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The Stable distributions

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بسم الله الرحمن الرحيم

﴿ اقرأ باسم ربك الذي خلق ﴿١﴾ خلق الإنسان من علق ﴿٢﴾ اقرأ
وربك الأكرم ﴿٣﴾ الذي علم بالقلم ﴿٤﴾ علم الإنسان ما لم يعلم
﴿٥﴾ ﴾

صدق الله العلي العظيم

[سورة العلق , الايات (1) _ (5)]

الأهـــداء

وصلت رحلتي الجامعية إلى نهايتها بعد تعب ومشقة..

وها أنا ذا أختم بحث تخرجي بكل همة ونشاط

وأمتن لكل من كان له فضل في مسيرتي وساعدني ولو باليسير،

إلى من أفضلها على نفسي، ولِمَ لا؛ فلقد ضحت من أجلي ولم تدَّخر جهدًا في سبيل إسعادي على الدوام

(أمي الحبيبة).

نسير في دروب الحياة، ويبقى من يُسيطر على أذهاننا في كل مسلك نسلكه

صاحب الوجه الطيب، والأفعال الحسنة.

فلم يبخل علي طيلة حياته

(والدي العزيز).

الى اخوتي) ونور عيوني ومصدر قوتي واحبائي

الى من وقف بجانبي وساندني بجميع الصعاب

صديقي ورفيق دربي صاحب القلب الطيب

(زوجي العزيز)

الى اساتذتي المبجلين... أهديكم بحث تخرجي..

شُكر وتقدير

أود أن أعبر عن خالص شكري وتقديري لكم على دعمكم وتوجيهاتكم القيمة أثناء عملي على بحثي التخرج. بفضل تعاونكم وإرشاداتكم المهمة، تمكنت من إكمال هذا العمل بنجاح.

أود أن أعبر عن امتناني العميق لكم على وقتكم وجهودكم المبذولة في مراجعة البحث وتقديم الملاحظات القيمة. كانت تعليقاتكم المفصلة والمنهجية ذات أهمية كبيرة في تحسين جودة البحث وتطويره. واستفدت كثيرًا من نصائحكم وملاحظاتكم في توجيه تفكيري وتحديد الاتجاهات المناسبة للدراسة.

كما أود أن أشكركم على التحفيز والتشجيع الذي قدمتموه لي طوال هذه الفترة. بفضل دعمكم المستمر وثقتكم في قدراتي، تمكنت من تخطي التحديات والعثور على حلول للمشكلات المعقدة التي واجهتني.

أنا ممتن جدًا للفرصة التي منحتوني إياها للعمل على هذا المشروع واكتساب المهارات والمعرفة القيمة التي ستساعدني في مستقبلي الأكاديمي والمهني.

أود أن أعرب أيضًا عن امتناني العميق للجهود التي بذلها أعضاء هيئة التدريس والمشرفين الذين ساعدوني خلال فترة البحث. لقد كنتم دعامة قوية ومصدر إلهام لي، ولن أنسى العطاء والتشجيع الذي قدمتموه.

أخيرًا، أتمنى أن يكون بحثي قد نال رضاكم وتقديركم، وأن يكون قد ألبى التوقعات المحددة.

Abstract :

Stable distributions are a class of probability distributions known for their stability properties under addition. They exhibit heavy-tailed behavior and are widely used to model phenomena such as financial returns, natural disasters, and other data with long-tailed characteristics. This abstract highlights the key features of stable distributions, including their parameterization, estimation methods, and simulation techniques. It emphasizes the importance of understanding their properties in various fields, such as finance, earthquake engineering, and telecommunications. The abstract also acknowledges the mathematical complexity associated with stable distributions and the computational advancements that have made their analysis and modeling more accessible. Overall, stable distributions provide a versatile framework for capturing and analyzing data with skewness, excess kurtosis, and outliers.

The Content

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Introduction

Stable distributions are a fascinating and fruitful area of research in probability theory; furthermore, nowadays, they provide valuable models in physics, astronomy, economics, and communication theory,

The general class of stable distributions was introduced and given this name by the French mathematician Paul Lévy in the early 1920's, see Lévy (1923,1924,1925).

Formerly, the topic attracted only moderate attention from the leading experts, though there were also enthusiasts, of whom the Russian mathemati- cian Alexander Yakovlevich Khintchine should be mentioned first of all. The inspiration for Lévy was the desire to generalize the celebrated Central Limit Theorem, according to which any probability distribution with finite variance belongs to the domain of attraction of the Gaussian distribution. The concept of stable distributions took full shape in 1937 with the appearance of Lévy's monograph soon followed by Khintchine's monograph

The theory and properties of stable distributions have been systematically presented by Gnedenko & Kolmogorov and Feller . These distribution are also discussed in some other classical books in probability theory including Lukacs (1960-1970), Feller (1966-1971), Breiman (1968-1992), Chung (1968- 1974) and Laha & Rohatgi (1979). Also treatises on fractals devote particular attention to stable distributions in view of their properties of scale invariance, see e.g. Mandelbrot (1982) and Takayasu (1990). It is only recently that monographs devoted solely to stable distributions and related stochastic processes have been appeared, i.e. Zolotarev (1983-1986), Janicki & Weron (1994), and Samorodnitsky & Taqqu (1994), Uchaikin & Zolotarev (1999), Nolan (????). In these books tables and/or graphs related to stable distributions are also exhibited. Previous sets of tables and graphs have been provided by Mandelbrot &

Zarnfaller (1959), Fama & Roll (1968), Bo'lshev & Al. (1968) and Holt & Crow (1973).

Stable distributions have three exclusive properties, which can be briefly summarized stating that they

1) are invariant under addition,

2) possess their own domain of attraction, and

3) admit a canonical characteristic function.

In the following sections let us illustrate the above properties which, providing necessary and sufficient conditions, can be assumed as equivalent definitions for a stable distribution. We recall the basic results without proof.

1- Define a Stable Distribution

A stable distribution is a class of probability distributions that possess a remarkable property called stability under addition. In other words, when you take a linear combination of independent random variables from a stable distribution, the resulting random variable also follows a stable distribution, albeit potentially with different parameters. This property makes stable distributions highly valuable in various fields, including finance, physics, and signal processing.

