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Solving Optimization Problems By Using Iteration Methods

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بسم الله الرحمن الرحيم

﴿يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ

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Dedication

To those who enlightened my mind others

My instructors

To the light that illuminates my path of success...

My mother and my brother

To those who have borne the burden of my scientific

Career....

My sons and my wife

To my companions and loved ones.......................

my brothers

Give this humble effort and hope that it will be....

accepted

Haider Neama

Abstract

Newton's method is an iterative numerical technique used to find approximate solutions to nonlinear equations. It relies on the concept of linear approximation by iteratively improving the initial estimate towards the root of the equation

The method is highly efficient when the initial guess is close to the actual root and when the function is well-behaved in the vicinity of the root. However, convergence is not guaranteed for all cases, and some roots might be missed if multiple roots exist.

The fixed-point iteration method is another approach to solving nonlinear equations by transforming the problem into finding fixed points of an iterative function.

Contents

Introduction

Newton's method is an iterative numerical method for finding the roots of a differentiable function. It was developed by Sir Isaac Newton in the 17th century, and is still widely used in various fields of science and engineering.

Isaac Newton was an English physicist and mathematician, who was the culminating figure of the Scientific Revolution of the 17th century. In optics, his discovery of the composition of white light integrated the phenomena of colures into the science of light and laid the foundation for modern physical optics. In mechanics, his three laws of motion, the basic principles of modern physics, resulted in the formulation of the law of universal gravitation. In mathematics, he was the original discoverer of the infinitesimal calculus.

The basic idea behind Newton's Method is to start with an initial guess for the root of the function, and then use the derivative of the function to refine the guess. This is done by constructing a tangent line to the function at the current guess, and finding where the tangent line intersects the x-axis. This intersection point becomes the new guess, and the process is repeated until the desired level of accuracy is achieved.[1]

Newton's Method provides a powerful and versatile approach for approximating solutions to nonlinear equations. Its effectiveness lies in its iterative nature and exploitation of tangent lines to navigate the complex landscape of nonlinear functions. Despite its limitations and sensitivity to initial guesses, Newton's Method remains a cornerstone of numerical analysis and plays a vital role in various scientific, engineering, and mathematical applications.[2]

Chapter One

Newton's Method

1.1 Introduction

The essence of Newton's method lies in its ability to iteratively refine an initial estimate of the optimal solution. It achieves this by leveraging the behavior of the function's derivative. By repeatedly adjusting the estimate using the function's value and derivative, the method converges towards a point where the derivative is nearly zero. This point corresponds to a potential maximum or minimum of the function, depending on the nature of the optimization problem.

1.2 The Idea of Newton's Method

Newton's Method is a technique for solving equations of the form $f(x) = 0$, by successive approximation. The idea is to pick an initial guess x_0 such that $f(x_1)$ is reasonably close to 0 . We then find the equation of the line tangent to $y = f(x)$ at $x = x_0$ and follow it back to the x-axis at a new (and improved!) guess x_1 The formula for this is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We then find the equation of the line tangent to $y = f(x)$ at $x = x_1$ and follow it back to the x-axis to get a new (and improved!) guess x_2 from the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We keep on refining our guesses until we are close enough for whatever application we have in mind. In general, we have the recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In typical situations, Newton's Method homes in on the answer extremely quickly, roughly doubling the number of decimal points in each round. So if your original guess is good to one decimal place, 5 rounds later you will have an answer good to 30+ digits

numerical methods, it is the most powerful method that is used to solve the equation $f(x) = 0$. If we take the Taylor's series expansion of the function $f(x)$ about the point x_1: [1] .

$$f(x) = f(x_1) + (x - x_1)\, f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + \cdots \qquad (1)$$

where f, and its first and second order derivatives, f' and f'' are calculated at x_1 . If we take the first two terms of the Taylor's series expansion, we get :

$$f(x) \approx f(x_1) + (x - x_1)\, f'(x_1) \qquad (2)$$

We are set (2) to zero (i.e. $f(x) = 0$) to find the root of the equation which gives us:

$$f(x_1) + (x - x_1)\, f'(x_1) = 0 \qquad (3)$$

By rearranging (3), we obtain the next approximation to the root, giving by :

$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad (4)$$

Thus, by generalizing (4), we obtain Newton's iterative method: [3]

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} \quad , \qquad i \in N$$

Example : 1.1

Using Newton's Method to Compute a Square Roots Calculators use Newton's Method to compute square roots. On this slide, we'll see how to compute $\sqrt{2}$ Finding the square root of 2 is the same thing as solving $x^2 - 2$ So we set [4]

$$F(x) = x^2 - 2$$

$$f'(x) = 2x$$

and apply the recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\,2}-2}{2x_n}$$

Equation solution table 1.1

X_0	$F(x) = x^2-2$	$f'(x) = 2x$	$x - \frac{F(x)}{f'(x)}$	$\sqrt{2} =$ 1.4142135623731
1	1	2	$1-\frac{-1}{2}=\frac{3}{2}$	1.5000000000000
$\frac{3}{2}$	$\frac{1}{4}$	3	$\frac{3}{2}-\frac{\frac{1}{4}}{2}=\frac{17}{2}$	1.4166666666667
$\frac{17}{12}$	$\frac{1}{144}$	$\frac{17}{6}$	$\frac{17}{12}-\frac{1/144}{17/6}=\frac{577}{408}$	1.4142156862745
$\frac{577}{408}$	$\frac{1}{166464}$	$\frac{577}{204}$	$\frac{665857}{470832}$	1.4142135623747

Example 1.2

$$F(x) = e^x + x - 4$$

Sol:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{e^{x_n} + x_n - 4}{e^{x_n} + 1}$$

$$F(x) = e^x + x - 4$$

$$f'(x) = e^x + 1$$

Equation solution table 1.2

X_0	$F(x) = e^x + x - 4$	$f'(x) = e^x + 1$	$x - \frac{f(x)}{f'(x)}$
1	-0.281718172	3.718281828	1.0757656
1.0757656	0.00800256201	3.93223696242	1.07373048307
1.07373048307	0.0000060681	3.926215865	1.07372493756
1.07372493756	0.00000000001		

