\ldots x_n; t_1, t_2, \ldots\ldots t_n)}{\partial x_1\, \partial x_2 \ldots\ldots \partial x_{1n}}$

Definition(1.3)

Let $(X_t)_{t\geq0}$ be a stochastic process

1.The mean of X_t is defined by μ_X(t) = E(X_t) where X_t is treated a random variable for a fixed value of t .In general μ_X(t)is a function of time, and it is often called the ensemble average of X_1

2.The characteristic function $\varphi_x(\gamma_1, \gamma_2, \ldots, \gamma_n)$of X_t is defined by

$\emptyset_x(\lambda_1, \lambda_2, \ldots, \lambda_n) = E(e^{i\sum_{k=1}^{n}\lambda_1} x_k)$

3. A measure of dependence among the random variables of X_t, is provided by its autocorrelation function, defined by R_x (t,s) = E(X_1X_s) and $R_x(\tau)$ = E(X_1,$X_{1+}\tau$)

Note that R_x (t,s) = R_x (s, t) and R_x (t,t) = E(X_1^2)

4. The auto covariance function of X, is defined by

Kx (t,s)=Cov (X_1,X_s) = E (($X_1 - \mu_x$ (t)) ($X_s - \mu_x$ (s))) = R_x (t,s)- μ_x (t) μ_x (s)

It is clear that if the mean of X_1 is zero, then *Kx (t,s)*= *R_x (t,s)*

Theorem (1.1)

Let $(X_1)_{t<0}$ be a stochastic process

1. $Rx(-\tau) = R_x(\tau)$

2. $|R_x(\tau)| \leq R_x(0)$ (3) $R_x(0) = E(X_1^2) \geq 0$

Example(1.6)

Consider a stochastic process $(X_t)_{t<0}$ defined by X_t = U cost λt , t≥0 where λ is a constant and U is a uniform random variable over (0,1).

Solution :

Since U ~u(0,1) ⇒ f (x)=1, 0<x <1 ⇒ $E(U)=\frac{1}{2}$, $E(U^2)=\frac{1}{3}$

Thus:

$E(X_t)=E(U\cos\lambda t) = E(U)\cos\lambda t=\frac{1}{2}\cos\lambda t$

$R_x(t,s)=E(X_1X_s)=E((U\cos\lambda t)(U\cos\lambda s)) = E(U^2\cos\lambda t\cos\lambda s)= E(U^2)\cos\lambda t\cos\lambda s=\frac{1}{3}\cos\lambda t\cos\lambda s$

$K_x(t,s)=R_x(t,s)-E(X_1)E(X_s)=\frac{1}{3}\cos\lambda t\cos\lambda s-(\frac{1}{2}\cos\lambda t)(\frac{1}{2}\cos\lambda s)=\frac{1}{3}\cos\lambda t\cos\lambda s-\frac{1}{4}\cos\lambda t\cos\lambda s=\frac{1}{12}\cos\lambda t\cos$

Definition(1.4)

The cross correlation of two stochastic processes $(X_1)_{t<0}$ and $(Y_t)_{t<0}$ is defined by

$R_{XY}(\tau) = R_{XY}(t,t+\tau) = E(X_1\, Y_{t+\tau})$

Two stochastic processes $(X_t)_{t<0}$ and $(Y_t)_{t<0}$ is called (mutually) orthogonal if

$R_{XY}(\tau) = 0$ for all τ

Theorem (1.2)

Let $(X_t)_{t<0}$ and $(Y_t)_{t<0}$ be two stochastic processes

1. $Rxy(-\tau) = Ryx(\tau)$ _ *2.* $|R_{XY}(\tau)|\leq\sqrt{Rx(0)Ry(0)}$

3. $\backslash R_{XY}(\tau)|\leq\frac{1}{2}(R_x(0)+R_y(0))$

Definition(1.5)

• A stochastic process $(X_t)_{t\in T}$er is said to be stationary or strict sense stationary if, for all n and for every set of time of time instants $\{t_1 \in T : i = 1,2,\cdots,n\}$,

$F_x(x_1, x_2,\ldots, x_n ; t_1, t_2,\ldots, t_n) = F_x(x_1, x_2,\ldots, x_n ; t_{1+}\tau, t_{21+}\tau,\ldots, t_{n+}\tau)$ (1)

For any τ. Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and X_t and $X_{t+\tau}$ will have the same distribution for any τ. Thus, for the first order distribution.