A random variable X is said to follow a stable distribution if its characteristic function has the form:

$\phi(t) = \exp(i\delta\, t - |\,\gamma\, \sigma t|\hat{\ }\alpha(1 - i\beta \mathrm{sign}(t)\tan(\pi\alpha/2)))$,

a random variable x is S (α , δ , x , β , 0)

$$X \overset{d}{=} \begin{cases} \gamma(Z - \beta \tan \frac{\pi\alpha}{2}) + \delta & \alpha \neq 1 \\ \gamma Z + \delta & \alpha = 1 \end{cases},$$

where i is the imaginary unit, μ represents the location parameter, σ is the scale parameter, α (0 < α ≤ 2) is the stability index, and β (-1 ≤ β ≤ 1) is the skewness parameter. The characteristic function ϕ(t) uniquely defines the distribution of a random variable. The parameters μ, σ, α, and β determine the location, scale, shape, and skewness of the distribution, respectively.

Definition 1.7 A random variable *X* is S(α,β ,γ ,δ ;0) if

$$X \overset{d}{=} \begin{cases} \gamma(Z - \beta \tan \frac{\pi\alpha}{2}) + \delta & \alpha \neq 1 \\ \gamma Z + \delta & \alpha = 1 \end{cases}, \qquad (1)$$

where $Z = Z(\alpha, \beta)$ is given by (1.2). X has characteristic function

$$E \exp(iuX) = \begin{cases} \exp\left(-\gamma^{\alpha}|u|^{\alpha}\left[1 + i\beta(\tan \frac{\pi\alpha}{2})(\operatorname{sign} u)(|\gamma u|^{1-\alpha} - 1)\right] + i\delta u\right) & \alpha \neq 1 \\ \exp\left(-\gamma|u|\left[1 + i\beta \frac{2}{\pi}(\operatorname{sign} u)\log(\gamma|u|)\right] + i\delta u\right) & \alpha = \end{cases} \qquad (2)$$

When the distribution is standardized, i.e. scale γ = 1, and location δ = 0, the symbol S(α,β ;0) will be used as an abbreviation for S(α,β ,1, 0;0).

Stable distributions exhibit a variety of intriguing properties. One of the most notable features is their heavy-tailed nature. Unlike many commonly encountered distributions (e.g., Gaussian or exponential), stable distributions can have tails that decay more slowly or even infinitely. This property makes them suitable for modeling extreme events, such as financial market crashes or natural disasters, where outliers and rare occurrences are of interest.

Additionally, stable distributions are closed under various operations, such as convolution and linear combinations. This means that if X and Y are independent random variables following stable distributions, then their sum (X + Y) also follows a stable distribution. This property has practical implications in areas such as portfolio theory, where the overall risk of a portfolio of assets can be determined using stable distribution models.

Stable distributions have been extensively studied in the field of probability theory and statistics. Researchers have derived numerous results and established relationships with other statistical distributions, such as the Gaussian, Cauchy, and Levy distributions. These connections provide a deeper understanding of the behavior and properties of stable distributions.

It is worth noting that stable distributions can be challenging to work with due to their complex mathematical expressions and limited closed-form solutions. However, numerical techniques, such as simulation methods or specialized algorithms, have been developed to estimate parameters and generate random samples from stable distributions.

To gain a more comprehensive understanding of stable distributions, I would recommend referring to textbooks on probability theory and statistics that cover this topic in detail. Some prominent references include "Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance" by Gennady Samorodnitsky and Murad S. Taqqu, "Stable Distributions" by John P. Nolan, and "Stable Probability Distributions: Models for Heavy Tailed Data" by Richard L. Tweedie and Giovanna Peccati. These resources provide extensive discussions, derivations, and applications of stable distributions, allowing readers to explore the topic further.

Please note that the availability of specific books or links may vary, and I recommend checking online libraries, bookstores, or academic databases for access to these references.

2- Parameterizations of stable laws

Definition 1.6 shows that a general stable distribution requires four parameters to describe:
an *index of stability* or *characteristic exponent* $\alpha \in (0,2]$, a skewness parameter $\beta \in [-1,1]$, a scale parameter and a location parameter. We will use γ for the scale parameter and δ for the location parameter to avoid confusion with the symbols σ and μ, which will be used exclusively for the standard deviation and mean. The parameters are restricted to the range$\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma \geq 0$ and $\delta \in R$. Generally $\gamma > 0$, although $\gamma = 0$ will sometimes be used to denote a degenerate distribution concentrated at δ when it simplifies the statement of a result. Since α and β determine the form of the distribution, they may be considered shape parameters.

There are multiple parameterizations for stable laws and much confusion has been caused by these different parameterizations.
The variety of parameterizations is caused by a combination of historical evolution, plus the numerous problems that have been analyzed using specialized forms of the stable distributions. There are good reasons to use different parameterizations in different situations. If numerical work or fitting data is required, then one parameterization is preferable. If simple algebraic properties of the distribution are
desired, then another is preferred. If one wants to study the analytic properties of strictly stable laws, then yet another is useful. This section will describe three parameterizations; in Section 3.4 eight others are described.
In most of the recent literature, the notation $S\alpha(\sigma, \beta, \mu)$ is used for the class of stable laws. We will use a modified notation of the form $S(\alpha, \beta, \gamma, \delta; k)$ for three reasons. First, the usual notation singles out α as different and fixed. In statistical applications, all four parameters $(\alpha, \beta, \gamma, \delta)$ are unknown and need to be estimated; the new notation emphasizes this. Second, the scale parameter is not the standard deviation (even in the Gaussian case), and the location parameter is not generally the mean. So we use the neutral symbols γ

for the scale (not σ) and δ for the location (not μ). And third, there should be a clear distinction between the different parameterizations; the integer *k* does that. Users of stable distributions need to state clearly what parameterization they are using, this notation makes it explicit.

It lets α and β determine the shape of the distribution, while γ and δdetermine scale and location in the standard way: if $X \sim S(\alpha,\beta ,\gamma ,\delta ;0)$, then $(X - \delta)/\gamma \sim S(\alpha,\beta ,1, 0;0)$. This is not true for the $S(\alpha,\beta ,\gamma ,\delta ;1)$ parameterization when $\alpha = 1$.