Example 1.2

F(x) = $x^3 - 7$

Sol:

$f'(x) = 3x^2$

Equation solution table 1.3

X_0	$F(x) = x^3 - 7$	$f'(x) = 3x^2$	$x - \frac{f(x)}{f'(x)}$
2	1	12	1.91666667
1.91666667	0.041088	11.02083372	1.879384584
1.879384584	-0.361851242	10.59625924	1.913533545
1.913533545	0.006614765		

1.3 The Importance Of Newton's Method

1. Efficient Convergence: Newton's Method exhibits fast convergence, often achieving quadratic convergence. This means that the number of accurate digits approximately doubles with each iteration, leading to rapid convergence towards the optimal solution. This efficiency makes it particularly valuable in complex optimization problems where finding the solution quickly is crucial.
2. Handling Multivariate Optimization: Newton's method can effectively handle optimization problems with multiple variables. By utilizing partial derivatives, it ca
3. identify critical points and determine whether they are local minima, maxima, or

saddle points. This capability makes it versatile for optimizing functions with multiple dimensions.

4. Widely Applicable: Newton's method finds application in various fields, including engineering, physics, economics, and computer science. Its versatility allows it to tackle diverse optimization problems, making it a valuable tool across multiple disciplines.

5- the significance of Newton's Method in solving optimization problems lies in its efficient convergence, ability to handle multivariate optimization, capability to incorporate constraints, and its wide applicability across various disciplines. These qualities make it a valuable and widely used technique for finding optimal solutions efficiently.

6- Newton's Method for solving equations is a powerful and effective method[5].

1.4 Disadvantages Of Newton's Method

1- Need for initial guess: Newton's Method requires an initial guess for the solution. It can be difficult to determine an appropriate initial guess, especially when there are multiple roots or potential solutions.

2- Need for repeated calculations: Using Newton's Method requires repeated calculations, which can negatively impact performance in some cases.

3- Need for linear algebra rules: Newton's Method requires knowledge of linear algebra rules, which can be challenging for some individuals.

4- Lack of stability: Newton's Method may be unstable in some cases and may require additional adjustments to achieve stability.

5- May not work well with some functions: Newton's Method can work well with some

functions, but may not be effective when dealing with complex and nonlinear

function [6].

Chapter Two

Linear Equations

2.1 Introduction

linear equation is a mathematical equation that represents a linear relationship between variables. It is an equation in which each term is either a constant or a product of a constant and a variable raised to the power of one. Linear equations can be expressed in the general form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $x_1, x_2, \ldots, x_n$ represent the variables, $a_1, a_2, \ldots, a_n$ are the coefficients, and b is a constant term.

The key characteristic of a linear equation is that the highest exponent of any variable is one. This means that the variables appear linearly, without being squared or raised to any other power.

Linear equations play a fundamental role in mathematics and have wide-ranging applications in various fields such as physics, engineering, economics, and computer science. They provide a way to model and analyze relationships between variables and solve problems involving proportional relationships, rates of change, and systems of equations .

The solutions to linear equations can vary. A linear equation may have a unique solution, where there is a specific set of values for the variables that satisfy the equation. It can also have no solution, indicating that there are no values for the variables that make the equation true (inconsistent equations). Additionally, a linear equation may have infinitely many solutions, suggesting that there are multiple sets of values that satisfy the equation (dependent equations).

Solving linear equations involves finding the values of the variables that make the equation true. Various methods, such as substitution, elimination, or matrix operations, can be employed to determine the solutions to linear equations. These methods allow for the manipulation and rearrangement of equations to simplify and isolate the variables, leading to the discovery of the desired solutions.[7].

Examples 2.2

1. The equation of a straight line: $y = mx + b$, where y is the dependent variable, x is the independent variable, m is the slope, and b is the y-intercept. Example: $y = 2x + 3$
2. The equation for calculating the total cost of an item: C = mP + b, where C is the total cost, P is the quantity purchased, m is the unit price, and b is the fixed cost. Example: $C = 5P + 10$
3. The equation for converting Celsius to Fahrenheit: $F = (9/5)C + 32$, where F is the temperature in Fahrenheit and C is the temperature in Celsius. Example: $F = (9/5)C + 32$
4. The equation for determining the distance traveled by an object with constant velocity: $d = vt$, where d is the distance, v is the velocity, and t is the time. Example: $d = 50t$
5. The equation for calculating the annual interest earned on an investment : $A = P(1 + rt)$, where A is the final amount, P is the principal investment, r is the interest rate, and t is the time in years. Example: $A = 1000(1 + 0.05t)$

These are just a few examples of linear equations, where the relationship between variables is a straight line.[7]

2.2 Newton's Method for solving linear equations

Newton's method is used to find the roots of a function, not necessarily to solve linear equations directly Newton's method, we need to find the root of the function $f(x)$, which means we need to solve for x such that $f(x) = 0$. The iterative formula for Newton's method is

