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On Chaotic Properties of Hyperbolic Dynamical Systems

A Dissertation

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By

Dunia Mohamed Kreem Hussein

Supervised by

Prof. Iftichar Mudhar Talb AL-Shara’a (PhD)

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بسم الله الرحمن الرحيم

﴿ وَاللَّهُ أَخْرَجَكُم مِّن بُطُونِ أُمَّهَاتِكُمْ لَا
تَعْلَمُونَ شَيْئًا وَجَعَلَ لَكُمُ السَّمْعَ وَالْأَبْصَارَ
وَالْأَفْئِدَةَ لَعَلَّكُمْ تَشْكُرُونَ ﴾

صدق الله العلي العظيم

من سورة النحل - الآية «٧٨»

Contents

Dedicated To

my all family ,

my teachers ,

and my friends

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In the name of Allah, the most merciful the most compassionate all praise be to Allah the lord of the worlds and prayers and peace be upon Mohamed his servant and messenger and to his good and pure household.

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ABSTRACT

This dissertation aims to study and develop the chaotic properties of hyperbolic dynamical systems .

Let $f_A: M \to M$ be Anosov diffeomorphism map of a manifold M. When M is T^n torus of dimension n, we found f_A achieved several definitions of chaos such as :

- Gulick definition
- L-chaotic definition
- W-chaotic definition
- Devaney definition
- Touhey definition

We explain around C^1-stable fitting shadowing property of a diffeomorphism f of C^∞(closed manifold M) of invariant closed set Λ in M, that $f_{|\Lambda}$ satisfies a C^1- stable fitting shadowing property .

In general, if partially hyperbolic diffeomorphism map can be achieved fitting shadowing property if the center is uniformly compact center foliation W^c.

We show us any chain transitive set Λ of C^1generic diffeomorphism f, has another type of shadowing property is called, the eventual fitting shadowing property .

We restrict the diffeomorphism $f|_\Lambda$ to explain f is Anosov diffeomorphism map if and only if Λ has the strongly fitting shadowing property, and also to find hyperbolic sets, which have the same property.

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List of Symbols

Symbols	Meaning	Page
T^n	Torus of dimension n	2
M	differentiable manifold	2
$\mathbb{R}^n$	The set of real numbers of dimension n	2
$\mathrm{E}^{\mathrm{s}}, \mathrm{E}^{\mathrm{u}}$	uniformly contracted stable (expanded unstable) subspace of M	2
$T_x M$	Tangent space is split for every $x \in M$	2
$\oplus$	Direct sum	2
$\mathrm{D}\, f(x)$	The differential of $f(x)$	3
Λ	Hyperbolic set	3
E^c	Center direction slower than in the stable and unstable subspaces of M	3
$\mathcal{U}, \mathcal{V}, \mho$	Open subsets	7
C^∞	The set of continuous and existing mixed k^{th}partial derivatives for all $k \in \mathbb{Z}^+$.	11
$\det(A)$	Determinate of a matrix A	11
$\mathrm{J}(f)$	Jacobian of f	12
$\mathfrak{J}$	The metric and topological space	13
A, A^{-1}	Hyperbolic matrix, inverse of hyperbolic matrix	14
$R(f)$	The set of recurrent points of f	21
$CR(f)$	Chain recurrent points	21
$L^{\pm}$	Lyapunov exponent	22
$\Omega(f)$	non-wandering set of f	22
$v = (v_1, \dots, v_n)$	Unit vectors space of f	23
$GL(n, Z)$	The set of all $n \times n$ matrices A, whose entries integer number with $\det(A) = \pm 1$ and eigenvalues not in unit circle	27

pr_i	the natural(coordinate) projection map of $\mathbb{R}^n$	27
Per(f)	the set of periodic points of f	30
$diff^r(M)$	The set of C^r-diffeomorphism from M to M	37
TM	Tangent bundle the set of all such vectors at all each $x \in M$	37
W^s	f-invariant submanifolds tangent at x to E^s	38
W^u	f-invariant submanifolds tangent at x to E^u	38
W^c	f-invariant submanifolds tangent at x to E^c	38
$Per_h(f)$	Hyperbolic periodic points of f	40
E^{sc}	Direct sum between E^c and E^s	41
E^{uc}	Direct sum between E^c and E^u	41
$\hbar$	Conjugacy between two maps	43
$\mathbb{S}$	The shift map	43
Q_μ	The Quadratic map	43
p_μ	Fixed point of Q_μ	47
$\mathcal{R}$	Residual subset of $diff(M)$	48
$F_A \approx_\hbar F_B$	Topological conjugate between two maps	55
$AP(f)$	The set of almost periodic points of f	69
$\aleph(\propto)$	Positive integer	76
$f_{\|\Lambda}$	Restrict the map f at a set Λ	76
$O(x)$	The orbits of x	78
$\epsilon(x)$	Number of periods of a point x	82
C(f)	The set of chain transitive	100
$\mathbb{B}$(f)	Birkhoff centre set of f	113

List of Abbreviations

abbreviations	Full Form	Page
TT	Topological transitive	13
DO	Dense orbit	13
SDIC	Sensitive depended on initial condition	17
L-chaotic	Lyapunov- chaotic	24
W-chaotic	Wiggins-chaotic	24
TAD	Transitive Anosov diffeomorphisms	77
SSP	Stable shadowing property	77
PHD	Partial hyperbolic diffeomorphism	84
PO	Pseud orbit	85
RNT	Robustly non-hyperbolic transitive	86
$\mathrm{SPH}_1(M)$	Strongly partially hyperbolic diffeomorphism with one dimensional of center space	87
$\mathrm{Per}^c(f)$	The set of all periodic center points for partially hyperbolic diffeomorphism.	89
$UCCF$	Uniformly compact center foliation	89
EFSP	Eventual fitting shadowing property	100
SFSP	strongly fitting shadowing property	109

Introduction

The origins of the hyperbolic dynamics can be traced back to Poincare, who hypothesised that homoclinic tangles might give rise to intricate dynamics [1]. Maz′ja, & Shaposhnikova [2] conducted similar investigations in the 1890s. Through the1960s advancements achieved by Smale, who was also concerned with structural stability [3].

In 2005 Bonatti & Wilkinson [4] proved that partially hyperbolic diffeomorphisms on 3-manifolds implies that the center foliation carries an expansion. In 2009, Saha, Mohanty & Bharti [5] used this hyperbolicity concept, hyperbolic fixed points and their stabilities to investigate three discrete models used for spreading the epidemic virus. In 2010, Sakai, Sumi, & Yamamoto [6] proved that $f_{|\Lambda}$ satisfies a C^1-stable specification property if and only if Λ is a hyperbolic elementary set.

In 2013, Burguet & Fisher [7] proved the existence of symbolic extensions for C^2 partially hyperbolic diffeomorphisms with a 2-dimensional center bundle. In 2018, Ovadia S.B [8] construct countable Markov partitions for non-uniformly hyperbolic diffeomorphisms on compact manifolds of any dimension.

In 2018, Regis Varao [9] who gave classify C^∞ conjugacy for conservative partially hyperbolic diffeomorphisms homotopic to a linear Anosov automorphism on the 3-torus by its center foliation behavior. In 2021, Zhang [10] proved that for any partially hyperbolic diffeomorphism there is neutral center behavior on a 3-manifold. In 2022, Fernando [11] gave some sufficient conditions for transitivity of

Anosov diffeomorphisms. In 2023, Jianhua [12] proved that a class of partially hyperbolic attractors in a Banach space have semi-horseshoes.

The idea of shadowing in the dynamical systems (DS) theory comes down to the question is it possible to approximate pseudo-orbit (PO) of a given dynamical systems by true orbits?

Naturally, the answer of this question depends on the type of the approximation. The "Shadowing Property" is the most reliable structural theorem. In particular, this approach can be used to prove that hyperbolic sets are structurally stable. Any time a mathematical solution to a differential equation (or map) remains in close proximity to an approximate solution obtained in the presence of noise or round-off error, we say that the mathematical solution "shadows" the noisy solution. The shadowing property introduced by the works of Anosov, Bowen, have used several types of shadowing properties, such as limit shadowing in 1979, weak shadowing in 2001, asymptotic average shadowing in 2007, inverse shadowing in 2008, orbital shadowing in 2011, periodic shadowing in 2019, and eventual shadowing in 2021 see in :[13],[14],[15],[16],[17].

Expanding and contracting directions for the derivative are hallmarks of hyperbolic dynamics. In this case, the differential alone gives us abundant information on the dynamics on a local, semi-local, or even a global scale. In fact, these directions when applied to a set of orbits, the exponential growth in their relative behaviour under iteration yields a wealth of information about the topology and quantifiability of the orbit structure.

The cat map is the prototypical illustration of a uniformly hyperbolic dynamical system. This specific case is easily generalised to any $n \times n$ -matrix with integer elements, a unit determinant, and no eigenvalues on the unit circle act on the n-torus T^n. Since they are algebraic automorphisms of the additive group $\mathbb{R}^n/\mathbb{Z}^n$, we call these toral automorphisms.

Smale built pictures showing transformations transfering onto a compact factor with the same property of expanding and contracting. Uniform hyperbolicity is characterised by the fact that the tangent space at every point can be partitioned into subspaces that are either contracting (or stable) or expanding (or unstable).

Assume there is a diffeomorphism map $f: M \to M$, and let M be a manifold; if f is uniformly hyperbolic, or an Anosov diffeomorphism map, then the tangent space is split for every $x \in M$.

$$T_x M = E^s(x) \oplus E^u(x)$$

and for any $n \in \mathbb{N}$, there exists $\lambda \in (0,1)$ and constants $C > 0$ such that

- $\|Df^n(v)\| \leq C\lambda^n \|v\|$,for all $v \in E^s(x)$
- $\|Df^{-n}(v)\| \leq C\lambda^n \|v\|$, for all $v \in E^u(x)$

Stable subspaces at x are denoted by $E^s(x)$, while unstable subspaces are denoted by $E^u(x)$.

The invariance of the stable and unstable subspaces and their continuous dependence on the point logically follow this concept. Not all manifolds allow for an Anosov diffeomorphism map because the directions $E^s(x)$ and $E^u(x)$ are swapped and when going from one map to its inverse, topological constraints are necessary since there must always be two possible directions at each given place [13].

Let $\Lambda \subset M$ be a hyperbolic set that is any $x \in \Lambda$ in the tangent space of f satisfies the contraction and expansion conditions. In the case of two points sufficiently close together, the local maximal of the hyperbolic set is the location where a little piece of the unstable manifold of one of the points overlaps a small piece of the stable manifold of the other point. Hyperbolic diffeomorphisms have dense periodic points in their non-wandering set.

To preserve uniformity without hyperbolicity, we can generalise the concept of uniform hyperbolicity by permitting a centre direction in which expansion or contraction is uniformly slower than in the unstable and stable subspaces. This is the basis for the theory of partially hyperbolic systems, that is the tangent space is split for every $x \in M$.

$$T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

In this work, we study other types of shadowing property for hyperbolic dynamical systems.

We dealt with many chaotic properties, and we noticed that the system achieved several definitions of chaos, namely Gulick, L-chaotic, W-chaotic, Devaney, and Touhey definitions. The current work is divided into four chapters:

Chapter One deals with the basic concepts and fundamental definitions that helped us achieve our goals in two sections.

Chapter Two contains two sections. The first section investigates the general properties of Anosov diffeomorphism map. Section two deals with chaotic properties and holding a comparison between chaotic tools of Anosov diffeomorphism map, and producing several definitions of the chaos.

Chapter Three contains three sections. Section one discusses the fitting shadowing property for hyperbolic set. Section two discusses the eventual fitting shadowing property for hyperbolic set. Section three discusses the strongly fitting shadowing property for hyperbolic set.

Finally, the conclusions and future works were presented in **Chapter Four**.

Publications

1. AL-Ftlawy, D.M.K.,& AL-Shara'a, I.M.T.(2022).Some chaotic properties of Anosov diffeomorphisms of n-dimension tours T^n.International Journal of Health Sciences, 6(S5),4913-4921.

2. AL-Ftlawy, D.M.K.,& AL-Shara'a, I.M.T.," The Hyperbolic Sets with The Fitting Shadowing Property", Journal of Interdisciplinary Mathmatics, (acceptable for publication).

3. AL-Ftlawy, D.M.K.,& AL-Shara'a, I.M.T.," Center Fitting Shadowing Property for Partial Hyperbolic Diffeomorphisms", 2023 Second International Conference on Electrical, Electronics, Information and Communication Technologies (ICEEICT), Trichirappalli, India, 2023, pp. 01-05, doi: 10.1109/ICEEICT56924.2023.10157665.

4. AL-Ftlawy, D.M.K.,& AL-Shara'a, I.M.T.," Strongly Fitting Shadowing Property for Hyperbolic Manifolds", 9th International Conference and Workshop on Basic and Applied Sciences of faculty of Education, TIU, Erbil, Iraq, Jurnal Teknologi, (acceptable for publication) (ICOWBAS 2023).

5. AL-Ftlawy, D.M.K.,& AL-Shara'a, I.M.T.," Eventual Fitting Shadowing Property for Hyperbolic Dynamical Systems", Journal of Al-Qadisiyah for Computer Science and Mathmatics, 15(3), (acceptable for publication 2023).

Chapter One

Preliminaries

1.1 Basic Definitions and Notation of Chaotic Dynamical Systems

We recall some necessary definitions needed for this work:

Definition 1.1.1 [15]

For any x in the domain, $f: \mathfrak{I} \to \mathfrak{I}$ ($\mathfrak{I}$ a metric space), defined the map f.

$f\ (x)$ = the first iterate of x for f.

$f^{\ 2}(x) = f\big(f\ (x)\big)$ = the second iterate of x for f.

more generally, n is any positive integer, and

$f^{\ n}(x) = f \text{o} f \text{o} \dots \text{o}\, f(x)$;(n −times composition).

also, when *n* is any negative integer

$f^{\ -n}(x) = f^{-1} \text{o} f^{-1}\ \text{o} \dots \text{o}\, f^{-1}(x)$;(-$n$ −times composition).

so that in general

$x_n\ =\ f^{\ n}(x)\ = f(x_{n-1})$

Definition 1.1.2 [14]

Let $\mho \in \mathbb{R}^n$ be an open set and $f{:}\mho \to \mathbb{R}^n$ be a C^1 diffeomorphism map. A point $x \in \mho$ is said to be a **fixed point** of f, if $f(x) = x$, we can see fixed points occur where the graph of $f(x)$ intersects the line $y = x$ in Fig (1.1).

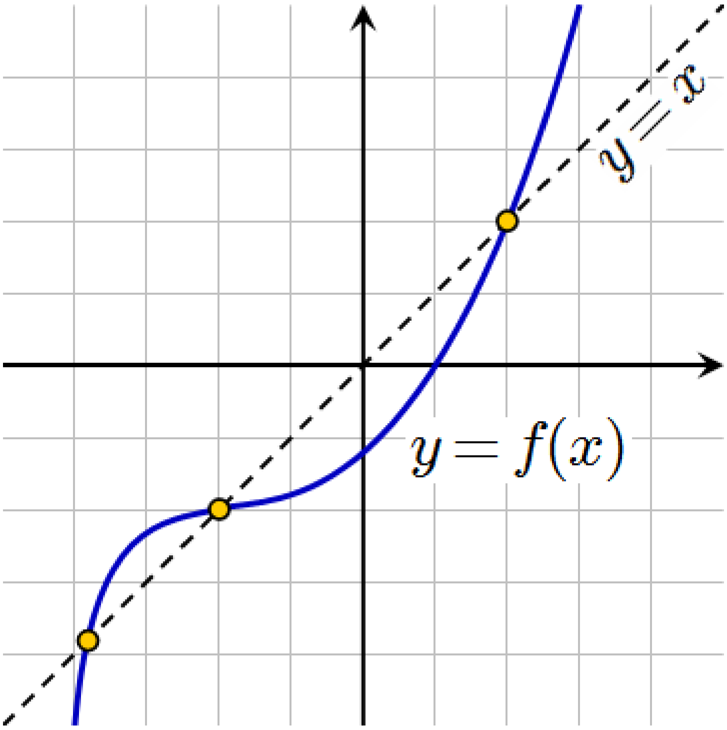


Figure (1.1): Fixed points occur where the graph of $f(x)$ intersects the line $y = x$

Example 1.1.3

If $f: \mathbb{C} \to \mathbb{C}$ is a map and x in the domain, f where $f(x) = x^5$. Then f has five fixed points :0,1,-1,$i, -i$.

Definition 1.1.4 [14]

Let $\mho \in \mathbb{R}^n$ be an open set and $f : \mho \to \mathbb{R}^n$ be a C^1 diffeomorphism onto its image. A point $x \in \mho$ with period $n \geq 1$ is referred to as a **periodic point**.

$$x, f(x), f^2(x), \ldots, f^{n-1}(x)$$

are distinct but $f^n(x) = x$

The set

$$\{x, f(x), f^2(x), \dots, f^{n-1}(x)\}$$

is called the **orbit** of x.

x has period n where $n \geq 1$, the orbit of x which is periodic orbit and called an $\boldsymbol{n}$**-cycle**.

Example 1.1.5

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be any map having the form $f\ (x) = 2x + 3$

consider the iterates of such maps:

$f^{\ 2}\ (x) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$

$f^{\ 3}(x) = 2(4x + 9) + 3 = 8x + 21$

In general

$$f^{\ n}(x) = 2^n x + 3.\,2^{n-1}(x) + 3.\,2^{n-2}(x) + \cdots + 2.\,(x)3^{n-1} + 3^n$$

Let $x \in \mathbb{R}$ and the set $x_n = \ f^{\ n}(x)$ then we have

$x_n = 2^n x + (2^{n-1}\ + 2^{n-2}\ + \cdots + 2\ + 1)(3)$

$$= 2^n x + \sum_{j=0}^{n-1} 2^{n-j-1}\,(3)$$

$$=2^n x + 3(\tfrac{2^n - 1}{2-1})$$

by Definition (1.1.2), the fixed points are periodic points with period 1.

Example 1.1.6

Let $f: \mathbb{C} \to \mathbb{C}$ be the map, where $f(x) = -x^5$. Then f has 2-cycle at $x = 1$ (since $f(1) = -1$ and $f^2(1) = 1$).

Example 1.1.7

Let $\boldsymbol{f}$: [0,1]→ $[\mathbf{0}, \mathbf{1}]$ be a map, recall that the tent map defined by

$$f(x) = \begin{cases} 2x & for\ 0 \leq x \leq \dfrac{1}{2} \\ 2 - 2x & for\ \dfrac{1}{2} < x \leq 1 \end{cases}$$

$f\left(\frac{2}{7}\right) = \frac{4}{7}, f\left(\frac{4}{7}\right) = \frac{6}{7}, f\left(\frac{6}{7}\right) = \frac{2}{7}$

confirming that $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ is a 3-cycle for f at $x = \frac{2}{7}$.

Definition 1.1.8 [19]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a metric space. A point $x \in f$ is not periodic, but for some $n > 0$ that is $f^{n+i}(x) = f^i(x)$ for $i > n$ that is, $f^i(x)$ is periodic, then x is an **eventually periodic** of n -period .

Example 1.1.9

Let $f(x) = x^2 - 1$, $x = 1$ is eventually periodic of period of f, since

$f(1) = 0, f^2(1) = f(0) = -1, f^3(1) = f(-1) = 0$

then, $x = 1$ is eventually periodic of 2-period.

Definition 1.1.10 [20]

A map $F: \mathbb{R}^n \to \mathbb{R}^n$ is $\boldsymbol{C^1}$, then there is a set of continuous and existing first partial derivatives. If f is $\boldsymbol{C^\infty}$, then it has a set of continuous and existing mixed k^{th}partial derivatives for all $k \in \mathbb{Z}^+$.

Example 1.1.11 [20]

Let $T_{a,b,c,d}: \mathbb{R}^2 \to \mathbb{R}^2$ be the Tinkerbell map is a two-dimensional map with four parameters $a, b, c, d \in \mathbb{R}$, where

$$T_{a,b,c,d}(x, y) = \begin{pmatrix} x^2 - y^2 + ax + by \\ 2xy + cx + dy \end{pmatrix}$$

since all first partial derivatives exist and continuous map

Note that

Let $f(x, y) = x^2 - y^2 + ax + by$, $\mathrm{G}(x, y) = 2xy + cx + dy$

$\frac{\partial f(x,y)}{\partial x} = 2x + a,\ \frac{\partial^2 f(x,y)}{\partial x^2} = 2, \frac{\partial^n f(x,y)}{\partial x^n} = 0$,$\forall\, n \in \mathbb{N}$ and $n \geq 3$

$\frac{\partial f(x,y)}{\partial \mathrm{y}} = -2y + b\boldsymbol{,} \frac{\partial^2 f(x,y)}{\partial y^2} = -2, \frac{\partial^n f(x,y)}{\partial y^n} = 0,\ \forall\, n \in \mathbb{N}$ and $n \geq 3$

$\frac{\partial G(x,y)}{\partial x} = 2y + c, \frac{\partial^2 G(x,y)}{\partial x^2} = 0, \frac{\partial^n G(x,y)}{\partial x^n} = 0$,$\forall\, n \in \mathbb{N}$ and $n \geq 2$

$\frac{\partial G(x,y)}{\partial y} = 2x + d, \frac{\partial^2 G(x,y)}{\partial y^2} = 0, \frac{\partial^n G(x,y)}{\partial x^n} = 0$,$\forall\, n \in \mathbb{N}$ and $n \geq 2$ we get that all its mixed k^{th} partial derivatives exist from Definition (1.1.10), $T_{a,b,c,d}$ is C^∞.

Definition 1.1.12 [14]

Let $\mho \subset \mathbb{R}^n$ be an open set, a point x in $\mho$ and f:$\mho \to \mathbb{R}^n$ be a map. Then all partials derivatives maps exist the linear map $Df(x)$ is the **differential** of f defined on $\mathbb{R}^n$ at x.

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

for very, Ʊ in $\mathbb{R}^n$. The **Jacobean** of f at x is the determinant of $Df(x)$ and denoted by $\mathrm{J}(x) = \det Df(x)$.

Definition 1.1.13 [20]

Let $\mathrm{F}:\mathbb{R}^n \to \mathbb{R}^n$ be a map and $x \in \mathbb{R}^n$. F is called **area-contracting** at x if $|\mathrm{JF}(x)|$ less than one, and is called **area-expanding** at x if $|\mathrm{JF}(x)|$ more than one.

Example 1.1.14

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a map, where $F\begin{pmatrix} x^2 - 2y \\ 2xy + 2x \end{pmatrix}$, then F is area-expanding map of a fixed point $(0,0)$ since

$DF(x,y) = \begin{pmatrix} 2x & -2 \\ 2y+2 & 2x \end{pmatrix}, DF(0,0) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

$\mathrm{J}(0,0) = |\det DF(0,0)| = 4 > 1,$

by Definition (1.1.13), F is area-expanding map.

Now, we will present some definitions of chaotic properties.

Definition 1.1.15 [12]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a map of the topological space $\mathfrak{I}$, f is **topological transitive**, denoted by TT :if every $\mho, \mathcal{V}$ are open subsets of $\mathfrak{I}$ in the following sense $f^k(\mho) \cap \mathcal{V} \neq \emptyset$, for some $k \in \mathbb{N}$.

Also, there is another definition of topological transitive f, if there exists $x \in \mathfrak{I}$ with orbit is dense ($\overline{O(x)} = \mathfrak{I}$), that is, there is a point having a dense orbit, and denote (DO), such that the orbit $\{x, f(x), \dots, f^k(x), \dots\}$ is dense in $\mathfrak{I}$.

If a map f possesses a dense orbit, then is TT. The converse is also true (for compact subsets of $\mathbb{R}$ or S^1).