On the other hand, if one is primarily interested in a simple form for the characteristic function and nice algebraic properties, the $S(\alpha,\beta ,\gamma ,\delta ;1)$ parameterization is favored. Because of these properties, this is the most common parameterization in use and we will generally use it when we are proving facts about stable distributions. The main practical disadvantage of the $S(\alpha,\beta ,\gamma ,\delta ;1)$ parameterization is that the location of the mode is unbounded in any neighborhood of $\alpha = 1$: if $X \sim S(\alpha,\beta ,\gamma ,\delta ;1)$ and $\beta > 0$, then the mode of X tends to $+\infty$ as $\alpha \uparrow 1$ and tends to $-\infty$ as $\alpha \downarrow 1$. Moreover, the $S(\alpha,\beta ,\gamma ,\delta ;1)$ parameterization does not have the intuitive properties desirable in applications (continuity of the distributions as the parameters vary, a scale and location family, etc.).

for densities in the 1-parameterization and Section 3.2.2 for more information on modes.

When $\alpha = 2$, a $S(2,0,\gamma ,\delta ;0) = S(2,0,\gamma ,\delta ;1)$ distribution is normal with mean δ , but the standard deviation is not γ . Because of the way the characteristic function is defined above, $S(2,0,\gamma ,\delta ;0) =N(\delta ,2\gamma 2)$, so the normal standard deviation $\sigma = \surd$

2γ . This fact is a frequent source of confusion when one tries to compare stable quantiles when $\alpha = 2$ to normal quantiles. This complication is not inherent in the properties of stable laws; it is

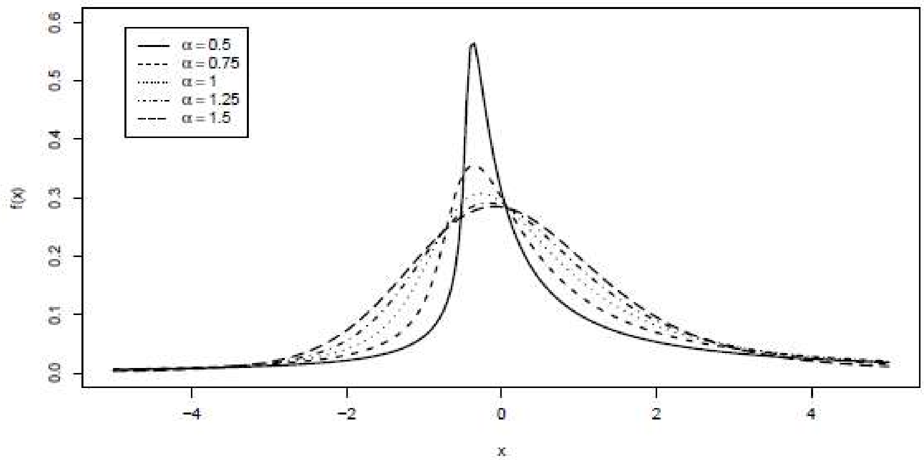

Figure 1: Stable densities in the $S(\alpha, 0.5, 1, 0; 0)$ parameterization, $\alpha = 0.5, 0.75, 1, 1.25, 1.5$.

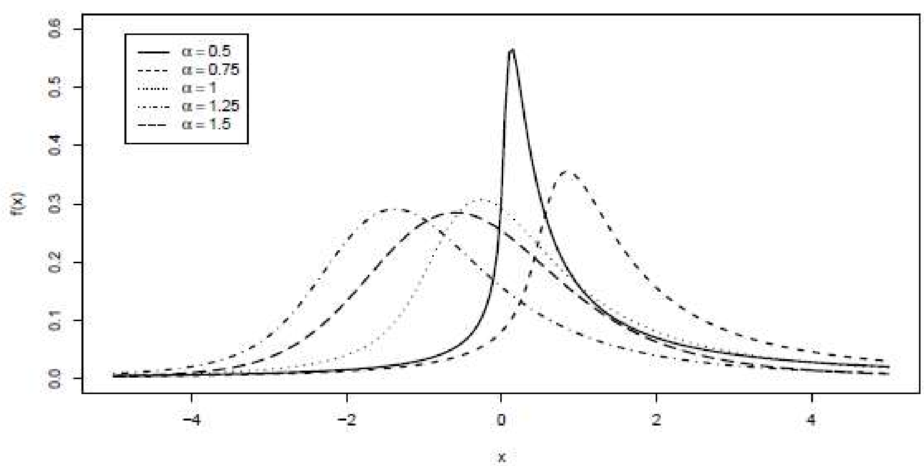

Figure 2: Stable densities in the $S(\alpha, 0.5, 1, 0; 1)$ parameterization, $\alpha = 0.5, 0.75, 1, 1.25, 1.5$.

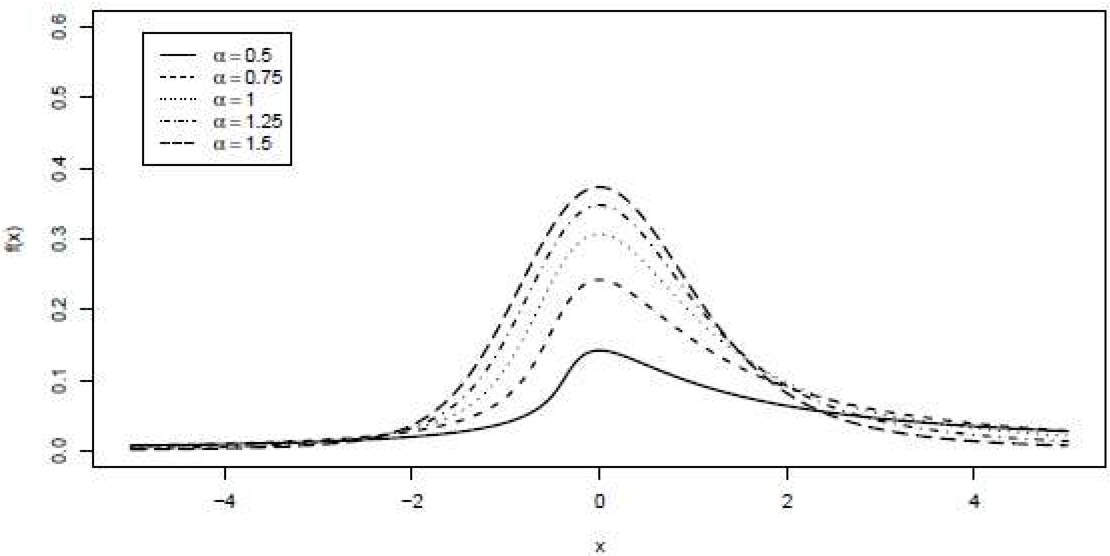

Figure 3 : Stable densities in the $S(\alpha, 0.5; 2)$ parameterization, $\alpha = 0.5, 0.75, 1, 1.25, 1.5$.

a consequence of the way the parameterization has been chosen. The 2-parameterization mentioned below rescales to avoid this problem, but the above scaling is standard in the literature. Also, when $\alpha = 2$, β is irrelevant because then the factor $\tan(\pi\alpha/2) = 0$. While you can allow any $\beta \in [-1,1]$, it is customary take $\beta = 0$ when $\alpha = 2$; this emphasizes that

the normal distribution is always symmetric.