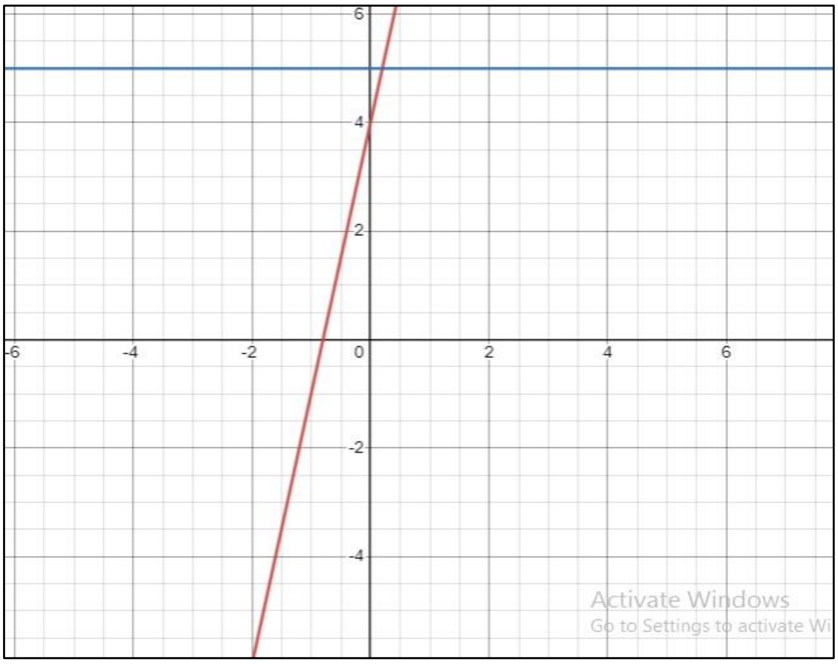
$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

where x_n is the current approximation and $f'(x_n)$ is the derivative of f(x) with respect to x $at\ x_n = x$[5]

Example2.2

$f(x) = 5x + 4$

$$f'(x) = 5$$



$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

$$Let\ x_0 = 1$$

$$x_1 = x_0 - f(x_0) \,/\, f'(x_0)$$

$$x_1 = 1 - \frac{5(1) + 4}{5} = 1 - \frac{9}{5} = \frac{-4}{5}$$

$$x_2 = \frac{-4}{5} - \frac{5\left(\frac{-4}{5}\right) + 4}{5} = \frac{-4}{5}$$

We note the appearance of the same product (root) when iterating, which achieves the solution when $f(x) = 0$

Please note that in this specific case of a linear equation, Newton's method is not necessary, as the root can be obtained directly by isolating x in the equation[5].

Example 2.4

$$f(x) = 2x - 6 = 0$$

$$f'(x) = 2$$

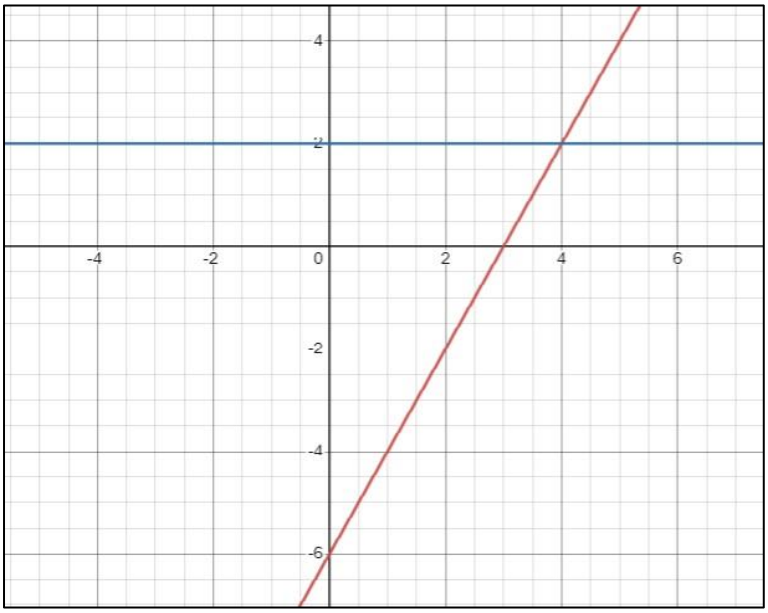
$$x_{n+1} = x_n - f(x_n) \,/\, f'(x_n)$$

Let $x_0 = 2$

$$x_1 = x_0 - f(x_0) \,/\, f'(x_0)$$

$$x_1 = 2 - \frac{2(2) - 6}{2} = 2 + 1 = 3$$

$$x_1 = 3 - \frac{2(3) - 6}{2} = 3$$



2.3 The main difference between Newton's Method and direct methods :

The main difference between Newton's Method and direct methods for solving linear equations lies in their approach and computational complexity

2.3.1 Newton's Method

Newton's Method is primarily used for finding the roots of nonlinear equations or functions. It is an iterative numerical method that relies on approximating the root by linearizing the function around an initial guess and then refining the approximation through successive iterations. Newton's method requires the computation of derivatives and can converge to a solution more quickly for well-behaved functions. However, it may suffer from convergence issues if the initial guess is far from the actual root or if the function has complex behavior .

In the context of linear equations, Newton's method is not typically used directly since linear equations have well-defined direct solution methods. Instead, Newton's method is more commonly employed for solving nonlinear systems of equations, where the equations are nonlinear and need to be linearized [6].

2.3.2 Direct Methods

Direct methods for solving linear equations provide a direct solution without the need for iterative approximation. These methods aim to manipulate the system of equations to obtain a unique solution or identify cases where no solution exists or the equations are dependent. Some common direct methods include substitution, elimination (e.g., Gaussian elimination), and matrix methods (e.g., inverse matrix, Cramer's rule).

Direct methods are more straightforward and conceptually simpler for solving linear equations. They involve performing arithmetic operations and manipulations on the equations to isolate the variables and obtain a solution. These methods are generally reliable and computationally efficient for small to moderately-sized linear systems.

However, the computational complexity of direct methods can be high for large-scale systems due to the need for matrix inversion and other operations.