Example 1.1.16 [21]

In Example (1.1.7), let $\mathfrak{I} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with usual metric and $g: \mathfrak{I} \to \mathfrak{I}$ defined $g(0)$ =0, $g(\frac{1}{n}) = \frac{1}{n+1}$, for n=1,2,…

To show that (TT) does not imply (DO), start with $I = [0,1]$ and the standard tent map $f: [0,1] \to [0,1]$, then $\mathfrak{I}$ the union of all periodic points of f and $g = f|_{\mathfrak{I}}$, the system $(\mathfrak{I}, g)$ does not satisfy the condition (DO) since $\mathfrak{I}$ is infinite (dense in I) which the orbit of any periodic point is finite. On other hand, every pair of subintervals of I shares a periodic orbit of the tent map and so $(\mathfrak{I}, g)$ satisfies (TT).

Definition 1.1.17 [20]

Let $f: \mathfrak{J} \to \mathfrak{J}$ be map of the topological space $\mathfrak{J}$, for every $\mho, \mathcal{V}$ non empty sets of $\mathfrak{J}$ and for some $k \in \mathbb{N}$ satisfy $f^n(\mho) \cap \mathcal{V} \neq \emptyset$; $\forall n \geq k$ then f is **topologically mixing**.

Example 1.1.18 [22]

Vladimir Arnold, in the 1960s, used a picture of a cat to illustrate the implications of his chaotic map of the torus into itself, earning the term Arnold's cat map.

The Arnold cat map, $f_A: T^2 \to T^2$, is the transformation obtained by thinking of the torus T^2 as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$.

$f_A(x, y) = (2x + y, x + y)$ mode 1

the automorphism of T^2 given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ mode 1

- The determinant of A is 1.
- (0,0) only fixed point of
- $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$ so, the eigenvalues of A are $\lambda_i = \frac{3\pm\sqrt{5}}{2}$,

  for $i = 1,2$, (0,0) is a saddle point
- For $i = 1,2$ and $\lambda_1 > 1, \lambda_2 < 1$ then (0,0) is hyperbolic fixed point.

  For each $x \in T^2$, the unstable manifold $E^u(x)$ of f is dense in T^2.

  For every $\varepsilon > 0$, the collection of balls of radius ε centered at point of $E^u(x)$ covers T^2, by compactness a finite subcollection of these balls also covers T^2.

Hence there is $\ell(\varepsilon) > 0$ such that every set X of length $\ell(\varepsilon)$ in an unstable manifold is ε −dense in T^2, that is

$$d(y, X) \leq \varepsilon, \text{ for all } y \in T^2$$

Let $\mho$ and $\mathcal{V}$ be non-empty open sets in T^2

Choose $y \in \mathcal{V}$ and $\varepsilon > 0$ such that $\overline{B(y, \varepsilon) \subset \mathcal{V}}$.

The open set $\mho$ contains a set of length $\delta > 0$ in some unstable manifold $E^u(x)$.

Let λ, $|\lambda| > 1$, be the expanding eigenvalue of f_A, and choose $k > 0$ such that $|\lambda|^k \delta \geq \ell(\varepsilon)$.

Then for any $n \geq k$, the image $f_A^n(\mho)$ contains a set of length at least $\ell(\varepsilon)$ in some unstable manifold, so $f_A^n(\mho)$ is ε −dense in T^2

Therefore $f_A^n(\mho) \cap \mathcal{V} \neq \emptyset$. Explanation of the map as in Fig (1.2).

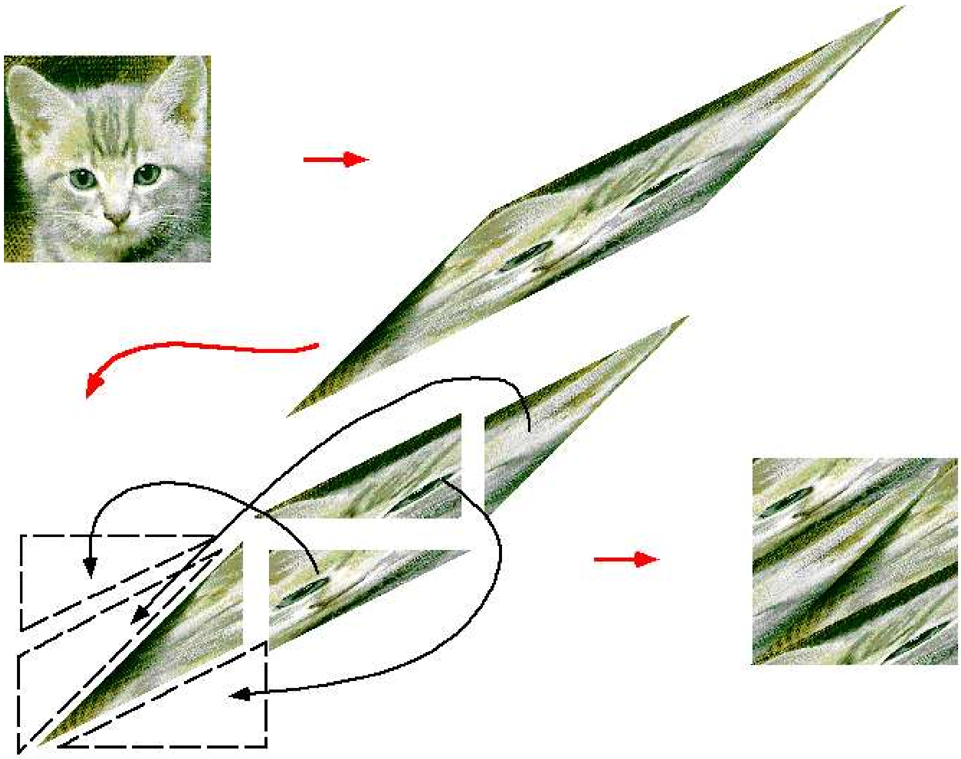


Figure (1.2): The cat map

Note 1.1.19

By Definition (1.1.17), topological mixing implies topological transitivity, but not vice versa.

Definition 1.1.20 [21]

Let $f:\Im \to \Im$ be a map of a topological space $\Im$, f is called **minimal** if all points are transitive .

Example 1.1.21 [19]

Let $f: S^1 \to S^1$, given by $f(\theta) = \theta + k$ where k is an irrational circle rotation, f is minimal and therefore it is, a topologically transitive map, but not topological mixing, since all points are remain the same distance apart after iteration.

Definition 1.1.22 [21]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a topological transitive of the topological space $\mathfrak{I}$, f^k is topologically transitive for any $k \geq 1$, so f is said to be **totally transitive** .

Definition 1.1.23 [19]

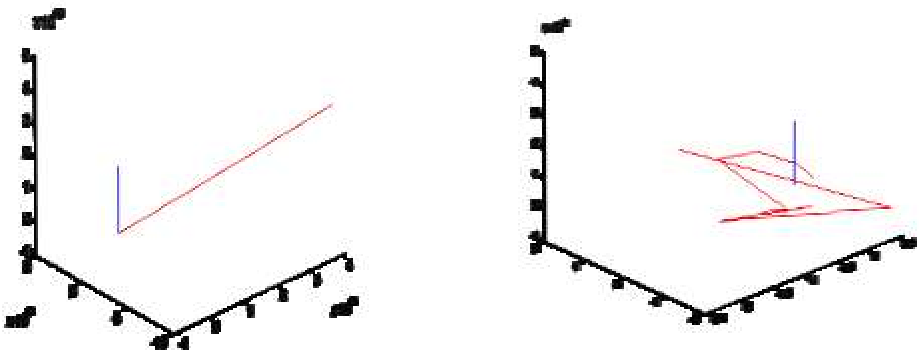
Let $f: \mathfrak{I} \to \mathfrak{I}$ be a map of the metric space $(\mathfrak{I}, \text{d})$, f has sensitive dependence on initial condition, if for any $x \in \mathfrak{I}$ and $\mho \subset \mathfrak{I}$ open set of x, there exists $y \in \mho$ and $k \in Z^+$, such that for all $\varepsilon > 0$, there exists $\delta > 0$, that is $d(x, y) < \delta$

then

$$d(f^k(x), f^k(y) > \varepsilon$$

Example 1.1.24 [20]

In Example (1.1.11), the Tinkerbell map has sensitive dependence on initial condition when $a = -0.3, b = -0.6, c = 2.2, d = 0$, we show that by MATLAB software the map has sensitive dependence in initial condition, see Fig (1.3) .



(a) (b)

Figure (1.3): The Tinkerbell map has sensitive dependent on initial condition, with initial condition (0,0) and (0.3,0), (a) of $a = 0.9, b = -0.6, c = 2.2, d = 0.5$ and (b) of $a = -0.3, b = -0.6, c = 2.2, d = 0$.

Example 1.1.25 [23]

The Kaplan York map was introduced in 1983, it has two dimensional dynamical system with one parameter form as:

$$K_\mu \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \bmod 1 \\ \mu y + cos4\pi x \end{pmatrix}, \text{ where } \mu \in \mathbb{R}$$

To find $K_\mu^n(x)$ by Induction Law to simplify the form:

$$K_\mu \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \bmod 1 \\ \mu y + cos4\pi x \end{pmatrix}$$

$$K_\mu^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^2 x \bmod 1 \\ \mu(\mu y + cos4\pi x) + cos4\pi x \end{pmatrix} = \begin{pmatrix} 4x \bmod 1 \\ \mu^2 y + \mu + 1 \end{pmatrix}$$

By Induction Law $K_\mu^n(x) = \begin{pmatrix} 2^n x \bmod 1 \\ \mu^n y + \mu^{n-1} + \mu^{n-2} + \cdots + 1 \end{pmatrix}$

$$K_\mu^n(x) = \begin{pmatrix} 2^n x \bmod 1 \\ \mu^n y + \dfrac{\mu^n - 1}{\mu - 1} \end{pmatrix}$$

If $|\mu| < 1$

then $\mu^n y + \frac{\mu^{n-1}}{\mu-1} \to 0 \;\; \forall\, n \to \infty$

And $2^n x \to \infty$ as $n \to \infty$

So the diverge in one side

If $|\mu| > 1$ and $n \to \infty$, then $2^n x \to \infty$

But $\mu^n y + \frac{\mu^{n-1}}{\mu-1} \to \infty$

So the diverge in two side

Let $\chi_1, \chi_2 \in \mathbb{R}^2$ such that $\chi_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\chi_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

thus $d\left(K_\mu(\chi_1), K_\mu(\chi_2)\right) = d\begin{pmatrix} 2(x_1 - x_2) \bmod 1 \\ \mu(y_1 - y_2) + cos4\pi(x_1 - x_2) \end{pmatrix}$

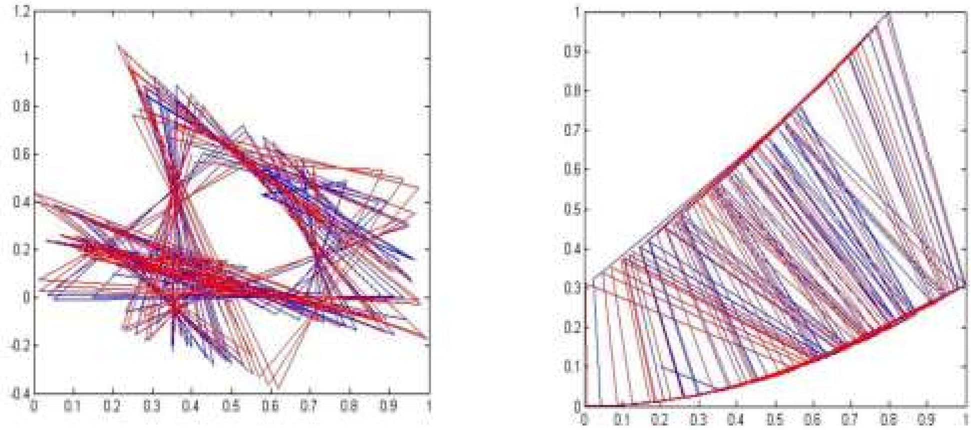
$$= d\begin{pmatrix} 2(x_1 - x_2) \bmod 1 \\ \mu(y_1 - y_2) \end{pmatrix}$$

$$d\left(K_\mu^n(\chi_1), K_\mu^n(\chi_2)\right) = \begin{pmatrix} 2^n(x_1 - x_2) \bmod 1 \\ \mu^n(y_1 - y_2) \end{pmatrix}$$

$$= \sqrt{2^n(x_1 - x_2)^2 + \mu^n(y_1 - y_2)^2}$$

$d\left(K_\mu^n(\chi_1), K_\mu^n(\chi_2)\right) \to \infty$ as $n \to \infty$

so the map has sensitive dependence on initial condition, illustration in Fig (1.4).



(a) (b)

Figure (1.4): Clarification sensitivity of the Kaplan York map by MATLAB software in (a) $\mu = 1.9$ with initial points (0.2,0.1), $(b)\mu = 0.9$ with initial points (0.2,0.1)

Definition 1.1.26 [6]

A map $f: \mathfrak{I} \to \mathfrak{I}$ of the metric space $\mathfrak{I}$ is said to be **equicontinuous** if there exists $\delta > 0$, for any $\varepsilon > 0$ and any x, y with $d(x, y) < \delta$ then $d\left(f^k(x), f^k(y)\right) < \varepsilon, \forall k \in \mathbb{N}$.

Example 1.1.27

Let $f\colon \mathfrak{I} \to \mathfrak{I}$ be a map of metric space $\mathfrak{I}$, $f_n(x) = \frac{cos(nx)}{n^2}$ is an equicontinuous on all of $\mathbb{R}$. Indeed, for each $x, y \in \mathbb{R}$, there exists

$\delta > 0$

$$|f_n(x) - f_n(y)| = \frac{1}{n^2} \, |cos(nx) - cos(ny)|$$

$$= \frac{1}{n^2} \left| \int_y^x n \, sin\,(nt)dt \right|$$

$$\leq \frac{1}{n} \, |x - y| < \delta$$

when

$$|x - y| < \delta$$

There are chaotic properties related to maps, as mentioned earlier, and chaotic properties related to the points of sets, which we will mention later.

Definition 1.1.28 [24]

Let $\mathfrak{I}$ be topological space of a map $f\colon \mathfrak{I} \to \mathfrak{I}$, for every neighbourhood $\mho$ of a point $x \in f$ there exists a positive integer $k > 0$ such that $f^k(\mho) \cap \mho \neq \emptyset$, then x is a **non-wandering point** of f, and the set of all **non-wandering** points in f is denoted by Ω .

Definition 1.1.29 [24]

The set $\sigma_1(x) = \sigma_{1_f}(x)$ of all σ_1**-limit points** of x, that is,

$$\sigma_1(x) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\kappa\geq n} f^{\kappa}(x).}$$

If f is an invariant, the $\boldsymbol{\sigma_2}$**-limit set** of x is $\sigma_2(x) = \sigma_{2_f}(x)$

$$\sigma_2(x) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\kappa\geq n} f^{-\kappa}(x).}$$

A point in $\sigma_2(x)$ is an σ_2-limit point of x. Both the σ_2- and σ_1-limit sets are closed and f- invariant .

Definition 1.1.30 [24]

Let $f\colon \mathfrak{J} \to \mathfrak{J}$ be a map, the set of **recurrent points** of f, denoted by $R(f)$, there exist a positive integer sequence $\{y_i\}_{i=1}^{\infty}$ such that $f^{y_i}(x) \to x$, as $y_i \to \infty$.

Definition 1.1.31 [25]

Let $f\colon \mathfrak{J} \to \mathfrak{J}$ be any map of a compact metric space $\mathfrak{J}$, an $\boldsymbol{\varepsilon}$ **−chain** from x to y for f is a sequence of points in $\mathfrak{J}$,

$x = x_0, x_1, \dots, x_n = y$, with $n \geq 1$, such that

$$d(f(x_i, x_{i+1}) < \varepsilon\ , \forall\ 0 \leq i \leq n-1$$

A point $x \in \mathfrak{J}$ is called **chain recurrent** if there is ε -chain from x to itself for every $\varepsilon > 0$, **chain recurrent set** of f is the set $CR(f)$ of all recurrent points, we say that f is **chain transitive** implies that every point is chain recurrent .

Example 1.1.32

See Example in [25], f is chain transitive.

Example 1.1.33

A periodic point such that in Definition (1.1.4), $f^n(x) = x$ for some $n > 0$ is obviously recurrent, the set $\sigma_1(x)$ is the finite periodic orbit.

Note 1.1.34 [21]

- The periodic points are recurrent points,
- Every recurrent point is non-wandering, in fact $\overline{R(f)} \subset \Omega(f)$,
- Every point is non-wandering, but not all points are recurrent,
- Every isolated points are recurrent.

Definition 1.1.35 [24]

Let $f: \Im \rightarrow \Im$ be any map, f is called **pointwise** non-wandering map, if each point is non-wandering

Definition 1.1.36 [23]

Consider the continuous differentiable map $F: \mathbb{R} \to \mathbb{R}$, for all $x \in \mathbb{R}$ in direction v, when the limit exists, we defined the Lyapunov exponent of a map F at $\mathbb{R}$ by $L^{\pm}(x, v) = \lim_{n\to\infty} \frac{1}{n} \ln\|DF_x^n v\|$.

The Lyapunov exponents of the map F in higher dimensions, such as $\mathbb{R}^n$, will have the form $L_1^{\pm}(x, v), L_2^{\pm}(x, v), \dots, L_n^{\pm}(x, v)$ for a system of n variables. This maximum number of Lyapunov exponents n, is the Lyapunov exponent.

$$L^{\pm}(x, v) = \max\{L_1^{\pm}(x, v), L_2^{\pm}(x, v), \dots, L_n^{\pm}(x, v)\}$$

Where $v = (v_1, v_2, \dots, v_n)$ are non zero vectors, $v \in \mathbb{R}^n$.

Example 1.1.37 [23]

By Example (1.1.25), the Kaplan York map K_μ has positive Lyapunov exponents, for all $\mu \neq 0 \in \mathbb{R}$

Since $L_1^{\pm}(x, v) = \lim_{n\to\infty} \frac{1}{n} \ln \left| DK_\mu \begin{pmatrix} x \\ y \end{pmatrix}, v \right|$

$$= \ln 2 > 0, \forall \mu \in \mathbb{R}$$

$$L_2^{\pm}(w, v) = \lim_{n\to\infty} \frac{1}{n} \ln \left| DK_\mu \begin{pmatrix} x \\ y \end{pmatrix}, v \right|$$

$$= \ln |\mu|, \qquad \mu \neq 1$$

$$L^{\pm}(w, v) = \max\{L_1^{\pm}(w, v), L_2^{\pm}(w, v)\} > 0$$

So, K_μ has positive Lyapunov exponents.

There are some definitions of chaos that, we have achieved in this work and these definitions are not equal in strength, we start from the weakest in sequence:

Definition 1.1.38 [21]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a contiuous map on a metric space $\mathfrak{I}$, is chaotic (in the sense that **Gulick**) if it satisfies at least one of the conditions listed below:

- Each point in f the domain of f that is not eventually periodic point has a positive Lyapunov exponent, hence f itself is not eventually periodic.
- f has exhibited on its domain sensitive depended on initial condition.

Definition 1.1.39 [21]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a contiuous map on a metric space $\mathfrak{I}$ is chaotic according to Lyapunov or **L-chaotic** if :

- f is topologically transitive.
- f has a positive Lyapunov exponent.

Definition 1.1.40 [20]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a contiuous map on a metric space $\mathfrak{I}$, Is the map chaotic (**Wiggins** or **W-chaotic**) if it meets these criteria:

- sensitive dependence on initial conditions SDIC
- topologically transitive TT

Example 1.1.41 [20]

By Example (1.1.11) the Tinkerbell map is chaotic satisfy Definitions (1.1.38), (1.1.39) and (1.1.40) .

Definition 1.1.42 [27]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a map, **Devaney** states that if a map of the metric space $\mathfrak{I}$ meets the following properties:

- Sensitive dependence on initial conditions SDIC
- Topologically transitive TT
- Density of periodic points in $\mathfrak{I}$.

In 1997 Touhey also proposed another definition of chaos, concerning about orbits of periodic points, which proved to be equivalent to Devany's definition.

Definition 1.1.43 [21]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a contiuous map on a metric space $\mathfrak{I}$ is chaotic map verification chaotic (**Touhey**) on $\mathfrak{I}$, there exists a periodic point $x \in \mho$ and nonnegative integer k, given $\mho$ and $\mathcal{V}$ non-empty open subsets of $\mathfrak{I}$, such that $f^k(x) \in \mathcal{V}$, where any open subsets of $\mathfrak{I}$ that are not empty share a periodic orbit .

1.2 Some Concepts of Topological Manifold and Hyperbolic Systems

In this section, are explained some general concepts of topological manifold, as well as the necessary concepts for hyperbolic systems.

The theorems taken from the references will be presented without proof, you can review a lot referred to.

Definition 1.2.1 [28]

A **topological manifold** is a topological space that locally resembles Euclidean space near each point. More precisely, an n-dimensional manifold, or n-manifold for short, is a topological space with the property that each point has a neighborhood that homeomorphic to an open subset of n-dimensional Euclidean space.

Theorem 1.2.2 [24]

If the 1-manifold is compact, then it is homeomorphic to S^1, and if it is not compact, then it is homeomorphic to $\mathbb{R}$.

Example 1.2.3 [28]

A one-dimensional shape whose boundary is a circle. A square is a 2-manifold with interior and boundary. A ball (a sphere plus its inside) is a boundary-enclosed 3-manifold**.**

However, a closed ball in $\mathbb{R}^n$ is not a manifold because a boundary point does not have a Euclidean neighbourhood, which is required for the existence of a boundary.

Definition 1.2.4 [28]

A differentiable maps $f: \mho \to \mathbb{R}^n$ for each $i = 1,2, \dots, n$, let $pr_i: \mathbb{R}^n \to \mathbb{R}$ be the natural(coordinate) projection map of $\mathbb{R}^n$ given by $pr_i(x_1, \dots, x_n) = x_i, i = 1,2, \dots, n$.

Let $\mho$ be an open set in $\mathbb{R}^n$ and let $f: \mho \to \mathbb{R}^n$ be the map

$f_i \coloneqq pr_i o f: \mho \to \mathbb{R}$, $i = 1,2, \dots, n$ are called (slot) **coordinate maps** of f.

Let M be a differentiable manifold the a differentiable map

$f: M \to M$. The **derivative** $Df_x$is linear map from $T_x M$ to $T_{f(x)} M$ provided by the f is partial derivatives matrix

Definition 1.2.5 [29]

Assume that $A \in GL(n, \mathbb{Z})$, where $GL(n, \mathbb{Z})$, is the collection of all $n \times n$ invertible matrices whose entries are in $\mathbb{Z}$. Each eigenvalues

$\lambda_i \in \mathbb{C}, i = 1, \dots, n$ of A is **hyperbolic** if and only if $|\lambda_i| \neq 1$. If the eigenvalue $|\lambda_i|$ decreases by more than one, we say that it expands, and vice versa.

$GL(n,\mathbb{Z})$is diffeomorphism since $GL(n,\mathbb{Z}) \subset GL(n,\mathbb{R})$ even so, F is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$f_A(\mathbb{Z}^n) = \mathbb{Z}^n$, Therefore, A induces a map by quotient $\mathbb{R}^n$ by $\mathbb{R}^n$.

$$\widetilde{A} : x + \mathbb{Z}^n \to A(x) + \mathbb{Z}^n$$

Definition 1.2.6 [20]

The **tangent space** is defined as the set of tangent vectors to a smooth compact Riemannian manifold M at $x \in M$, denote by $T_x M$, and it is a real vector space.

