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Monotone Rational $L_p, p < 1$, Approximation

A Thesis

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By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

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Dedication

Lover Of Allah ,Mohammed and kinsfolk his home ...

My Brothers

I dedicate this work

Acknowledgments

Thanks for God, Lord of the Worled, who enabled me to accomplish *this work after I sought to achieve it .*

 First, I would like to thank my supervisor, Dr. "Eman S .Bhaya", for her support and help during my work and I would like to extend special thanks to all professors in the Mathematics Department.

Thanks to my family for supporting me in making this project a success. Finally, thanks to the esteemed panelists.

Abstract

Many papers were introduced the approximation of monotone functions using monotone polynomials, but we sometimes need to approximate monotone function by rational monotone polynomials. In this thesis, we fill this gap to estimate the degree of best constrained approximation using rational monotone polynomials and rational r-monotone polynomials.

Moreover, many researchers related to the degree of unconstrained approximation to constrained approximation, and proved: for a continuous function f on closed interval we have

$$E_n(f) \leq C \, E_n^{(r)}(f) \qquad\qquad (1.1)$$

C is positive constant. The converse of the relation (1.1) is not achieved, so we will obtain the converse of the relation (1.1)under some conditions on f which belonging to quasi normed spaces. We relate the degree of best monotone approximation to the degree of best approximation of functions in $L_{p[0,1]}$. And relate the degree of r-monotone approximation to the unconstrained approximation.

Recently many direct theorems introduced for monotone, convex, 3-monotone approximation, r-monotone for function in L_p quasi normed space. Now there is a question: Can we strength these results in terms of moduli of smoothness of higher order?. The answer is no, and that what we see in our work here. Here we introduce negative theorem for the r-monotone approximation of function in L_p quasi normed space for $0 < p < 1$.

List of Symbols

$c(p)$	*constant depending only on p, and may different from one to other*
L_p	*Lebesgue space*
$\mathcal{R}$	*Set of the real number*
$\|f(x)\|_p$	*The quasi norm*
P_n	*Polynomial of degree at most n*
$\Delta_h^r(f)$	*The rth symmetric difference of f.*
$E_n(f)_p$	*The degree of best approximation*
$E_n^{(r)}(f)_p$	*The degree of best r- monotone approximation*
$\omega_r(f,\delta)_p$	*The usual modules of smoothness of order r.*
x_+^0	*Signum function*
P_n^r	*The set of all polynomials of $D_n \in P_n$ such that $D_n^{(r)}(x) \geq 0$ on the interval $0 \leq x \leq 1$.*
$\sim$	*Equivalence relation*
D_n	*The set of all Polynomial of degree at most n*
Sup	*The least upper bound*
inf	*The greatest lower bound*

Publications

1 W. A. Ajel and E. S. Bhaya " r-Monotone Approximation and The Unconstrained Approximation for Function in $L_{p[0,1]}$, 0<p<1", Journal of Journal of Results in Nonlinear Analysis, Date: 24-05-2023.

2-" W. A. Ajel and E. S. Bhaya " The Relation Between the Degree of Monotone Approximation and the Degree of Unconstrained Approximation ", Mustansiriyah Journal of Pure and Applied Sciences, Date: 19-06-2023.

Introduction

We wish sometimes to approximate monotone function defined on a finite interval, while preserving certain shape properties such us monotonicity, convexity and r-monotonicity. That is what is called constrained approximation. Sometimes we need to approximate a monotone function using a rational monotone approximation that is what we called monotone rational approximation.

The L_p space for $0 < p < 1$

$$L_{p[0,1]} = \left\{ f: [0,1] \to \mathcal{R}, with \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

On $L_{p[0,1]}$ we define the quasi norm[6]

$$\|f(x)\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}, p < 1.$$

Let

$$H_n = \{D_n = a_0 + a_1 x + a_2 x^2 + \cdots \cdots + a_n x^n \, of \, degree \, \leq n\},$$

we specify

$$M_n = \left\{ D_n \in H_n \; : D_n^{(r)}(x) \geq 0, \quad 0 \leq x \leq 1 \right\}.$$

The degree of best approximation of $f \in L_p$ from H_n

$$E_n(f)_p = \inf_{D_n \in H_n} \|f - D_n\|_{L_{p[0,1]}},$$

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$$\Delta^r = \{ f \in L_{p[0,1]} , f \text{ is monotone} \}$$

the degree of best monotone approximation of $f \in L_p \cap \Delta^r$

$$E_n^r(f)_p = \inf_{D_n \in M_n} \| f - D_n \|_p .$$

A function $f : [0,1] \to \mathcal{R}$ is said to be r-monotone , $r \geq 1$,

on $[a, b]$ if and only if for all choices of $r + 1$ distinct points

$x_0, x_1, , \cdots \cdots , x_r$ in $[a, b]$ the inequality

$$f(x_0, x_1, , \cdots \cdots , x_r) > 0$$

$$f(x_0, x_1, , \cdots \cdots , x_r) = \sum_{j=0}^{r} \frac{f(x_j)}{\omega'(x_j)} ,$$

is the r-divided difference of f at $x_0, x_1, , \cdots \cdots , x_r$, where

$$\omega(x) = \prod_{j=0}^{r} (x - x_j).$$

We will use moduli of smoothness which are connected with

difference of higher orders. The rth symmetric difference for f is

defined by

$$\Delta_h^r(f, x, [a, b]) := \Delta_h^r(f, x) :$$

$$= \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f \left(x - \frac{rh}{2} + ih \right), x \pm \frac{rh}{2} \in [a, b] \\ \\ 0, \end{cases} ,$$

thus the rth usual modules of smoothness of $f \in L_{p[0,1]}$ is

defined by

$$\varpi_r(f, \delta)_p = \varpi_r(f, \delta, [a, b])_p = \sup_{0 < h \leq \delta} \| \Delta_h^r(f, \cdot) \|_{L_{p[0,1]}}, \delta \geq 0.$$

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In [27] Petrushev and V.A Popov studied the degree of best rational approximation for function in Soblev spaces. In [2] B. Gao D.J. Newman, V.A, Popov and S. P. Zhou studied the convex and monotone approximation of function in Soblev spaces. In [7] E. S. Bhaya, and D.B.Alasmawy studied Monotone rational approximation of functions in L_p space for $0 < p < 1$.

In chapter one, we estimate the degree of best approximation of monotone functions in $L_{p[0,1]}$, for $0 < p < 1$, using monotone rational polynomials of degree $\leq n$.

Many researchers work on the upper bound of $E_n^{(r)}(f)$ which is the case $r = 1$, it means $f'(x) \geq 0$ for $a \leq x \leq b$, $a, b \in \mathcal{R}$. In [14] Lorentz and Zeller proved that for a continuous function on the interval $[a, b]$, that satisfies

$$f^{(r)}(x) \geq 0 \text{ that } E_n(f) \leq E_n^{(r)}(f).$$

In [16] Lorenz proved that if $f^{(r_i)}(x) \geq 0$ for any $a \leq x \leq b$ and $i = 1,2,3, \cdots \cdots, q$, then there exists $c > 0$ such that for positive consistent $E_n^{(r_i)}(f) \leq c\, E_n(f)$. In[20] Roulier, put conditions on (f) to insure for $n \to \infty$,

$$E_n^{(r_i)}(f) = E_n(f).$$

In [19] Roulier gave some conditions to get $\frac{E_n^{(r)}(f)}{E_n(f)} \leq c(f')$, where $c(f')$is a constant which depends of f', in [22], [32], [23], [24], [25] K.A. Kopotun, D. Leviatan, and I.A. Shevchuk, discussed, are the degrees of best (co)convex and unconstrained polynomial approximation the same? In[6] E. S. Bhaya & B.K. Hussein, studied

a new modulus of smoothness for uniform approximation, in[9]E. S.Bhaya, H. Almurieb, discussed Nearly monotone neural approximation with quadratic activation function.