$F_x(x;t) = F_x(x;t+\tau) = F_x(x)$ and $f_x(x;t) = f_x(x)$

Then

$\mu_x(t) = E(X_t) = \mu$ and $Var(X_t) = \sigma^2$

Where μ and σ^2 are constants. Similarly, for the second order distribution,

$F_x(x_1,x_2;t_1,t_2) = F_x(x_1,x_2;t_{-}t_2)$ and $f_x(x_1,x_2;t_1,t_2) = fx(x_1,x_2;t_{-}t_2)$

Nornstationary processes are characterized by distribution depending on the points $t_1, t_2,\ldots, t_n$

- A stochastic process $(X_t)_{t\in T}$is said to be stationary to order k, if stationary

condition (1) not hold for all n but holds for $n \leq k$.

• A stochastic process (X_t), er is said to be wide sense stationary (WSS) or weak stationary if $(X_t)_{t\in T}$is stationary to order 2

Remark

If $(X_t)_{t \in T}$ is WSS stochastic process, then we have

1. $E(X_t) = \mu$ (constant)
2. $R_x(t,s) = E(X_t X_s) = R_x(|s - t|)$

Note that a strict sense stationary process is also a wss process, but, in general, the converse is not true.

Example (1.2)

Let $\{X_n, n \geq 0\}$ be a sequence of independent and identically distributed random variables with mean zero and variance 1. show that $\{X_n, n>0\}$ is a WSS process.

solution :

Since $E(X_n)=0$ (constant) for all n

$R_x(n,n+k) = E(X_n X_{n+k}) = \begin{cases} E(Xn)E(X_{n+k}), & k \neq 0 \\ E(X_n^2), & k \neq 0 \end{cases}$ which depends on k .

Thus {X} is WSS

Definition(1.6)

A stochastic process $(X_1)_{t<0}$ is called independent stochastic process if $X_{t_1}, X_{t_2}, \ldots X_{t_n}$, are independent random variables for n = 2,3,..., i.e. if

$F_x(x_1, x_2, \cdots, x_n; t_1, 1_2, \cdots, t_n) = \prod_{i-1}^{n} F_x(x_i, t_i)$

Thus, a first order distribution is sufficient to characterize an independent stochastic process X_i.

Theorem (1.3)

If $\{X_n\}$ is a sequence of independent random variables and $S_n = \sum_{k=1}^{n} X_k X$, then

$\emptyset_{S_n}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{k=1}^{n} \emptyset_{X_n}(\lambda_k)$

Definition(1.7)

A stochastic process $(X_t)_{t<0}$ is said to have

(1) Independent increments if for n≥ 1 and time points $0 \leq t_0, < t_1 < \cdots < t_n$, the increments :

X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1},…, X_{t_n} - $X_{t_{n-1}}$ are independent random variables.

(2) Stationary independent increment if $\{X_t\}_{t\geq 0}$independent increments and $X_1 - X_s$ has the same distribution as X_{t+h}-X_{s+h}. for all s, t, h ≥ 0 , s <t

Theorem(1.4)

Let $(X_t)_{t\geq 0}$ be a stochastic process with stationary independent increment and assume that $X_0 = 0$.

Show that

1. $E(X_t) = \mu_1 t$ 2. $Var(X_t)=\sigma_1^2$ 3. $V\,ar(X_1 - X_s)= \sigma_1^2\ (t-s)$, t>s 4. $K_x(t,s) = \sigma_1^2\ \min\{t,s\}$, where $\mu_1 = E(X_1)$ and σ_1^2 =Vare (X_t)

Chapter Tow

Some Special Stochastic Processes

In this section we introduce some special stochastic process, Markov process, Bernoulli process, random walk, counting processes, Poisson processes, normal process, and Winer process.