Since multiple parameterizations are used for stable distributions, it is perhaps worthwhile to ask if there is another parameterization where the scale and location parameter have a more intuitive meaning. Section 3.4 defines the $S(\alpha,\beta ,\gamma ,\delta ;2)$ parameterization so that the location parameter is at the mode and the scale parameter agrees with the standard scale parameters in the Gaussian and Cauchy cases. While technically more cumbersome, this parameterization may be the most intuitive for applications. In particular, it is useful in signal processing and in linear regression problems when there is skewness. Figure 1.4 shows plots of the densities in this parameterization.

A stable distribution can be represented in any one of these or other parameterizations.

For completeness, Section 3.4 lists eleven different parameterizations that can be used, and the relationships of these to each other. We will generally use the S(α,β ,γ ,δ ;0) and S(α,β ,γ ,δ ;1) parameterizations in what follows to avoid (or at least limit) confusion. In these two parameterizations, α, β and the scale γ are always the same, but the location parameters will have different values. The notation $X \sim S(\alpha,\beta,\gamma,\delta_k; k)$ for $k = 0,1$ will be shorthand for $X \sim S(\alpha,\beta,\gamma,\delta_0;0)$ and $X \sim S(\alpha,\beta,\gamma,\delta_1;1)$ simultaneously. In this case, the parameters are related by (see Problem 1.9)

$$\delta_0 = \begin{cases} \delta_1 + \beta\gamma\tan\frac{\pi\alpha}{2} & \alpha \neq 1 \\ \delta_1 + \beta\frac{2}{\pi}\gamma\log\gamma & \alpha = 1 \end{cases} \qquad \delta_1 = \begin{cases} \delta_0 - \beta\gamma\tan\frac{\pi\alpha}{2} & \alpha \neq 1 \\ \delta_0 - \beta\frac{2}{\pi}\gamma\log\gamma & \alpha = 1 \end{cases} \qquad (3)$$

In particular, note that in (1.2), $Z(\alpha,\beta) \sim \mathbf{S}(\alpha,\beta,1,\beta\tan\frac{\pi\alpha}{2};0) = \mathbf{S}(\alpha,\beta,1,0;1)$ when $\alpha \neq 1$ and $Z(1,\beta) \sim \mathbf{S}(1,\beta,1,0;0) = \mathbf{S}(1,\beta,1,0;1)$ when $\alpha = 1$.

3- Invariance under addition

A random variable X is said to have a stable distribution P(x) = Prob {X ≤ x} if for any n>2, there is a positive number cn and a real number dn such that

X1+ X2+...+Xn Cn X + dn,

(2.1)

where X1, X2,...Xn denote mutually independent random variables with common distribution P(x) with X. Here the notation denotes equality in distribution, i.e. means that the random variables on both sides have the same probability distribution.

When mutually independent random variables have a common distribution [shared with a given random variable X], we also refer to them as independent, identically distributed (i.i.d) random variables (independent copies of X]. In

general, the sum of i.i.d. random variables becomes a random variable with a distribution of different form. However, for independent random variables with a common stable distribution, the sum obeys to a distribution of the same type, which differs from the original one only for a scaling (c) and possibly for a shift (dn). When in (A.1) the d2 = 0 the distribution is called strictly stable.

It is known, see [20], that the norming constants in (2.1) are of the form(1)

$$c_n = n^{1/\alpha} \quad \text{with} \quad 0 < \alpha \leq 2 . \qquad (4)$$

The parameter a is called the characteristic exponent or the index of stability of the stable distribution. We agree to use the notation X ~ Pa(x) to denote that the random variable X has a stable probability distribution with characteristic exponent a. We simply refer to Pa(x), Pa(x) = dPa(x)/dx (probability density functions = pdf) and X as a-stable distribution, density, random variable, respectively.

Definition (2.1) with theorem (2.2) can be stated in an alternative version that needs only two i.i.d. random variables. see also Lukacs (1960-1970). A random variable X is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that

$$A\,X_1 + B\,X_2 \stackrel{d}{=} C\,X + D\,, \qquad (5)$$

(2) where X1 and X2 are independent copies of X. Then there is a number a € (0,2] such that the number C in (2) satisfies Ca = A + Ba.

For a strictly stable distribution Eq. (2) holds with D = 0. This implies that all linear combinations of i.i.d. random variables obeying to a strictly stable distribution is a random variable with the same type of distribution.

A stable distribution is called symmetric if the random variable --X has the same distribution. Of course, α symmetric stable distribution is necessarily strictly stable.

Noteworthy examples of stable distributions are provided by the Gaussian (or normal) law (with α = 2) and by the Cauchy-Lorentz law (α = 1). The corresponding pdf's are known to be

$$p_G(x;\sigma,\mu) := \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\,, \quad x \in \mathbf{R}\,, \qquad (6)$$

where σ2 denotes the variance and μ the mean,

$$p_C(x;\gamma,\delta) := \frac{1}{\pi}\frac{\gamma}{(x-\delta)^2+\gamma^2}, \quad x \in \mathbf{R}, \qquad (7)$$

where γ denotes the semi-interquartile range and δ the "shift". The corresponding (cumulative) distribution functions are

$$P_G(x;\sigma,) := \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x}{\sqrt{2}\,\sigma}\right)\right], \quad x \in \mathbf{R}, \qquad (8)$$

and

$$P_C(x;\gamma,0) = \frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x}{\gamma}\right), \quad x \in \mathbf{R}. \qquad (9)$$

4- Properties Of Stable Distribution

- Stability: Stable distributions exhibit stability under addition. This means that if two independent random variables follow stable distributions, their sum (or linear combination) will also follow a stable distribution. This property makes stable distributions suitable for modeling the sum of a large number of random variables.
- Heavy Tails: Stable distributions have heavy tails, which means they have a higher probability of observing extreme values compared to other distributions such as the normal distribution. This property makes stable distributions useful for modeling phenomena that exhibit rare or extreme events, such as financial market returns or natural disasters.
- Infinite Variance: Stable distributions have infinite variance, except for the special case of the normal distribution. This implies

that stable distributions can have high variability and do not have a finite second moment. The lack of finite moments makes them suitable for modeling processes with long-range dependence and outliers.