In chapter two, we obtain an estimate for the degree of best approximation once for $r = 1$ and another for any r. Then we relate $E_n^{(r)}(f)$ to $E_n(f)$ by a constant independent on f'. These all for Lebesgue integrable functions.

In [3] Jackson is the first mathematician, that proved direct theorem for best approximation of a continuous function. Then in [17] Lorentz proved inverse theorem for the direct theorem of Jackson. In [33] Malozemov, relate the approximation degree of function to its derivative. In [21] J.Roulier proved direct (Jackson) theorem for the monotone approximation. In [11] E. S . Bhaya, S. A. Abdalredha, studied Piecewise 3-monotone approximation for 3-monotone functions in L_p-spaces for $p < 1$. In [26] O.shisha studied this problem which will be proven as follows.

If $1 \leq r \leq m$ and a continuous function $f \in L_\infty[0,1]$

$$f^{(r)}(x) \geq 0, \left| f^{(m)}(x) \right| \leq M \quad for \quad 0 \leq x \leq 1 ,$$

the for all integer $n(\geq m)$,

$$E_n^{(r)}(f)_\infty \leq \frac{C}{n^{m-r}} \omega \left(f^{(m)}, \frac{1}{n} \right)_\infty ,$$

such that C depends upon m and r. $\omega(f,h)$ which is the modulus of continuity of the function f.

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In chapter three here we relate the degree of best monotone approximation to the degree of best approximation at function in $L_{p[0,1]}$. And relate the degree of r-monotone approximation to the unconstrained approximation [31].

Several, researchers devoted their researches in studying monotone and co monotone approximation of continuous function by algebraic polynomials, particularly to Jackson type estimates (cf., for example, $[28], [29], [30] - [5], [15], [10][12] - [34]$). In[8]E. S. Bhaya, G. Abdullah, introduced Negative theorem for $L_p, 0 < p < 1$ monotone approximation. Many researchers, including Lorentz and Zeller in [15] and Shvedov in [1], from their opposite results, it appears that the estimates of the Jackson type for the criteria of the higher degree of smoothness are not correct in the monotone approximation. [35], imagine that there is a continuous function which is r-monotone such that for $r \geq 1$.

$$\lim_{n \to \infty} \frac{E_n^{(r)}(f)}{\omega_{2r+1}\left(f, \frac{1}{n}\right)} = +\infty.$$

In [35], the authors got weaker result, which showed the existence of an r-monotone continuous function such that

$$\lim_{n \to \infty} \sup \frac{E_n^{(r)}(f)}{\omega_{2r+1}\left(f, \frac{1}{n}\right)} = +\infty.$$

For $r \geq 2$

$$\lim_{n \to \infty} \sup \frac{E_n^{(r)}(f)}{\omega_4\left(f, \frac{1}{n}\right)} = +\infty.$$

In chapter four, we shall prove a negative theorem for the best r-monotone approximation of function in $L_{p[0,1]}$ space .Let $r \geq 1$. Then there exists a function $f \in L^r_{p[0,1]} \cap \Delta^r$ such that

$$E^{(r)}_{n_l}(f)_p > \omega_{r+3}\left(f, \frac{1}{n}\right)_p.$$

Chapter One

L_pBound for the Degree of Monotone Rational Approximation

Many papers were introduced for the approximation of monotone functions using monotone polynomials. But we sometimes need to approximate monotone function by rational monotone polynomials. Here we fill this gap, and estimate the degree of best constrained approximation using rational polynomials.

1.1 Introduction

Let P_n be the collection of all algebraic polynomials of degree $\leq n$, $P_n = \{ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n \leq n \}$ and Q_n be the set of all rational polynomials $r = {}^S\!/_q$, where $s, q \in P_n$, $q \neq 0$,

$$Q_n = \left\{ r(x) = \frac{s_n(x)}{q_n(x)} , s(x), q(x) \in P_n, q(x) \neq 0 \right\}.$$

$$L_{p[0,1]} = \left\{ f: [0,1] \rightarrow \mathcal{R}, with \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

On $L_{p[0,1]}$ we define the quasi norm[6]

$$\|f(x)\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}, p < 1.$$

The degree of best approximation of $f \in L_{p[0,1]}$, using rational function is defined as:

$$E_n(f)_p = \inf_{r_n \in Q_n} \|f - r_n\|_p.$$

Let

$$\Delta' = \{f \in L_{p[0,1]}, f \text{ is monotone} \}.$$

The degree of best monotone rational approximation of $f \in L_{p[0,1]} \cap \Delta'$ is

$$E_n^1(f)_p = \inf_{r_n \in Q_n \cap \Delta'} \|f - r_n\|_p.$$

The rth symmetric difference for f is defined by

$$\Delta_h^r(f, x, [a, b]) := \Delta_h^r(f, x):$$

$$= \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{rh}{2} + ih\right), & x \pm \frac{rh}{2} \in [a, b] \\ 0, & \end{cases} \quad [3]$$

thus the rth usual modules of smoothness of $f \in L_{p[0,1]}$ is

$$\mathcal{O}_r(f, \delta)_p = \mathcal{O}_r(f, \delta, [a, b])_p = \sup_{0 < h \le \delta} \left\| \Delta_h^r(f, \cdot) \right\|_{Lp[a,b]}, \delta \ge 0.$$

During this chapter, we will seek to solve a problem that has been dealt with R. A. Devore in several lectures within the15 years. The problem we want to deal with monotone function, whether increasing or decreasing, belongs to $L_{p[0,1]}$. We want to estimate the degree of best approximation of the function f that belongs to $L_{p[0,1]}$, by using a polynomial that is monotone and rational.

In [27] Petrushev and V.A Popov studied the degree of best rational approximation for function in Soblev spaces. In [2],[32] B. Gao D.J. Newman, V.A, Popov and S. P. Zhou studied the convex and monotone approximation of function in Soblev spaces. In [7] E. S. Bhaya, and D.B.Alasmawy studied Monotone rational approximation of functions in L space for 0 <p< 1. In this chapter we estimate the degree of best approximation of monotone functions in $L_{p[0,1]}$, for $0 < p < 1$, using monotone rational polynomials of degree$\le n$. To prove our main results, we make use of proof strategies from [18].

1.2. Direct Theorem for Monotone Rational Approximation

In this section we shall introduce our main results for the degree of best monotone rational approximation.

First let us introduce the following auxiliary lemma from [4]:

Lemma 1.2.1.[4]

For all even $m > N_0$ and integer $n > m^2$ we take

$$Q_{n,m}(x) = \frac{1 + R_{[\frac{n}{m}]+1,m}(x)}{2} \qquad (1.1)$$

$$x_+^0 := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Put

$$R_{n,m}(x) := \frac{2}{A} \int_0^x T_n^m(t)\, dt + H_{n,m}(x).\frac{R_m^*(x)}{1+b}$$

$$R_m^*(x) := \frac{1}{1 + (nx)^{m^5+m}}$$

$$T_n(x) := \sum_{k=0}^{3n} (-1)^k n^{2k+1} \frac{x^{2k}}{(2k+1)!}$$

$$H_{n,m}(x) := \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A} \int_0^x P_m^m\, dt,$$

$$P_m(x) := \sum_{k=0}^{m} (-1)^k n^{2k+1} \frac{x^{2k}}{(2k+1)!} - n^{2m+1} x^{2m} \left(\frac{2}{5}\right)^{2m},$$

$$R_m(x) := \frac{N(x) - N(-x)}{N(x) + N(-x)}.$$

The Newman rational function, where

$$N(x) := \prod_{i=1}^{m}(x + a^i)\ and\ a := e^{-1/\sqrt{m}},$$

And $Q_{n,m} = p_{n,m} + q_{n,m},$

where $P_{n,m} \in Q_{7n},\ \text{q}_{n,m} \in Q_{2m^6},$

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$$\left| Q_{n,m}(x) - x_+^0 \right| < e^{-\sqrt{m}/4} , \quad \frac{e^{-\sqrt{m}/4}}{n} \le \left| x \right| \le 1, \tag{1.2}$$

$$\left| Q_{n,m}(x) - x_+^0 \right| < \frac{e^{-\sqrt{m}/4}}{j^2}, \frac{j-1}{n} \le \left| x \right| \le \frac{j}{n}, j = \overline{2,n} \tag{1.3}$$

$$\left| Q_{n,m}(x) - x_+^0 \right| < 2, \left| x \right| \le 1.$$

Theorem 1.2.1.