Note 1.2.7

Recall that the derivative of f at x is a linear map from the tangent space at x to the tangent space at $f(x)$, see Fig (1.5), illustrate the tangent space at a point x of a manifold M.

$$D_x f: T_x M \rightarrow T_{f(x)} M$$

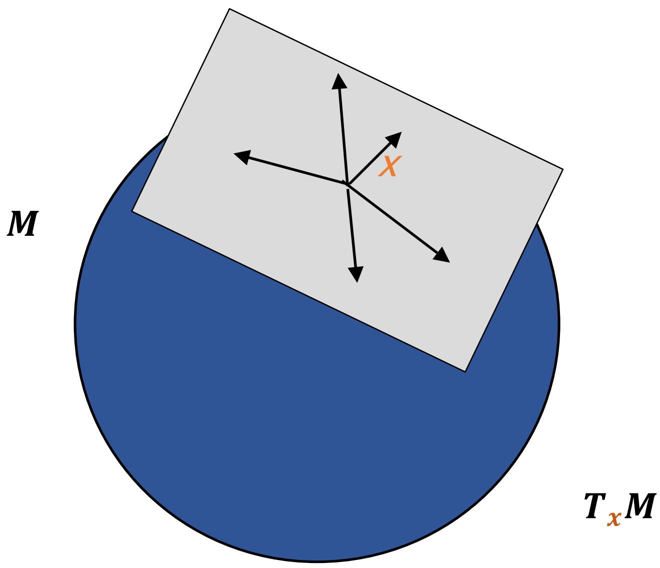


Figure (1.5) : Illustrate the tangent space $T_x M$ at a point x of a manifold M

Definition 1.2.8 [14]

Let $\mho \in \mathbb{R}^n$be an open set and $f: \mho \rightarrow \mathbb{R}^n$ be a C^1 diffeomorphism map. The term **hyperbolic fixed point** of f refers to the point $x \in \mho$ satisfy $f(x) = x$, moreover, $Df(x)$ has eigenvalues that are not on the unit circle, then we say that x is a hyperbolic periodic point of f with n −period, can you see in Fig (1.6).

Stable (or unstable) subspaces, represented by E^s(resp. E^u), are the union of the generalized eigenspaces corresponding to the eigenvalues inside (resp. outside) the unit circle.

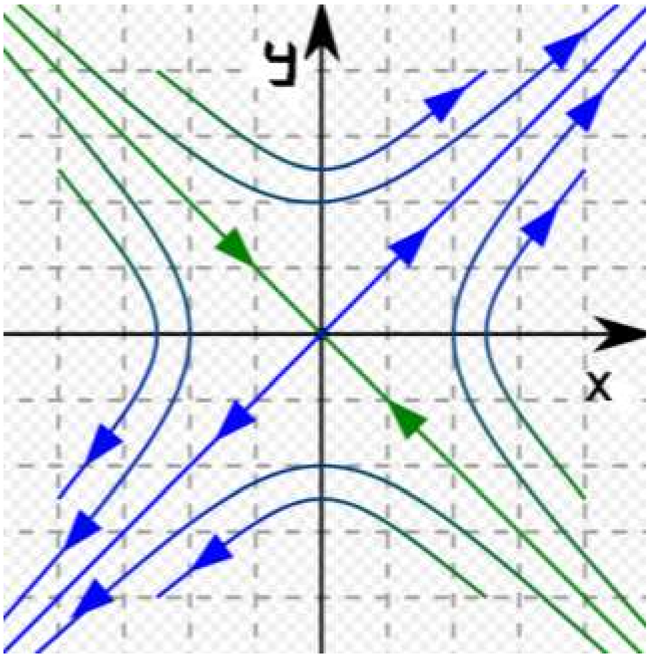


Figure (1.6): Hyperbolic fixed point

Definition 1.2.9 [30]

The **stable subspace** E^s, is spanned by the generalized eigenvector associated with the eigenvalues less than one of A .

Definition 1.2.10 [30]

The **unstable subspace** E^u, is spanned by the generalized eigenvector associated with the eigenvalues more than one of A .

One implication is that as time t tends to $\pm\infty$, all solutions converge to zero or infinity, respectively.

Let's review what we already know about stable and unstable sets:

$$E^s(x) = \{y \in M: d(f^n(x), f^n(y)) \to 0, as\ n \to \infty\}$$

$$E^u(x) = \{y \in M: d(f^n(x), f^n(y)) \to 0, as\ n \to -\infty\}$$

Where $per_h(f)$ is a hyperbolic periodic points of x.

Example 1.2.11

By Example (1.1.18), illustration of how the modulo operation expands and rearranges the unit square as a result of the linear map, see Fig (1.7).

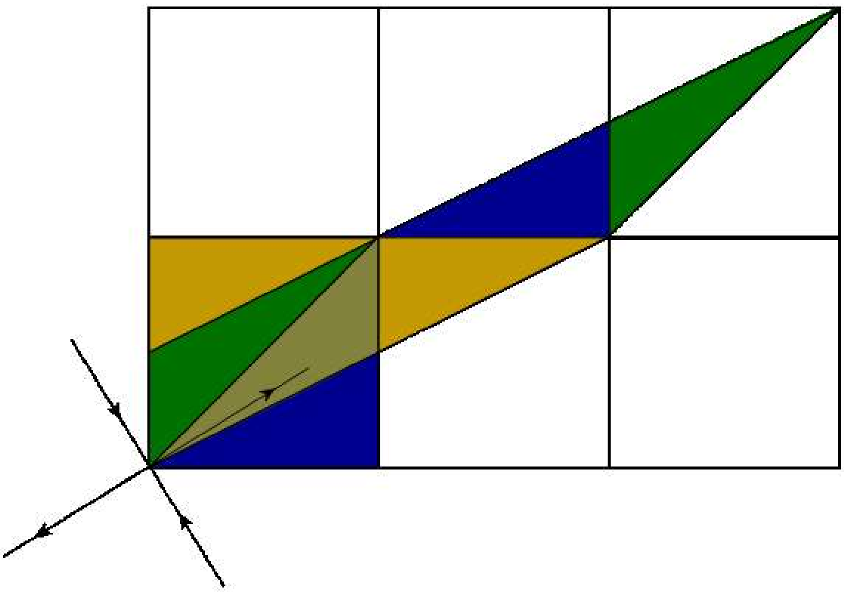


Figure (1.7): Illustration of how the modulo operation expands and rearranges the unit square as a result of the linear map. The orientation of the contracting and expanding eigenspaces is indicated by the arrowed lines.

Example 1.2.12

Consider the map $H_{a,b}: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$H_{a,b} = \binom{1-ax^2+y}{-bx}, \text{ for all } a, b \in \mathbb{R}$$

to find the fixed points of the Henon map, we have to solve the equations, there are two fixed points if $a \neq 0$ and $(b+1)^2 > 4a$, one fixed point if $a \neq 0$ and $(b+1)^2 = 4a$, if $a = 0$ and $b \neq -1$ and no fixed point otherwise, these fixed points are hyperbolic except when $b = 1$ and $-3 < a < 1$ or when $b \neq -1$ and $a = \frac{-3(b+1)^2}{4}$ or $\frac{(b+1)^2}{4}$.

Non hyperbolic fixed points, when $|f'(x)| = 1$, the stability criteria for non hyperbolic fixed points are more involued, were x is a fixed point of f .

Definition 1.2.13 [30]

A diffeomorphism map $f: M \to M$ of a compact Riemannian manifold M, we called that a closed, invariant subset $\Lambda \subset M$ **hyperbolic** if the tangent space $T_x M$ of M at x, can be represented as the direct sum of subspaces E^s, E^u such that

- $T_\Lambda M = \mathrm{E}^{\mathrm{s}} \oplus \mathrm{E}^{\mathrm{u}}$
- $Df_x(E_x^s) = E_{f(x)}^s, Df_x(E_x^u) = E_{f(x)}^u$

and there is $C > 0$, $\lambda \in (0,1)$ that is for any $n \geq 0$,

- $\|Df_x^n(v)\| \leq C\lambda^n\|v\|$, when $v \in E_x^s$
- $\|Df_x^n(v)\| \leq C\lambda^{-n}\|v\|$, when $v \in E_x^u$

Definition 1.2.14 [30]

The collection of all such vectors at all points $x \in M$ is called the **tangent bundle** of M and is denoted

$$TM = \{(x, v): x \in M, v \in T_xM\}$$

Definition 1.2.15 [30]

Let $f: M \to M$ be a diffeomorphism map is said to be **uniformly hyperbolic** or **Axiom A** satisfy:

- The tangent bundle restricted to Ω has a hyperbolic spliting or the Ω (f) has hyperbolic structure.
- The set of all periodic points, denoted by Per (f), is dense in $\Omega(f)$.

Definition 1.2.16 [24]

A diffeomorphism map $: M \to M$ of Reimanian manifold M; is called **uniformly hyperbolic**, or an **Anosov** diffeomorphism map, then the tangent bundle TM has continuous splitting into direct sum DT −invariant subbundles for every $x \in M$.

$$TM = E^s(x) \oplus E^u(x)$$

and for any $n \in \mathbb{N}$, there exists $\lambda \in (0,1)$ and constants $C > 0$ such that

$\|Df^n(v)\| \leq C\lambda^n\|v\|$ for all $v \in E^s(x)$ is unit vector

$\|Df^{-n}(v)\| \leq C\lambda^n\|v\|$ for all $v \in E^u(x)$ is unit vector

Stable subspaces at x are denoted by $E^s(x)$, while unstable subspaces are denoted by $E^u(x)$

Example 1.2.17 [24]

Every Anosov diffeomorphism is consistent with Axiom A. In this instance, the entire M manifold is hyperbolic. .

It is important to notice that this According to Definition (1.2.15), all periodic and fixed points of f must be hyperbolic, since

Per$(f) \subset \Omega(f)$ for any diffeomorphism f. Manifolds corresponding to f^n are those under which the hyperbolic fixed point x is an invariant.

Note 1.2.18

Let M be a smooth Riemannian manifold, then the diffeomorphism map $f: M \to M$ is an Anosov map if $\Lambda = M$.

Example 1.2.19

In Example (1.1.18), Arnold's cat map is the simplest example of an Anosov diffeomorphism, automorphism map of T^2

Note 1.2.20

More generally any linear hyperbolic automorphism of the n-torus $f_A: T^n \to T^n$ is Anosov diffeomorphism map.

In terms of a pair of splitting of the tangent manifold TM into invariant subbundles E^s and E^u, where E^s and E^u are both C^r for each point in the manifold.

Definition 1.2.21 [24]

Suppose that a closed f-invariant set Λ subset of M, and $f_{|\Lambda}$ the restriction of f to the set Λ. A compact neighborhood Ʊ of Λ, that is

$$\Lambda_f(\mho) = \cap_{n\in\mathbb{Z}} f^n(\mho)$$

A set Λ is called **locally maximal** , there is Ʊ of Λ such that

$$\Lambda = \Lambda_f(\mho)$$

and denote by (LM).

Example 1.2.22 [24]

Smale Horseshoe is example of locally maximal hyperbolic sets, since $E^s(x) \cap E^u(x) = \{0\}$, the local stable and unstable manifolds of x intersect at x transversely. By continuity, this transversality extends to a neighborhood of the diagonal in $\Lambda \times \Lambda$, see Fig (1.8).

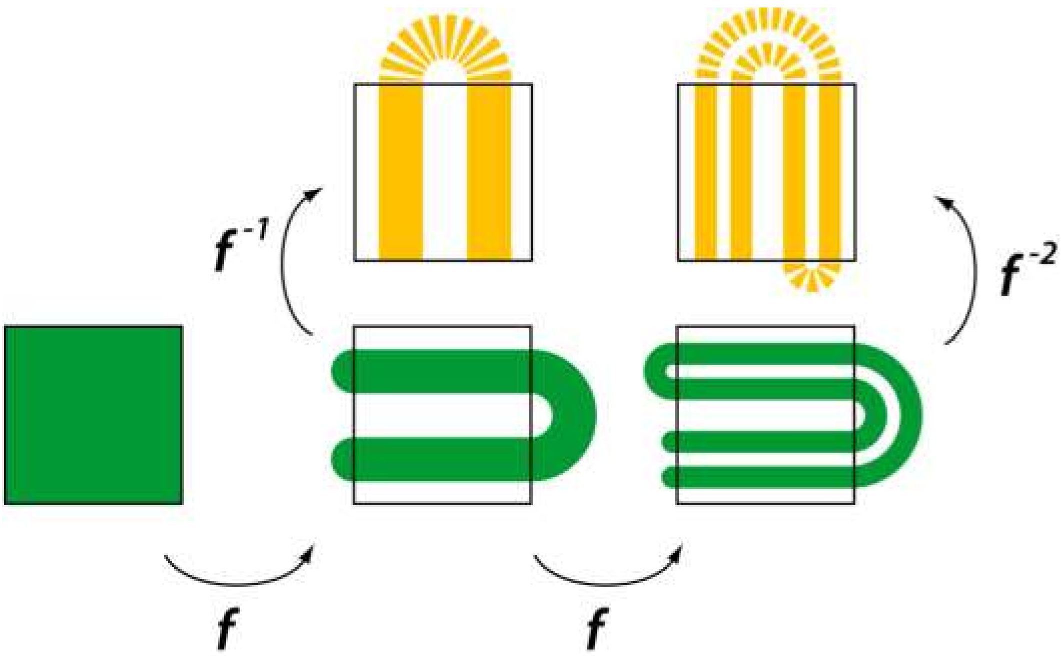


Figure (1.8): Generating a horseshoe.

Definition 1.2.23 [24]

Let $f: \mathfrak{I} \to \mathfrak{I}$ be a map has topological transitive, a subset $X \subset \mathfrak{I}$ has no isolated points this condition is equivalent to the existence of a point whose σ_1 −limit set is dense in X, then X is called **transitive set**.

Theorem 1.2.24 [7]

If Λ is hyperbolic set with non empty interior, then f is Anosov (Λ = M), satisfy:

- Λ is an transitive set.
- Λ is locally maximal.

Definition 1.2.25 [7]

A diffeomorphism $f: M \rightarrow M$ on a smooth compact manifold M, we denote the set of $\boldsymbol{C^r}$ **diffeomorphism** from M to M by $diff^r(M)$.

Definition 1.2.26 [7]

Let $f: M \rightarrow M$ be a diffeomorphism map on a closed manifold M and Λ be any f −invariant set. A continuous splitting

$$T_x M = E_1(x) \oplus \ldots \oplus E_k(x)$$

$x \in \Lambda$ of the tangent bundle over Λ is **dominated splitting** if it is invariant under the derivative Df.

Now, we will know another case of hyperbolic systems:

Definition 1.2.27 [7]

Let $f: M \rightarrow M$ be a diffeomorphism map on a closed manifold M, we say that f is a **partially hyperbolic** diffeomorphism map satisfy:

$$TM = E_x^u \oplus E_x^c \oplus E_x^s$$

where the subspaces E_x^s , E_x^u, E_x^c are stable, unstable and center subspaces at x respectively

- $D\,f(x)E_x^u = E^u f(x)$,
- $D\,f(x)E_x^c = E^c f(x)$
- $Df(x)E_x^s = E^s f(x)$,

for all unit vector $v^s \in E_x^s, v^c \in E_x^c$ and $v^u \in E_x^u$ we have

$$(Df(x)v^s) < 1 < (Df(x)v^u)$$

$$(Df(x)v^s) < (Df(x)v^c) < (Df(x)v^u)$$

that content:

- There is a dominant splitting.
- E^u is uniformly expanded , E^s is uniformly contracted, There is a non-trivial one of them.
- The set of partially hyperbolic diffeomorphism map is C^1-open in $diff^1(M)$.
- The **subspace** E^c, is spanned by the generalized eigenvector associated with the eigenvalues equal to one of A .

Example 1.2.28 [32]

Let $f: T^3 \to T^3$ be a partially hyperbolic (but not Anosov), and can be decomposed into two or three invariant subbundles.

- $TM = E^s \oplus E^c$, the one-dimensional uniformly hyperbolic space E^s, and the two-dimensional non-uniformly hyperbolic space E^c.
- $TM = E^u \oplus E^c$, the one-dimensional uniformly hyperbolic (expanding) space E^u, and the two-dimensional non-uniformly hyperbolic space E^c
- $TM = E^s \oplus E^c \oplus E^u$, the one-dimensional uniformly hyperbolic (contracting and expanding respectively) spaces E^s, E^u, and the one-dimensional non-uniformly hyperbolic space E^c .

Definition 1.2.29 [30]

A **foliation** is a partition of a subset of the ambient space into smooth submanifold, that one calls leaves, of the foliation, all with the same dimension and varying continuously from one point to the other.

Definition 1.2.30 [39]

We designate the **unstable**, **stable**, and **center foliation** manifolds for all x of Riemannian manifold M, by $\mathrm{W}^{\mathrm{u}}(x), \mathrm{W}^{s}(x)$and $\mathrm{W}^{\mathrm{c}}(x)$respectively, and for $\varepsilon > 0$, denote by $\mathrm{W}^{\mathrm{u}}_{\varepsilon}(x), \mathrm{W}^{s}_{\varepsilon}(x)$ and $\mathrm{W}^{\mathrm{c}}_{\varepsilon}(x)$ are the local leaves of size ε, that is,

- $\mathrm{W}^{\mathrm{u}}_{\varepsilon}(x) = \{\mathrm{y} \in \mathrm{W}^{\mathrm{u}}(x) | \mathrm{d}(f^{-\kappa}(x), (f^{-\kappa}(\mathrm{y})) < \varepsilon, \kappa \geq 0\}$

- $W_{\varepsilon}^{s}(x) = \{y \in W^{s}(x) | d(f^{\kappa}(x), (f^{\kappa}(y)) < \varepsilon, \kappa \geq 0\}$
- $W_{\varepsilon}^{c}(x) = \{y \in W^{c}(x) | d(f^{\kappa}(x), (f^{\kappa}(y)) < \varepsilon, \kappa \in \mathbb{Z}\}$

The conjugacy $\hbar$ image of E^{s} is a region close to the origin inside E^{s}, and it has the same dimensions as E^{s}. In addition, when time approaches infinity, a fixed point in W^{s}, the orbits of the points truly converge.

$y \in W^{s}(x)$ if and only if $f^{n}(y) \in W_{\varepsilon}^{s}(x), \forall n \geq 0$

Corollary 1.2.31 [20]

If a set Λ is hyperbolic under a diffeomorphism map f , then

- $E^{s}(x) = \cup_{k\geq 0} f^{-k}(E_{\varepsilon}^{s}(f^{k}(x)))$,
- $E^{u}(x) = \cup_{k\geq 0} f^{-k}(E_{\varepsilon}^{u}(f^{-k}(x)))$

Note 1.2.32

If f is Anosov diffeomorphism map, then E^{s}, E^{u} are uniformly contracted (expanded) respectively and $E^{c} = \{0\}$, but if f is partially hyperbolic diffeomorphism map then either $E^{s}$is uniformly contracted or E^{u} is uniformly expanded and $E^{c} \neq \{0\}$

Proposition 1.2.33 [34]

Let $f: M \to M$ be a map, on a manifold M, $f \in diff^r(M)$, for some $r \geq 1$, and $x, y \in M$ are hyperbolic periodic orbits having a cycle, then

- $E^u(x) \cap E^s(\mathrm{y}) \neq \emptyset$,
- $E^s(x) \cap E^u(\mathrm{y}) \neq \emptyset$.

Definition 1.2.34 [35]

Let $f: M \to M$ be C^r-diffeomorphism map, f is called **robustly transitive**, there exists a C^r-neighborhood $\mho(f)$ of f such that every $g \in \mho(f)$ is hyperbolic transitive map.

Example 1.2.35

See Example in [35], f is robustly transitive.

Corollary 1.2.36 [36]

Let f be a robustly transitive diffeomorphism map which has a hyperbolic periodic point $Per_h(f)$ with index one. Then one of the following properties holds:

- f could be Anosov diffeomorphism map.
- f could be a partially hyperbolic diffeomorphism map.

Definition 1.2.37 [24]

Let $f: M \to M$ be C^r-diffeomorphism map, f is called **robustly non hyperbolic transitive** map, there exists a C^r-neighborhood $\mho(f)$ of f such that every $g \in \mho(f)$ is non hyperbolic transitive map, denote by RNTD.

Example 1.2.38 [24]

See example (5.2) in [22], f is robustly non hyperbolic transitive.

In general, if $f \in diff(M)$ is robustly transitive map then f is transitive, vice versa is not true, but in [22], it has been proven that the opposite is true if f is transitive and uniformly hyperbolic then it is robustly transitive, see Fig (1.9). Indeed, if $f \in diff(M)$ is robustly transitive map and $\dim M \leq 3$ then f is partially hyperbolic diffeomorphism map.

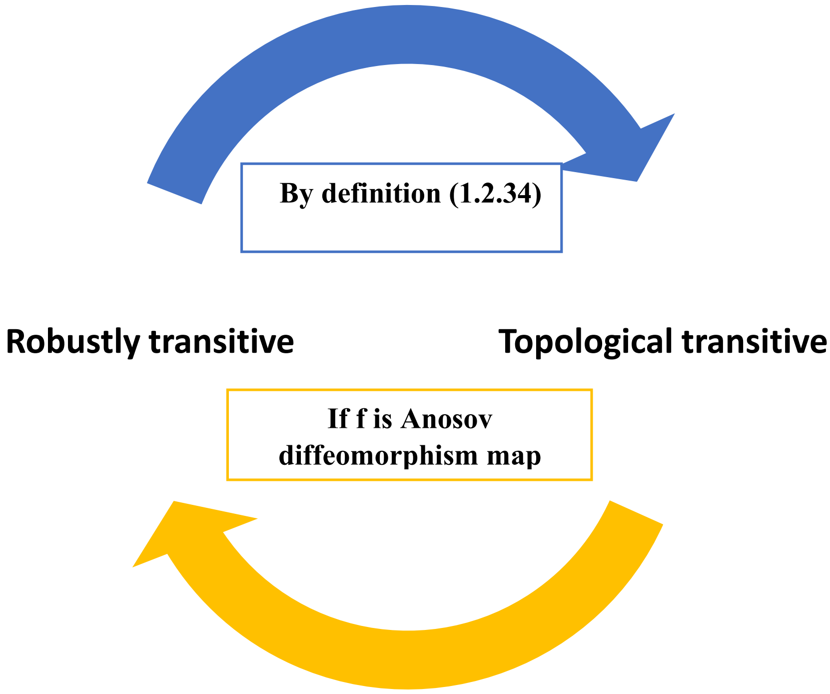


Figure (1.9): Relation between transitive and robustly transitive in $diff(M)$

Definition 1.2.39 [7]

Let $f: M \to M$ be a diffeomorphism map is called **strongly partially** hyperbolic if there is a Df −invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

such that E^s and E^u are uniformly contracting and uniformly expanding respectively, and both of them are not trivial. A diffeomorphism is hyperbolic if it is strongly partially hyperbolic and E^c is trivial, we say that E^c is the central direction of the splitting.

Definition 1.2.40 [30]

Let $f \in diff\ (M)$ be a map from a manifold M to itself and $g \in diff\ (M')$ be a map from a manifold M' to itself f, g are called **topologically conjugate** if there is a homeomorphism $\hbar: M \to M'$ such that $\hbar \circ f = g \circ \hbar$, a map $\hbar$ is called a **conjugacy**.