1. Markov Process

A stochastic process $((X_t)_{t\in T}$ is said to be a Marko process if

$$P\{X_{t_{n+1}} \leq X_{n+1} \mid X_0=x_0, X_1=x_1, \ldots, X_{nt}=x_n\} = P\{X_{t_{n+1}} \leq x_{n+1} \mid X_{t_n}=x_n\} \quad (1)$$

Whenever $t_1 < t_2 < \cdot \cdot . \cdot < t_n < t_{n+1}$ •

A discrete state Markov Process is called a Markov chain. For a discrete parameter Markov chain $\{X_n, n \geq 0\}$, we have for every n

$$P\{X_{t_{n+1}} \leq X_{n+1} \mid X_0=x_0, X_1=x_1, \ldots, X_n=x_n\} = P\{X_{t_{n+1}} \leq x_{n+1} \mid X_{t_n}=x_n\}$$

Equation (1) or equation (2) is referred to as the Markov property (which is also known as the memoryless property). This property of a Markov process states that the future state of the process depends only on the present state and not on the past history. Cleary, any process with independent increments is a Markov process.

Example (2.1)

Using the Markov property, the nth-order distribution of a Markov process X_1 can be expressed as

$$F_x(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n) = F_x(x_1, t_1) \prod_{k=2}^{n} P\,\{X_{t_k} \leq x_k\ X_{t_{k-1}} = x_{k-1}\}$$

Thus, all finite order distribution of a Markov Process can be expressed in terms of the second order distributions.

2. Bernoulli Process

A stochastic process $\{X_n, n \geq 1\}$ is called a Bernoulli process if $X_1, X_2, \ldots$ are independent Bernoulli random variables with $p\{X_n = 1\} = p$ and $p\{X_n = 0\} = q$, where $q = 1-p$

Thus

The Bernoulli process $\{X_1, n \geq 1\}$ is a discrete time, discrete state process and the state space is $E = \{0,1\}$, and the index set is $T = \{1,2,\ldots\}$

3. Random Walk

A stochastic process $(X_n, n > 0\}$ is called the simple random walk process if $X_0 = 0$,

$X_n = \sum_{k=1}^{n} Y_k$ Where $Y_1, Y_2, \ldots$ are independent identically distribution random variables with

$P\{Y_n = 1\} = p$, $P\{Y_n = -1\} = 1-p$ for all n.

Thus. The simple random walk process $\{X_n, n \geq 0\}$

Is a discrete time discrerete state process and the state space is $E = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and the index set $T = \{0, 1, 2, \ldots\}$

Theorem (2.7)

Let $\{X_n, n\geq 0\}$ be a simple random walk. Then

1. $E(X_n) = n(p-q)$, $q=1-p$
2. $Var(X_n) = 4npq$, $q=1-p$
3. $R_x(n,m) = \begin{cases} m + (n-1)m(p-q)^2, & m < n \\ n + (m-1)n(p-q)^2, & n < m \end{cases}$

Definition(2.1)

Counting Processes

Let t represent a time variable. Suppose an experiment begins at $t = 0$. Events of a particular kind occur randomly, the first at T_1, the second at T_2, and so on. The random variable T, denotes the time at which ith event occurs, and the values t_1 of T_i, $i = 1,2,...$ are called points of occurrence.

Definition (2.2)

A stochastic process $(X_t)_{t\geq 0}$is said to be a counting process if X_1 represents the total number of "events" that have occurred in the interval (0,1).

From its definition, we see that for a counting process, X, must satisfy the following conditions

1. $X_t \geq 0$ and $X_0 = 0$
2. X, is integer valued
3. $X_s \leq X_t$ if $s < t$
4. $X_t - X_s$ equals the number of events that occurred on the interval (s,t)

Theorem (2. 2)

A counting process $\{X_t\}_{t\geq 0}$is a Poisson process with rate (intensity) λ (>0) if

1. $X_0 = 0$
2. X_t has independent and stationary increments
3. $f\,(X_{t+\Delta t}$ -X_1 = 1) = λ Δt +0(Δt)
4. $f\,(X_{t+\Delta t}$ -X_1 ≥1) =0(Δt)

Where 0(A) is a function of At which goes to zero faster than does Δt, i.e. $\lim_{\Delta t\to 0}\frac{0(\Delta t)}{\Delta t}$ =0.