In summary, the key distinction lies in the approach and nature of the equations being solved. Newton's method is an iterative numerical method primarily used for nonlinear equations or systems, whereas direct methods provide direct solutions for linear equations or systems. Direct methods are typically more suitable and efficient for linear equations, while Newton's method is more applicable to nonlinear equations or systems where linearization is required.[8]

comparison table:

Aspect	Direct Method	Newton's Method
Approach	Iterative	Iterative
Basic Idea	Directly substitute values	Approximate using tangent line
Formula	Equation to solve	$x_1 = x_0 - f(x_0)/f'(x_0)$
Initialization	Initial guess required	Initial guess required
Convergence	May converge slowly	Typically fast convergence
Advantages	Simple, intuitive	Faster convergence
Disadvantages	Slower convergence	Requires derivative calculation
Example Equation	$2x^2 - 5x + 2 = 0$	$x^2 - 5 = 0$
Iteration 1	$x = 1.0$	$x_1 = 3.0$
Iteration 2	$x = 0.25$	$x_2 = 2.4$
Iteration 3	$x = 1.53125$	$x_3 = 2.23846$
Solution	$x \approx 1.53125$	$x \approx 2.23607$

In this example, we're solving the equation $2x^2 - 5x + 2 = 0$ using the direct method and the equation $x^2 - 5 = 0$ using Newton's method. The comparison highlights how Newton's method generally converges faster compared to the direct method, but it requires the calculation of the derivative, which the direct method doesn't.

Chapter Three

Nonlinear System of Equations

3.1 Introduction :

A nonlinear equation is a mathematical equation that does not follow a linear relationship between the variables involved.

In a nonlinear equation, the dependent variable is not directly proportional to the independent variable(s) and may exhibit complex and non-uniform behavior . Nonlinear equations often involve powers, exponentials, logarithms, trigonometric functions, or other mathematical operations that introduce nonlinearity into the equation.

These equations can have multiple solutions or no solutions at all, and their solutions may require numerical methods or approximation techniques to find. Nonlinear equations are essential in various fields of science, engineering, and mathematics, where linear relationships are inadequate to model complex phenomena accurately [9] .

Examples 3.1

1. The equation for compound interest: $A = P(1 + r/n)^{nt}$, where A is the final amount, P is the principal investment, r is the annual interest rate, n is the number of times interest is compounded per year, and t is the time in years. This equation involves an exponential function. Example: $A = 1000(1 + 0.05/12)^{12t}$
2. The equation for a circle: $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) represents the centre of the circle and r is the radius. This equation involves squared terms.
   Example: $(x - 2)^2 + (y + 1)^2 = 9$
   The equation for population growth with a carrying capacity:
   $$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where P represents the population, t is time, r is the growth rate, and K is the carrying capacity. This equation involves a nonlinear term $(1 - P/K)$ that introduces a limitation on growth.

Example : $\frac{dP}{dt} = 0.1\,P(1 - P/1000)$

3. The equation for the motion of a pendulum :

$$\theta'' + (g/L)\sin(\theta) = 0$$

where θ represents the angle of displacement, g is the acceleration due to gravity, and L is the length of the pendulum . This equation involves a trigonometric function.

Example: $\theta'' + (9.8/2)\sin(\theta) = 0$

4. The equation for the logistic growth model :

$$dP/dt = rP(1 - P/M)$$

where P represents the population, t is time, r is the growth rate, and M is the carrying capacity. This equation introduces a nonlinear term $(1 - P/M)$ that reflects limited resources and competition.

Example: $dP/dt = 0.05P(1 - P/5000)$

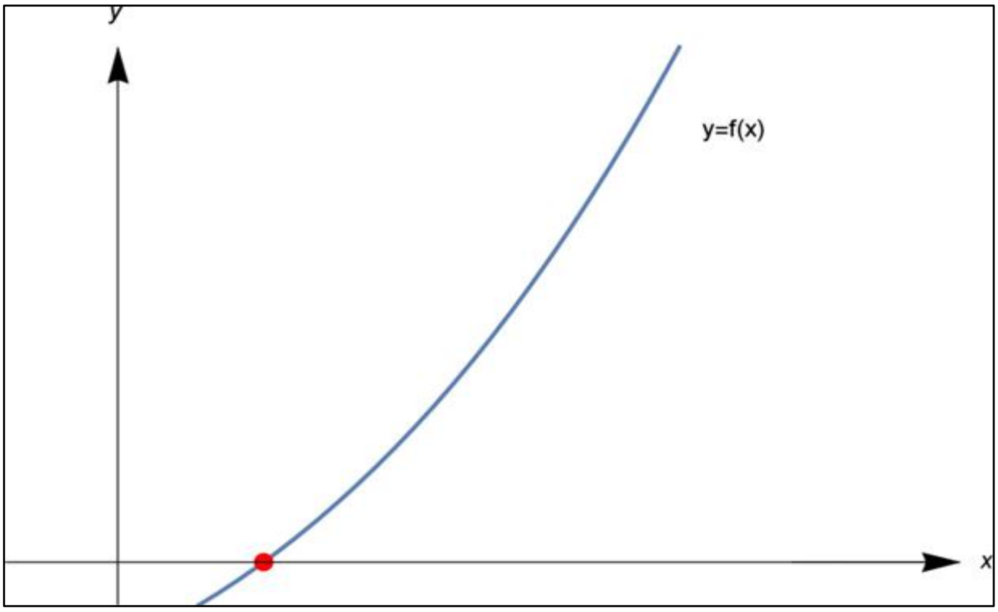
These examples demonstrate how nonlinear equations involve mathematical operations such as exponentials, trigonometric functions, or terms with powers, leading to more complex and diverse relationships between variables.[9]

3.2 Finding a Solution With Geometry

Newton's Method for solving equations is another numerical method for solving an equation f(z) = 0. It is based on the geometry of a curve, using the tangent lines to a curve. As such, it requires calculus, in particular differentiation. [10]