Example 1.2.41 [27]

Let $Q_\mu: C \to C$ be Quadratic map, where $Q_\mu = \mu x(1-x)$ when μ is sufficiently large. Recall that all points in $\mathbb{R}$ tend to $-\infty$ under iteration of Q_μ with the exception of those points in the Cantor set C, and the Shift map $\mathbb{S}: \Sigma_2 \to \Sigma_2$, if $\mu > 2+\sqrt{5}$ then there exist a homeomorphism $\hbar: C \to \Sigma_2$ satisfy $\hbar \circ Q_\mu = \mathbb{S} \circ \hbar$. A point x in C may be defined uniquely by the nested sequence of intervals, $\bigcap_{n\geq 0} I_{s_0 s_1 \dots s_n \dots}$ determined by $\hbar(x)$. Now

$$I_{s_0 s_1 \dots s_n} = I_{s_0} \bigcap Q_\mu^{-1}(I_{s_1}) \bigcap \dots \bigcap Q_\mu^{-n}(I_{s_n})$$

so that $Q_\mu(I_{s_0 s_1 \dots s_n})$ may be written

$$I_{s_1} \cap Q_\mu^{-1}(I_{s_2}) \cap \dots \cap Q_\mu^{-n+1}(I_{s_n}) = I_{s_1 \dots s_n} \text{ ,}$$

since $Q_\mu(I_{s_0}) = I$. Hence

$$\hbar \circ Q_\mu(x) = \hbar\, Q_\mu\left(\bigcap_{n=0}^{\infty} I_{s_0 s_1 \dots s_n}\right)$$

$$= \hbar\left(\bigcap_{n=1}^{\infty} I_{s_1\ldots s_n}\right)$$

$$= s_1 s_2 \ldots . = \mathbb{S} \circ \hbar(x)$$

Definition 1.2.42 [27]

A C^1diffeomorphism $f: M \to M$ is called **structurally stable**, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $g \in Diff^1(M)$ and $d_1(g, f) < \delta$, then there is a homeomorphism $\hbar: M \to M$ for which $f \text{o} \hbar = \hbar \text{o} g$ and $d_0(\hbar, Id) < \varepsilon$.

Example 1.2.43

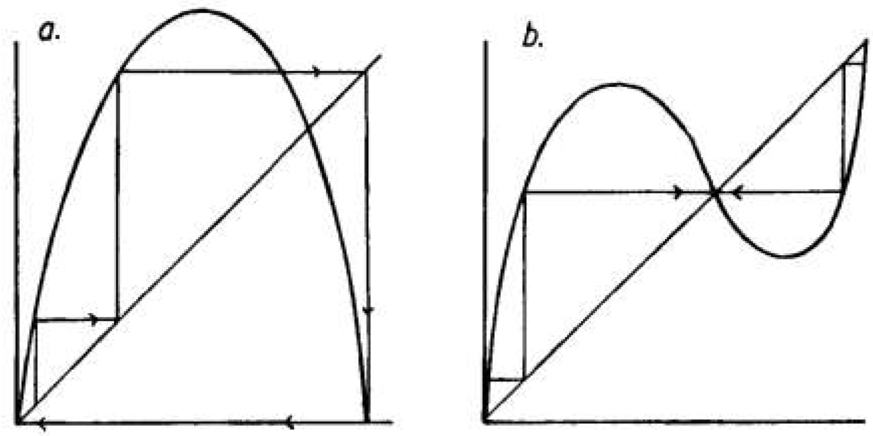
Let $f_A: T^n \to T^n$ be Anosov diffeomorphism map are structurally stable, see Theorem (2.1.10).

Definition 1.2.44 [26]

Let $x \in M$ be a periodic point for $f: M \to M$, where M is topological manifold. A **homoclinic point** to x is a point $y \neq x$ which lies in $E^s(x) \cap E^u(x)$, where E^s is stable set and E^u unstable.

Example 1.2.45 [36]

By Example (1.2.43), $Q_\mu(x) = \mu x(1-x)$, with $\mu > 4$, where $Q_\mu: [0,1] \to [0,1]$, Q_μ has two fixed points, 0 and $p_\mu = \frac{\mu-1}{\mu}$ both of which admit infinitely many homoclinic points, see Fig (1-10).



Figure(1-10): Illstrate the Quadratic map has infinitely many homoclinic points

If $\mu = 4$, $\frac{1}{2}$ lies on homoclinic orbit to 0, we see below that homoclinic orbits lead to chaotic behavior, at least on some invariant subset, while homoclinic orbits often lead to complicated bifurcations when a parameter is varied

Definition 1.2.46 [17]

Let $f: M \to M$ be a diffeomorphism of a smooth manifold and $diff(M)$ be the space of diffeomorphisms of M endowed with the C^1

topology. We say that a subset $\mathcal{R} \subset diff(M)$ is **residual** if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of $diff(M)$; in this case, $\mathcal{R}$ is dense in $diff(M)$.

Definition 1.2.47 [21]

We call a map $F: R^n \to R^n$ is **lift** of f if the canonical projection $pr_i: R^n \to T^n$ is onto, continuous (that is $pr_i of = Fopr_i$). For a given map $f: T^n \to T^n$ a **lift** $F: R^n \to R^n$ will not be unique.

Chapter Two

On Anosov Diffeomorphism Maps

This chapter discuses some general properties and topological metrical chaoticity properties of Anosov diffeomorphism maps $f_A: T^n \to T^n$ on the torus T^n, it plays nice and an important role in dynamical systems, where represent of more perfect type of global hyperbolic behavior it give examples of dynamical systems with structural stability.

The first section investigates general properties of Anosov diffeomorphism maps f_A from the n-torus to itself.

Section two presents topological and metrical chaoticity properties of Anosov diffeomorphism map.

2.1 General Properties of Anosov Diffeomorphism Maps

This work aims to study the general properties of Anosov diffeomorphism map. A diffeomorphisms on compact manifolds, we consider the case where an n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a diffeomorphism of a smooth Riemannian manifold M termed Anosov if and only if M is a hyperbolic set of f.

$F: \mathbb{R}^n \to \mathbb{R}^n$ induce of Anosov diffeomorphism map on T^n, then F_A be the lift of f_A , where F_A has a unique saddle fixed point in each square unit. We study a class of a map of the torus. Let A be an $n \times n$ matrix with integer entries. Then A acts linearly on $\mathbb{R}^n$, we can use this linear to define a map $f_A: T^n \to T^n$ on the torus T^n.

If det $(A) = \pm 1$, then det $(A^{-1}) = \frac{1}{\det(A)}$, A^{-1} is a gain matrix with integer entries, so it induces a well-defined map $f_A: T^n \to T^n$ then f_A is an invertible and the inverse is given by $(f_A)^{-1} = f_{A^{-1}}$.

We will present some lemmas and we prove some Theorems to show general properties of Anosov diffeomorphisms.

Proposition 2.1.1

If $f_A, f_B : T^n \to T^n$ are two maps that induce matrices by means of Anosov diffeomorphism maps then, $f_A \circ f_B$ again is Anosov diffeomorphism map .

Proof: Let $x \in T^n$; where $x = (x_i); i = 1, \dots n$ without losing generality of the proposition, if we take $n = 2$,

$$f_A \circ f_B \ (x_i) = f_A\big(B(x_i)\big) = AB(x_i);\ x_i \in T^n, i = 1, \dots n$$

Since f_A and f_B are two Anosov diffeomorphism maps,

det($A.B$)=det(A).det(B) then det(AB)= ± 1 ,we know that then $A\ and\ B$ have the interiors of AB are integer number

Let the eigenvalues of A be$|\lambda_i| < 1, |\lambda_j| > 1$, where

$i = 1, \dots, k\ and\ j = k + 1, \dots, n$

and the eigenvalues of B are

$$|\beta_i| < 1, |\beta_j| > 1, where\ i = 1, \dots k\ and\ j = k + 1, \dots, n$$

Then the eigenvalues of $A.B$ is$|\lambda_i\beta_i| < 1$and $|\lambda_j\beta_j| > 1$, such that

$$|\lambda_i|.|\lambda_j|.|\beta_i|.|\beta_j| = 1$$

Thus $f_A \circ f_B$ Anosov diffeomorphism map requirements must be met so, $f_A \circ f_B$ is Anosov diffeomorphism map. ■

Corollary 2.1.2

Given an Anosov diffeomorphism map $f_A: T^n \to T^n$, we say that f_A^m is also an Anosov diffeomorphism map, for all m.

Proof: Assume f_A^k is Anosov diffeomorphism map, $k = 1, \dots, n-1$
By Proposition (2.1.1), $f_A^{k+1} = f_A^k \circ f_A$, $k = 1, \dots, n-1$ is Anosov diffeomorphism map,
By Proposition (2.1.1) f_A^{k+1} is Anosov diffeomorphism map.
so, by Induction Law f_A^m is Anosov diffeomorphism map ■

Remark 2.1.3

- If f_A and f_B are both Anosov diffeomorphism maps, then $f_A + f_B$ need not be one as well.
- $f_A \circ f_B$ is not an Anosov diffeomorphism map if and only if f_A (or f_B) is an Anosov diffeomorphism map and f_B (or f_A) is not.

Example 2.1.4

Let A, B be two matrix assailed to Anosov diffeomorphism maps such that:

$$B = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

Since $\det(A+B) \neq \pm 1$, then $A+B$ does not induce of to Anosov diffeomorphism maps

If $C = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$ then $B \circ C$ also does not induce of to Anosov diffeomorphism maps.

Since $B \circ C = \begin{pmatrix} 14 & 23 \\ 18 & 30 \end{pmatrix}$

Proposition 2.1.5

If f_A and f_B are two Anosov diffeomorphism maps, then $f_A \times f_B$ is Anosov diffeomorphism map.

Proof: Let A and B be matrices such that

$$A = \begin{pmatrix} a_{11} & \dots\dots\dots\dots & a_{1n} \\ \vdots & \dots\dots\dots\dots & \vdots \\ a_{n1} & \dots\dots\dots\dots & a_{nn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \dots\dots\dots\dots & b_{1n} \\ \vdots & \dots\dots\dots\dots & \vdots \\ b_{n1} & \dots\dots\dots\dots & b_{nn} \end{pmatrix}$$

of f_A , f_B respectively, since f_A *and* f_B are both Anosov diffeomorphism maps so the eigenvalues of f_A and f_B is not equal to 1. The eigenvalues of A is $|\lambda_i| = \frac{1}{\prod_{\substack{j=1 \\ j\neq i}}^{n} |\lambda_j|} > 1, where\ i = 1, \dots\dots, n$

and the eigenvalues of B is $|\beta_i| = \frac{1}{\prod_{\substack{j=1 \\ j\neq i}}^{n} |\beta_j|} > 1, where\ i = 1, \dots\dots, n$ the eigenvalues of $A \times B$ not equal one , then the map from f_A to f_B is an Anosov diffeomorphism map . ■

Note 2.1.6

If f_A is an Anosov diffeomorphism map, then f_A^{-1} is again Anosov diffeomorphism map.

Since $A \in GL(n,\mathbb{Z})$ and f is diffeomorphism map, then $A^{-1} \in GL(n,\mathbb{Z})$

Since f and f^{-1} are bijective and continuous

Then f_A^{-1} is also Anosov diffeomorphism map.

Proposition 2.1.7 [21]

Let $f_A : T^n \to T^n$ be Anosov diffeomorphism map of T^n with corresponding matrix A has λ_i, where $i = 1, \ldots\ldots, n$, then the set of rational points of f_A correspond precisely with the set of rational points of T^n.

$\{\left(\frac{p_i}{q}\right)_{i=1}^{n} + \mathbb{Z}^n, p_i, q \in \mathbb{N} \; and \; 0 \le p_i < q\}$

Proposition 2.1.8

Let $f_A : T^n \to T^n$ be Anosov diffeomorphism map, then F_A is the lift of f_A, F_A is not expanding.

Proof: Since f_A is Anosov diffeomorphism map then the Jacobean matrix of f_A have $\det(F_A) = \pm 1$ such that, $\prod_{i=1}^{n} |\lambda_i| = 1$

There exist $k = 1, \ldots, j$ $|\lambda_k| = \frac{1}{|\lambda_{k+1}|} \neq 1$, $|\lambda_k|$ less than one

and $k = j+1, \ldots, n$, $|\lambda_k|$ more than one,

then by Definition (1.1.13), F_A is not expanding. ■

Lemma 2.1.9

The lift of f_A is not contracting if and only if $f_A : T^n \to T^n$ is an Anosov diffeomorphism map, as required by the Definition (1.1.13).

Proof: The proof of Lemma has the same of Proposition (2.1.8), as required by the Definition (1.1.13).

Theorem 2.1.10

If $f_A : T^n \to T^n$ is Anosov diffeomorphism of T^n, then f_A are structural stability .

Proof: Let $x \in T^n$, since F_A is diffeomorphism such that $F_A \approx_{\hbar} F_B$ and Let $\mho(x)$ be neighborhood of x, we claim that any $\delta > \frac{1}{2}$ and we will defined $d(f_A, f_B) = sup_{x \in T^n} \|f_A(x) - f_B(x)\|$ then we must have $f_B(x) \in \mho(x)$ we mean that is neighborhood of the fixed point and δ such that if a map F_B is $c - \delta$ closed to F_A on this neighborhood then F_A is topologically conjugate to F_B on this neighborhood, by definition of F_A contain n-eigenvalues $|\lambda_k| > 1$where $k = 1, \dots, j$ and $|\lambda_k| < 1$where $k = j + 1, \dots, n$

Let $$B = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & & \cdots & \lambda_n \end{pmatrix}, F_A^n \approx F_B^n,$$

$$F_A^n(x) = E^{-1} F_B^n(x) E$$

$$B^n = \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & & \cdots & \lambda_n^n \end{pmatrix},$$

$$B^n E(v) = \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & & \cdots & \lambda_n^n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda_1^n x_1 \\ \vdots \\ \vdots \\ \lambda_n^n x_n \end{pmatrix}$$

Since $|\lambda_k| > 1$, where $k = 1, \dots, j$ and $|\lambda_k| < 1$, where $k = j + 1, \dots, n$

we find that $\|B^n E(v)\| = \left\| \begin{pmatrix} \lambda_1^n x_1 \\ \vdots \\ \vdots \\ \lambda_n^n x_n \end{pmatrix} \right\| = \sqrt{\sum_{i=1}^{n} \lambda_i^{2n} x_i^2} \to 0$

Then $f_A^n(v) = E^{-1}(B^n E(v)) \to 0$

That is $d(f_A^n(x), f_B^n(x)) < \delta$, for all $\delta > 0$

Then F_A structural stability. ■

Remark 2.1.11

Since $F_A: \mathbb{R}^n \to \mathbb{R}^n$ is induce Anosov diffeomorphism map of T^n, we refer to $f_A: T^n \to T^n$ as having structural stability.

2.2 Some Topological and Metrical Chaoticity Properties of Anosov Diffeomorphism Map

In this section, some topological and metrical properties of chaoticity are studied for Anosov diffeomorphisms maps.

Theorem 2.2.1 [11]

If a map $f_A : T^n \to T^n$ is Anosov diffeomorphisms map, then the lift of f_A is F_A , each non eventually periodic point in its domain has a positive Lyapunov exponent.

Theorem 2.2.2

Let $f_A : T^n \to T^n$ be Anosov diffeomorphism map and F_A be the lift of f_A .Then, the set of F_A periodic points is a dense set.

Proof: Let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{p_1}{q} \\ \vdots \\ \frac{p_n}{q} \end{pmatrix} \in \mathbb{R}^n$, has rational coordinates such that

$p_i \in \mathbb{N}$,$q \in \mathbb{N}/\{0\}$, $0 < p_i < q$, then $F_A^n(x_i) = \frac{p_i^n}{q}$ where p_i^n are integers ,$0 < p_i^n < q$, for all $i = 1, \dots, n$ which are given by

$$\frac{p_i^n}{q} = A^n(\frac{p_i^n}{q}) \bmod \mathbb{Z}^n$$

Since there are at most q^nchoices for (p_i^n) there exists

$0 < \ell \neq n \leq q^n + 1$ such that $F_A^n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = F_A^{\ell} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ this means x_i is eventually periodic and

since f_A is an invertible, this implies that x_i is periodic point ,

Conversely, if x_i is a periodic point , then there is n such that

$F_A^n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

So by definition of F_A this means that there exists $l_i \in \mathbb{Z}$, for all $i = 1, \dots, n$ such that

$$A^n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix},$$

Then $(A^n - I) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}$ where

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

Let us check that $(A^n - I)$ is an invertible since f_A is hyperbolic.

A has no eigenvalues equal to 1, so A^n has no eigenvalues equal to 1.

So, the solution vector v cannot be non zero. $(A^n - I)v = 0$ and

This shows that $(A^n - I)$ is an invertible.

Hence, we can solve $\begin{pmatrix} x_1 \\ \vdots \\ x_i \end{pmatrix} = (A^n - I)^{-1}$, for all $i = 1, \ldots, n$,since A has entries in $\mathbb{Z}$, $(A^n - I)$ has entries in $\mathbb{Z}$ and $(A^n - I)^{-1}$ has entries in $\mathbb{Q}$,

So any x_i are rational numbers, for all $= 1, \ldots, n$.

Since the set of rational numbers in $\mathbb{R}^n$ are dense. ■

Remark 2.2.3

Since $\mathrm{F}_A: \mathbb{R}^n \to \mathbb{R}^n$ induces an Anosov diffeomorphism map on T^n, thus we say the set of all periodic points of f_A are dense.

Proposition 2.2.4

Let $f_A : T^n \to T^n$ be an Anosov diffeomorphism map. Then the homoclinic points are dense in T^n .

Proof: We assert that E^s is a line with an irrational slope in $\mathbb{R}^n$, assuming that E^s is the stable subset in $\mathbb{R}^n$.

Since A is an integer matrix, E^s must go through a point with coordinates $(x_i)_{i=1}^n$since $x_i \in \mathbb{Z}$. This means that every iteration of $(x_i)_{i=1}^n$ will have integer coordinates. However, given $f^n(x_i)_{i=1}^n \longrightarrow 0$ as $n \longrightarrow \infty$, this is impossible.

The x-coordinate of the E^s and $\mathrm{y} = \mathrm{z}$ point is x_n.

Keep in mind that the irrational slope of E^s is the reciprocal of x_1.

Since x_2=2x_1, and $x_{\mathrm{n}} = \mathrm{n}_{x_1}$ in general, the projection of $(x_{\mathrm{j}}, \mathrm{j})$ onto the plane is $[\ell_{\mathrm{j}}, 0]$ for any j such that $0 \leq \ell_{\mathrm{j}} < 1$ are the iterative pictures of [0] under an irrational translation of the circle defined by the line y=0 in *T*.

By Definition (1.1.12) these points are dense in T^n.■

Theorem 2.2.5

Let $f_A: T^n \to T^n$ be an Anosov diffeomorphism map, then f_A is topologically transitive.

Proof: Let $\mho$ and $\mathcal{V}$ be open subsets of T^n,

by Proposition (2.2.4), there exists x a homoclinic point and $i > 0$, since the set of a homoclinic points is dense such that

$f_A^i(\mathrm{u}) = x$ where $\mathrm{u} \in \mho$, $f_A^k(w) = x$ where, $\mathrm{k} > i$ and $w \in \mathcal{V}$

$f_A^i(u) = x = f_A^k(w)$.

Choose $\mathrm{m} = i + \mathrm{k}$ such that $x \in f_A^m(\mho) \cap \mathcal{V} \neq \emptyset$

Then by Definition (1.1.15), f_A is topological transitive. ■

Theorem 2.2.6

If $f_A : T^n \to T^n$ is an Anosov diffeomorphism map and F_A be the lift of f_A then F_A has sensitive dependence on initial condition.

Proof: By Theorem (2.2.2), Per (f_A) is dense in T^n

Let $x_i, \forall i = 1, \ldots, n.$ be periodic points , and

$\mho$ be an open set such that $x_i \in \mho, \forall i = 1, \dots, n$

$$\bigcap_{i=1}^{n} \mho(x_i) = \emptyset$$

Let x_1, x_2 be two periodic points such that

$$\delta_D = d(\mho(x_1), \mho(x_2)), \text{ where } \delta = \frac{\delta_D}{8}, \delta_D > 0$$

and for every $y \in T^n$ other

$$d\left(y, \mathbb{Q}_{f_A}(x_1)\right) > \frac{\delta_D}{2}$$

Assume $\mho$ any open set that includes y and $\mathbb{N}_\delta(y)$ is neighborhood of y with radius δ

Let z be a periodic point in $\mathcal{H} = \mho \cap \mathbb{N}_\delta(y)$ with n-period from this we conclude that one of the point x_1, x_2 dented$x$ has an orbit for which

$$d\left(y, \mathbb{Q}_{f_A}(x)\right) > 4\delta, \text{ where } \mathbb{Q} \text{ are rational numbers}$$

Let us define $V = \bigcap_{i=1}^{n} f(\mathbb{N}_\delta(f^i(x))$, where V is non-empty set, because $x \in V$, and V is an open.

From Theorem (2.2.5), of f_A then there exists $p \in \mathcal{H}$ and k is integer number such that $f_A^k(p) \in V$

Let I be integer part of $\frac{k}{n} + 1$.

Consequently $\frac{k}{n} + 1 = \mathrm{I} + r$, where r is the rest, $0 \le r \le 1$

Then $n\,\mathrm{I} - k = n - rn$, it follows that $0 < n\,\mathrm{I} - k \le n$ by construction

$$f_A^m(p) = f_A^{m-k}\left(f_A^k(x)\right) \in f_A^{m-k} \subset \mathbb{N}_\delta(f_A^{m-k}(x)), \text{ where } m > k.$$

Let $a = f_A^m(p)$ and $b = f_A^{m-k}(x)$, note that

$$d(a,b) < \delta$$

Let use the Triangle Inequality for points z, a, b and $y, z, b \in \mathbb{Z}$

$$d(z,b) \leq d(z,a) + d(a,b),$$

$$d(y,b) \leq d(y,z) + d(z,b)$$

where

$$d(y,b) \leq d(y,z) + d(z,a) + d(a,b) \text{ or}$$

$$d(z,a) \geq d(y,b) - d(y,z) - d(a,b)$$

By construction

$$d(y,b) = d\big(y, f_A^{m-k}(x)\big) \geq d\big(y, \mathbb{Q}_{f_A}(x)\big) \geq 4\delta$$

Since $z \in \mathbb{N}_\delta(y)$, then $d(y,z) < \delta$

It follows that

$$d(z,a) > 4\delta - \delta - \delta \text{ or}$$

$$d(f_A^m(z), f_A^m(p)) > 2\delta$$

Applying the Triangle Inequality to the following point $f_A^m(y), f_A^m(z), f_A^m(p)$ we get that

$$d(f_A^m(y), f_A^m(z)) > \delta \text{ or}$$

$$d(f_A^m(y), f_A^m(p)) > \delta \quad \blacksquare$$

Lemma “Franks” 2.2.7 [37]

Let $f \in diff(M)$ and let $\mho(f)$ be in Definition (1.2.42), then

$\varepsilon > 0$, there is $\delta > 0$ such that for a finite set $\{y_i\}$ where $i = 1, \ldots., N$ a neighborhood U of $\{y_i\}$ and linear map $\xi: T_{y_i}M \to T_{f(y_i)}M$ satisfying $d(\xi, D_{y_i}f) < \delta$, for all $1 \le i \le N$ and $g \in \mho(f)$, such that

- $g(\mathrm{y}) = f(y)$ if $y \in M \setminus$ U and
- $g(\mathrm{y}) = exp_{f(y_i)} \circ \xi \circ exp_{(y_i)}{}^{-1}(y)$ if $y \in B_\varepsilon(y_i)$, for all

$1 \le i \le N$, observe that assertion implies that $g(\mathrm{y}) = f(y)$ if $y \in \{y_1, \ldots, y_N\}$ and that $D_{y_i}g = \xi$, for all $1 \le i \le N$.

Corollary 2.2.8

By Theorem (2.2.5), f_A is a topological transitive and by Theorem (2.2.1), f_A has positive Lyapunov exponents , that is, f_A is chaos according to Definition (1.1.39).