Example (2.2)

The autocorrelation function R_x (t,s) and the autocovariance function K_x (t,s) of a Poisson process X_t with rate λ are given by R_x (t,s) = λ min{t,s}+λ^2ts and K_s (t,s) = λ min{t,s}

Definition (2.3)

A stochastic process $\{X_t\}_{t\in T}$is said to be a normal(or Gaussian) process if for any integer n and any subset { $t_1,t_2,\cdots,t_n$ } of T, then the random variables X_{t_1}- $X_{t_2},\dots.,X_{t_n}$- are jointly normally distributed in the sense that their joint characteristic function is given

$\varphi X_{t_1}, X_{t_2},\dots.,X_{t_n}(\lambda_1,\lambda_{2,\dots,}\lambda_n)$= E(exp(i$\sum_{k=1}^{n}\lambda_1 X_{tk}$))= exp(i$\sum_{k=1}^{n}\lambda_1 E(X_{tk})$-$\frac{1}{2}\Sigma$)

Example(2.3)

Let B, be a Brownian motion with drift 0. Then for any level x > 0 and any time t>0,

Chapter Three

Limiting Distributions

In this section we introduce laws of large numbers, convergence in distribution and the central limit theorem. First we prove some results from real analysis that will be needed.

Theorem (3.1)

Let $A=[a_{ij}]$ be an infinite matrix of real numbers such that $a_{nj} \rightarrow 0$ for each fixed j, and that for some non-negative real number c, $\sum_{j=}^{\infty} |a_{nj}| \leq c$ for all n. If $\{x_n\}$ is a bounded sequence of real numbers, define

$$Y_n = \sum_{j=}^{\infty} a_{nj\,x_j},\ n = 1, 2, ...$$

Then

1. If $x_n \rightarrow 0$, then $y_n \rightarrow 0$
2. If $\sum_{j=}^{\infty} a_{nj} \rightarrow 1$ and $x_n \rightarrow x$, $x \in R$, then $y_n \rightarrow x$

Theorem (3.2) Toeplitz Lemma

Let$\{a_n\}$ be a sequence of non-negative real numbers, and let $b_n = \sum_{j=1}^{n} a_j$ assume b,,>0 for all n, and $b \rightarrow \infty$ as $n \rightarrow \infty$. If $\{x_n\}$ is a sequence of real numbers such that $x_n \rightarrow x$, $x \in R$, then $\frac{1}{b_n} \sum_{j=1}^{n} a_j\, x_j \rightarrow x$ as $n \rightarrow \infty$.

heorem(3.3) Kronceker Lemma

Let $\{b_n\}$ be an increasing sequence of positive real numbers with $b_n \rightarrow \infty$, and let$\{x_x\}$ be a sequence of real numbers with $\sum_{j=1}^{n} x_n = x$, (finite), then

Theorem (3.4) Kolmogorov's Inequality

Let $X_1, X_2, \ldots, X_n$ be independent random variables with finite expectation, and let

$S_k = \sum_{i=1}^{k} X_i$ Then for any $\varepsilon > 0$

$$P\{\max_{i \le j \le n} |S_j E(S_j)| \ge \varepsilon\} \le \frac{var(S_n)}{\varepsilon^2}, \ 1=1,2,\ldots\ldots,n$$

Theorem (3.5)

Let $X_1, X_2, \ldots$ be independent random variables with finite expectation. If $\sum_{n=1}^{\infty} \text{Var}(X_n) < \infty$ then

$\sum_{n=1}^{\infty} (X_n - E(X_n))$ converges a.e.

Theorem(3.6) Kolmogorov Strong law of large numbers.

Let $X_1, X_2, \ldots$ be independent random variables with finite expectation and variance, and let $\{b_n\}$ be an increasing sequence of positive real numbers with $b_n \to \infty$.

If $\sum_{n=1}^{\infty} \frac{Var(X_n)}{b_n^2}$, then $\frac{S_{n-E(S_n)}}{b_n} \longrightarrow 0$ a.e. where $S_n = \sum_{i=1}^{n} X_1$

Theorem (3.7)

If Y is a nonnegative random variable, then

$$\sum_{n=1}^{\infty} P\{Y \ge n\} < E(Y) \le 1 + \sum_{n=1}^{\infty} P\{Y \ge n\}$$

Remark

Recall that, the random variables X_n are called identically distributed if all have same distribution. The phrase " independent and identically distributed' will be abbreviated i.i.d.