Let f be a monotone function in $L_{p[0,1]}$. Then

$$E_n^1(f)_p \le c(p) \frac{\omega_1 \left(f, \frac{1}{n} \right)_p}{n^{\frac{1}{p}}}.$$

proof:

Let $0 = x_0 < x_1 < \ldots\ldots < x_k \le 1$, be a partition for $[0,1]$ such that

$$\left| x_{i+1} - x_i \right| = \frac{c}{n} \text{ and } \left| f(x_i) - f(x_{i+1}) \right| = \frac{c}{n}.$$

Let f is a non-decreasing function of $[0,1]$, and $\frac{K}{n} \le f(1) - f(0) \le c$.

Put

$$S(x) = f(x_0) + \frac{1}{n} \sum_{i=1}^{k-1} (x - x_i)_+^0.$$

$$\left\| S(x) - f(x) \right\|_p = \left\| f(x_0) + \frac{1}{n} \sum_{i=1}^{k-1} (x - x_i)_+^0 - f(x) \right\|_p$$

$$\le \frac{1}{n}.$$

Define an even number greater then n_0, as

$$m_i = 2n_0 + 16 \left[ln^2 + \left(n^{-1} max \{ (x_{i+1} - x_i)^{-1}, (x_i - x_{i-1})^{-1} \} \right) \right]. \quad \text{Since}$$

m_i are even $m_i > n_0$, we have,

$$\left\| Q_{n,m}(x) - x_+^0 \right\| \le e^{-\sqrt{m}/4}.$$

Thus

$$e^{-\sqrt{m}/4} < n \, min\left[x_{i+1} - x_i, \; x_i - x_{i-1}, \frac{1}{n}\right].$$

So

$$m_i \leq 2N_{\circ} + 16ln^2 \frac{1}{n^{p-1}}, \; so \left(2N_{\circ} + 16ln^2 \frac{1}{n^{p-1}}\right)^2 < n, \quad since \quad n > m^2,$$

when $p < 1$, for every $i = 1,2,3,4, \ldots \ldots, k - 1$.

For $Q_{n,m}$ using (1.1), to get $Q_{n,m}(x) = \dfrac{1 + R_{\left[\frac{n}{m}\right]+1, \; m(x)}}{2}$,

put

$$R(x) = f(x_o) + \frac{1}{n}\sum_{i=1}^{k-1} Q_{n,m_i} \; (x - x_i).$$

And all Q_{n,m_i} are non-decreasing function, and therefore we have by

$$n\, \|f(x_i) - R(x_i)\|_p \leq c(p)\left(n\,\|f(x_i) - S(x_i)\|_p + n\,\|S(x_i) - R(x_i)\|_p\right).$$

$$2\,c(p)n \cdot \frac{2}{n} + n\left\|f(x_0) + \frac{1}{n}\sum_{i=1}^{k-1}(x - x_i)_+^0 - (f(x_0)\right.$$

$$\left. + \frac{1}{n}\sum_{i=1}^{k-1} Q_{n,m_i}(x - x_i)\right\|_p$$

$$n\|f(x) - R(x)\| \leq n\left(\|f(x) - S(x) + S(x) - R(x)\|_p\right)$$

$$= n\left(\left\|f(x) - f(x_0) - \frac{1}{n}\sum_{i=1}^{k-1}(x - x_i)_+^0 + f(x_0) + \frac{1}{n}\sum_{i=1}^{k-1}(x - x_i)_+^0\right.\right.$$

$$\left.\left. - f(x_0) - \frac{1}{n}\sum_{i=1}^{k-1} Q_{n,m_i}(x - x_i)\right\|_p\right)$$

$$\leq n\left(\|\,|f(x) - f(x_0)|\, + \frac{1}{n}(k - 1) + \frac{1}{n}\sum_{i=1}^{k-1}|(x - x_i)_+^0 - Q_{n,m_i}(x\right.$$

$$\left. - x_i)\|_p\right)$$

$$\leq c(p)n\left\| f(x) - f(x_0) - \frac{1}{n}\sum_{i=1}^{k-1}(x - x_i)\right\|_p + \frac{1}{n}\sum_{i=1}^{k-1}\left\| (x - x_i)\right.$$

$$\left. - Q_{n,m_i}(x - x_i)\right\|_p.$$

So

$$n\left\| f(x) - R(x)\right\|_p \leq c(p)n\left\| \frac{1}{n} + \frac{2}{n}\sum_{i=1}^{k-1}(x - x_i)_+^0 - \sum_{i=1}^{k-1}Q_{n,m_i}(x - x_i)\right\|_p$$

$$n\left\| f(x) - R(x)\right\|_p = c(p)\left\| 3 + \sum_{i=1}^{k-1}(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)\right\|_p$$

$$n\left\| f(x) - R(x)\right\|_p = c(p)n\left\| f(x) - S(x)\right\|_p + n\left\| S(x) - R(x)\right\|_p$$

$$\leq 2 + \left\| \sum_{i=1}^{k-1}(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)\right\|_p$$

$$= 2 + \left\| \sum_{i:\,|x-x_i| \leq \frac{1}{n}e}^{-\sqrt{m_i}/4}(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)\right\|_p$$

$$+ \left\| \sum_{\substack{i:\frac{1}{n}e \quad |x-x_i| \leq \frac{1}{n}}}^{-\sqrt{m_i}/4}(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)\right\|_p$$

$$+ \left\| \sum_{j=2}^{n}\sum_{i:\frac{j-1}{n} < |x-x_i| \leq \frac{j}{n}}(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)\right\|_p.$$

Using (1.2) and (1.3) to get

$$n\|f(x) - R(x)\|_p \leq 2 + c(p) \, sup \left| (x - x_i)_+^o - Q_{n,m_i}(x - x_i) \right|$$

$$+ \quad c(p) \, sup \left| (x - x_i)_+^o - Q_{n,m_i}(x - x_i) \right|$$

$$+ c(p) \sup \left| (x - x_i)_+^o \ - \ Q_{n,m_i}(x - x_i) \ \right|$$

$$j = 2 \, , i : \quad \frac{j-1}{n} < \left| x - x_i \right| \leq \frac{j}{n}$$

$$\leq 2 + \sum_{i: \left| x - x_i \right| \leq \frac{1}{n} \, e^{-\sqrt{m_i}/_4}} \left| (x - x_i)_+^o - Q_{n,m_i}(x - x_i) \right|$$

$$+ \sum_{j=1 \ i \ : \frac{j-1}{n} < \left| x - x_i \right| \leq \frac{j}{n}}^{n} \sum \frac{1}{j^2} e^{-\sqrt{m_i}/_4} \, .$$