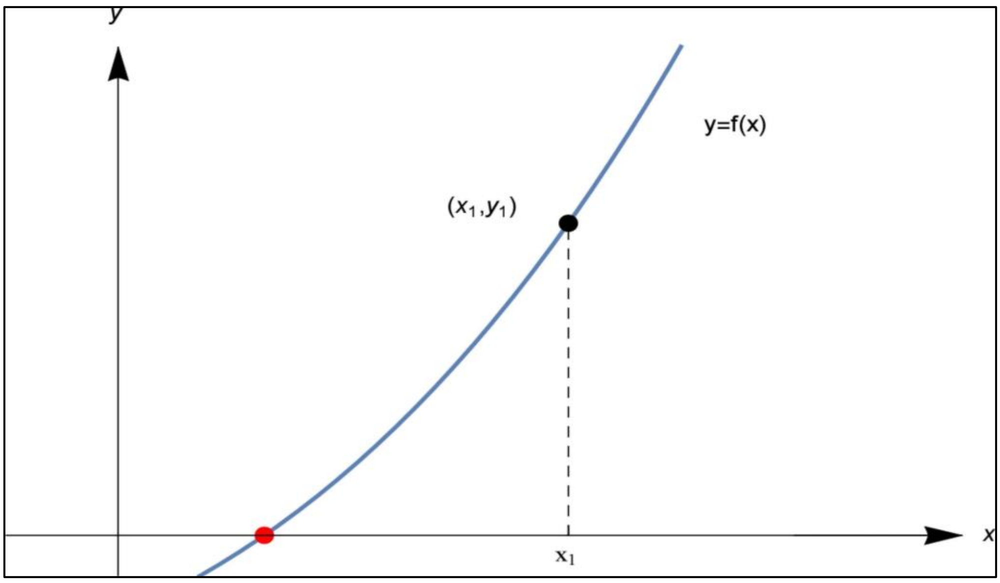
Roughly, the idea of Newton's method is as follows. We seek a solution to f(x) = 0. That is, we want to find the red dotted point in the picture below.[10]

figure 1



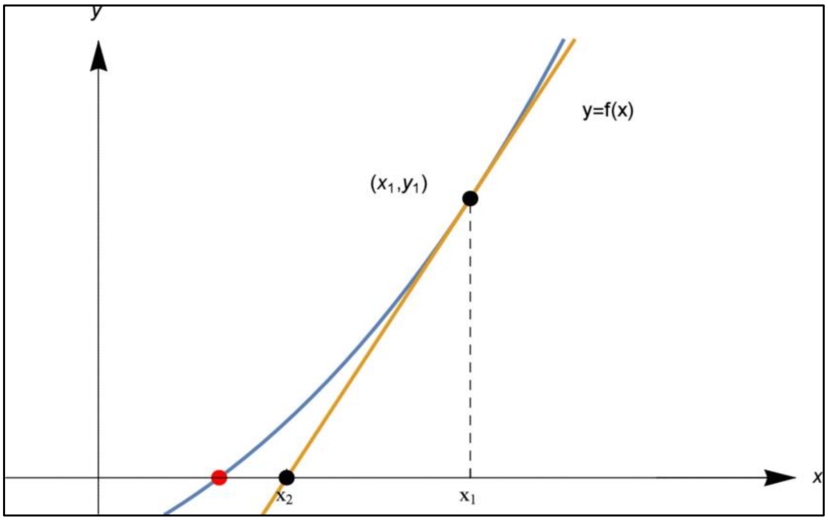
We start with an initial guess x_1 . We calculate $f(x_1)$. If $f(x_1) = 0$, we are very lucky, and have a solution. But most likely $f(x_1)$ is not zero. Let $f(x_1) = y_1$, as shown.

figure 2



We now try for a better guess. How to find that better guess? The trick of Newton's method is to draw a tangent line to the graph $y = f(x)$ at the point (x_1, y_1). See below.[11]

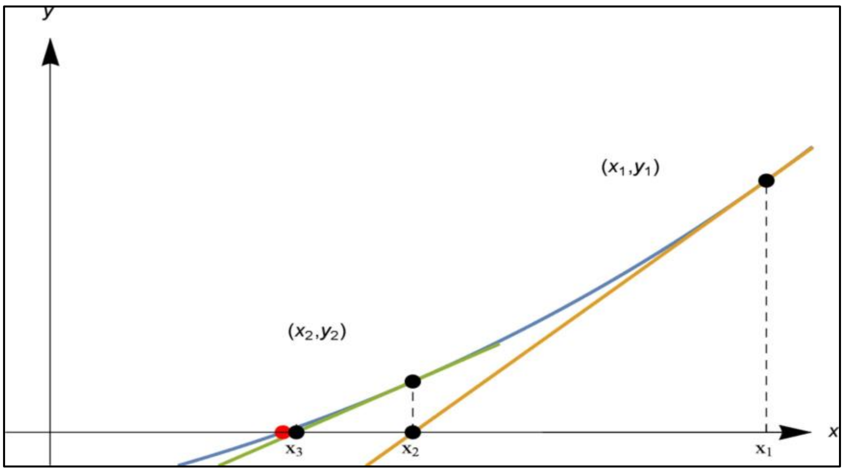
figure 3



This tangent line is a good linear approximation to $f(x)$ near x_1 so our next guess x_2 is the point where the tangent line intersects the x-axis, as shown above .

We then proceed using the same method. We calculate $y_2 = f(x_2)$; if it is zero, we're finished. If not, then we draw the tangent line to $y = f(x)$ at (x_2, y_2), and our next guess x_3 is the point where this tangent line intersects the x-axis :

figure 4



In the figure shown, x_1, x_2, x_3 rapidly approach the red solution point! Continuing in this way, we find points x_1, x_2, x_3, x_4, -----x_n approximating a solution.