Proposition 2.2.9

Let $f_A: T^n \to T^n$ be an Anosov diffeomorphism map. Then the map f_A satisfy totally transitive.

Proof: Let $\mho$ and $\mathcal{V}$ be subsets of T^n, if $\mho \cap \mathcal{V} \neq \emptyset$ then it is trivial.

Now if $\mho \cap \mathcal{V} = \emptyset$ since f_A is a topological transitive

and $\mho \in T^n$ then $f_A(\mho) \subset T^n$,

f_A^2 is Anosov diffeomorphism map, then f_A^2 is topological transitive

so, $f_A\big(f_A(\mho)\big) \cap \mathcal{V} \neq \emptyset$, by Induction Law $f_A^k(\mho) \cap \mathcal{V} \neq \emptyset$,

for all $\mathrm{k} \in \mathbb{N}$ thus f_A is a totally transitive. ■

Note 2.2.10

The converse holds true for maps with f_A Anosov diffeomorphism map by Proposition (2.2.9), every totally transitive is topological transitive by Definition (1.1.22), illustrate that in Fig (2.1).

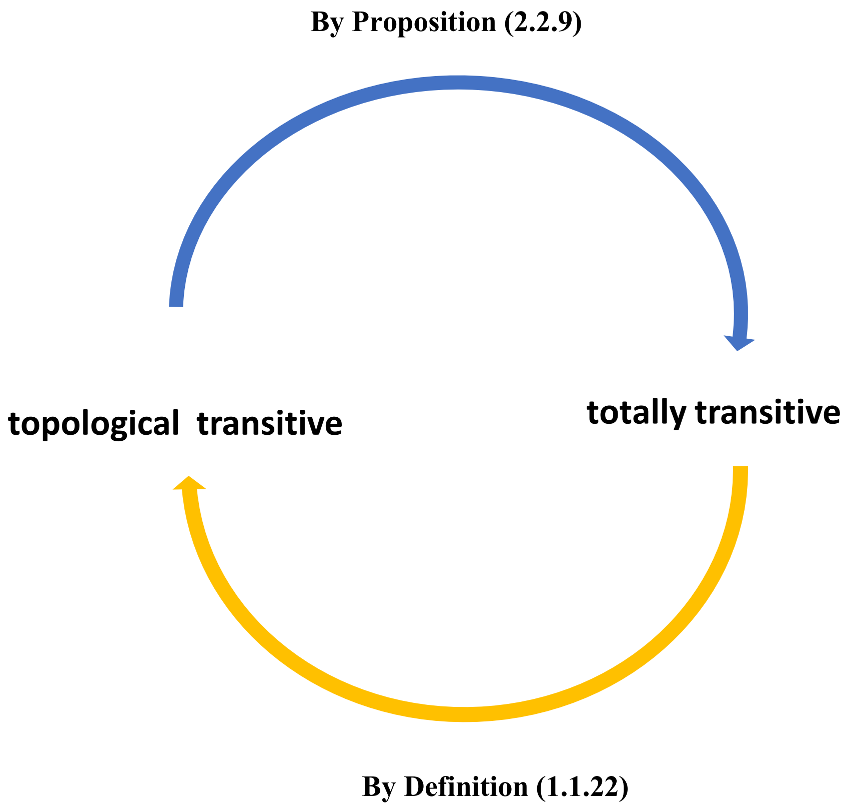


Figure (2.1): The relation between topological transitive and totally transitive.

Through previous theories, we showed that the Anosov diffeomorphism map achieves several definitions of chaos according to its gradation from the weakest to the strongest from definitions of chaos, which is the definition of Devaney.

Corollary 2.2.11

By Theorem (2.2.6), Anosov diffeomorphism map has sensitive dependence on initial condition and has positive Lyapunov exponent at each point in its domain that is not eventually periodic by Theorem (2.2.1) then f_A satisfy the chaos by Definition (1.1.38), that is , f_A is chaos in sense of Gulick.

Corollary 2.2.12

The Anosov diffeomorphism map is W-chaotic corresponding to Definition (1.1.40), since by Theorem (2.2.5) Anosov diffeomorphism map is topological transitive and Theorem (2.2.6) Anosov diffeomorphism map has sensitive dependence on initial conditions.

Theorem 2.2.13

Let $f_A: T^n \rightarrow T^n$ be an Anosov diffeomorphism map. Then f_A is a chaotic in sense of Touhey

Proof: Let $\mho$ and $\mathcal{V}$ be any two open sets in T^n, that the preimages under pr_i of $\mho$ and $\mathcal{V}$ in T^n are open in $\mathbb{R}^n$.

Let x be an periodic point with m-period , we will produce $f_A^{\ell}(x) \in \mathcal{V}$ for all a point $x \in \mho$ and an integer ℓ , where $\ell < m$.

We may select points $y \in \mho$ and $z \in \mathcal{V}$ that are homoclinic to 0.

Now, let $\varepsilon > 0$, choose an open interval $J_u$of length $\delta > 0$ in $E^u(0)$ and containing y.

Similarly, choose J_s containing z in $E^s(0)$

f_A^k expands J_u by a factor of $|\lambda_u|^k$ and f_A^{-k} expands J_s by the same factor, we choose k large enough so that:

- $d(f_A^k(y), 0) < \frac{\varepsilon}{2}$
- $d(f_A^{-k}(z), 0) < \frac{\varepsilon}{2}$
- $|\lambda_u|^k \, \delta > \varepsilon$

Here, d is the distance of the Euclidean distance defined in neighborhood of 0.

Since $f_A^{-k}(J_s)$ and $f_A^k(J_u)$ are parallel to $E^s(0)$ and $E^u(0)$ respectively, it follows that

$$f_A^k(J_u) \cap f_A^{-k}(J_s) \neq \emptyset$$

Let x^* be a point in this intersection.

Then $x = f_A^k(x^*) \in \mho$ and $f_A^k(x^*) \in \mathcal{V}$.

Consequently $f_A^{2k}(x) \in \mathcal{V}$, giving the required point.

Where $\ell > m$, let x' be an periodic point with m -period, there is

$$\alpha \ni \ell = \alpha + m, \qquad \alpha < m$$

$$f_A^\ell(x') = f_A^\alpha f_A^m(x')$$

$$f_A^\ell(x') = f_A^\alpha\,(x')$$

Then $f_A^\ell(x') \in \mathcal{V}$. ■

Proposition 2.2.14

Let $f_A: T^n \to T^n$ be Anosov diffeomorphism map, then f_A is topologically mixing.

Proof: for each $x \in T^n$, the unstable manifold $E^u(x)$ of f_A is dense in T^n.

For every $\varepsilon > 0$, the collection of balls of radius ε centered at point of $E^u(x)$ covers T^n, by compactness a finite subcollection of these balls also covers T^n,

hence, there is $\ell(\varepsilon) > 0$ such that every set X of length $\ell(\varepsilon)$ in an unstable manifold is ε −dense in T^n, that is

$$d(y, X) \leq \varepsilon, \text{ for all } y \in T^n$$

Let $\mho$ and $\mathcal{V}$ be non-empty open sets in T^n,

choose $y \in \mathcal{V}$ and $\varepsilon > 0$ such that $\overline{B(y,\varepsilon) \subset \mathcal{V}}$.

The open set $\mho$ contains a set of length $\delta > 0$ in some unstable manifold $E^u(x)$.

Let λ_i, $|\lambda_i| > 1$, for all $i = 1, \dots, j$ be the expanding eigenvalue of f_A, and choose $k > 0$ such that $|\lambda_i|^k \delta \geq \ell(\varepsilon)$.

Then for any $n \geq k$, the image $f_A^n(\mho)$ contains a set of length at least $\ell(\varepsilon)$ in some unstable manifold, so $f_A^n(\mho)$ is ε −dense in T^n.

Therefore $f_A^n(\mho) \cap \mathcal{V} \neq \emptyset$. ■

Note 2.2.15

In general, every topological mixing is topological transitive , but not the opposite, however when f_A is Anosov diffeomorphisms, we say the converse hold, see Fig (2.2).

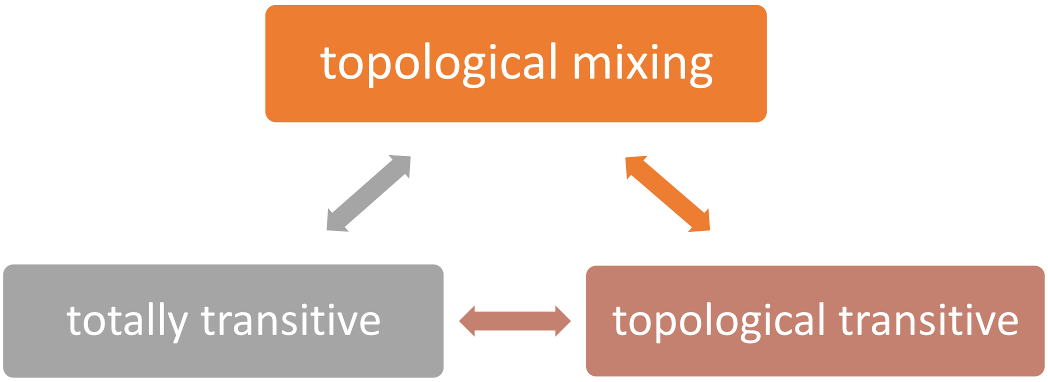


Figure (2.2): The relations between topological mixing, totally transitive and topological transitive of Anosov diffeomorphism map.

Remark 2.2.16

By Definition (1.1.20), we say the map is minimal if each orbit is dense, since the periodic orbit is a finite set, so the map has no periodic points.

Proposition 2.2.17

Let $f_A : T^n \to T^n$ be Anosov diffeomorphism map, then f_A is not minimal.

Proof: By Theorem (2.2.2) the set of periodic points of Anosov diffeomorphism map are dense
Then by Definition (1.1.20) f_A is not minimal. ■

We want to get relation between the sets $Per(f_A), AP(f_A), R(f_A), \Omega(f_A)$ denoted the set of periodic points, the set of almost periodic points, recurrent points , non-wandering points of f_A respectively of Anosov diffeomorphisms map, every periodic point is non-wandering point for any map.

In [21] Jong and Seong proved that relation between the sets for any map f :

$$\text{Per}(f) \subseteq AP(f) \subseteq R(f) \subseteq \Omega(f) \ldots\ldots\ldots (2.1)$$

Now, to prove the non-wandering set Ω of Anosov diffeomorphism map are equal in the space T^n.

Proposition 2.2.18

Let $f_A: T^n \to T^n$ be Anosov diffeomorphism map, then the set of all non-wandering point equal to T^n

Proof: By equation (2.1), $\Omega(f_A) \subseteq T^n$ now, we want to prove

$T^n \subset \Omega(f_A)$,

Let $x \in$ T then there is Ʊ open set such that $x \in$ Ʊ, By Theorem (2.2.13), f_A is chaotic in sense of Touhay ,

Thus for all $\mho \subset T^n$, there is k ∈ ℕ and

y ∈ $\mho$ a periodic point of period m, for all f_A^k(y) ∈ $\mho$

then $f_A^k(\mho) \cap \mho \neq \emptyset$

$f^{\text{mk}}(\mho) \cap \mho \neq \emptyset$, and so $T^n \subseteq \Omega(f_A)$

That is $\Omega(f_A) = T^n$. ■

Theorem 2.2.19 [38]

f is pointwise non wandering map if each point is non-wandering by Definition (1.1.35), on compact metric space to itself, f^n is pointwise non- wandering map for each $n \geq 1$.

Proposition 2.2.20

If f_A is Anosov diffeomorphism map then $\Omega(f) = \Omega(f_A^n)$.

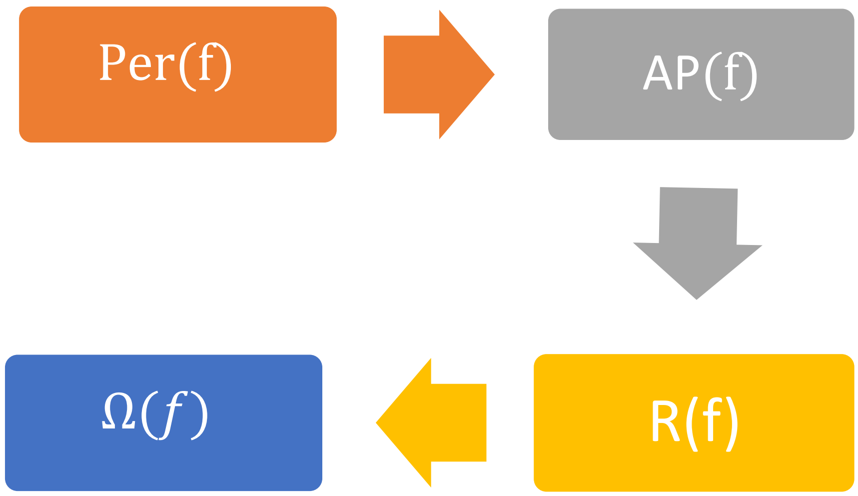
Proof: By Proposition (2.2.18), $\Omega(f_A) = T^n$ and f_A is Anosov diffeomorphism map then by Definition (1.1.35) , f_A is pointwise non-wandering and by Theorem (2.2.19), f_A^n is pointwise non-wandering then $T^n = \Omega(f_A^n)$ therefore $\Omega(f_A) = \Omega(f_A^n)$. ■

Lemma 2.2.21 [21]

On compact metric space if f_A is an equicontinuous then $\Omega(f_A) = AP(f_A)$.

Proposition 2.2.22 [21]

A continuous map $f: S^1 \to S^1$ of the circle. Then



Proposition 2.2.23

If f_A is Anosov diffeomorphism map then $\Omega(f_A) = AP(f_A)$

Proof: By Lemma (2.2.21) and since f_A is equicontinuous on T^n and by Proposition (2.2.22), then

$\text{Per}(f_A) \subset AP(f_A) = \Omega(f_A) = T^n$.■

Proposition 2.2.24

If f is Anosov diffeomorphism map then $\Omega(f_A) = \text{R}(f_A)$

Proof: By Proposition (2.2.23) and by Proposition (2.2.18), $\Omega(f_A) = T^n$ we get

$\mathrm{R}(f_A) = T^n$ thus $\Omega(f_A) = R(f_A)$.

Proposition 2.2.25

If $f_A: T^n \to T^n$ is Anosov diffeomorphism map then

$\mathrm{AP}(f_A) = \mathrm{AP}(f_A^n)$

Proof: By Proposition (2.2.23), $\Omega(f_A) = \mathrm{AP}(f_A) = T^n$ and since

f_A is Anosov diffeomorphism map

f_A^n is continuous, and by Theorem (2.2.19)

Then

$$\Omega(f_A^n) = \mathrm{AP}(f_A^n)\ , \forall\ n \in \mathbb{N}$$

Therefore $\mathrm{AP}(f_A) = \mathrm{AP}(f_A^n)$. ■

Corollary 2.2.26

Since Anosov diffeomorphism map is transitive by Theorem (2.2.5), sensitive dependence on initial condition by Theorem (2.2.6), and the set of all periodic point are dense by Theorem (2.2.2) then the Anosov diffeomorphism map satisfy Devaney definition.

Chapter Three

Some Types of Shadowing Property in Hyperbolic Dynamical Systems

This chapter aims to investigate some sorts of shadowing property in hyperbolic dynamical systems.

In the first section, we discuss the fitting shadowing property for hyperbolic set, under sufficient conditions, we can achieve the fitting shadowing property for non- hyperbolic set.

The two section deals with the eventual fitting shadowing property for hyperbolic set.

The three section investigate the strongly fitting shadowing property for hyperbolic set.

3.1 Fitting Shadowing Property for Hyperbolic Set

we recall the definitions and we formulate new definitions for later application for different types of the shadowing property:

Definition 3.1.1

Let $(\mathfrak{I}, d)$ be a metric space, and $f: \mathfrak{I} \to \mathfrak{I}$ be a map, a sequence $\{y_\kappa\}_{\kappa=0}^{\infty} \in \mathfrak{I}$ is called a **δ –fitting pseudo-orbit** of f, if there exists $\delta > 0$ and positive integer $\aleph = \aleph(\delta)$ such that for all m$\geq \aleph$ and $\kappa \in \mathbb{N}$, we have

$$\sum_{\kappa=0}^{m-1} d(f(y_\kappa), y_{\kappa+1}) < \delta .$$

Definition 3.1.2

A map f has the **fitting shadowing property**, denote (FSP) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ –fitting pseudo-orbit is ε −shadowed in fitting by the orbit of some point $z \in \mathfrak{I}$, that is

$$\lim_{m\to\infty} sup \sum_{\kappa=0}^{m-1} d(f^{\kappa}(z), y_{\kappa}) < \varepsilon$$

Let M be a closed C^{∞} manifold, and f be a diffeomorphism of M, let $\Lambda \subset$ M be a closed f-invariant set, then denote by $f|_{\Lambda}$ the restriction of f to a set Λ.

Let U $\subset M$ be a compact neighborhood of Λ and

$$\Lambda_f(\text{U}) = \bigcap_{n\in\mathbb{Z}} f^n(\text{U})$$

Let $f \in diff(M)$, and the set of all periodic points of f is $Per(f)$, symbolize for $O_f(x)$, if the periodic f-orbit of $x \in Per(f)$ type of hyperbolic saddle with period $\epsilon(x) > 0$, then there are the local stable (unstable) manifolds $E^s_{\epsilon}(x)$, $E^u_{\epsilon}(x)$ of x respectively for some

$\epsilon = \epsilon(x) > 0$.

In this work, we determine that $E^s(x)$ is the dimension of the stable manifold , we symbolizes it by index(x).

"Shadowing Lemma" 3.1.3 [14]

If Λ is a hyperbolic set for f, then there is a neighbourhood $\mho$ of Λ which has shadowing property.

Here C^1- stable means that the shadowing property under consideration is preserved by C^1- perturbation of the original map.

Every transitive Anosov diffeomorphism map of M is topologically mixing if M is connected,. Thus we have the following corollary.

Corollary 3.1.4 [14]

Transitive Anosov diffeomorphisms (TAD) are defined as the set of diffeomorphisms of M satisfying the C^1- stable shadowing property (SSP).

Proposition 3.1.5

Let $x, y \in \Lambda \cap \mathrm{Per}(f)$ be hyperbolic saddles points. If $f|_\Lambda$ has the stable fitting shadowing property (SFSP), then

$$E^s\left(O_f(x)\right) \cap E^u\left(O_f(y)\right) \neq \emptyset.$$

Proof: Let $x, y \in \Lambda \cap \mathrm{Per}(f)$ be hyperbolic saddles points, and let $\epsilon(x)$ and $\epsilon(y) > 0$.

Put $\epsilon = \min\{\epsilon(x), \epsilon(y)\}$. And let $\aleph = \aleph(\epsilon) \ > 0$ be the number of the stable fitting shadowing property SFSP of $f|_\Lambda$, there is $x_i \in \Lambda$ such that

i. $d(f^k(x_i), f^k(f^{-i}(x)) \leq \epsilon \,, for \;\; 0 \leq k \leq i \,,$

ii. $d(f^k(x_i), f^k(f^{-\aleph-i}(y)) \leq \epsilon, for\ \aleph + i \leq k \leq \aleph + 2i$

implies (i) that

$$d\left(f^{-n}\left(f^i(x_i)\right), f^{-n}(x)\right) \leq \epsilon, for\ \ 0 \leq n \leq i$$

and (ii) implies

$$d\left(f^n\left(f^\aleph\left(f^i(x_i)\right)\right), f^n(y)\right) \leq \epsilon, for\ \ 0 \leq n \leq i$$

put $z_i = f^i(x_i)$ and let $z = \lim_{i\to\infty} z_i$ if necessary, we take a subsequene if necessary. Then

since $$d(f^{-k}(z_i), f^{-k}(x)) \leq \epsilon \leq \epsilon(x),$$

and

$$d(f^n(f^\aleph(z_i), f^n(y)) \leq \epsilon \leq \epsilon(y) \text{ for } 0 \leq n \leq i$$

we have that

$$z \in E^u_{\epsilon(x)}(x) \subset E^u(x) \text{ and } f^\aleph(z) \in E^s_{\epsilon(y)}(y), \text{ that is,}$$

$$z \in E^s\left(f^{-\aleph}(y)\right).$$

Hence

$z \in E^u(x) \cap E^s\left(f^{-\aleph}(y)\right) \subset E^u\left(O_f(x)\right) \cap E^s\left(O_f(y)\right)$.

$E^s\left(O_f(x)\right) \cap E^u\left(O_f(y)\right) \neq \emptyset$. ∎

Definition 3.1.6 [6]

A diffeomorphism map f is said to be **Kupka-Smale** if the periodic points of f are hyperbolic, and if $x, y \in Per(f)$, then $E^s(x)$ is transversal to $E^u(y)$. The set of all Kupka-Smale diffeomorphism map is C^1-residual in $diff(M)$.

Definition 3.1.7 [6]

A set Λ is locally maximal in U if there is a compact neighborhood U of Λ such that Λ =Λ_f(U), we say that $f|_{\Lambda_f(\mathrm{U})}$ fulfills the C^1-stable fitting shadowing property (C^1-SFSP), if there are a compact neighborhood U of Λ and a C^1- neighborhood $\mho(f)$ of f such that Λ is locally maximal in U and for any $g \in \mho(f)$, $g|_{\Lambda_g(\mathrm{U})}$ satisfies the fitting shadowing property. Here

$$\Lambda_g(U) = \cap_{n \in \mathbb{Z}} g^n(\mathrm{U}) \text{ ....} \qquad (3.1)$$

is called the **continuation** of $\Lambda_f(U) = \Lambda$, in the case $\Lambda = \mathrm{M}$, we just say that f satisfies the C^1-SFSP.

If $r \in Per(f)$ is hyperbolic, then for any $g \in diff(M)$ is C^1- near by f, there exists a unique hyperbolic periodic point $x_g \in Per(g)$ near by x such that $\epsilon(x_g) = \epsilon(x)$ and $index\ (x_g) = index(x)$. Such that x_g is called the **continuation** of x.

Proposition 3.1.8

Suppose $f|_{\Lambda_f(\mathrm{U})}$ is verification the C^1- stable fitting shadowing property, let $\mho(f)$ be as in Equation (3.1). Every hyperbolic saddles points $x, y \in \Lambda_g(U) \cap \mathrm{Per}(g)$ $(g \in \mho(f))$, $index\ (x) = index(y)$.

Proof: Let $f|_{\Lambda_f(\mathrm{U})}$ be verification the C^1stable fitting shadowing property and $x, y \in \Lambda_g(U) \cap \mathrm{Per}(g)$ $(g \in \mho(f))$be hyperbolic saddle points.

Then, a C^1 −neighborhood exists $V(g) \subset \mho(f)$ of g such that for any $Q \in V(g)$, there are the continuation $x_Q$and y_Q(of x and y)in $\Lambda_Q(\mathrm{U})$, respectively recall that since $\Lambda_f(\mathrm{U}) = \Lambda \subset \mathrm{int\ U}$,

we may assume that $\Lambda_g(\mathrm{U}) \subset$int U , for any $g \in \mho(f)$ reducing $\mho(f)$. To proof the Proposition by contradiction.

Let index(x) be less than index(y), and thus

dim $E^s(x, g)$ +dim $E^u(y, g)$ are less than dim M

(also dim $E^u(x, g)$ +dim $E^s(y, g)$ <dim M).

Now $E^s(x, g)$ and $E^u(y, g)$ are stable and the unstable manifolds of x and y with respect to g, by Definition (3.1.6)

$Q \in V(g)$.Then $E^s(x_Q, Q) \cap E^u(y_Q, Q) = \emptyset$.