Theorem (3.8)

Kolmogorov Strong law of large numbers, i.i.d. case.

If $X_1, X_2, \ldots$ are i.i.d. random variables with finite expectation μ, then

$\frac{1}{n}S_n \to \mu$ a.e. where $S_n = \sum_{i=1}^{\infty} X_1$

Remark

If $E(X_i)$ exists but is not necessarily finite in above theorem, the result still holds. To see this, first assume that the X_i, are nonnegative, with infinite expectation.

If $M > 0$ and $S_n^1 = \sum_{i=1}^{n} X_i \, I_{\{X_i \le M\}}$,then almost everywhere,

$\liminf_{n\to\infty} \frac{S_n}{n} \geq \ \liminf_{n\to\infty} \frac{S_n}{n} \ = E(X_1 I_{\{X_1 \le M\}})) = \to E(X_1) = \infty$ as $M \to \infty$

Therefore $n^{-1}S \to \infty$ a.e. .The general case is handled by splitting the random variables

X, into positive and negative parts.

Definition (3.1)

Let μ, μ_1, $\mu_2$be finite measures on β(R), weak convergence of μ_n to μ means that b$\int_R$ f d $\mu_n \rightarrow \int_R$ f d μ for every bounded continuous function f : R→R.

If corresponding (bounded) distribution functions are F, F_1,F_2, the equivalent condition is that F_n (a,b]→ F(a,b] at all continuity points of F.

The Central Limit Theorem

Let X_1,X_2 be independent random variables, with each X_k having finite mean μ_k and finite variance σ_k^2. Let $S_n = \sum_{i=1}^{n} X_1$ and c_n^2 = Var(S_n), n=1,2 then

$$E(S_n)=\sum_{i=1}^{n} \mu_i \text{ and } c_n^2 = \text{Var}(S_n) = \sum_{i=1}^{n} \sigma_1^2$$

We consider the normalized sum $T_n = \frac{S_n - E(S_n)}{c_n}$ which has mean 0 and variance 1.

If X^* is a random variable having normal distribution with mean 0 and variance 1; i.e. $X^* \sim N(0,1)$, so that the distribution function of $X^*$is

$$F^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{1}{2}t^2}\, dt$$

We ask for condition under which T_n converges in distribution to X^*

Theorem (3.9) Lindeberg's Theorem

Let $X_1, X_{,2}$ be independent random variables, each X_k with finite expectation μ_k and finite variance σ_k^2. Let $T_n = \frac{S_{n-E(S_n)}}{c_n}$ where c_n^2 =Var (S_n), $S_n = \sum_{i=1}^{n} X_i$ and let F_k be the distribution function of X. If for every ε>0,

$$\frac{1}{c_n^2}\sum_{k=1}^{n}\int_{\{x:\left|x-\mu_k\right|\geq\varepsilon c_n\}}(x-\mu_k)^2 dF_k(x) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Then $T_n \xrightarrow{d} X^*$ that is normal with mean 0 and variance 1 (N (0,1))

Theorem (3.11)

Let X_1, X_2, be independent random variables, with each X_k having finite meanμ_k and finite varianceσ_k^2. Then the Lindeberg condition

$$\frac{1}{c_n^2}\sum_{k=1}^{n}\int_{\{x:x-\mu_k\geq\varepsilon c_n\}}(x-\mu_k)^2 dF_k(x) \longrightarrow 0 \text{ for all}$$

ε>0 holds iff $T_n \xrightarrow{d} X^*$ and the $\frac{X_k-\mu_k}{c_n}$ are uniform asymptotic negligibility

References

1. Apostol. T.M, " Mathematical Analysis", 1974

2. Ash, R. B., "Real Analysis And Probability", 1972.

3. Ash, R.B," Probability and Measure Theory " 2000

4. Athreva. K. B., "Measure Theory and Probability Theory" Springer,2006.

5. Chung, K. L., " A Course In Probability Theory", 1968.

6. Dembo. A " Stochastic Processes" 2008