- Parameterization: Stable distributions are characterized by four parameters: alpha (α), beta (β), gamma (γ), and delta (δ). The parameter α, known as the stability index, controls the shape of the distribution. The parameters β and γ determine the location and scale, respectively. The parameter δ represents a shift in the distribution.
- Levy's Theorem: Stable distributions satisfy Levy's theorem, which states that the sum of a large number of independent and identically distributed random variables, properly normalized, converges to a stable distribution. This property makes stable distributions a fundamental concept in the theory of stochastic processes.
- Levy Flight: Stable distributions are often associated with Levy flights, which are random walks characterized by long jumps. Levy flights have been observed in various fields, including physics, biology, and finance. Stable distributions provide a mathematical framework to model Levy flights and analyze their properties.

It's worth noting that stable distributions encompass a wide range of shapes and behaviors, depending on the values of their parameters. Different choices of parameters lead to different types of stable distributions, such as the Cauchy distribution, the Levy distribution, and the Gaussian distribution (for $\alpha = 2$). Each type has its own characteristics and applications.

5- Characteristic Function Representation

From a practitioner's point of view the crucial drawback of the stable distribution is that, with the exception of three special cases, its probability density function (PDF) and cumulative distribution function (CDF) do not have closed form expressions. These exceptions include the well known Gaussian ($\alpha = 2$) law, whose density

function is given by:

$$f_G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad (10)$$

and the lesser known Cauchy ($\alpha = 1$, $\beta = 0$) and Lévy ($\alpha = 0.5$, $\beta = 1$) laws.

Hence, the α-stable distribution can be most conveniently described by its characteristic function $\phi(t)$ - the

inverse Fourier transform of the PDF. However, there are multiple parameterizations for α-stable laws and much confusion has been caused by these different representations. The variety of formulas is caused by a combination of historical evolution and the numerous problems that have been analyzed using specialized forms of the stable distributions. The most popular parameterization of the characteristic function of $X \sim S_\alpha(\sigma, \beta, \mu)$, i.e. an α-stable random variable with parameters α, σ, β and μ, is given by ([90,98]):

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \left\{1 - i\beta \mathbf{sign}(t) \tan \frac{\pi\alpha}{2}\right\} + i\mu t\,, & \alpha \neq 1\,, \\ -\sigma|t| \left\{1 + i\beta \mathbf{sign}(t) \frac{2}{\pi} \log|t|\right\} + i\mu t\,, & \alpha = 1\,. \end{cases} \qquad (11)$$

Note, that the traditional scale parameter σ of the Gaussian distribution is not the same as σ in the above representation. A comparison of formulas (1.2) and (1.3) yields the relation: $\sigma_{\mathbf{Gaussian}} = \sqrt{2}\sigma$.

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \left\{1 - i\beta \mathrm{sign}(t) \tan \frac{\pi\alpha}{2}\right\} + i\mu t\,, & \alpha \neq 1\,, \\ -\sigma|t| \left\{1 + i\beta \mathrm{sign}(t) \frac{2}{\pi} \log |t|\right\} + i\mu t\,, & \alpha = 1\,. \end{cases} \tag{12}$$

Note, that the traditional scale parameter of the Gaussian distribution is not the same as in the above representation. A comparison of formulas (13) and (14) yields the relation: . $\sigma_{\text{Gaussian}} = \sqrt{2}\sigma.$

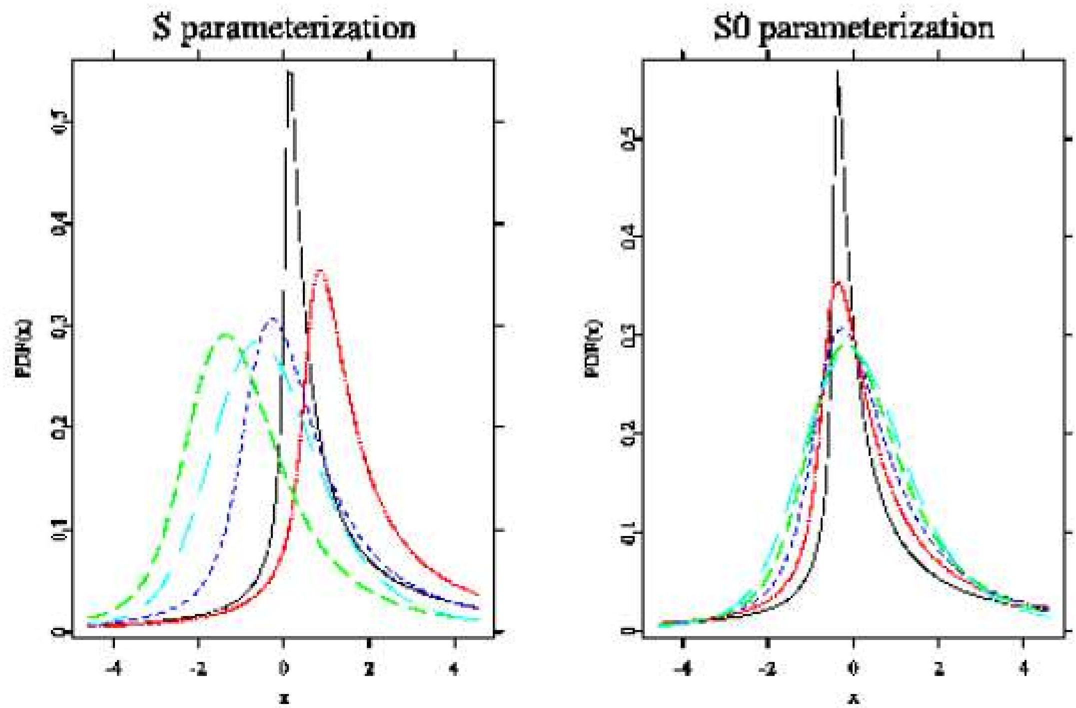

Figure 4 : Comparison of S and S^0 parameterizations: α-stable probability density functions for $\beta = 0.5$ and $\alpha = 0.5$ (*solid*), 0.75 (*dotted*), 1 (*short-dashed*), 1.25 (*dashed*) and 1.5 (*long-dashed*)
(Q: STFstab04)