Since dim $E^s(x, g)$ =dim $E^s(x_Q, Q)$ and

dim $E^u(y, g)$ =dim $E^u(y_Q, Q)$. On the other hand, since $Q \in \mathrm{V}(g)$,

$g \in \mho(f)$ so $Q|_{\Lambda_Q(\mathrm{U})}$satisfies the fitting shadowing property

so that

$E^s(x_Q, Q) \cap E^u(y_Q, Q) \neq \emptyset$

by Proposition (3.1.5), it is not possible, this is contradiction by above assumption. ■

Proposition 3.1.9

Let Λ be locally maximal in U, and $\mho(f)$ be in Equation (3.1). If $x \in \Lambda_g(\mathrm{U}) \cap \mathrm{Per}(g)$ $(g \in \mho(f))$ is non- hyperbolic set .Then there exists $\mathrm{Q} \in \mho(f)$ possessing of x_1 and x_2 in $\Lambda_{\mathrm{Q}}(\mathrm{U})$are hyperbolic periodic points with index $(x_1) \neq$ index (x_2).

Proof: For a non-hyperbolic case of $x \in \Lambda_g(\mathrm{U}) \cap \mathrm{P}er(g)$

$(g \in \mho(f))$,

Put $\mathrm{V}(g) \subset \mho(f)$, then we prove that there is $\mathrm{Q} \in \mathrm{V}(g)$ possessing of a Q^{m}-invariant C^1-curve in U ($\forall$ $\mathrm{m} > 0$) whose endpoints are both hyperbolic with index $(x_1) \neq$ index (x_2).

By applying Lemma (2.2.7) and a small modification the map g with respect to the C^1-topology,

we may assume that $\mathrm{D}_x g^{\epsilon(x)}$ has only one eigenvalue λ , with modulus one (other eigenvalue of $\mathrm{D}_x g^{\epsilon(x)}$ are with modulus less than 1 or greater than 1).

Denote by E_x^c the eigenspace (all of the eigenvectors that correspond to some eigenvalue λ), by E_x^s the eigenspace corresponding to the eigenvalues with modulus less than 1, and by E_x^u the eigenspace corresponding to the eigenvalues with modulus greater than 1, Thus

$$T_x M = E_x^s \oplus E_x^c \oplus E_x^u.$$

The proof consists of two cases with respect to dimension E_x^c:

Case 1 : dim $E_x^c = 1$. The eigenvalue λ is real with modulus is one, we suppose further that $\lambda = 1$, (the other case is similar)

Then by Lemma (2.2.7), $\exists\ \varepsilon > 0$ and $\mathrm{Q} \in \mathrm{V}(g)$ such that

$\mathrm{Q}^{\epsilon(x)}(x) = g^{\epsilon(x)}(x) = x$

and

$$\mathrm{Q(z)} = \exp_{g^{\mathrm{k}+1}(x)} \circ \mathrm{D}_{g^{\mathrm{k}}(x)} g \circ \exp^{-1}_{g^{\mathrm{k}}(x)}(\mathrm{z})$$

If $\mathrm{z} \in \mathrm{B}_\varepsilon\left(g^{\mathrm{k}}(x)\right)$ for $0 \le \mathrm{k} \le \epsilon(x) - 2$,

and

$$\mathrm{Q(z)} = \exp_x \circ \mathrm{D}_{g^{\epsilon(x)-1}(x)}\, g \circ \exp^{-1}_{g^{\epsilon(x)-1}(x)}(\mathrm{z})\ .$$

If $\mathrm{z} \in \mathrm{B}_\varepsilon\left(g^{\epsilon(x)-1}\,(x)\right)$.

Since λ of $\mathrm{D}_x\, g^{\epsilon(x)}_{|\mathrm{E}_x^{\mathrm{c}}}$ is 1, there is a small arc

$\mathrm{I}_x \subset \mathrm{B}_\varepsilon(x) \cap \exp_x \mathrm{E}_x^{\mathrm{c}}(\varepsilon)$ with its center at x such that

$$\mathrm{Q}^{\epsilon(x)}(\mathrm{I}_x) = \mathrm{I}_x.$$

Here $\mathrm{E}_x^{\mathrm{c}}(\varepsilon)$ is the ε-ball in $\mathrm{E}_x^{\mathrm{c}}$ with its center at the origin 0_x .

Let $\mathrm{I}_x \subset \Lambda_{\mathrm{Q}}(\mathrm{U})$, reducing both $\mho(f)$ and ε if necessary (observe that Λ is locally maximal). Denote by x_1 and x_2 are endpoints of I_x

observe that

$$\mathrm{D}_{x_{\mathrm{k}}} \mathrm{Q}^{\epsilon(x)}_{|\mathrm{E}_x^{\mathrm{c}}} = \mathrm{D}_x\, g^{\epsilon(x)}_{|\mathrm{E}_x^{\mathrm{c}}} = 1\ ;\ \text{for all } \mathrm{k} = 1{,}2.$$

Hence, by Lemma (2.2.7), with C^1-modification of the map Q at the end points, we may have that both points are hyperbolic with

$$\text{index } (x_1) \neq \text{index } (x_2) \text{ with respect to Q}$$

Case 2 : dim $E_x^c = 2$, and the corresponding eigenvalues that is, the λ are complex conjugate with modulus equal to one, in the proof the second case to avoid notational complexity, we consider only the case $g(x) = x$

As in the first case, by Lemma (2.2.7), there is $\varepsilon > 0$ and $Q \in V(g)$ such that

$$Q(x) = g(x) = x$$

and

$$Q(z) = \exp_{g(w)} \circ\ D_w g\ \circ\ \exp_w^{-1}(z)$$

If $z \in B_\varepsilon(x)$. With a small modification of the map $D_x g$, we may suppose that there is $\tau > 0$ is the minimum number such that, $D_x g^\tau(y) = y$

for every $y \in \exp_x^{-1}(E_x^c(\varepsilon))$ by Lemma(2.2.7) .

Take $y_\circ \in \exp_x^{-1}(E_x^c(\varepsilon))$ such that $\|y_\circ\| = \frac{\varepsilon}{4}$ and set

$$\gamma_x = \exp_x\left(\left\{t.y_\circ: 1 \leq t \leq 1 + \frac{\varepsilon}{4}\right\}\right)$$

Then $\gamma_x \subset \Lambda_Q(U)$ is an arc such that

$$Q^k(\gamma_x) \cap Q^j(\gamma_x) = \emptyset \text{ if } 0 \leq k \neq j \leq \tau - 1,$$

$Q^\tau(\gamma_x) = \gamma_x$ and $Q^\tau_{|\gamma_x}$ is identity map.

As in the first case with a C^1-modification of the map at the endpoints x_1 and x_2 of γ_x.

We have that both points are hyperbolic with

index $(x_1) \neq$ index (x_2). ■

Theorem 3.1.10

If $f|_{\Lambda_f(U)}$ satisfies the C^1- stable fitting shadowing property (C^1-SFSP), then there is a C^1-neighborhood $\mho(f)$ of f, for any $g \in \mho(f)$, any periodic points $x_1, x_2 \in \Lambda_g(U)$ are hyperbolic and $index\ (x_1) = index\ (x_2)$. .

Proof: Let $f|_{\Lambda f(U)}$ be satisfies the C^1-SFSP, and $\mho(f)$ be in Equation (3.1), to get the conclusion,

it is enough to show that every $x \in \Lambda_g(U) \cap Per(g)$ $(g \in \mho(f))$ is hyperbolic.

Then by Proposition (3.1.8), we prove this Theorem by contradiction.

Let $x \in \Lambda_g(U) \cap Per(g)$ $(g \in \mho(f))$ be non- hyperbolic,

by Proposition (3.1.9), there is $Q \in \mho(f)$, possessing hyperbolic periodic points x_1 and x_2 in $\Lambda_Q(U)$ with different indices .

This is impossible again by Proposition (3.1.8) since $f|_{\Lambda_f(U)}$ satisfies C^1-stable fitting shadowing property SFSP. ■

We aim to explain the stable fitting shadowing property (SFSP) for partially hyperbolic diffeomorphism (PHD), contain x_i, where $i = 1,2$ saddle points with different indices, then f does not satisfy the fitting shadowing property.

Note 3.1.11

we assume that the center direction with one dimensional is open in $diff(M)$ for the strongly partially hyperbolic diffeomorphism map $f \in \mathrm{SPH}_1(M)$defined by Proposition (1.2.33), there exist x, y a hyperbolic periodic points have not equal indices. Then f does not achieve the fitting shadowing property.

Theorem 3.1.12 [39]

There exists an open and dense set $\mathcal{M}$ of $diff^1(M)$, such that all $f \in \mathcal{M}$ satisfies: if for any C^1-neighborhood $\mho$ of f there is $\mathcal{M}' \in \mho$ has two distinct hyperbolic periodic points with different indices, then f has two distinct hyperbolic periodic points with different indices.

Theorem 3.1.13 [7]

Every robustly transitive set has a dominated splitting.

Proposition 3.1.14

Suppose that a C^1-dense open $\mathcal{M} \subset$ RNT(robustly non-hyperbolic transitive) diffeomorphisms such that, for every $f \in \mathcal{M}$ does not satisfy the fitting shadowing property.

Proof: Let M be closed manifold of dimension greater or equal three,

by Theorem (3.1.12), there is a dense open subset $\mathcal{M}' \subset$ RNT, has two saddles points with different indices. for any $f \in \mathcal{M}$ such that then f has two distinct hyperbolic periodic points with different indices.

since $f \in \mathcal{M}$ is robustly transitive by Theorem (3.1.13), then f has a partially hyperbolic splitting

$$E^s \oplus E^c \oplus E^u.$$

by Theorem (3.1.12), there exists x, y saddles points with different indices,

then f does not achieve fitting shadowing property. ■

Proposition 3.1.15

There is a C^1-dense open $\mathcal{M}$ in $RNT \cap SPH_1(M)$ such that every $f \in \mathcal{M}$ does not achieve the fitting shadowing property.

Proof: By Theorem (3.1.12) satisfy, there is an open and dense subset $\mathcal{M}'$ in RNT such that every diffeomorphism in $\mathcal{M}'$ has two saddles with different indices.

Put $\mathcal{M} = \mathcal{M}' \cap SPH_1(M)$. Then by the openness of $SPH_1(M)$, $\mathcal{M}'$ is open and dense in $RNT \cap SPH_1(M)$. Let $f \in \mathcal{M}$, then there are two saddles x, y so that $W^u(x)$ is less than from dimension $W^u(y)$.

Since dim $E^c = 1$, we see that dim $W^u(x)$ =dim E^u, then f does not satisfy the fitting shadowing property. ■

Proposition 3.1.16

Let $f: M \to M$ be satisfy fitting shadowing property. Then for every $x \in M$ is hyperbolic periodic point, then $W^s(x)$ and $W^u(x)$ the stable (unstable) manifolds respectively are dense in M.

Proof: Given x is hyperbolic periodic point (x is fixed point) for f, we prove that the unstable manifold $W^u(x)$ is dense in M. Assume that $(x) = x$, and f^{ϵ} where ϵ is the period of x.

Suppose $W^u_{\delta_1}(x)$ is local unstable manifold for any $\delta_1 > 0$, take any $x_1 \in M$ and $\delta_2 > 0$, it is sufficient to show that there exists $x_2 \in M$, to show that $d(x_1, x_2) \leq \delta_2$ and $x_2 \in W^u(x)$.

Put $\delta := \frac{1}{2} \min\{\delta_1, \delta_2\}$, and assume $\ell \geq \aleph(\delta)$ any integer.

Since f satisfy fitting shadowing property, for any $\sigma \geq 1$.

$f^{\ell}(B_{-\sigma}(x, \delta)) \cap B(x_1, \delta) \neq \emptyset$, where $B(x_1, \delta)$ stands for the closed ball of radius δ a round x_1 , since the previous intersection is a strictly decreasing family of closed sets.

By compactness of M there exist a point $x_2 \in M$, such that

$$x_2 \in \bigcap_{\sigma=1}^{\infty} f^{\ell}(B_{-\sigma}(x, \delta)) \cap B(x_1, \delta)$$

Then we have

$$d(x_2, x_1) \leq \delta_2 \text{ and } d(f^{-\sigma}\left(f^{-\ell}(x_2)\right), f^{-\sigma}(x)) \leq \delta_1 \text{ for any } \sigma \geq 1.$$

The letter implies that $x_2 \in f^{\ell}\left(\mathrm{W}^{\mathrm{u}}_{\delta_1}(x)\right) = f^{\ell}\left(\mathrm{W}^{\mathrm{u}}_{\delta_1}\left(f^{-\ell}(x)\right)\right)$.

Thus we have $x_2 \in W^u(x)$ and $d(x_1, x_2) \leq \delta_2$. ■

If $\Lambda = M$, then we say that f has the fitting shadowing property.

Note 3.1.17

It follows that C^1-generically a non-hyperbolic diffeomorphism map does not have fitting shadowing property .On other hand, maps with the fitting shadowing property could be dense in the complement of the uniformly hyperbolic diffeomorphism map. Even for some dynamical systems with partial hyperbolic.

Center Fitting Shadowing Property for Partial Hyperbolic Diffeomorphisms:

Let $f: M \to M$ be a partially hyperbolic diffeomorphism map on a closed (that is, boundaryless and compact) Riemannian manifold M with uniformly compact center foliation W^c.

To use a central pseudo orbit (PO) for explain that any pseudo orbit (PO) of a partially hyperbolic diffeomorphism (PHD) can fitting shadowing property by a central pseudo orbit.

Generally, for a partially hyperbolic diffeomorphism$f$, the stable and unstable foliations always exists while the center bundle might not exist. In this section, we restrict ourselves to the case that the center foliation exists and uniformly compact .

It is first shown that if a compact locally maximal invariant center set Λ is center topologically mixing then $f|_\Lambda$ has the center fitting shadowing property (any fitting shadowing property FSP with a large spacing can be center shadowed by a periodic center leaf with a fine precision .

Let $\mathrm{Per}^c(f)$be the set of all periodic center points for partially hyperbolic diffeomorphism.

Definition 3.1.18 [27]

An δ – fitting pseudo-orbit $\{y_\kappa\}_{\kappa=0}^{\infty}$ is called central if for any $\kappa \in \mathbb{Z}$ the inclusion $y_\kappa \in W_\delta^c(y_{\kappa+1})$.

Lemma 3.1.19 [16]

Let f be partially hyperbolic diffeomorphism on M with uniformly compact center foliation W^c. Then there exists $\delta > 0$,

for all $0 < \varepsilon < \delta$, there exists $\aleph = \aleph(\varepsilon) > 0$ verification the following: for all $x_1, x_2 \in M$ with $d(x_1, x_2) < \aleph$, if $x_1' \in W^c(x_1)$ then there exists $x_2' \in W^c(x_2)$, such that $W_\varepsilon^s(x_1') \cap W_\varepsilon^u(x_2') \neq \emptyset$ has exactly one point.

"Center Closing Lemma" 3.1.20 [28]

Let f be a partially hyperbolic diffeomorphism map on M with a uniformly compact center foliation. Then for any $\varepsilon > 0$, there exist $\delta \in (0, \varepsilon)$such that for any $x \in M$ and $n \in \mathbb{N}$ with $d(x, f^n(x)) < \delta$, there exist a periodic center leaf $W^c(x')$ of period ϵ satisfying

$$d_H(W^c(x), W^c(x')) < \varepsilon$$

Proposition 3.1.21

Let f be partially hyperbolic diffeomorphism map on M with a uniformly compact center foliation and Λ be a compact locally maximal invariant center set of f , then

$\overline{\mathrm{Per^c}(f)|_\Lambda} = \Lambda$. Moreover, for any periodic center leaf $W^c(x)$in Λ, we have $W^{cu}(x)$ is dense in Λ, that is $\overline{W^{cu}(x)} \supset \Lambda$.

Proof: Since Λ is center topological mixing, then from Lemma (3.1.20), we get $\forall x^* \in \Lambda$ and any $\varepsilon > 0$

there exists periodic center leaf $W^c(x)$ such that

$$d_H(W^c(x^*), W^c(x)) < \varepsilon.$$

Where $\mathrm{d}_H(0,0)$ denoted the Hausdorff distance given by

$$\mathrm{d}_H(\mathrm{A},\mathrm{B}) = \max_{a\in A} \min_{b\in B} d(a,b)$$

For closed subsets $\mathrm{A},\mathrm{B} \subset \mathrm{M}$.

Therefore, every center leaf which lies in Λ can be approximated by a sequence of periodic center leaves and hence the periodic center leaves are dense in Λ. Given a periodic center leaf $W^c(x)$ in Λ.

Since $\overline{\mathrm{Per^c}(f)|_\Lambda} = \Lambda$ we only need to prove that for any y which lies in a periodic center leaf, and for all $\varepsilon > 0$, we can find a point

$y' \in W^{\mathrm{cu}}(x)$, such that

$$\mathrm{d}(\mathrm{y}', y) < \varepsilon$$

Since the center foliation W^c is uniformly compact set and Λ is center topological mixing, for any open center sets $\mho$ and $\mathcal{V}$ which contains $W^c(x)$ and $W^c(y)$ respectively,

there exists $\ell \in \mathbb{N}$ such that $f^i(\mho) \cap \mathcal{V} \neq \emptyset$ for any $i \geq \ell$.

take $\delta > 0$, such that the corresponding result in Lemma (3.1.19) holds.

By continuity for the center foliation W^c we obtain

$$\mho \subset B_{d_H}(W^c(x), \delta) \text{ and } \mathcal{V} \subset B_{d_H}(W^c(y), \varepsilon) \ldots\ldots\ldots \quad (3.2)$$

suppose $W^c(x)$ of period ϵ ,

we take a number $i \in \mathbb{N}$, such that $\epsilon i \geq \ell$, and $f^{\epsilon i}(\mho) \cap \mathcal{V} \neq \emptyset$.

We can choose $x \in \Lambda$ such that $W^c(x^*) \subset \mho$ and

$$f^{\epsilon i}\big(W^c(x^*)\big) = W^c(f^{\epsilon i}(x^*)) \subset \mathcal{V}$$

By Equation (3.2), we can take a point

$x^{**} \in W^c\left(f^{\epsilon i}(x^*)\right)$ such that

$$d(x^{**}, y) < \varepsilon$$

note that

$$d_H(W^c(x), W^c(x^*)) < \delta < \varepsilon$$

and

$$f^{-\epsilon i}(x^{**}) \text{ lies in } W^c(x^*)$$

if $W^c(x^*) = W^c(x)$ then

$$x^{**} \in W^c\left(f^{\epsilon i}(x^*)\right) = W^c\left(f^{\epsilon i}(x)\right) = W^c(x)$$

Hence $y' = x^{**}$ is a desired point.

Now, suppose $W^c(x^*) \neq W^c(x)$

take $y'' \in W^c(x)$ such that

$$d(y'', f^{-\epsilon i}(x^{**})) < \delta$$

By Lemma (3.1.19) , $W_\varepsilon^u(y'') \cap W_\varepsilon^s\left(f^{-\epsilon i}(x^{**})\right)$ has only one point, say y‴

Then $y' = f^{\epsilon i}(y''')$ is a desired point. ■

In the Proposition (3.1.21), s ince the periodic center leaves are dense in Λ, we get that for each point in Λ , its center-unstable manifold is dense in Λ.

In addition, the following Proposition tells us that the above density is even uniform.

Proposition 3.1.22

Let f be partially hyperbolic diffeomorphism map on M with a uniformly compact center foliation and Λ be a compact locally maximal invariant center set of f. Then for all $\gamma > 0$, there exsits $m \in \mathbb{N}$, such that for any $x, y \in \Lambda$ and $n \geq m$, we have

$$f^n(W^c\left(W_\gamma^u(x)\right)) \cap W_\gamma^s(y) \neq \emptyset,$$

Where $W^c\left(W_\gamma^u(x)\right) = \cup_{x' \in W_\gamma^u(x)} W^c(x', f).$

Proof: By Lemma (3.1.19), we have there is a $\delta > 0$ such that for any $0 < \varepsilon < \delta$, there exists $\aleph = \aleph(\varepsilon) > 0$ satisfies the following property, for any $x, y \in M$ with

$$d(x,y) < \aleph,$$

if $x'' \in W^c(x)$then there is $y'' \in W^c(y)$ such that

$$W^s_\varepsilon(x'') \cap W^u_\varepsilon(y'') \neq \emptyset$$

has exactly one point by Lemma (3.1.19) .

Given $\gamma > 0$, since the center foliation is continuous and uniformly compact, there exists $0 < \vartheta < \min\{\delta, \gamma/2, \aleph(\gamma/2)/3\}$, such that

$$d(x,y) < \vartheta$$

that is implies

$$d_H(W^c(x), W^c(y)) < \aleph(\gamma/2)/3.$$

To choose m take a ϑ-dense set $\{p_k\}$ where $k = 1,2,\dots,r$ of points each of which lies in a periodic center leaf (with period ϵ) by Proposition (3.1.21), for all $1 \leq k \leq r, W^{cu}(p_k)$is dense in Λ, and hence

there exists ℓ_k such that $f^{\ell\epsilon}(W^c(W^u_\vartheta(p_k)))$ is $\aleph(\vartheta)$-dense for any $\ell \geq \ell_k, k = 1,\dots,r$ that is

for any $x' \in \Lambda$, there is $x''' \in f^{\ell\epsilon}(W^c(W^u_\vartheta(p_k)))$

such that

$$d(x', x''') < \aleph(\vartheta)$$

Let $m = \prod_{k=1}^{r} \ell_k \epsilon$ and note that $f^m(W^c(W^u_\vartheta(p_k)))$ is $\aleph(\vartheta)$-dense for all k

Now, we show that m is as desired

For $x, y \in \Lambda$, take α such that

$$d(x, p_\alpha) < \vartheta$$

hence by Lemma (3.1.20),

$$d_H(W^c(x), W^c(p_\alpha)) < \aleph\,(\gamma/2)/3$$

$z \in f^m\left(W^c\left(W^u_\vartheta(p_\alpha)\right)\right)$ such that

$$d(y,z) \leq \aleph(\vartheta)$$

and $$w \in W^u_\vartheta(z) \cap W^s_\vartheta(y),$$

that is

$$d(f^{-m}(w), W^c(p_\alpha)$$

$$\leq\ d\big(f^{-m}(w), (f^{-m}(z)) + d(f^{-m}(z)), W^c(p_\alpha)\big)$$

$$\leq \vartheta + \aleph\,(\gamma/2)/3$$

$$< 2\aleph\,(\gamma/2)/3$$

Therefore by Triangle Inequality, we have

$$d(f^{-m}(w), W^c(x)) \leq\ d(f^{-m}(w), W^c(p_\alpha)) + d_H(W^c(p_\alpha)), W^c(x))$$

$$< \aleph(\gamma/2)$$

So, there exists $s \in W^c\left(W^u_{\gamma/2}(x)\right) \cap W^s_{\gamma/2}(f^{-m}(w)\,)$

by Lemma (3.1.19) and

$$f^m(s) \in f^m\left(W^c\left(W^u_{\gamma/2}(x)\right)\right) \cap W^s_{\gamma/2}(w)$$

$$\subset f^m\left(W^c\left(W^u_\gamma(x)\right)\right) \cap W^s_\gamma(y) \neq \emptyset$$

since $w \in W^s_\vartheta(y)$ and $\vartheta \leq \gamma/2,$ for any $x, y \in \Lambda$,$n \geq m$ note that

$$f^n(W^c\big(W^u_\gamma(x))\big) \cap W^s_\gamma(y)$$

$$\supset f^m(W^c\big(W^u_\gamma(f^{n-m}(x)))\big) \cap W^s_\gamma(y) \neq \emptyset \qquad \blacksquare$$

Theorem 3.1.23

Let f be a partially hyperbolic diffeomorphism map on M with a uniformly compact center foliation and Λ be a compact locally maximal invariant center set of f . If Λ is center topological mixing, then $f|_\Lambda$ has the center fitting shadowing property .