We get $\left| x_i - x_{i\pm 1} \right| > n^{-1} \, e^{-\sqrt{m_i}/_4}$, be a satisfactory result $\left| x - x_i \right| <$

$$n^{-1} \, e^{-\sqrt{m_i}/_4} \ ,$$

$$\sum_{i: \left| x - x_i \right| \leq \frac{1}{n} \, e^{-\sqrt{m_i}/_4}} \left| (x - x_i)_+^0 - Q_{n,m_i}(x - x_i) \right| < 4$$

$$\|f(x) - R(x)\|_p$$

$$\leq \ c(p) \ \left(\frac{6}{n} + \sum_{j=1}^{n} \frac{1}{j^2} \sum_{i: \frac{j-1}{n} < \left| x - x_i \right| \leq \frac{j}{n}} \min \left\{ (x_{i+1} - x_i), \frac{1}{n} \right\} \right).$$

$$\left\| f\,(x) - f\,(x_0) + \frac{1}{n} \sum_{i=1}^{k-1} Q_{n,m_i}\,(x - x_i\,) \right\|_p$$

$$\leq \frac{6}{n} + \sum_{j=1}^{n} \frac{1}{j^2} \sum_{i\,:\,\frac{j-1}{n} < |x-x_i| \leq \frac{j}{n}} \min\left\{ (x_{i+1} - x_i), \frac{1}{n} \right\}$$

$$\left\| f(x) - f(x_0) + \frac{1}{n} \sum_{i=1}^{k-1} Q_{n,m_i}(x - x_i) \right\|_p \leq \frac{6}{n} + \sum_{j=1}^{n} \frac{1}{j^2} \cdot \frac{4}{n}.$$

So

$$\left\| f(x) - R(x) \right\|_p \leq c(p)\left(\frac{6}{n} + \frac{1}{n} \sum_{j=1}^{n} 4 \cdot \frac{1}{j^2} \right) \leq \frac{c(p)}{n}.$$

Recall $0 = x_0 < x_1 < \cdots\cdots <= x_k = 1$ be a partition for $[0,1]$, such that

$$\left| x_i - x_{i+1} \right| = \frac{c}{n}, \text{ and } \left| f\,(x_{i+1}) - f\,(x_i) \right| = \frac{c}{n}.$$

Then

$$\frac{c(p)}{n} \leq \left\| f' \right\|_{L_p[x_i\,,\,\,x_{i+1}]} = \left(\int_{x_i}^{x_{i+1}} |f'(x)|^p\,\,dx \right)^{\frac{1}{p}}$$

$$\leq \left\| f(x_{i+1}) - f(x_i) \right\|_p \left| x_{i+1} - x_i \right|$$

$$\leq \frac{1}{n^{1/p}} \omega_1\left(f, \frac{1}{n} \right)_p.$$

So that

$$\left\| f(x) - R(x) \right\|_p \leq \frac{c(p)}{n} \leq c(p)\,\omega_1\left(f, \frac{1}{n} \right)_p.$$

Recall

$$R(x) = f(x_0) + \frac{1}{n} \sum_{i=1}^{k-1} Q_{n,m_i}(x - x_i),$$

by using the facts

$$R(x) = f(x_0) + \frac{1}{n} \sum_{i=1}^{k-1} \left(P_{n,m_i}(x - x_i) + q_{n,m_i}(x - x_i) \right)$$

$P_{n,m_i} \in Q_{7n}$, $q_{n,m_i} \in Q_{2m^6}$,

P_{n,m_i} is a polynomial of degree $\leq 7n$,

q_{n,m_i} is a polynomial of degree $\leq 2m_i^{\,6}$

So degree of $R(x)$ is $\leq 7n + \sum_{i=1}^{k-1} 2m_i^6$

$\leq 7n + 2(k-1)^6 \leq 7n \leq 2n^6$

$\leq 7n^6.$

Chapter Two

The Relation Between the Degree of Monotone Approximation and the Degree of Unconstrained Approximation

Many researchers related to the degree of unconstrained approximation to constrained approximation, and proved the inequality: For a continuous function f on closed interval we have

$$E_n(f) \le C \, E_n^{(r)}(f) \qquad (2.1)$$

where C is a positive constant. The converse of the relation (2.1) is not a hold, so we will obtain the converse of the relation (2.1)under some conditions on f belonging to quasi normed spaces.

2.1.Introduction

Let the L_p space for $0 < p < 1$

$$L_{p[0,1]} = \left\{ f \colon [0,1] \to \mathcal{R}, with \; \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Denote $L_{p[0,1]}$ we define the quasi norm

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, we have $x \in [0,1], [6]$

$$\|f\| = \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

Before we define the degree of best approximation of $f \in L_{p[0,1]}$, let us introduce

$$\mathcal{P}_n = \{P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n \; of \; degree \; \le n\},$$

and let

$$M_n = \{P_n(x) \in \mathcal{P}_n : P_n(x_0, x_1, x_2, \cdots\cdots, x_r) \ge 0\},$$

let

$$M = \{f \in L_{p[0,1]} : f(x_0, x_1, x_2, \cdots\cdots, x_r) \geq 0\},$$

The space of r-monotone function see [13].

$$f(x_0, x_1, x_2, \cdots\cdots, x_r) = \sum_{j=0}^{r} \frac{f(x_j)}{\omega'(x_j)},$$

is the r-divided difference of f

$$\omega(x) = \prod_{i=0}^{r} (x - x_i).$$

We define for $f \in M$.

$$E_n^{(r)}(f)_p = \inf_{P_n \in M_n} \|f - P_n\|_{p'}$$

where $P_n \in M_n$ and $f \in M$, and let

$$E_n(f)_p = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_{p'}$$

be the degree of unconstrained best approximation.

For $p = \infty$, we shall denote

$$E_n(f)_p = E_n(f)$$

$$E_n^{(r)}(f)_p = E_n^{(r)}(f).$$

The ordinary modulus of continuity of $f \in L_{p[0,1]}$ is

$$\omega_1(f, \delta)_p = \sup_{0 \leq |h| \leq \delta} \|f(x + h) - f(x)\|_p. \ [13]$$

Many researchers work on the upper bound of $E_n^{(r)}(f)$ which is the case $r = 1$, it means $f'(x) \geq 0$ for $a \leq x \leq b$, $a, b \in \mathcal{R}$. In [14] Lorentz and Zeller proved that for a continuous function on the interval $[a, b]$, that satisfies

$$f^{(r)}(x) \geq 0 \text{ that } E_n(f) \leq E_n^{(r)}(f).$$

In [16] Lorenz proved that if $f^{(r_i)}(x) \geq 0$ for any $a \leq x \leq b$ and $i = 1,2,3, \cdots\cdots, q$, then there exists $c > 0$ such that for positive consistent $E_n^{(r_i)}(f) \leq c\, E_n(f)$. In[20] Roulier, put conditions on (f) to insure for $n \to \infty$,

$$E_n^{(r_i)}(f) = E_n(f).$$

In [19] Roulier gave some conditions to get $\frac{E_n^{(r)}(f)}{E_n(f)} \leq c(f')$, where $c(f')$is a constant depends of f', in [22], [23] [24], [25]K.A. Kopotun, D. Leviatan, and I.A. Shevchuk , discussed , are the degrees of best (co)convex and unconstrained polynomial approximation the same? In[6] E. S. Bhaya & B.K. Hussein, studied anew modulus of smoothness for uniform approximation, in[9]E. S. Bhaya, H. Almurieb, discussed Nearly monotone neural approximation with quadratic activation function.

We obtain an estimate for the degree of best approximation once for $r = 1$ and another for any r. Then we relate $E_n^{(r)}(f)\ to\ E_n(f)$ by a constant independent on f'. These are all for Lebesgue integrable functions.