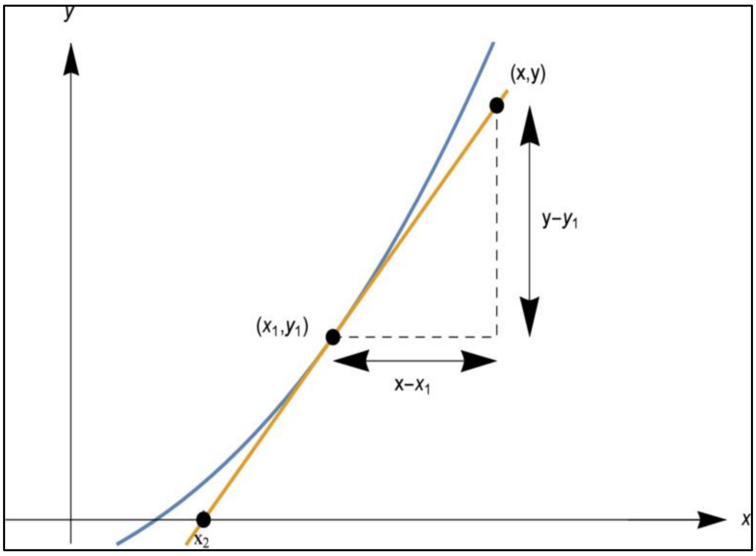
This method for finding a solution is Newton's method. As we'll see, Newton's Method can be a very efficient method to approximate a solution to an equation when it works[12].

The key calculation As our introduction above just showed, the key calculation in each step of Newton's method is to find where the tangent line to y = f(x) at the point (x1, y1) intersects the x-axis. (x_1, y_1) and has gradient f(x_1) Recall that, given the gradient rn of a line, and a point (x0, y0) on it, the line has equation

$$\frac{y-y_1}{x-x_1} = m, \text{or equivalently}, \mathrm{y} = \mathrm{m}(\mathrm{x}-x_0)+y_0$$

In our situation, the line has gradient f (x_1), and passes through (x_1, y_1), so has equation See the diagram bellow.

figure 5



Setting y = 0, we find the x-intercept as

$$x = x_1 - \frac{y_1}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

3.3 Newton's algorithm:

We can now describe Newton's method algebraically. Starting from x_1, the above calculation shows that if we construct the tangent line to the graph:[11]

1- $y = f(x)$ at $x = x_1$

this tangent line has x-intercept given by

2- $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Then, starting from x_2 we perform the same calculation, and obtain the next approximation x_3 as

3- $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

The same calculation applies at each stage: so from the n'th approximation x_n , the next approximation is given by

4- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Definition 3.4

Let f: R ⟶ R be a differentiable function. We seek a solution of $f(x) = 0$ starting from an initial estimate $x = x_1$.

At the n'th step, given x_n , compute the next approximation $x_{n=1}$ by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ and repeat .[12]

3.5 Algorithm properties

1- Often, Newton's method works extremely well, and the x_n converge rapidly to a solution. However, it's important to note that Newton's Method does not always work. Several things can go wrong, as we will see shortly.

2- Note that if $f(x_n) = 0$, so that x_n is an *exact* solution of $f(x) = 0$, then the algorithm gives $x_{n+1}=x_n$, and in fact all of x_n , x_{n+1} , x_{n+2} ,$x_{n=3}$ ... will be equal. If you have an exact solution, Newton's method will stay on that solution!

3- While the bisection method only requires f to be continuous, Newton's method requires the function jf to be *differentiable.* This is necessary for f to have a tangent line.[13]

Example 3.2

Use Newton's Method to determine an approximation to the solution to $cos(x) = x$ that lies in the interval [0, 2]. Find the approximation to six decimal places.

Solution :

First note that we weren't given an initial guess. We were however, given an interval in which to look. We will use this to get our initial guess. As noted above the general rule of thumb in these cases is to take the initial approximation to be the midpoint of the interval. So, we'll use $x_0 = 1$ as our initial guess .

Next, recall that we must have the function in the form f (x) = 0. Therefore, we first rewrite the equation as,

$$cos(x) - x = 0$$

We can now write down the general formula for Newton's Method. Doing this will often simplify up the work a little so it's generally not a bad idea to do this.

$$x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$$

Let's now get the first approximation.

$$x_1 = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} = 0.7503638679$$

At this point we should point out that the phrase “six decimal places” does not mean just get x_1 to six decimal places and then stop. Instead it means that we continue until two successive approximations agree to six decimal places. Given that stopping condition we clearly need to go at least one step farther.

$$x_2 = 0.7503638679 - \frac{\cos(0.7503638679) - 0.7503638679}{-\sin(0.7503638) - 1} = 0.7391128909$$

Alright, we’re making progress. We’ve got the approximation to 1 decimal place.

Let’s do another one, leaving the details of the computation to you.

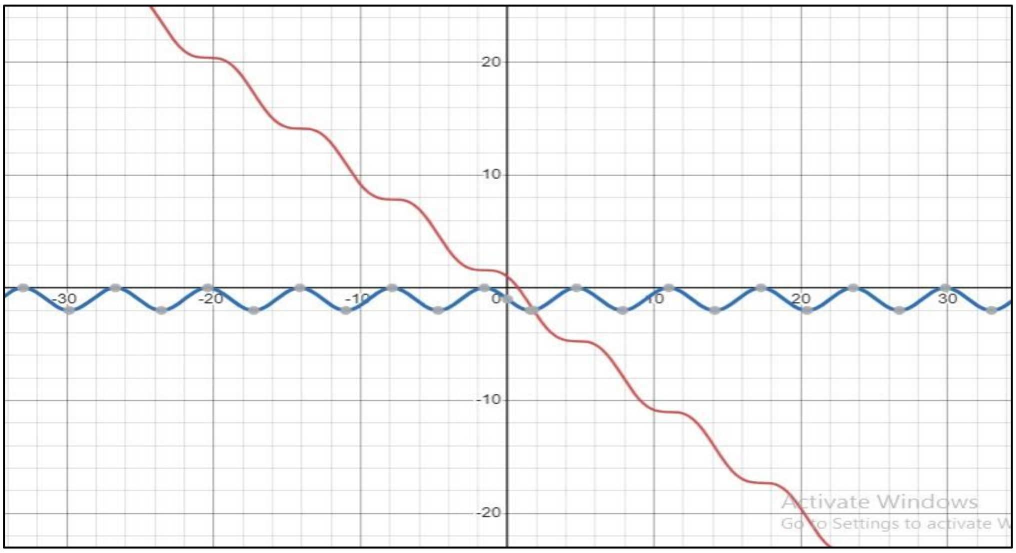
$$x_3 = 0.7390851334$$

We’ve got it to three decimal places. We’ll need another one.

$$x_4 = 0.7390851332$$

And now we’ve got two approximations that agree to 9 decimal places and so we can stop. We will assume that the solution is approximately.

$$x_5 = 0.7390851332$$



In this last example we saw that we didn't have to do too many computations in order for Newton's Method to give us an approximation in the desired range of accuracy.