Proof: Suppose that $0 < \varepsilon < \delta$ where δ from Lemma (3.1.19), we assume that the local stable and unstable manifolds satisfy that

$$x \in W_\vartheta^s(z) \text{ so, } f(x) \in W_{\lambda\vartheta}^s(f(z)), \forall 0 < \lambda < 1 \ldots\ldots \quad (3.3)$$

and $x^* \in W_\vartheta^u(z^*)$, so $\quad f^{-1}(x^*) \in W_{\lambda\vartheta}^u(f^{-1}(z^*))$…….. (3.4)

$\forall\ 0 < \vartheta < \varepsilon$, take $\gamma < \varepsilon/4$ such that

$$d(x, z) < 4\,\gamma$$

$$d_H(W^c(x), W^c(z)) < \varepsilon/4$$

For this γ, take the corresponding m we get from Proposition (3.1.23), If necessary increase m such that $\lambda^m < 1/4$, where λ is the contraction rate in the Definition (1.2.13).

Let $\Phi = (\Gamma, \Psi)$ be of f on Λ in which

$\Gamma = \{\ell_k = [a_k, b_k]: 1 \le k \le r\}$ and Ψ is the corresponding map on Γ

$\Psi: \Gamma \coloneqq \bigcup_{k=1}^r \ell_k \to \Lambda$, for each $\ell \in \Gamma$

Put $x_1 = f^{-a_1}(\Psi(a_1))$ and define $x_2, x_3, \ldots . x_r$ it follows:

Given x_k, by Proposition (3.1.22), there exists x_{k+1} , such that

$$f^{a_{k+1}}(x_{k+1}) \in f^{a_{k+1}-b_k}\left(W^c(W_\gamma^u\left(f^{b_k}(x_k)\right))\right) \cap W_\gamma^s\left(\Psi(a_{k+1})\right)$$

since by assumption $a_{k+1} - b_k \ge m.$

To explain that Φ can be center ε-fitting shadowed by $x =: x_r$

since for each $k \in [1, r], f^{a_k}(x_k) \in W_\gamma^s(\Psi(a_k))$ by construction,

$$d(f^n(x_k), \Psi(n)) = d(f^{n-a_k}(f^{a_k}(x_k)), f^{n-a_k}(\Psi(a_k))) < \gamma$$

by Equation (3.4) , we have

$$d_H(W^c(f^n(x_k)), W^c(\Psi(n))) \leq \varepsilon/4 \text{ for } n \in \ell_k$$

we prove

$$d_H\left(W^c(f^n(x)), W^c(f^n(x_k))\right) \leq \varepsilon/4 \ \ldots\ldots\ (3.5)$$

for all $n \in \ell_k, k \in [1, r]$, then we get the desired result by the Triangle Inequality. Now we show Equation (3.5).

For $k = r$ and $n \in \ell_r$ Equation (3.5) hold obviously since $x = x_r$ so we begin at $k = r - 1,\ n \in \ell_{r-1}$

Since

$$f^{b_{r-1}}(x) \in W^c\left(W_\gamma^u\left(f^{b_{r-1}}(x_{r-1})\right)\right)$$

by construction, we can take $x_{p-1}^* \in W^c(x)$ such that

$$f^{b_{r-1}}(x_{r-1}^*) \in W_\gamma^u\left(f^{b_{r-1}}(x_{r-1})\right) \cap W^c\left(f^{b_{r-1}}(x)\right)$$

then by Equation (3.4), we have

$$d_H\left(W^c\left(f^{b_{r-1}}(x_{r-1}^*)\right), W^c\left(f^{b_{r-1}}(x_{r-1})\right)\right) \leq \varepsilon/4$$

and hence by Equation (3.3), we have

$$d_H\left(W^c(f^n(x_{r-1}^*)), W^c(f^n(x_{r-1}))\right) < \varepsilon/4 \ \ldots\ldots\ (3.6)$$

for $n \in \ell_{r-1}$ and

$$f^{a_{r-1}}(x_{r-1}^*) \in W^u_{\gamma\lambda^{b_{r-1}-a_{r-1}}}\left(f^{a_{r-1}}(x_{r-1})\right) \cap W^c\left(f^{a_{r-1}}(x)\right)$$

Now consider if $k = r-2$ and $\in \ell_{r-2}$, note that

$$f^{b_{r-2}}(x_{r-1}^*) \in W^c\left(W^u_{\gamma\lambda^{b_{r-1}-b_{r-2}}}\left(f^{b_{r-2}}(x_{r-1})\right)\right)$$

by Equation (3.3)

$$f^{b_{r-2}}(x_{r-1}) \in W^c(W^u_\gamma\left(f^{b_{r-2}}(x_{r-2})\right)$$

we have

$$f^{b_{r-2}}(x_{r-1}^*) \in W^c\left(W^u_{\gamma+\gamma\lambda^{b_{r-1}-b_{r-2}}}\left(f^{b_{r-2}}(x_{r-2})\right)\right)$$

Take $x_{r-2}^* \in W^c(x_{r-1}^*)$ such that

$$f^{b_{r-2}}(x_{r-2}^*) \in W^u_{\gamma+\gamma\lambda^{b_{r-1}-b_{r-2}}}\left(f^{b_{r-2}}(x_{r-2})\right) \cap W^c\left(f^{b_{r-2}}(x_{r-1}^*)\right)$$

then by Equations (3.4),(3.5), $b_{r-1} - b_{r-2} > m$, we have

$$d_H(W^c\left(f^{b_{r-2}}(x_{r-2}^*)\right), W^c\left(f^{b_{r-2}}(x_{r-2})\right)) \le \varepsilon/4$$

and hence by Equation (3.3), we have

$$\mathrm{d}_H(W^c(f^n(x_{r-2}^*)), W^c(f^n(x_{r-2}))) < \varepsilon/4 \ldots.. \quad (3.7)$$

for $n \in \ell_{r-2}$. Inductively, we get $x_1^*, x_2^*, \ldots, x_{r-3}^*$ such that for each $3 \le j \le r-1$, we have $x_{r-j}^* \in W^c\left(x_{r-j+1}^*\right)$ such that

$$f^{b_{r-1}}(x_{r-j}^*) \in W^u_{\gamma(1+\sum_{i-1}^{j-1}\lambda^{b_{r-i}-b_{r-j}}}\left(f^{b_{r-j}}\left(x_{r-j}\right)\right) \cap W^c\left(f^{b_{r-j}}\left(x_{r-j+1}^*\right)\right)$$

and

$$d_H\left(W^c\left(f^n\left(x_{r-j}^*\right)\right), W^c\left(f^n\left(x_{r-j}\right)\right)\right) < \varepsilon/4 \ldots. \quad (3.8)$$

for $n \in \ell_{r-j}$. Note that

$$W^c(x_1^*) = W^c(x_2^*) = \cdots = W^c(x_{r-1}^*) = W^c(x)$$

by Equation (3.6),(3.7) and (3.8), we have

$$\mathrm{d}_H\left(W^c(f^n(x)), W^c(f^n(x_k))\right) < \varepsilon/4$$

for all $n \in \ell_k, k \in [1, r]$, that is Equation (3.5) holds. Now, let $f(\Phi) \coloneqq f(\Gamma)$ and $\mathrm{L}(\Phi) \coloneqq \mathrm{L}(\Gamma) \coloneqq b_r - a_1$,then Φ is called center ε −fitting shadowed by $x \in M$.

By Lemma (3.1.21), there is $0 < \rho < \varepsilon/4$, such that for any $x \in M$ and $n \in \mathbb{N}$ with $d_H\left(W^c(x), W^c(f^n(x))\right) < \rho$, there exists a periodic center leaf $W^c(w)$ of period n satisfying

$$d_H\left(W^c\left(f^j(w)\right), W^c\left(f^j(x)\right)\right) \le \varepsilon/4 \ldots\ldots \quad (3.9)$$

for all $1 \le j \le n$

We increase m if necessary such that any m −spaced can be center $\varepsilon/4$ −fitting shadowed by some point, now we define another fitting shadowing $\Phi^* = (\Gamma^*, \Psi^*)$ with $\Gamma^* = \Gamma \cup \{\{\mathrm{a}_1 + \epsilon\}\}$ and

$\Psi^*|_{f(\Gamma)} = \Psi, \Psi^*(\mathrm{a}_1 + \epsilon) = \Psi(\mathrm{a}_1)$, which is m −spaced, we obtain a point $x^* \coloneqq f^{a_1}(x) \in \Lambda$ such that

$$d_H\left(W^c(x^*), W^c(f^\epsilon(x^*))\right) \le$$

$$d_H\left(W^c(x^*), W^c(\Psi(a_1))\right) + d_H\left(W^c(f^\epsilon(x^*), W^c(\Psi(a_1))\right) \le \rho$$

Therefore, by Equation (3.9), there exists a periodic center leaf $W^c(w)$ of period ϵ such that

$$d_H\left(W^c(f^{n+a_1}(w),W^c\big(f^n(x^*)\big)\right) \le \varepsilon\ /\ 4, \forall\ n \in [0,\epsilon]$$

Since Λ is locally maximal, we obtain $W^c(w) \subset \Lambda$.

$$\lim_{m\to\infty} sup \sum_{n=0}^{m-1} d_H\left(W^c(f^{n+a_1}(w),W^c\big(f^n(x^*)\big)\right) \le \varepsilon/4$$

the fitting shadowing Φ is center ε −shadowed the period− ϵ center leaf $W^c(w)$

By the Triangle Inequality, then $f|_\Lambda$ has the center fitting shadowing property . ■

3.2 Eventual Fitting Shadowing Property for Hyperbolic Set

In this section, we will study another type of shadowing property is it called eventual fitting shadowing property if and only if f^k has the eventual fitting shadowing property for all $k \in \mathbb{Z} \setminus \{0\}$ on locally maximal chain transitive set, then $f: M \to M$ is hyperbolic. Let Λ be a closed invariant set of f, if f has the eventual shadowing property, then f has the eventual shadowing property on Λ.

It is denoted a hyperbolic periodic point of x by $Per_h(f)$.

Definition 3.2.1

Let $(\mathfrak{I}, \mathrm{d})$ be a metric space, and $f: \mathfrak{I} \to \mathfrak{I}$ be a map, then f is called has the **eventual fitting shadowing property** (EFSP) on Λ if there is

$\delta > 0$, for every $\varepsilon > 0$, such that, for any δ −pseudo-orbit

$\{y_\kappa\} \in \mathfrak{I}$, $\kappa \in \mathbb{Z}$ there exist $z \in M$ and $\aleph = \aleph(\delta) \in \mathbb{N}, \forall m \geq \aleph$, such that:

$$\lim_{m\to\infty} \sup \sum_{\kappa=0}^{m-1} \mathrm{d}(f^\kappa(z), y_\kappa) < \varepsilon \ , \forall \kappa \geq \ \aleph$$

$$\lim_{m\to\infty} \sup \sum_{\kappa=0}^{m-1} \mathrm{d}(f^\kappa(z), y_\kappa) < \varepsilon \ , \forall \kappa \leq -\aleph$$

If $\Lambda = M$, then we say that f has the eventual fitting shadowing property.

Lemma 3.2.2[17]

There is residual set $\mathcal{R}_1 \subset diff(M)$, such that for all $f \in \mathcal{R}_1$, and any chain transitive set $C(f)$, there is a sequence $O_f\ (z_n)$ of periodic orbits of f such that

$$\lim_{n\to\infty} O_f\ (z_n)\ = C(f)$$

Proposition 3.2.3

For every chain transitive set $C(f)$ of $f \in \mathcal{R}_1$, if $C(f)$ is locally maximal, then

$$C(f) \cap \mathrm{Per}(f) \neq \emptyset\ .$$

Proof: Suppose $f \in \mathcal{R}_1$, and let a chain transitive set $C(f)$ of f be locally maximal in Ʊ.

To proof this Proposition by contradiction, if $C(f) \cap \mathrm{Per}(f) = \emptyset$

Because $C(f)$ is compact, there is $\varepsilon > 0$ such that

$$C(f) \subset \mathrm{B}_\varepsilon\big(C(f)\big) \subset Ʊ$$

By Lemma (3.2.2), there is a periodic orbit sequence $O_f\ (z_n)$ of f such that for sufficiently large n, we have $\mathrm{d}_H(O_f\ (z_n)\ , C(f)) < \frac{\varepsilon}{2}$

Then it is $O_f\ (z_n)\ \subset \mathrm{B}_\varepsilon\big(C(f)\big) \subset Ʊ$.

Since $C(f)$ is locally maximal in Ʊ, $f^{\mathrm{n}}(O_f\ (z_n)\ \subset f^{\mathrm{n}}(Ʊ), \forall\ n \in \mathbb{Z}$

If $C(f)$ is locally maximal, then $C(f) \cap \mathrm{Per}(f) \neq \emptyset$,

which is a contradiction, so $C(f) \cap \mathrm{Per}(f) \neq \emptyset$ ■

Proposition 3.2.4

Assume Λ is a compact f-invariant set of f. If f has the eventual fitting shadowing property on locally maximal Λ, then the eventual fitting shadowing points are take from Λ.

Proof: Take a locally maximal neighborhood $\mho$ of Λ,

since Λ is compact, there is $\varepsilon > 0$ such that

$$\Lambda \subset B_{\varepsilon}(\Lambda) \subset \mho.$$

Let $0 < \delta \leq \varepsilon$ be the number of the eventual fitting shadowing property, and

Suppose $\{y_\kappa\}_{\kappa\in\mathbb{Z}} \subset \Lambda$ a δ −pseudo-orbit of f

by Definition (3.2.1) on Λ, there is $z \in M$, and $m \geq \aleph$, $\aleph \in \mathbb{Z}$ such that

$$\lim_{m\to\infty} \sup \sum_{\kappa=0}^{m-1} d(f^{\kappa}(z), y_\kappa) < \varepsilon \ , \forall \kappa \geq \aleph \quad \text{and}$$

$$\lim_{m\to\infty} \sup \sum_{\kappa=0}^{m-1} d(f^{-\kappa}(z), y_{-\kappa}) < \varepsilon \ , \forall -\kappa \leq -\aleph$$

Then, we have that

for all $\kappa \geq \aleph$, $f^{\kappa}(z) \in B_{\varepsilon}(\Lambda)$ and

for all $-\kappa \leq -\aleph$, $f^{-\kappa}(z) \in B_{\varepsilon}(\Lambda)$ and so,

$$f^{\kappa}\left(f^{\aleph}(z)\right) \in B_{\varepsilon}(\Lambda) \text{ and } f^{-\kappa}(f^{-\aleph}(z)) \in B_{\varepsilon}(\Lambda)$$

Since Λ is locally maximal, we know that

$$\bigcap_{n\in\mathbb{Z}} f^{n}(f^{\aleph+\kappa}(z)) \in \bigcap_{n\in\mathbb{Z}} f^{n}(B_{\varepsilon}(\Lambda)) \subset \bigcap_{n\in\mathbb{Z}} f^{n}(\mho) = \Lambda$$

Then we have $f^{\aleph+\kappa}(z)) \in \Lambda$. Since Λ is an f −invariant set,

$$z \in f^{-\aleph-\kappa}(\Lambda) = \Lambda$$

Thus the eventual fitting shadowing point z is take from Λ. ■

Lemma 3.2.5 [40]

Let $C(f)$ be a chain transitive set of $f: M \to M$ is a diffeomorphism of smooth manifold, then $C(f)$ has neither attracting nor repelling points.

Proposition 3.2.6

If f has the eventual fitting shadowing property on a locally maximal $C(f)$, then for all hyperbolic $x_1, x_2 \in C(f) \cap \mathrm{Per}(f)$, we have

$$E^{s}(x_1) \cap E^{u}(x_2) \neq \emptyset \text{ and } E^{u}(x_1) \cap E^{s}(x_2) \neq \emptyset$$

Proof: By Lemma (3.2.5), f does not contain attracting or repelling since $C(f)$ is a chain transitive set of f.

Thus, every periodic point in $C(f)$ is saddle.

Let x_1, x_2 be Per_h of f,

take ε =min {ε(x_1),ε(x_2)} and let $0 < \delta \leq \varepsilon$ be the number of the eventual fitting shadowing property for f.

To simple expression, we may assume that

$$f(x_1) = x_1 \quad \text{and} \quad f(x_2) = x_2$$

Since f is chain transitive, there is $\{y_\kappa\}_{\kappa=0}^{n}$ $(n \geq 1) \subset C(f)$ is a finite δ-pseudo-orbit such that $y_0 = x_1$ and $y_n = x_2$ and

$$d(f(y_\kappa), y_{\kappa+1}) < \delta \ , \quad \forall\, 0 \leq \kappa \leq n-1$$

Take $y_\kappa = f^\kappa(x_1),\ \forall\ \kappa \leq 0$ and $y_{\kappa+n} = f^\kappa(x_2),\ \forall\ \kappa \geq 0,$

Then the sequence

$$\{\ldots, x_1(= y_{-1}), x_1(= y_0), y_1, y_2, \ldots, x_2(= y_n), x_2(= y_{n+1}), \ldots\}$$

$$= \{y_\kappa\}_{\kappa\in\mathbb{Z}} \subset C(f)$$

is δ-pseudo-orbit of f

by, the eventual fitting shadowing property on $C(f)$,

there are $z \in M$, and $m \geq \aleph$, $\aleph \in \mathbb{Z}$ such that

$$d(f^\kappa(z), y_\kappa) < \varepsilon,\ \forall\ \kappa \geq \aleph$$

and $$d(f^\kappa(z), y_\kappa) < \varepsilon,\ \forall\ \kappa \leq -\aleph$$

that is,

$$\limsup_{m\to\infty} \sum_{\kappa=0}^{m-1} d(f^\kappa(z), y_\kappa) < \varepsilon \quad \forall\ \kappa \geq \aleph \text{ and}$$

$$\limsup_{m\to\infty} \sum_{\kappa=0}^{m-1} d(f^\kappa(z), y_\kappa) < \varepsilon \ \forall\, \kappa \leq -\aleph$$

Since $y_{-\kappa} = x_1 = f^{-\kappa}(x_1),\ \forall\ \kappa \geq 0$ and

$$y_{n+\kappa} = x_2 = f^{n+\kappa}(x_2),\ \forall\ \kappa \geq 0$$

If $\aleph \geq n,$ then we know

$$f^{-\aleph}(z) \in B_\varepsilon(y_{-\aleph}) = B_\varepsilon(x_1)$$

and

$$f^{\aleph}(z) \in B_\varepsilon(y_{\aleph}) = B_\varepsilon(x_2)$$

Thus for all $\kappa \geq \aleph$

$$f^{\kappa+\aleph}(z) = f^{\kappa}\left(f^{\aleph}(z)\right) \in B_\varepsilon(y_{\aleph+\kappa}) = B_\varepsilon(x_2) \quad \ldots.. \quad (3.10)$$

By Equation (3.10), we have

$$d(f^{\kappa}(f^{\aleph}(z), x_2) < \varepsilon, \quad \forall \kappa \geq 0$$

and for all $-\kappa \leq -\aleph$

$$f^{-\aleph-\kappa}(z) \in B_\varepsilon(y_{-\aleph-\kappa}) = B_\varepsilon(x_1) \quad \ldots.. \quad (3.11)$$

By Equation (3.11), we have

$$d(f^{-\kappa}(f^{-\aleph}(z), x_1) < \varepsilon, \quad \forall \kappa \geq 0$$

Then $f^{\aleph}(z) \in E^s_\varepsilon(x_2)$, $f^{-\aleph}(z) \in E^u_\varepsilon(x_1)$

and so $z \in f^{\aleph}(E^u_\varepsilon(x_1))$ and $z \in f^{-\aleph}(E^s_\varepsilon(x_2))$

Since $f^{\aleph}(E^u_\varepsilon(x_1)) \subset E^u(x_1)$, and $f^{-\aleph}(E^s_\varepsilon(x_2)) \subset E^s(x_2)$,

we have, $z \in E^u(x_1) \cap E^s(x_2)$.

Thus , $E^u(x_1) \cap E^s(x_2) \neq \emptyset$.

Now, to prove other case when $E^s(x_1) \cap E^u(x_2) \neq \emptyset$

In fact, the proof of this case has the same to the above case. ■

Proposition 3.2.7

There is a residual set $\mathcal{R}_2 \subset diff(M)$ such that given any chain transitive set $C(f)$ of $f \in \mathcal{R}_2$, if f has the eventual fitting shadowing property on locally maximal chain transitive $C(f)$, then for any $x_1, x_2 \in C(f) \cap Per(f)$, we have $index\ (x_1) = index\ (x_2)$.

Proof: Let $f \in \mathcal{R}_\mathbf{2} = \mathcal{R}_\mathbf{1} \cap \mathcal{KS}$ where $\mathcal{KS}$ in Definition (3.1.6) and let $C(f)$ be a locally maximal chain transitive set of f.

Assume that f has the eventual fitting shadowing property on $C(f)$ since $C(f)$ is locally maximal of f and by Proposition (3.2.3)

$$C(f) \cap \mathrm{Per}(f) \neq \emptyset$$

to proof by contradiction .

Assume that there exist are two hyperbolic periodic points $x_1, x_2 \in C(f)$ such that.

$$index\ (x_1) \neq index\ (x_2)$$

Since $index\ (x_1) \neq index\ (x_2)$, by Proposition (3.1.8)

$$\dim \mathrm{E}^s(x_1) + \dim \mathrm{E}^u(x_2) < \dim M$$

or

$$\dim \mathrm{E}^u(x_1) + \dim \mathrm{E}^s(x_2) < \dim M$$

Then, take the case in which $\dim \mathrm{E}^s(x_1) + \dim \mathrm{E}^u(x_2) < \dim M$, the other case has the same proof.

Since $f \in \mathcal{KS}$ and $\dim \mathrm{E}^s(x_1) + \dim \mathrm{E}^u(x_2) < \dim M$,

$$\mathrm{E}^s(x_1) \cap \mathrm{E}^u(x_2) = \emptyset$$

this is contradiction, since f has the eventual fitting shadowing property on $C(f)$,

By Proposition (3.2.6), $\mathrm{E}^{\mathrm{s}}(x_1) \cap \mathrm{E}^{\mathrm{u}}(x_2) \neq \emptyset$,

Thus, if $f \in \mathcal{R}_2$ has the eventual fitting shadowing property on a locally maximal chain transitive set $C(f)$,

then for any $x_1, x_2 \in C(f) \cap Per(f)$ have

$index\ (x_1) = index\ (x_2)$. ■

Lemma 3.2.8 [40]

Let $f \in diff(M)$ be has eventally fitting shadowing property, then f is chain transitive .

Theorem (Theorem Mane) 3.2.9 [40]

There is a residual subset $\mathcal{R} \subset diff(M)$ satisfies one of the following:

- f is Axiom A without cycles;
- f has neither attracting nor repelling points.

From above Theorem, we have the following Theorem, which is a main result of this section.

Theorem 3.2.10

There is a residual subset $\mathcal{R} \subset diff(M)$ such that for any $f \in \mathcal{R}$, if f has eventally fitting shadowing property, then f is Anosov diffeomorphism map.

Proof: Let $f \in \mathcal{R}$ has an eventally fitting shadowing property.
According to Lemma (3.2.8), f is chain transitive.

By Theorem (3.2.9), f has neither attracting nor repelling points.

According to Theorem (3.2.9), f is Axiom A without cycles; finally, we show that $\Omega(f) = M$. Since f is Axiom A without cycles, we has the non-wandering set $\Omega(f)$ is hyperbolic and by Proposition (2.2.24),

$\Omega(f) = R(f)$.