2.2- The Main Results.

In this section we present two theorem that we need in our work.

Theorem 2.2.1

If *f is monotone in $L_{p[0,1]}$,* and satisfy the condition

$$(f(x_2) - f(x_1)) \geq M(x_2 - x_1),$$

where M is a positive a constant, for $0 \leq x_1 < x_2 \leq 1$. Then

$$E_n(f)_p \leq \frac{c(p)}{E_n(f)_p} \left(\omega \left(\frac{3E_n}{M}, \frac{1}{n} \right)_p + E_n(f)_p \right)$$

$$E_n^1(f)_p \leq c(p)E_n(f)_p,$$

where $c(p)$ is constant depending only on p.

Proof.

Choose Q_n be a best approximation to $f \in L_{p[0,1]}$, i.e

$$E_n(f)_p = \|f - Q_n\|_p$$

$$Q_n(x_2) - Q_n(x_1)$$
$$\geq f(x_2) - f(x_1) - |f(x_1) - Q_n(x_1)| - |f(x_2) - Q_n(x_2)|$$
$$\geq f(x_2) - f(x_1) - \|f(x_1) - Q_n(x_1)\|_p - \|f(x_2) - Q_n(x_2)\|_p$$
$$= f(x_2) - f(x_1) - 2E_n(f)_p$$
$$\geq M(x_2 - x_1) - 2E_n(f)_p.$$

If $\left(M(x_2 - x_1) - 2E_n(f)_p\right) \geq E_n(f)_p$, then

$$M(x_2 - x_1) - 3E_n(f)_p \geq 0$$

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$$(x_2 - x_1) \geq \frac{3}{M} E_n(f)_p.$$

$$Q_n(x_2) - Q_n(x_1) > M \left(\frac{3}{M} E_n(f)_p \right) - 2E_n(f)_p$$

$$= E_n(f)_p > 0.$$

Define

$$P_n(x) = \frac{M}{3E_n(f)_p} \int\limits_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1- \frac{3E_n(f)_p}{M} + \frac{3E_n(f)_p}{M}\right)x} Q_n(t)dt,$$

we have

$$0 \leq \left(1 - \frac{3E_n(f)_p}{M} + \frac{3E_n(f)_p}{M}\right)x \leq 1, \text{ and } 0 \leq \left(1 - \frac{3}{M}E_n(f)_p\right)x \leq 1.$$

For $1 > x > 0,$ we have $P_n'(x) > 0.$

If

$$1 - 3\frac{E_n(f)_p}{M} \leq t \leq \left(1 - \frac{3}{M}E_n(f)_p + 3E_n(f)_p\right)x,$$

definition of the modulus of continuity implies

$$\| f(x) - f(t) \|_p \leq \omega \left(\frac{3}{M} E_n(f)_p, \ |x - t| \right)_p.$$

$$\|f - P_n\|_p = \|f - \frac{M}{3E_n(f)_p} \int\limits_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)_p}{M}+\frac{3E_n(f)_p}{M}\right)x} \frac{3}{M}Q_n(t)dt \|_p$$

$$= \| \frac{M}{3E_n(f)_p} \int\limits_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)_p}{M}+\frac{3E_n(f)_p}{M}\right)x} (f(x) - Q_n(t))\, dt \|_p$$

$$= \frac{M}{3E_n(f)_p} \| \int\limits_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)_p}{M}+\frac{3E_n(f)_p}{M}\right)x} (f(x) - f(t) + f(t) - Q_n(t))\, dt \|_p.$$

$$E_n(f)_p \leq \|f - P_n\|_p \leq \frac{M}{3E_n(f)_p} \| \int\limits_0^x (f(x) - f(t) + f(t) - Q_n(t))\, dt \|_p$$

$$\leq \frac{M\, 2^{\frac{1}{p}-1}}{3E_n(f)_p} \left(\| \int\limits_0^x f(x) - f(t)\|_p + \| \int\limits_0^x f(t) - Q_n(t)\|_p \right)$$

$$\leq \frac{M2^{\frac{1}{p}-1}}{3E_n(f)_p} \left(\left(\int\limits_0^1 | \int\limits_0^x (f(x) - f(t))\, dt |^p\, dx \right)^{\frac{1}{p}} \right.$$

$$\left. + \left(\int\limits_0^1 | \int\limits_0^x |f(t) - Q_n(t)|\, dt |^p\, dx \right)^{\frac{1}{p}} \right)$$

Using the fact $\int_a^b f(x)dx \cong \frac{c(a,b)}{n}\sum_{i=1}^n f(x_i)$.

$$E_n(f)_p \leq \frac{c(p)\ 2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\left(\sum_{i=1}^n \frac{1}{n}\left|\int_0^x f(x_i)-f(t)dt\right|^p\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_0^1\left|\int_0^x |f(t)-Q_n(t)|dt\right|^p dx\right)^{\frac{1}{p}}\right)$$

$$\leq \frac{c(p)\ 2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\sum_{i=1}^n \frac{1}{n}\left|\frac{1}{n^p}\sum_{i=1}^n (f(x_i)-f(y_i))\right|^p\right)^{\frac{1}{p}}$$

$$+\left(\int_0^1\left|\int_0^x |f(t)-Q_n(t)|dt\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \frac{c(p)\ 2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\left(\sum_{i=1}^n \frac{1}{n.n^p}\sum_{i=1}^n |f(x_i)-f(y_i)|^p\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_0^1 |f(t)-Q_n(t)|^p dt\right)^{\frac{1}{p}}\right),$$

so

$$\|f - P_n\|_p \leq \frac{c(p)\, 2^{\frac{1}{p}-1}}{3E_n(f)_p} \left(\omega \left(\frac{3E_n}{M}, \frac{1}{n}\right)_p + E_n(f)_p \right).$$

This implies

$$E_n^1(f)_p \leq c(p)E_n(f)_p.$$

Theorem 2.2.2.

Let $f \in L_{p[0,1]}$ be a monotone function and assume that f is a positive bounded function on $L_{p[0,1]}$. Then if f is not polynomial, then

$$\lim_{n \to \infty} \frac{E_n^1(f)_p}{E_n(f)_p} \leq 4.$$

Proof :-

Assume there exists $b > 0$ *such that* $f(x) > b$. Let α be a positive constant greater than 1. Choose k so large that

$$E_k(f)_p < \frac{b}{(3 + \alpha)}.$$

Let P_k be the polynomial from $\mathcal{P}_k$ of best approximation to $f \in L_{p[0,1]}$.