For numerical purposes, it is often useful to use Nolan's (1997) parameterization:

$$\log \phi_0(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \left\{1 + i\beta \mathbf{sign}(t) \tan \frac{\pi\alpha}{2} [(\sigma|t|)^{1-\alpha} - 1]\right\} + i\mu_0 t\,, & \alpha \neq 1\,, \\ -\sigma|t| \left\{1 + i\beta \mathbf{sign}(t) \frac{2}{\pi} \log(\sigma|t|)\right\} + i\mu_0 t\,, & \alpha = 1\,. \end{cases} \quad (13)$$

The $S_\alpha^0(\sigma, \beta, \mu_0)$ representation is a variant of Zolotarev's 1986 (M)-parameterization, with the characteristic function and hence the density and the distribution function jointly continuous in all four parameters, see Fig. (15) particular, percentiles and convergence to the power-law tail vary in a continuous way as α and β vary. The location parameters of the two representations are related by $\mu = \mu_0 - \beta\sigma \tan(\pi\alpha/2)$ for $\alpha \neq 1$ and $\mu = \mu_0 - \beta\sigma(2/\pi) \log \sigma$ for $\alpha = 1$.

Then the follwing laws are satified for the stable distributions

- The Mean δ when α > 1 , otherwise undefined

- Median δ when ß = 0 otherwise not analytically

- **Mode** δ when ß = 0 otherwise not analytically

- **Variance** 2δ 2 when α = = 2, otherwise infinite

- Skewness 0 when α = = 2, otherwise undefined

- Ex. Kurtosis 0 when α = = 2, otherwise undefined

- **Entropy** not analytically expressible, except for certain parameter values

- **MGF** exp (tu+c2t2) when α = 0 , otherwise undefined

6- Relations with other Distributions

Stable distributions are a class of probability distributions that exhibit certain properties. They have interesting relationships with other well-known distributions, including the normal (Gaussian) distribution, Cauchy distribution, and Levy distribution. Here are some of the key relationships:

1. Relationship with the Normal Distribution: The normal distribution is a special case of the stable distribution when the stability index α equals 2. In this case, the stable distribution reduces to the familiar bell-shaped Gaussian distribution. The Gaussian distribution is characterized by finite mean and variance, while stable distributions (except for the normal distribution itself) have infinite variance.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty.$$

2. Relationship with the Cauchy Distribution: The Cauchy distribution is another well-known stable distribution. It can be considered a special case of the stable distribution when α equals 1 and β (the skewness parameter) equals 0. The Cauchy distribution has heavy tails and does not have a finite mean or variance. It is often used to model extreme events or outliers.

$$f(x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2} \quad -\infty < x < \infty.$$

3. Relationship with the Levy Distribution: The Levy distribution is another special case of the stable distribution. It occurs when α equals 1/2, and it is characterized by even heavier tails than the Cauchy distribution. The Levy distribution is commonly used to model phenomena with extreme variability, such as financial market returns.

$$f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\delta)^{3/2}} \exp\left(-\frac{\gamma}{2(x-\delta)}\right), \quad \delta < x < \infty.$$

Note that some authors use the term Lévy distribution for all sum stable laws; we shall only use it for this particular distribution. Problem 1.4 shows a Lévy distribution is stable with parameters $\alpha = 1/2$, $\beta = 1$ and Problem 1.5 gives the d.f. of a Lévy distribution. □

4. Stable Distribution as a Limiting Distribution: Stable distributions can arise as limiting distributions for certain stochastic processes. For example, the sum of a large number of independent and identically distributed random variables, properly normalized, converges to a stable distribution according to Levy's theorem. This property makes stable distributions important in the theory of stochastic processes and provides a framework to model various complex phenomena.
5. Stable Distributions in Multivariate Analysis: Stable distributions also have implications in multivariate analysis. The multivariate stable distribution generalizes the concept of stable distributions to multiple dimensions. It is characterized by a stability index α and a matrix of skewness parameters. Multivariate stable distributions have been employed in fields such as finance, telecommunications, and image processing.

7- Maximum Likelihood Method

The maximum likelihood (ML) estimation scheme for α -stable distributions does not differ from that for otherlaws, at least as far as the theory is concerned. For a vector of observations $x = (x_1, \ldots, x_n)$, the ML estimateof the parameter vector $\theta = (\alpha, \sigma, \beta, \mu)$ is obtained by maximizing the log-likelihood function:

$$L_\theta(x) = \sum_{i=1}^{n} \log \tilde{f}(x_i; \theta),$$

$\tilde{f}(\cdot; \theta)$

where is the stable density function. The tilde denotes the fact that, in general, we do not know theexplicit form of the stable PDF and have to approximate it numerically. The ML methods proposed in theliterature differ in the choice of the approximating algorithm. However, all of them have an appealing commonfeature - under certain regularity conditions the maximum likelihood estimator is asymptotically normal with thevariance specified by the Fischer information matrix ([30]). The latter can be approximated either by using theHessian matrix arising in maximization or, as in [81], by numerical integration.Because of computational complexity there are only a few documented attempts of estimating stable lawparameters via maximum likelihood. [29] developed an approximate ML method, which was based on groupingthe data set into bins and using a combination of means to compute the density (FFT for the central values of x and series expansions for the tails) to compute an approximate log-likelihood function. This function was thennumerically maximized.

Applying Zolotarev's (1964) integral formulas, [17] formulated another approximate ML method, however, onlyfor symmetric stable random variables. To avoid the discontinuity and nondifferentiability of the symmetricstable density function at $\alpha = 1,$ the tail index was restricted to be greater than one.

Much better, in terms of accuracy and computational time, are more recent maximum likelihood estimationtechniques. [76] utilized the FFT approach for approximating the stable density function, whereas [81] used thedirect integration method. Both approaches are comparable in terms of efficiency. The differences inperformance are the result of different approximation algorithms, see Sect. 1.2.2.

As [83] observes, the ML estimates are almost always the most accurate, closely followed by the regression-typeestimates, McCulloch's quantile method, and finally the method of moments. However, as we have already saidin the introduction to this section, maximum likelihood estimation techniques are certainly the slowest of all thediscussed methods. For example, ML estimation for a sample of 2000 observations using a gradient search routine which utilizes the direct integration method needs 221 seconds or about 3.7 minutes

The calculationswere performed on a PC equipped with a Centrino 1.6 GHz processor and running STABLE ver. 3.13 (see alsoSect. 1.2.2 where the program was briefly described). For comparison, the STABLE implementation of theKogon-Williams algorithm performs the same calculations in only 0.02 seconds (the XploRe quantlet stabregneeds roughly four times more time, see Table 1.1). Clearly, the higher accuracy does not justify the applicationof ML (x, α, β) estimation in many real life problems, especially when calculations are to be performed on-line. For thisreason the program STABLE also offers an alternative - a fast quasi ML technique. It quickly approximatesstable densities using a -dimensional spline interpolation based on pre-computed values of the standardizedstable density on a grid of values.