Since, f is chain transitive, then $R(f) = M$ is hyperbolic, and so f is Anosov diffeomorphism map. ■

3.3 Strongly Fitting Shadowing Property for Hyperbolic Manifolds .

In this section, hyperbolic sets is identified as having the strongly fitting shadowing property and we show that the diffeomorphism $f|_{\Lambda}$ is an Anosov diffeomorphism map if and only if Λ has strongly fitting shadowing property.

Let $f: M \to M$ be a C^1-diffeomorphism map on a compact smooth manifold M, we say that $\Lambda \subset M$ is a hyperbolic manifold for f if Λ is a C^1 compact invariant submanifold of M with a hyperbolic structure subset of M, then f is said to act on a hyperbolic manifold of M. If M is hyperbolic for f then f is called Anosov diffeomorphism map.

And another aim of this section, is to find hyperbolic sets that have the strongly fitting shadowing property, and proving that the diffeomorphism $f|_{\Lambda}$ is an Anosov diffeomorphism map, if and only if Λ has this property.

Definition 3.3.1

Let $(\mathfrak{I}, \mathrm{d})$ be a metric space and $f: \mathfrak{I} \to \mathfrak{I}$ be a map. Then f is said to be have the **strongly fitting** shadowing property (SFSP): if for every $\varepsilon > 0$, there is $\delta > 0$, that is, $\forall \{\mathrm{y}_{\kappa}\} \in \mathfrak{I}$, $\kappa \in \mathbb{Z}$ be sequence, there is $\aleph = \aleph(\delta) \in \mathbb{N}$ that is for every m $\geq \aleph$, and for all z $\in \mathfrak{I}$, to get:

$$\lim_{m\to\infty}\sup\sum_{\kappa=0}^{m-1} d(f^{\kappa}(z), y_{\kappa}) < \varepsilon,\ \forall\kappa \geq \aleph$$

and

$$\lim_{m\to\infty}\sup\sum_{\kappa=0}^{m-1} d(f^{\kappa}(z), y_{\kappa}) < \varepsilon,\ \forall\kappa \leq -\aleph$$

Remark 3.3.2

If Λ is a hyperbolic manifold for f, then the set Λ has the shadowing property by Lemma (3.1.3). Note that, in general, the shadowing point z need not belong to Λ .

Here we show that if Λ is a hyperbolic manifold for f with the strongly fitting shadowing property then $f|_{\Lambda}$ is an Anosov diffeomorphism map.

A diffeomorphism f on M is Anosov diffeomorphism map if f has a hyperbolic structure on M.

Theorem 3.3.3

Let $\Lambda \subset M$ be a hyperbolic manifold for f that possesses the strongly fitting shadowing property. Then $f|_{\Lambda}$: $\Lambda \to \Lambda$ is Anosov diffeomorphism map.

Proof: Since Λ is hyperbolic, there exists $\varepsilon > 0$ such that if

$$d(f^{\kappa}(y), f^{\kappa}(z)) < \varepsilon$$

for any $y \in \Lambda, z \in M$ and $\forall\ \kappa \in \mathbb{Z}$ then $y = z$

choose $0 < \delta < \varepsilon$ such that any δ-pseudo orbit in Λ is

$\varepsilon/4$-shadowed by a point $y \in \Lambda$.

assume $0 < \theta < \frac{\delta}{4}$ such that $d(y, z) < \theta$, implies

$$d(f(y), f(z)) < \frac{\delta}{4}$$

assume U is compact neighborhood of Λ satisfying

$$\mathrm{U} \subset B(\Lambda, \theta/4)$$

we show that $\Lambda = \bigcap_{\kappa\in\mathbb{Z}} f^{\kappa}(\mathrm{U})$,

it is known that $\Lambda \subset \bigcap_{\kappa\in\mathbb{Z}} f^{\kappa}(\mathrm{U})$ since Λ is invariant

to show that $\bigcap_{\kappa\in\mathbb{Z}} f^{\kappa}(\mathrm{U}) \subset \Lambda$, let $z \in \bigcap_{\kappa\in\mathbb{Z}} f^{\kappa}(\mathrm{U})$.

then we have $z \in f^{\kappa}(\mathrm{U}), \forall \kappa \in \mathbb{Z}$, and so $f^{\kappa}(z) \in \mathrm{U}$

for each $\kappa \in \mathbb{Z}$ choose $y_{\kappa} \in \Lambda$ such that

$$d(y_{\kappa}, f^{\kappa}(z)) < \theta$$

then the sequence $\{y_{\kappa}\}_{\kappa\in\mathbb{Z}}$ is an δ-pseudo orbit for f , we have

$$d(f(y_{\kappa}), y_{\kappa+1}) \leq d(f(y_{\kappa}), f^{\kappa+1}(z)) + d(f^{\kappa+1}(z), y_{\kappa+1})$$

$$< \frac{1}{4}\delta + \theta < \delta$$

for each $\kappa \in \mathbb{Z}$ since Λ has the strongly fitting shadowing property, there is $y \in \Lambda$ such that $\{y_\kappa\}$ is $\varepsilon/4$-shadowed by the point $y \in \Lambda$, then, we have

$$d(f^\kappa(y), f^\kappa(z)) \leq d(f^\kappa(y), y_\kappa) + d(y_\kappa, f^\kappa(z))$$

$$< \frac{1}{4}\varepsilon + \theta$$

$$< \varepsilon$$

for each $\kappa \in \mathbb{Z}$ consequently, we get $y = z$, and $z \in \Lambda$.

We show that $f|_\Lambda$ is structurally stable. Let $g \in diff^1(\Lambda)$ be C^1 near to $f|_\Lambda$.

Then we can find $g^* \in Diff^1(M)$ such that $g^*$is C^1 near to f and

$g^*|_\Lambda = g$

If the maximal hyperbolic sets enjoy a type of structural stability, then we can find a homeomorphism.

$$\hbar: \bigcap_{\kappa\in\mathbb{Z}} f^\kappa(\mathrm{U}) \to \bigcap_{\kappa\in\mathbb{Z}} g^{*\kappa}(\mathrm{U})$$

such that

- $g^* o\, \hbar = \hbar o f$ on $\Lambda = \bigcap_{\kappa\in\mathbb{Z}} f^\kappa(\mathrm{U})$, and
- $\hbar$ is $C^0 -$ near to the identity map on Λ

because $g^*(\Lambda) = \Lambda$, we have $\Lambda \subset \bigcap_{\kappa\in\mathbb{Z}} g^{*\kappa}(\mathrm{U})$

take $\mathrm{K} = \hbar^{-1}|_\Lambda$.

since Λ is a compact manifold and K is $C^0 -$ near to the identity map on Λ, K is surjective.

Hence, we have $\hbar(\Lambda) = \hbar(\mathrm{K}(\Lambda)) = \Lambda$

This means that $f|_{\Lambda}$ is structurally stable

then we can see that $f|_{\Lambda}\colon \Lambda \to \Lambda$ is Anosov diffeomorphism map. ■

Corollary 3.3.4

Assume $\Lambda \subset M$ a hyperbolic manifold for f, a set Λ has the strongly fitting shadowing property, if and only if $f|_{\Lambda}\colon \Lambda \to \Lambda$ is an Anosov diffeomorphism map.

Proof: Let $f|_{\Lambda}\colon \Lambda \to \Lambda$ be an Anosov diffeomorphism map, it follows $\Lambda \subset M$ a hyperbolic manifold for f.

By Lemma (3.1.3), there is a neighborhood Ʊ of Λ which has shadowing property, then Λ has the strongly fitting shadowing property.

Now, we prove, by Theorem (3.3.3) is hold the other way.■

Now we wish to find hyperbolic sets which have the strongly fitting shadowing property.

Definition 3.3.5

Let $f\colon M \to M$ be a C^1-diffeomorphism map of a compact smooth manifold M, we say $\mathbb{B}(f)$ is the **Birkhoff centre** of f; that is

$$\mathbb{B}(f)=\overline{\{x \in M\colon x \in \sigma_1(x) \cap \sigma_2(x)\}}$$

Where $\sigma_1(x), \sigma_2(x)$ denote the positive and negative limit set of x. Then $\mathbb{B}(f)$is a nonempty closed invariant subset of M.

Then we have the following inclusions where $CR(f)$ is called **chain recurrent** in Definition (1.1.31), and $\Omega(f)$ is **non-wandering point** of f in Definition (1.1.28):

$$\overline{Per(f)} \subset \mathbb{B}(f) \subset \Omega(f) \subset CR(f) \text{……….. (3.12)}$$

Lemma "Stable manifold Theorem" 3.3.6 [41]

Let $\Lambda \subset M$ be a hyperbolic set for $f: M \to M$. Then there exist $\varepsilon > 0$ constants and $0 < \lambda < 1$ such that for all $x \in \Lambda$

- $W_\varepsilon^s(x) = \{z \in M: d(f^\kappa(z), f^\kappa(x)) < \varepsilon, \kappa \geq 0\}$ is C^1 submanifold of M with $T_x W_\varepsilon^s(x) = E_x^s$;
- If $y, z \in W_\varepsilon^s(x)$ then $d(f^\kappa(z), f^\kappa(y)) \leq \lambda^\kappa d(z, y)\ \forall\ \kappa \geq 0$

Lemma 3.3.7 [41]

Let $\Lambda \subset M$ be a hyperbolic set for $f: M \to M$, and $\varepsilon > 0$ a constants as in Lemma (3.3.6). Then

- There exists a constants $\delta > 0$ such that;

1. If $d(x, z) < \delta$, $\forall x, z \in \Lambda$, then $W_\varepsilon^s(x) \cap W_\varepsilon^u(z)$is a single point set;
2. The map $\gamma: \mho_\delta(\Lambda) \to M$ given by $\gamma(x, z) = W_\varepsilon^s(x) \cap W_\varepsilon^u(z)$ is continuous, where
   $\mho_\delta(\Lambda) = \{(x, z) \in \Lambda \times \Lambda: d(x, z) < \delta\}$; and

- There exists a constants $\delta^* > 0$ such that if $d(x,z) < \delta^*$, for all $x, z \in \Lambda$ then

$$d(x, \gamma(x,z)) < \delta \text{ and } d(z, \gamma(x,z)) < \delta$$

Theorem 3.3.8

If the set $\mathbb{B}(f)$ is hyperbolic for f then it has the strongly fitting shadowing property.

Proof: By apply Lemmas (3.3.6), (3.3.7) and (2.2.7), for the hyperbolic set $\mathbb{B}(f)$.

Then we obtain positive constants $\delta, \lambda < 1, \varepsilon, \varepsilon'$ and a neighborhood $\mho$ of $\mathbb{B}(f)$ in M which satisfy the results of the Lemmas (3.3.6), (3.3.7) and (2.2.7), and the inclusion

$$B(\mathbb{B}(f), \delta) \subset \mho$$

first we show that if

$$d(w_1, w_2) < \varepsilon \ \forall w_1, w_2 \in \mathbb{B}(f)$$

then the point $\gamma(w_1, w_2)$belongs to $\mathbb{B}(f)$

where γ is the map obtained in Lemma (3.3.7)

assume $\sigma_2 > 0$ arbitrary

$$\gamma(w_1, w_2) = x \text{ and } \gamma(w_2, w_1) = y$$

since $x \in W^s_\delta(w_1)$and $w_1 \in \sigma_1(w_1)$, there exists $a > 0$ satisfying

$$d(f^a(x), f^a(w_1)) < \frac{1}{4}\sigma_2$$

and

$$d(f^{a+1}(w_1),(w_1)) < \frac{1}{4}\,\sigma_2$$

since $w_1 \in \sigma_2(w_1)$ and $y \in W_\delta^u(w_1)$, we can choose $b > 1$such that

$$d(f(w_1), f^{-b}(w_1)) < \frac{1}{4}\,\sigma_2$$

and

$$d(f^{-b+1}(w_1), f^{-b+1}(y)) < \frac{1}{4}\,\sigma_2$$

since $y \in W_\delta^s(w_2)$ and $w_2 \in \sigma_2(w_2)$, there is $c > 0$ satisfying

$$d(f^c(y), f^c(w_2)) < \frac{1}{4}\,\sigma_2$$

and

$$d(f^{c+1}(w_2),(w_2)) < \frac{1}{4}\,\sigma_2$$

since $w_2 \in \sigma_2(w_2)$ and $x \in W_\delta^u(w_2)$, we can get $d > 1$ such that

$$d(f(w_2), f^{-d}(w_2)) < \frac{1}{4}\,\sigma_2$$

and

$$d(f^{-d+1}(w_2), f^{-d+1}(x)) < \frac{1}{4}\,\sigma_2$$

then the sequence

$$\{x, f(x), \dots, f^{a-1}(x), f^a(w_1), w_1, f^{-b}(w_1), f^{-b+1}(y), \dots, f^{-1}(y), y,$$

$$f(y), \dots, f^{c-1}(y), f^c\,(w_2), w_2, f^{-d}(w_2), f^{-d+1}(x), \dots, f^{-1}(x), x\}$$

is a periodic $\sigma_2 -$ pseudo orbit for f contained in $\mho$

Let $\ell > 0$ be a constant satisfy the results of Lemma (2.2.7), and $\tau > 0$ be arbitrary to show that $x \in \mathbb{B}(f)$.

By the first step of the proof, we can choose a periodic $\tau/4$- pseudo orbit $\{x = x_i = x\}, \forall\, 0 \le i \le n$ in $\mho$ from x to x.

assume that there exists a continuous map $g: M \to M$ satisfying

- $g(x_i) = x_{i+1}$, $\forall\, 0 \le i \le n-2$; $g(x_{n-1}) = x$;
- $d_0(f, g) < \tau$

If we apply Lemma (2.2.7), then we can choose a continuous map

$\hbar: \mho \to M$ satisfying the conditions

1. $f o\, \hbar = \hbar o g$ on ξ; and
2. $d_0(\hbar, Id_{\mho}) < \ell$

assume $\hbar(x) = \bar{x}$, then we have

$f^n(\bar{x}) = f^n(\hbar(x)) = \hbar(g^n(x)) = \hbar g(x_{n-1}) = \hbar(x) = \bar{x}$

This means that $x \in \mathbb{B}(f)$.

Next we show $\mathbb{B}(f)$ has the strongly fitting shadowing property

let $\beta > 0$ be arbitrary.

choose positive constants $\delta, \lambda < 1, \varepsilon, \varepsilon'$ satisfying the results of Lemmas (3.3.6) and (3.3.7), and assume that $\frac{4\varepsilon}{1-\lambda} < \beta$ and

$4\delta < \varepsilon' < \varepsilon$. Let $\xi = \{w_{1_\kappa}\}_{\kappa\in\mathbb{Z}}$ be an δ –pseudo orbit in $\mathbb{B}(f)$,

let $\xi_n^+ = \{w_{1_\kappa}\}_{\kappa=0}^{n}$

set $w_{1_0} = w_{2_0}$ and define w_{2_j} respectively by

$$w_{2_j} = \gamma\left(w_{1_j}, f\left(w_{2_{j-1}}\right)\right), 0 \le j \le n$$

since $w_{2_{j-1}} \in W_\delta^s(w_{1_{j-1}})$and $d\left(w_{1_j}, f\left(w_{1_{j-1}}\right)\right) < \delta$

we have

$$d\left(w_{1_j}, f\left(w_{2_{j-1}}\right)\right) < d\left(w_{1_j}, f\left(w_{1_{j-1}}\right)\right) + d\left(f\left(w_{1_{j-1}}\right), f\left(w_{2_{j-1}}\right)\right)$$

$$< \delta + \delta < \varepsilon', \forall 1 \le j \le n$$

hence Definition (3.3.1), is valid, and the points w_{2_j} belong to $\mathbb{B}(f)$.

since

$$w_{2_j} \in W_\delta^s(w_{1_j}) \cap W_\delta^u(f\left(w_{2_{j-1}}\right))$$

by Lemma (3.3.6), we have

$$d(w_{1_j}, w_{2_j}) < \varepsilon$$

and

$$d\left(w_{2_j}, f\left(w_{2_{j-1}}\right)\right) < \varepsilon, \qquad \forall 1 \le j \le n$$

put $\overline{w_2} = f^{-n}(w_{2_n})$. Then the set ξ_n^+ is β −shadowed by the point $\overline{w_2} \in \mathbb{B}(f)$, we have

$$d\left(f^j(\overline{w_2}), w_{2_j}\right) = d(f^{j-n}(w_{2_n}), w_{2_j}) \le$$

$$d(w_{2_j}, f^{-1}(w_{2_{j+1}})) + d(f^{-1}(w_{2_{j+1}}), f^{-2}(w_{2_{j+2}}) + \cdots$$
$$+ d(f^{j-n+1}(w_{2_{n-1}}), f^{j-\mathrm{n}}(w_{2_n}))$$

$$=\textstyle\sum_{\kappa=1}^{n-j} d(f^{-\kappa+1}(w_{2_{j+\kappa-1}}), f^{-\kappa}(w_{2_{j+\kappa}}))$$

$$= \sum_{\kappa=1}^{n-j} d(f^{-\kappa}(f(w_{2_{j+\kappa-1}})), f^{-\kappa}(w_{2_{j+\kappa}}))$$

$$\leq \sum_{\kappa=1}^{n-j} \lambda^{\kappa} d(f(w_{2_{j+\kappa-1}}), w_{2_{j+\kappa}})$$

$$< \sum_{\kappa=1}^{\infty} \lambda^{\kappa} \varepsilon = \frac{\varepsilon}{1-\lambda}$$

consequently, $\forall 1 \leq j \leq n$ we have

$$d\left(f^{j}(\overline{w_2}), w_{1_j}\right) \leq d\left(f^{j}(\overline{w_2}), w_{2_j}\right) + \mathrm{d}\left(w_{2_j}, w_{1_j}\right)$$

$$< \frac{\varepsilon}{1-\lambda} + \varepsilon$$

$$< \beta$$

for any $1 \leq j \leq n$, similarly we can show that every finite pseudo orbit

$\xi_n = \{w_{1_{-n}}, \dots, w_{1_{-1}}, w_{1_0}, w_{1_1}, \dots, w_{1_n}\}$ of ξ is β-shadowed by a point $\overline{w_{2_n}} \in \mathbb{B}(f)$, for each $n \geq 1$

let $\lim_{n\to\infty} \overline{w_{2_n}} = \overline{w_1}$, then to show that ξ is β-shadowed by a point

$\overline{w_1} \in \mathbb{B}(f)$. This means that

$\mathbb{B}(f)$ has the strongly shadowing property.■

Note 3.3.9

- If the chain recurrent set $CR(f)$ is hyperbolic then we have $CR(f) = \mathbb{B}(f)$,
- Note that $\mathbb{B}(f) \neq \Omega(f)$ even if $\Omega(f)$ is hyperbolic.

Corollary 3.3.10

If the set $CR(f)$ is hyperbolic manifold for f, then it has the strongly fitting shadowing property.

Proof: By Theorem (3.3.8),$\mathbb{B}(f)$ has the strongly fitting shadowing property, like that by Equation (3.12), satisfying $\mathbb{B}(f) \subset CR(f)$.

Since $\mathbb{B}(f)$and $CR(f)$are hyperbolic sets, then $\mathbb{B}(f)$= $CR(f)$

So, $CR(f)$has the strongly fitting shadowing property.

Note that $\mathbb{B}(f) \neq \Omega(f)$ even $\Omega(f)$ is hyperbolic. ■

Theorem 3.3.11

If $\mathbb{B}(f)(or\ CR(f))$ is hyperbolic manifold for f then, $f|_{\mathbb{B}(f)}$(or $f|_{CR(f)}$) is Anosov diffeomorphism map, respectively.

Proof: Let $\mathbb{B}(f)(or\ CR(f))$be hyperbolic manifold for f.

By Theorem (3.3.8)(or by Corollary (3.3.10)), $\mathbb{B}(f)(or\ CR(f))$ has the strongly fitting shadowing property.

Since $\mathbb{B}(f)(or\ CR(f))$ is hyperbolic set and has the strongly fitting shadowing property.

Then, by Theorem (3.3.3),

$$f|_{\mathbb{B}(f)}: \mathbb{B}(f) \to \mathbb{B}(f)(\text{or } f|_{CR(f)}: CR(f) \to CR(f)$$

is Anosov diffeomorphism map . ■

Chapter Four

Conclusions and Future Work

4.1 Conclusions

In this section, we list the most important results obtained during this work.

- Let A be a $n \times n$ matrix with integer entries. Then A acts linearly on $\mathbb{R}^n$, and we claim that this linear can be used to define a map $f_A: T^n \to T^n$on the torus T^n. This work investigated the general properties of Anosov diffeomorphism map f_A.

- If $f_A: T^n \to T^n$ is Anosov diffeomorphism map of T^n, thus we call f_A structural stability.

- In this work, the researcher investigated the topological and metrical chaoticity properties of Anosov diffeomorphism map f_A.

- It was proved that the Anosov diffeomorphism map f_A fulfills several definitions of chaos, Culick definition, Lypunov definition, Wiggin definition, Devaney definition and Touhey definition.

- We proved that Anosov diffeomorphisms map $f: M \to M$, where M, the closed C^∞ manifold, has fitting shadowing property.

- The sufficient conditions have been established to prove the partial hyperbolic map $f: \mathrm{M} \to \mathrm{M}$ on a closed $\mathrm{C}^\infty$of M (that is, boundaryless and compact) has fitting shadowing property.

- We proved cases where fitting shadowing property is not related to partial hyperbolic map.

- The study proved eventual fitting shadowing property of hyperbolic set.

- The researcher proved strongly fitting shadowing property of hyperbolic set.

4.2 Future work

We would like to expand for the future work investigating the following topics on hyperbolic dynamical systems.

1. Entropy for partially hyperbolic diffeomorphisms map with a uniformly compact center foliation W^c with dimension two.

2. Searching for other types of hyperbolic systems and study their chaotic behavior.

3. Finding other practical applications for these maps in various other sciences.

4. Studying other types of shadowing property in hyperbolic dynamical systems.

5. Examining stability in hyperbolic dynamical systems.

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الخلاصة

تهدف هذه الاطروحة إلى دراسة وتطوير الخصائص الفوضوية للأنظمة الديناميكية الزائدية.

طوال هذا العمل ، ليكن $f_A: M \to M$ تسمى دالة Anosov المتشاكلة للمنفولد M ، عندما تكون M هي التورس T^n للبعد n ، نجد ان دالة Anosov المتشاكلة تحق عدة تعاريف للفوضى مثل :

- تعريف Gulick
- تعريف L-chaotic
- تعريف W-chaotic
- تعريف Devaney
- تعريف Touhey

وضحت حالة الاستقرار لخاصية التظليل الملائمة C^1 لتشاكل f حول C^∞ (للمنفولد المغلق M للمجموعة Λ المغلقة المتغايرة الجزئية من M، فان $f_{|\Lambda}$ يحقق حالة الاستقرار لخاصية التظليل الملائمة) C^1 .

بشكل عام ، إذا كان من الممكن تحقيق دال التشاكل الزائدي الجزئيً خاصية التظليل الملائمة الا إذا كان المركز W^c uniformly compact center foliation .

بين ان أي مجموعة لسلسلة متعدية Λ بشكل عام C^1، لتشاكل f يمتلك نوع آخر من خاصية التظليل تسمى خاصية التظليل الملائمة النهائية .

نقيد التشاكل $f|_\Lambda$ لشرح اذا كانت دالة Anosov اذا وفقط اذا Λ تمتلك خاصية التظليل الملائمة بقوة ، كذلك وجدنا مجموعات زائدية تمتلك تلك الخاصية .

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أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة في جامعة بابل كجزء

من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل

دنيا محمد كريم حسين

بإشراف

أ.د. أفتخار مضر طالب الشرع

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