Let

$$h(x) = f(x) - P_k(x) + (1 + \alpha)E_k(f)_p.$$

Then we have

$$\alpha E_k(f)_p \leq f(x) - P_k(x) + (1 + \alpha)E_k(f)_p \leq (2 + \alpha)E_k(f)_p . \qquad (2.2)$$

Now let

$$\phi(x) = \int_0^x h(t)dt$$

$$\phi(x) = f(x) - Q_{k+1}(x), \qquad\qquad (2.3)$$

where

$$Q_{k+1}(x) = f(0) + \int_0^x \big(P_k(t) - (1+\alpha)E_k(f)_p\big)dt.$$

$$\phi(x) = f(x) - f(0) + \int_0^x \big(P_k(t) - (1+\alpha)E_k(f)_p\big)dt$$

$$Q_{k+1}(x) = P_k(x) - (1+\alpha)E_k(f)_p$$

$$= P_k(x) - f(x) + f(x) - (1+\alpha)E_k(f)_p$$

$$\geq f(x) - (2+\alpha)E_k(f)_p$$

$$\geq b - (2+\alpha)\frac{b}{(3+\alpha)}$$

$$> \frac{b}{(3+\alpha)}.$$

From (2.2)

$$\alpha E_k(f)_p \leq \phi(x) \leq (2+\alpha)E_k(f)_p.$$

From Theorem 2.2.1 we get for n sufficiently large

$$E_n^1(\phi)_p \leq \left(\frac{2+\alpha}{\alpha} + 1\right)E_n(\phi)_p.$$

That is, for n sufficiently large,

$$E_n^1(\phi)_p \le 2\left(1 + \frac{1}{\alpha}\right) E_n(\phi)_p.$$

$n \ge k + 1$, we get by (2.3) and the monotonicity of Q_{k+1}

$$E_n(\phi)_p = E_n(f)_p$$

$$E_n(f)_p = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_p$$

$$E_n^1(f)_p = \inf_{P_n \in M_n} \|f - P_n\|_{p'}$$

$$E_n^1(\phi)_p \ge E_n^1(f)_p.$$

$$E_n^1(\phi)_p \ge \inf_{P_n \in M_n} \|f - P_n\|_p$$

$$\frac{E_n^1(f)_p}{E_n(f)_p} \le 2\left(1 + \frac{1}{\alpha}\right)$$

for n sufficiently large

$$\frac{E_n^1(f)_p}{E_n(f)_p} \le 2(1 + 1) = 4.$$

Chapter Three

r-Monotone Approximation and The Unconstrained Approximation for Function in $L_{p[0,1]}, 0 < p < 1.$

Here we relate the degree of the best monotone approximation to the degree of best approximation at function in $L_{p[0,1]}$. And relate the degree of r-monotone approximation to the unconstrained approximation.

3.1.Introduction

Let $P_n = \{D_n = a_0 + a_1x + a_2x^2 + \cdots\cdots + a_nx^n \, of \, degree \leq n\}$. We specify $P_{n,r}, n = 0,1,2,\ldots\ldots,r = 0,1,2,\ldots\ldots,n,$ to be the set of each polynomials of $D_n \in P_n$ such that $D_n^{(r)}(x) \geq 0$ on the interval $0 \leq x \leq 1$.

We specify $P_{n,r} = \left\{D_n \in P_n : D_n^{(r)}(x) \geq 0, \, 0 \leq x \leq 1 \right\}$, where $n = 0,1,\ldots,r = 0,1,2,\ldots,n,$

$$P_{n,r} = \left\{D_n \in P_n : D_n^{(r)}(x) \geq 0, \qquad 0 \leq x \leq 1\right\}.$$

For $f \in L_{p[0,1]}$, the best of degree approximation to f by polynomials from P_n is

$$E_n(f)_p = \inf_{D_n \in P_n} \|f - D_n\|_{L_{p[0,1]}},$$

where $\| \cdot \|_{L_{p[0,1]}} = \|f\|_p$, is the L_p quasi norm,

In a similar way, if f is r-monotone in $L_{p[0,1]}$ see [31]. Write

$$E_n^{(r)}(f)_p = \inf_{D_n \in P_{n,r}} \|f - D_n\|_{L_{p[0,1]}}, [13]$$

the degree of approximation to f through the polynomials of $\mathcal{P}_{n,r}$.

Definition of the modulus of continuity implies

$\Delta_h^r(f, x, [0,1]) := \Delta_h^r(f, x) :$

$$= \left\{ \begin{array}{l} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{rh}{2} + ih\right), x \pm \frac{rh}{2} \in [0,1] \\ \\ 0, \end{array} \right\} \quad [13]$$

thus the rth usual modules of smoothness of $f \in L_{p[0,1]}$ is

$$\mathcal{O}_r(f, \delta)_p = \mathcal{O}_r(f, \delta, [0,1])_p = \sup_{0 < h \le \delta} \left\| \Delta_h^r(f, \cdot) \right\|_{Lp[0,1]}, \delta \ge 0. \, [6]$$

The work of the researcher in this chapter is to find the upper limits for $E_n^{(r)}(f)_p$.

We define Lipschitz class as :

$$Lip_{M,\alpha} = \left\{ f \in L_{p[0,1]}, \omega_M \left(f, \frac{1}{n} \right)_p \le n^{-\alpha} \right\}. \, [4]$$

In [17] Lorentz proved inverse theorem for the direct theorem of Jackson. Then in [3] Jackson in the first mathematician, that proved direct theorem for the best approximation of a continuous function. In[33]Malozemov, relate the approximation degree of function to its derivative. In [21] J.Roulier proved direct (Jackson) theorem for the monotone approximation. In[11] E. S . Bhaya, S. A. Abdalredha, studied Piecewise 3-monotone approximation for 3-monotone

functions in L_p-spaces for $p < 1$. In [26] O. Shisha studied this problem that will be proven as follows.

If $1 \leq r \leq m$ and a continuous function $f \in L_\infty[0,1]$

$$f^{(r)}(x) \geq 0, \left| f^{(m)}(x) \right| \leq M \quad for \quad 0 \leq x \leq 1 ,$$

the for all integer $n(\geq m)$,

$$E_n^{(r)}(f)_\infty \leq \frac{C}{n^{m-r}} \omega \left(f^{(m)}, \frac{1}{n} \right)_\infty$$

such that C depends upon m and $r. \omega(f,h)$ is the modulus of continuity of the function f . Here we relate the degree of best approximation of functions to the degree of best monotone and r-monotone approximation of function in $L_{p[0,1]}$ quasi normed spaces.

3.2-The Main Results.

In this section we shall introduce our main results.

Theorem 3.2.1.

Let $f \in L_{p[0,1]}$, have the property

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq q > 0 \quad if \quad 0 \leq x_1 < x_2 \leq 1. \tag{3.1}$$

Then

$$E_n^1(f)_p \leq \omega \left(\frac{3E_n(f)_p}{q} \right) + E_n(f)_p$$

(ω is the modulus of continuity of $f \in L_{p[0,1]}$)

Proof.

Choose D_n be a best approximation to $f \in L_{p[0,1]}$, it means

$$E_n = E_n(f)_p$$

$$E_n(f)_p = \|f - D_n\|_{L_{p[0,1]}}.$$

Where $D_n \in P_n$.

So that

$$E_n = \|f - D_n\|_{L_{p[0,1]}}.$$

Using (3.1) and the definition of D_n, we get then,

$$D_n(x_2) - D_n(x_1)$$

$$\geq f(x_2) - f(x_1) - |f(x_1) - D_n(x_1)| - |f(x_2) - D_n(x_2)|$$

$$\geq f(x_2) - f(x_1) - \|f(x_1) - D_n(x_1)\|_p - \|f(x_2) - D_n(x_2)\|_p$$

$$= f(x_2) - f(x_1) - 2E_n(f)_p$$

$$\geq q(x_2 - x_1) - 2E_n(f)_p.$$

If $\left(q(x_2 - x_1) - 2E_n(f)_p\right) \geq E_n(f)_p$, then

$$q(x_2 - x_1) - 3E_n(f)_p > 0$$

$$(x_2 - x_1) \geq \frac{3E_n(f)_p}{q}.$$

$$D_n(x_2) - D_n(x_1) \geq q\left(\frac{3E_n(f)_p}{q}\right) - 2E_n(f)_p$$

$$= E_n(f)_p > 0.$$

So

$$D_n(x_2) - D_n(x_1) \geq q(x_2 - x_1) - 2E_n(f)_p > E_n(f)_p > 0 \; if \, (x_2 - x_1)$$

$$\geq \frac{3E_n(f)_p}{q}, \qquad (3.2)$$

Put in your mind, the polynomial of degree $\leq n$.