At the cost of a large array of coefficients, the interpolation is highlyaccurate over most values of the parameter space and relatively fast – 0.26 seconds for a sample of 2000 observations.

Table 1.2: α-stable and Gaussian fits to 2000 returns of the Dow Jones Industrial Average (DJIA) index from the period January 2, 1985 - November 30, 1992. Values of the Anderson-Darling and Kolmogorov goodness-of-fit statistics suggest a much better fit of the 1.66-stable law. Empirical and model based (α-stable and Gaussian) VaR numbers at the 95 % and 99 % confidence levels are also given. The values in parentheses are the relative differences between model and empirical VaR estimates (Q: CSAfin05)

Parameters	α	σ	β	μ
α-stable fit	1.6596	0.0053	0.0823	0.0009
Gaussian fit		0.0115		0.0006
Test values	Anderson-Darling		Kolmogorov	
α-stable fit	1.0044		0.8641	
Gaussian fit	+INF		4.5121	
VaR estimates ($\times 10^{-2}$)	95 %		99 %	
Empirical	1.5242		2.8922	
α-stable fit	1.3296	(12.77 %)	2.7480	(4.98 %)
Gaussian fit	1.8350	(20.39 %)	2.6191	(9.44 %)

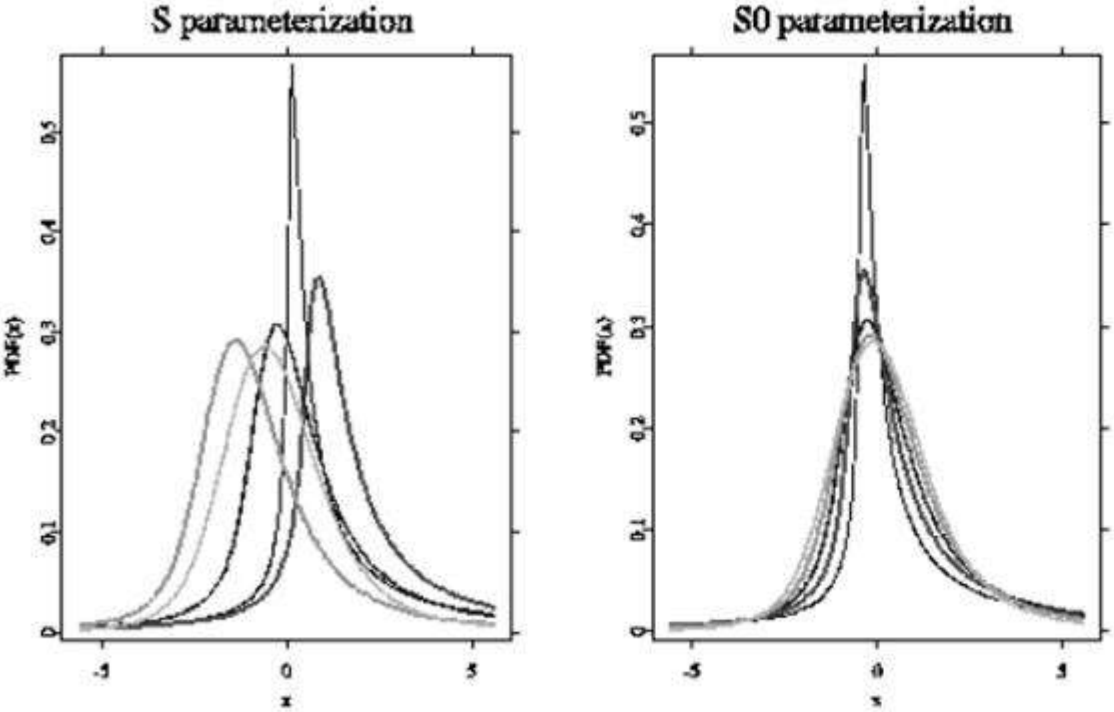

Figure 5 : 1.66-stable (*solid grey line*) and Gaussian (*dashed line*) fits to the DJIA returns (*circles*) empirical cumulative distribution function from the period January 2, 1985 - November 30, 1992. For better exposition of the fit in the central part of the distribution ten largest and ten smallest returns are not illustrated in the *left panel*. The *right panel* is a magnification of the left tail fit on a double logarithmic scale. *Vertical lines* represent the 1.66-stable (*solid grey line*), Gaussian (*dashed line*) and empirical (*solid line*) VaR estimates at the 95 % (*filled circles, triangles* and *squares*) and 99 % (*hollow circles, triangles* and *squares*) confidence levels (Q: CSAfin05)

8- Stable Distributions Applications

Many techniques in modern finance rely heavily on the assumption that the random variables under investigation follow a Gaussian distribution. However, time series observed in finance - but also in other applications - often deviate from the Gaussian model, in that their marginal distributions are heavy-tailed and, possibly, asymmetric. In such situations, the appropriateness of the com- monly adopted normal assumption is highly questionable.

It is often argued that financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving al- most continuously in time. Hence, in the presence of heavy-tails it is natural to assume that they are approximately governed by a stable non-Gaussian dis- tribution. Other leptokurtic distributions, including Student's t, Weibull, and hyperbolic, lack the attractive central limit property.

Stable distributions have been successfully fit to stock returns, excess bond returns, foreign exchange rates, commodity price returns and real estate returns (McCulloch, 1996; Rachev and Mittnik, 2000). In recent years, however, several studies have found, what appears to be strong evidence against the stable model (Gopikrishnan et al., 1999; McCulloch, 1997). These studies have estimated the

Table 1.2: Fits to 1635 Boeing stock price returns from the period July 1, 1997 – December 31, 2003. Test statistics and the corresponding p-values based on 1000 simulated samples (in parentheses) are also given.