This will not always be the case. Sometimes it will take many iterations through the process to get to the desired accuracy and on occasion it can fail completely. The following example is a little silly but it makes the point about the method failing.

Example3.3

use $x_0 = 1$ to find the approximation to the solution to $\sqrt[3]{x} = 0$.

solution

Yes, it's a silly example. Clearly the solution is x = 0, but it does make a very important point. Let's get the general formula for Newton's method.

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}\ x_n^{\frac{-2}{3}}} = x_n - 3x_n = -2x_n$$

In fact, we don't really need to do any computations here. These computations get farther and farther away from the solution, $x = 0$ with each iteration. Here are a couple of computations to make the point.

$$x_1 = -2$$

$$x_2 = 4$$

$$x_3 = -8$$

$$x_4 = 16$$

etc.

So, in this case the method fails and fails spectacularly. So, we need to be a little careful with Newton's method. It will usually quickly find an approximation to an equation. However, there are times when it will take a lot of work or when it won't work at all.[14]

Example3.4

$F(x) = \cos(x) - x^3$

Consider the problem of finding the positive number x with $\cos(x) = x^3$. We can rephrase that as finding the zero of $F(x) = \cos(x) - x^3$. We have . Since for all and for , we know that our solution lies between 0 and 1.

For example, with an initial guess $x_0 = 0.5$ the sequence given by Newton's method is (note that a starting value of 0 will lead to an undefined result, showing the importance of using a starting point that is close to the solution) [15]

Chapter Four

The fixed- point iteration method

4. The fixed- point iteration method

4.1 Introduction

The fixed-point iteration method is a numerical technique used to solve nonlinear equations. It involves transforming the original equation into a fixed-point form, where the equation is written as x = g(x).

The goal is to find a fixed point of the function g(x), which is a value of x that satisfies the equation. [16]

Here's a step-by-step explanation of the fixed-point iteration method:

1. Start with an initial guess for the solution, let's say x_0.
2. Define a function g(x) that is derived from the original equation, such that $x = g(x)$

   Iterate the following process:

   - Compute the next approximation using the equation: xn+1 = $g(x_n)$
   - Update x by setting x_{n+1} as the new value of x.
   - Repeat this process until the desired level of accuracy is achieved or convergence is obtained.
3. The final value of x, denoted as x*, will be an approximation of the solution to the original equation.

   Now, let's illustrate the fixed-point iteration method with a simple

Example 4.1

Consider the equation: $x^2 - 5x + 4 = 0$

We can rewrite this equation in fixed-point form by isolating x:

$$x = g(x) = \frac{(x^2 + 4)}{5}.$$

Let's start with an initial guess $x_0 = 2$.

Using the fixed-point iteration method:

Iteration 1: $x_1 = g(x_0) = \frac{(2^2 + 4)}{5} = 1.6$

Iteration 2: $x_2 = g(x_1) = \frac{(1.6^2 + 4)}{5} = 1.312$

Iteration 3: $x_3 = g(x_2) = \frac{(1.312^2 + 4)}{5} = 1.144.$

Iteration 4: $x_4 = g(x_3) = \frac{(1.144^2 + 4)}{5} = 1.062.$

Iteration 5: $x_5 = g(x_4) = \frac{(1.062^2 + 4)}{5} = 1.025.$

Iteration 6: $x_6 = g(x_5) = \frac{(1.025^2 + 4)}{5} = 1.005$

The iterations continue until the desired level of accuracy is achieved or convergence is obtained. In this example, the iterations converge to a fixed point of approximately 1 which is an approximation of the solution to the original equation.

Remember, the success of the fixed-point iteration method depends on choosing a suitable function g(x) and starting with a reasonable initial guess. Additionally, it's important to ensure that the iterations converge to a fixed point for the chosen function g(x). [15]

To find the other root we take the first approximation $x_0 = 5$

Using the fixed-point iteration method:

Iteration 1: $x_1 = g(x_0) = \frac{(5^2 + 4)}{5} = 5.8$

Iteration 2: $x_2 = g(x_1) = \frac{(5.8^2 + 4)}{5} = 7.528$

Iteration 3: $x_3 = g(x_2) = \frac{(7.528^2 + 4)}{5} = 12.12$

We notice that instead of the sequence of numbers converging from root $x = 4$, they

And for every test

diverge set the function f(x)

It can be done in several multiple ways

f(x) We get a sequence of numbers, so if the resulting sequence converges, its end is the required root.

4.2 comparison table

comparison table between the Newton's method and the fixed-point iteration method for solving nonlinear equations:

comparison table

	Newton's Method	Fixed-Point Iteration Method
Approach	Iterative numerical method	Iterative numerical method
Approximation	Linear approximation using derivatives	Nonlinear approximation using fixed-point form
Compatibility	Requires differentiability of the function	Requires continuity of the function
Speed and Convergence	Fast convergence, especially with close guesses	Slower convergence compared to Newton's method
Additional Requirements	Requires computing derivatives	No need to compute derivatives
Flexibility	May encounter convergence issues in some cases	Works in a wider range of cases

Note 4.1

Please note that the specific advantages and disadvantages of each method can vary depending on the equation being solved and the initial guess provided. It's important to consider the characteristics of the equation and the requirements of the problem before choosing a particular method. [17]

Examples 4.2

Examples of nonlinear equations that may not be solved using Newton's method:

1. Transcendental equations: Equations that involve transcendental functions like exponential functions, logarithmic functions, trigonometric functions, or their inverses may not have explicit solutions and may be challenging to solve using Newton's method. For example:

$$a. e^{x} = x^{2}$$

$$b. sin(x) + x = 0$$

2. Algebraic equations of higher degrees: Equations with polynomial terms of degree higher than two often have multiple roots and can be difficult to solve using Newton's method. For instance:

$$a. x^{4} - 4x^{2} + 3 = 0$$

$$\text{b. } x^{5} - 2x^{3} + x = 0$$

3. Singular or discontinuous equations: Equations that have singular points or discontinuities can pose challenges for Newton's method. An example is the equation: $1/x = 0$

4- Systems of equations: Newton's method is commonly used for solving single-variable equations. It becomes more complex and computationally intensive when applied to systems of nonlinear equations. Systems like these may not yield accurate solutions using Newton's method alone. An example is the system of equations:

$$a.\, x^2 + y^2 = 1$$

$$b.\, x^2 - y^2 = 0$$

These examples demonstrate scenarios where Newton's method may encounter difficulties or not provide accurate solutions. Other numerical methods or analytical techniques may be more suitable for solving such nonlinear equations. [18]

Examples 4.3

example to highlight the differences in the solutions obtained through Newton's method and the fixed point method. We'll solve the equation:

$$\boldsymbol{F(x) = e^{-x} - x}$$

using both methods.

Comparison table:

Iteration	Newton's Method	Fixed Point Method
0	Initial guess x_0	Initial guess x_0
	$x_0 = 0.5$	$x_0 = 0.5$
1	Calculate $f(x_0)$ and $f'(x_0)$	Calculate $f(x_0)$
	$f(x_0) = e^{0.5} - 0.5 \approx -0.106$	$f(x_0) = e^{0.5} - 0.5 \approx -0.106$
	$f'(x_0) = -e^{0.5} - 1 \approx -1.606$	
	Update $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$	Update $x_1 = g(x_0)$
	$x_1 \approx 0.5 - \frac{0.106}{1.606} \approx 0.560$	$x_1 \approx 0.5 - 0.106 \approx 0.394$
2	Calculate $f(x_1)$ and $f'(x_1)$	Calculate $f(x_1)$
	$f(x_1) = e^{0.560} - 0.560 \approx -0.034$	$f(x_1) = e^{0.394} - 0.394 \approx -0.003$
	$f'(x_1) = -e^{0.560} - 1 \approx -1.560$	
	Update $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$	Update $x_2 = g(x_1)$
	$x_2 \approx 0.560 - \frac{0.034}{1.560} \approx 0.581$	$x_2 \approx 0.394 - 0.003 \approx 0.391$
...	Repeat iterations as needed	Repeat iterations as needed
n	Converges to a root of $f(x)$	Converges to a fixed point of $g(x)$

In this example, you can observe that both methods are converging to solutions, but the values and convergence rates differ. Newton's method converges relatively faster compared to the fixed point method. The choice of initial guess and the characteristics of the functions involved play a crucial role in determining the convergence behavior for both methods.

Conclusions

After conducting research and analyzing the results, the following conclusions can be drawn regarding the effectiveness of Newton's method for solving optimization problems.

1-Fast Convergence: Newton's method exhibits fast convergence, often achieving quadratic convergence. This rapid convergence rate allows for efficient determination of the optimal solution in a relatively small number of iterations.

2- Multivariate Optimization: Newton's method is effective in handling optimization problems with multiple variables. By utilizing partial derivatives, it can identify critical points and determine their nature (minima, maxima, or saddle points) in a multidimensional space.

3-Incorporating Constraints: Newton's method can be extended to handle constrained optimization problems. By incorporating constraints into the objective function, it can find critical points that satisfy both the objective and the constraints.

4-Sensitivity to Initial Guess: Newton's method relies on an initial guess that is reasonably close to the optimal solution. If the initial guess is far from the solution or falls into a region of poor convergence, the method may fail to converge or converge to a suboptimal solution.

5- Sensitivity to Function Behavior: Newton's method may face convergence issues when dealing with non-convex optimization problems or functions that are poorly conditioned. In such cases, alternative methods or modifications to Newton's method may be necessary.

6- Overall Efficiency: Despite its limitations, Newton's method remains a powerful and widely used technique for solving optimization problems. Its fast convergence and ability to handle multivariate optimization make it particularly valuable in various fields, including engineering, physics, economics, and computer science.

In conclusion, Newton's Method demonstrates effectiveness in solving optimization problems by providing fast convergence, handling multivariate optimization, and incorporating constraints. However, its sensitivity to the initial guess and function behavior should be taken into consideration. It is recommended to carefully assess the problem at hand and consider alternative methods if needed to ensure the most efficient and accurate solution.

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الخلاصة

طريقة نيوتن هي تقنية عددية تكرارية تستخدم لإيجاد حلول تقريبية للمعادلات غير الخطية يعتمد على مفهوم التقريب الخطي عن طريق تحسين التقدير الأولي بشكل متكرر نحو جذر المعادلة.

تكون الطريقة عالية الكفاءة عندما يكون التخمين الأولي قريبًا من الجذر الفعلي وعندما تكون الوظيفة حسنة التصرف بالقرب من الجذر.

ومع ذلك فإن التقارب ليس مضمونًا لجميع الحالات وقد يتم تفويت بعض الجذور في حالة وجود جذور متعددة .

طريقة التكرار ذات النقطة الثابتة هي طريقة أخرى لحل المعادلات غير الخطية عن طريق تحويل المشكلة إلى إيجاد نقاط ثابتة لوظيفة تكرارية .

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قسم الرياضيات

حل مشاكل الامثلية باستخدام الطرق التكرارية

بحث مقدم إلى

قسم الرياضيات - كليه التربية للعلوم الصرفة - جامعه بابل وهو جزء من متطلبات نيل درجة الدبلوم العالي تربية / الرياضيات

من قبل

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