Define

$$P_n(x) = \frac{q}{3E_n(f)_p} \int_{\left(1 - \frac{3}{q}E_n(f)_p\right)x}^{\left(\left(1 - \frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} D_n(t)dt,$$

where $\delta(x) = \left(1 - \frac{3E_n(f)_p}{q}\right)x$ and $\beta(x) = \left(1 - \frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}$.

We have $0 \leq \left(1 - \frac{3E_n(f)_p}{q}\right)x < \left(1 - \frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q} \leq 1.$

If $0 \leq x \leq 1$. Using (3.2), we will see

$$P_n'(x) = \frac{q}{3E_n(f)_p}\left(1 - \frac{3E_n(f)_p}{q}\right)\left[D_n(\beta(x)) - D_n(\delta(x))\right]$$

$$\geq \frac{q}{3E_n(f)_p}\left(1 - \frac{3E_n(f)_p}{q}\right)E_n(f)_p > 0 \; for \, 0 \leq x \leq 1.$$

So $P_n'(x) > 0$.

If $\left(1 - \dfrac{3E_n(f)_p}{q}\right)x \le t \le \left(1 - \dfrac{3E_n(f)_p}{q}\right)x + \dfrac{3E_n(f)_p}{q}$, then by

definition of the modulus of continuity, we get

$$\left\| f(x) - f(t) \right\|_{L_{p[0,1]}} \le \omega\left(f, \frac{3E_n(f)_p}{q}\right)_p . \tag{3.3}$$

From (3.3) and the definition of D_n ,we get

$$\left\| f - P_n \right\|_{L_{p[0,1]}} = \left\| f - \frac{q}{3E_n(f)_p} \int\limits_{\left(1-\frac{3E_n(f)_p}{q}\right)x}^{\left(\left(1 - \frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} D_n(t)dt \right\|_{L_{p[0,1]}}$$

$$= \left\| \frac{q}{3E_n(f)_p} \int\limits_{\left(1-\frac{3E_n(f)_p}{q}\right)x}^{\left(\left(1 - \frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} [f(x) - D_n(t)]dt \right\|_{L_{p[0,1]}}$$

$$= \frac{q}{3E_n(f)_p} \quad \left\| \int_{\left(1-\frac{3E_n(f)_p}{q}\right)x}^{\left(\left(1-\frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} [f(x) - D_n(t)]dt \right\|_{L_{p[0,1]}}$$

$$= \frac{q}{3E_n(f)_p} \quad \left\| \int_{\left(1-\frac{3E_n(f)_p}{q}\right)x}^{\left(\left(1-\frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} [f(x) - f(t) + f(t) - D_n(t)]dt \right\|_{L_{p[0,1]}}$$

$$\leq c(p) \frac{q}{3E_n(f)_p} \left\| \int_{\left(1-\frac{3E_n(f)_p}{q}\right)x}^{\left(\left(1-\frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} f(x) - f(t)dt \right\|_{p}$$

$$+ c(p) \frac{q}{3E_n(f)_p} \left\| \int_{\left(1-\frac{3E_n(f)_p}{q}\right)x}^{\left(\left(1-\frac{3E_n(f)_p}{q}\right)x + \frac{3E_n(f)_p}{q}\right)} f(t) - D_n(t)dt \right\|_{p}$$

$$\leq \omega\left(f, \frac{3E_n(f)_p}{q}\right)_p + E_n(f)_p.$$

Theorem 3.2.2.

Let $f \in L_{p[0,1]}$ and $f(x) \geq q > 0$ on $0 \leq x \leq 1$. Then

$$E_n^1(f)_p \leq \frac{c(p)}{n^{\frac{1}{2}}} E_{n-1}(f)_p$$

Proof.

Let $D_{n-1} \in P_{n-1}$ be the polynomial of the best approximation to

f on $0 \leq x \leq 1, n = 1,2,3, \ldots$ Take n so large that $E_n^1(f)_p \leq \frac{q}{2}$.

Then define

$$Q_{n-1}(x) = D_{n-1}(x) - E_{n-1}(f)_p \geq 0 \text{ on } 0 \leq x \leq 1,$$

$$\left\| f - Q_{n-1}(x) \right\|_{L_{p[0,1]}} = 2E_{n-1}(f)_p.$$

$$\left\| f - D_{n-1}(x) - E_{n-1}(f)_p \right\|_{L_{p[0,1]}} = 2E_{n-1}(f)_p.$$

Define

$$\phi(x) = f(x) - \int_0^x Q_{n-1}(t)dt.$$

Then

$$\phi'(x) = f'(x) - [Q_{n-1}(x) - Q_{n-1}(0)].$$

Then, put

$$\phi'(x) = f'(x) - Q_{n-1}(x) \geq 0 \quad on \quad 0 \leq x \leq 1,$$

$$\left\| \phi'(x) \right\|_{L_{p[0,1]}} = 2E_{n-1}(f)_p, \text{ to get}$$

$$\left\| f'(x) - Q_{n-1}(x) \right\|_{L_{p[0,1]}} = 2E_{n-1}(f)_p.$$

Hence, $\phi(x) \in Lip_N 1, where \ N = 2E_{n-1}(f)_p$. Then $\phi(x) \in Lip_{2E_{n-1}(f)_p} 1,$

$$\left\| \phi(x) - B_n(\phi) \right\|_{L_{p[0,1]}} \leq \frac{c(p)}{n^{\frac{1}{2}}} \cdot 2E_{n-1}(f)_p,$$

$$\left\| f(x) - \int_0^x Q_{n-1}(t)dt - B_n(\phi) \right\|_{L_{p[0,1]}} \leq \frac{c(p)}{n^{\frac{1}{2}}} \cdot 2E_{n-1}(f)_p.$$

By the direct theorem of a function in $L_{p[0,1]}$, there exists a polynomial

D_n such that $\left\| f(x) - D_n \right\|_{L_{p[0,1]}} \leq E_n^1(f)_p \leq c(p)\omega_1 \left(f, \frac{1}{n} \right)_p.$

That is,

$$\left\| f(x) - D_n \right\|_{L_{p[0,1]}} \leq E_n^1(f)_p$$

$$\leq c(p)\omega_1 \left(f, \frac{1}{n} \right)_p$$

$$\leq c(p)\frac{1}{n^{\frac{1}{2}}} \cdot E_{n-1}(f)_p.$$

Where

$$D_n(x) = B_n(\phi, x) + \int_0^x Q_{n-1}(t)dt.$$

Where B_n is a polynomial of degree.

$$\left\| f(x) - B_n(\phi, x) + \int_0^x Q_{n-1}(t)dt \right\|_{L_{p[0,1]}} \le \frac{c(p)}{n^{\frac{1}{2}}} \cdot E_{n-1}(f)_p.$$

But $B_n'(\phi, x) \ge 0$ on $0 \le x \le 1$.

Now, we get

$$D_n'(x) = Q_{n-1}(x) + B_n'(\phi, x) \ge 0 \text{ on } 0 \le x \le 1.$$

$$D_n'(x) = D_{n-1}(x) - E_{n-1}(f)_p + B_n'(\phi, x) \ge 0.$$

This gives

$$E_n^1(f)_p \le \frac{c(p)}{n^{\frac{1}{2}}} E_{n-1}(f)_p.$$

Theorem 3.2.3.

Let $r \ge 2$ be an integer, and suppose that $f^{(r)} \in L_{p[0,1]}$ *and* $f^{(r)}(x) \ge q > 0$ on $0 \le x \le 1$. Then

$$E_n^{(r)}(f)_p \le \frac{c(p)}{n} E_{n-r}\left(f^{(r)}\right)_p \qquad\qquad (3.4)$$

Proof.