Parameters:	α	σ	β	μ
α-stable fit	1.7811	0.0141	0.2834	0.0009
Gaussian fit		0.0244		0.0001

Tests:	Anderson-Darling	Kolmogorov
α-stable fit	0.3756	0.4522
	(0.18)	(0.80)
Gaussian fit	9.6606	2.1361
	(<0.005)	(<0.005)

tail exponent directly from the tail observations and commonly have found a that appears to be significantly greater than 2, well outside the stable domain. Recall, however, that in Section 1.5.1 we have shown that estimating a only from the tail observations may be strongly misleading and for samples of typical size the rejection of the a-stable regime unfounded. Let us see ourselves how well the stable law describes financial asset returns.

In this section we want to apply the discussed techniques to financial data. Due to limited space we chose only one estimation method – the regression approach of Koutrouvelis (1980), as it offers high accuracy at moderate computational time. We start the empirical analysis with the most prominent example the Dow Jones Industrial Average (DJIA) index, see Table 1.1. The data set covers the period February 2, 1987 - December 29, 1994 and comprises 2000 daily returns. Recall, that it includes the largest crash in Wall Street history the Black Monday of October 19, 1987. Clearly the 1.64-stable law offers a much better fit to the DJIA returns than the Gaussian distribution. Its superiority, especially in the tails of the distribution, is even better visible in Figure 1.6.

To make our statistical analysis more sound, we also compare both fits through Anderson-Darling and Kolmogorov test statistics (D'Agostino and Stephens, 1986). The former may be treated as a weighted Kolmogorov statistics which

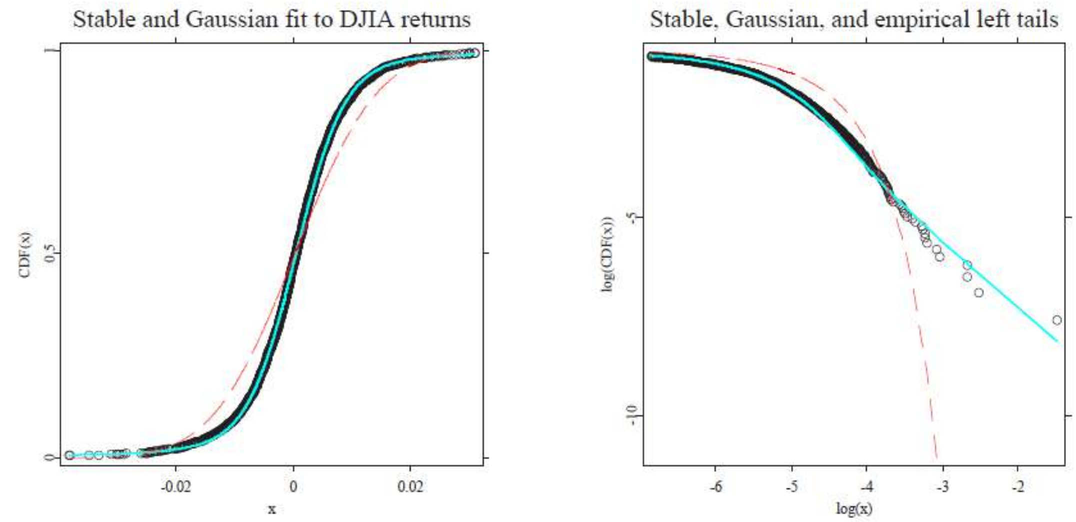

Figure 6: Stable (cyan) and Gaussian (dashed red) fits to the DJIA returns (black circles) empirical cdf from the period February 2, 1987 – December 29, 1994. Right panel is a magnification of the left tail fit on a double logarithmic scale clearly showing the superiority of the 1.64-stable law.

puts more weight to the differences in the tails of the distributions. Although no asymptotic results are known for the stable laws, approximate p-values for these goodness-of-fit tests can be obtained via the Monte Carlo technique. First the parameter vector is estimated for a given sample of size n, yielding Ø, and the test statistics is calculated assuming that the sample is distributed according to F(x; Ô), returning a value of d. Next, a sample of size n of F (x; 0)- distributed variates is generated. The parameter vector is estimated for this simulated sample, yielding 01, and the test statistics is calculated assuming that the sample is distributed according to F(x; 01). The simulation is repeated as many times as required to achieve a certain level of accuracy. The estimate of the p-value is obtained as the proportion of times that the test quantity is at least as large as d.

For the a-stable fit of the DJIA returns the values of the Anderson-Darling and Kolmogorov statistics are 0.6441 and 0.5583, respectively. The corresponding approximate p-values based on 1000 simulated samples are 0.02 and 0.5 allowing us to accept the a-stable law as a model of DJIA returns. The values of the test statistics for the Gaussian fit yield p-values of less than 0.005 forcing us to reject the Gaussian law, see Table 1.1.

Next, we apply the same technique to 1635 daily returns of Boeing stock prices from the period July 1, 1997 - December 31, 2003. The 1.78-stable distribution fits the data very well, yielding small values of the Anderson-Darling (0.3756) and Kolmogorov (0.4522) test statistics, see Figure 1.7 and Table 1.2. The approximate p-values based, as in the previous example, on 1000 simulated samples are 0.18 and 0.8, respectively, allowing us to accept the a-stable law as a model of Boeing returns. On the other hand, the values of the test statistics for the Gaussian fit yield p-values of less than 0.005 forcing us to reject the Gaussian distribution.

9- Conclusion

stable distributions are a class of probability distributions that possess several distinctive properties. These distributions exhibit stability under addition, making them suitable for modeling the sum of a large number of random variables. They have heavy tails, indicating a higher probability of extreme values compared to other distributions like the normal distribution. Stable distributions have infinite variance, except for the special case of the normal distribution, making them appropriate for modeling processes with long-range dependence and outliers.

Stable distributions are characterized by four parameters: alpha (α), beta (β), gamma (γ), and delta (δ), which control the shape, location, scale, and shift of the distribution, respectively. They satisfy Levy's theorem, which states that the sum of a large number of properly normalized independent and identically distributed random variables converges to a stable distribution.

Stable distributions have relationships with other well-known distributions. The normal distribution is a special case of the stable distribution when α equals 2. The Cauchy distribution is another special case when α equals 1 and β equals 0, while the Levy distribution arises when α equals 1/2.

These distributions find applications in various fields, including finance, physics, biology, and telecommunications. They are used to model phenomena with heavy-tailed behavior, extreme events, and complex dependencies. The study of stable distributions provides a mathematical framework to understand and analyze these phenomena.

In summary, stable distributions are valuable tools in probability theory and statistical modeling due to their unique properties, broad applicability, and connections with other distributions.

Refrences

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