If $r = 2$ suppose that $f'' \in L_{p[0,1]}$ and $f''(x) \ge q > 0$ on

$0 \le x \le 1$, and prove that for each sufficiently large n,

$$E_n^2(f)_p \leq \frac{c(p)}{n} E_{n-2}(f'')_p. \qquad\qquad (3.5)$$

Let Q_{n-2}^* be the polynomial of the best approximation from P_{n-2} to f'' on $0 \leq x \leq 1$.

Choose an n large enough to secure it $E_{n-2}(f'') \leq \frac{q}{2}$.

Define

$$Q_{n-2}(x) = Q_{n-2}^* - E_{n-2}(f')_p.$$

$Q_{n-2}(x) \geq 0$ on $0 \leq x \leq 1$, and

$$\left\| f''(x) - Q_{n-2}(x) \right\|_{L_{p[0,1]}} = 2E_{n-2}(f'')_p.$$

$$\left\| f''(x) - Q_{n-2}^* - E_{n-2}(f')_p \right\|_{L_{p[0,1]}} = 2E_{n-2}(f'')_p.$$

Define

$$\phi_1(x) = f'(x) - \int_0^x Q_{n-2}(t)dt.$$

$$\phi_1'(x) = f''(x) - [Q_{n-2}(x) - Q_{n-2}(0)] \geq 0 \text{ 0n } 0 \leq x \leq 1,$$

$$\phi_1'(x) = f''(x) - Q_{n-2}(x) \geq 0 \text{ 0n } 0 \leq x \leq 1,$$

$$\phi_1'(x) = f''(x) - Q_{n-2}^* - E_{n-2}(f')_p \geq 0 \text{ 0n } 0 \leq x \leq 1,$$

$$\left\| \phi_1'(x) \right\|_{L_{p[0,1]}} = 2E_{n-2}(f'')_p.$$

$$\left\| f''(x) - Q_{n-2}(x) \right\|_{L_{p[0,1]}} = 2E_{n-2}(f'')_p.$$

$$\left\| f''(x) - Q_{n-2}^* - E_{n-2}(f')_p \right\|_{L_{p[0,1]}} = 2E_{n-2}(f'')_p.$$

Now $\phi(x) \in Lip_N 1$, where $N = 2E_{n-2}(f'')_p$.

$$\| \phi(x) - B_n(\phi) \|_{L_{p[0,1]}} \leq \frac{c(p)}{n^{\frac{1}{2}}} \cdot 2E_{n-2}(f'')_p \frac{1}{n^{\frac{1}{2}}} = \frac{c(p)}{n} E_{n-2}(f'')_p,$$

where

$$\phi(x) = \int_0^x \phi_1(t)dt = f(x) - A_n(x),$$

and $A_n(x)$ is a polynomial of degree $\leq n$ with the property $A_n''(x)$.

$$\| \int_0^x \phi_1(t)dt - B_n(\phi) \|_{L_{p[0,1]}} \leq \frac{c(p)}{n^{\frac{1}{2}}} \cdot 2E_{n-2}(f'')_p \frac{1}{n^{\frac{1}{2}}} = \frac{c(p)}{n} E_{n-2}(f'')_p,$$

$$\| f(x) - A_n(x) - B_n(\phi) \|_{L_{p[0,1]}} \leq \frac{c(p)}{n^{\frac{1}{2}}} \cdot 2E_{n-2}(f'')_p \frac{1}{n^{\frac{1}{2}}}$$

$$= \frac{c(p)}{n} E_{n-2}(f'')_p,$$

Now,

$$\| f - D_n \|_{L_{p[0,1]}} \leq \left(\frac{c(p)}{n}\right) E_{n-2}(f'')_p,$$

$$\| f - A_n(x) + B_n(\phi, x) \|_{L_{p[0,1]}} \leq \left(\frac{c(p)}{n}\right) E_{n-2}(f'')_p,$$

where $D_n(x) = A_n(x) + B_n(\phi, x)$.

But, $D_n''(x) = A_n''(x) + B_n''(\phi, x) \geq 0$ on $0 \leq x \leq 1$.

We take (3.5). This indicates that for n sufficiently large there is a polynomial D_{n-r+2} such that $D_{n-r+2}''(x) \geq 0$ on $0 \leq x \leq 1$ and

$$\left\| f^{(r-2)}(x) - D_{n-r+2}(x) \right\|_{L_{p[0,1]}} \leq \frac{c(p)}{n} E_{n-r}\left(f^{(r)}\right)_p,$$

$$\leq \frac{c(p)}{n} E_{n-r}\left(f^{(r)}\right)_p.$$

Now followed by merging $r - 2$ times, that

$$\left\| f - Q_n \right\|_{L_{p[0,1]}} \leq \left(\frac{c(p)}{n}\right) E_{n-r}\left(f^{(r)}\right)_p,$$

where Q_n is some polynomial of degree $\leq n$.

Chapter Four

Negative Theorem for r-Monotone Approximation.

Recently many direct theorems introduced for monotone, convex, 3-monotone approximation, r-monotone for functions in L_p quasi normed space. Now there is a question; can we strength these results in terms of moduli of smoothness of higher order? The answer is no, and that what we see in our work here. Here we introduce negative theorem for the r-monotone approximation of function in L_p quasi normed space for $0 < p < 1$.

4.1. Introduction

We mean by $L_{p[0,1]}^M, 0 < p < 1$ the class of all function which has M derivatives in $L_{p[0,1]}$, and π_n the class of algebraic polynomials of degree $\leq n$.

The degree of the best r-monotone approximation is

$$E_n^{(r)}(f)_p = \inf\left\{\|f - p_n\|_p : p_n \in \pi_n \cap \Delta^r\right\},$$

for $f \in L_{p[0,1]}$,

$$\omega_m(f, \delta)_p = \sup_{0 < |h| \leq \delta} \|\Delta_h^m f(x)\|_p,$$

for $m \in \mathcal{N}, \ \delta > 0$.

Several, researchers devoted their researches in studying monotone and co monotone approximation of continuous function by algebraic polynomials, particularly to Jackson type estimates (cf., for example, $[28], [29], [30] - [5], [15], [10], [12] - [34]$). In[8] E. S. Bhaya, G. Abdullah, introduced Negative theorem for $L_P, 0 < P < 1$ monotone approximation. Many researchers, including Lorentz and Zeller in [15] and Shvedov in [1], from their opposite results, it

appears that the estimates of the Jackson type for the criteria of the highest degree of smoothness are not correct in the monotone approximation. [35], imagine that there is a continuous function which is r-monotone such that for $r \geq 1$.

$$\lim_{n \to \infty} \frac{E_n^{(r)}(f)}{\omega_{2r+1}\left(f, \frac{1}{n}\right)} = +\infty.$$

In [35], the authors got a weaker result, which showed the existence of an r-monotone continuous function such that

$$\lim_{n \to \infty} \sup \frac{E_n^{(r)}(f)}{\omega_{2r+1}\left(f, \frac{1}{n}\right)} = +\infty.$$

For $r \geq 2$

$$\lim_{n \to \infty} \sup \frac{E_n^{(r)}(f)}{\omega_4\left(f, \frac{1}{n}\right)} = +\infty.$$

Here we improve the results above for function in L_P quasi normed spaces. We shall prove a negative theorem for the best r-monotone approximation of function in $L_{p[0,1]}$ space. It means; we introduce the result.

4.2. The Main Result

Theorem.4.2.1

Let $r \geq 1$. Then there exists a function $f \in L_{p[0,1]}^r \cap \Delta^r$ such that

$$E_{n_l}^{(r)}(f)_p > \omega_{r+3}\left(f, \frac{1}{n}\right)_p.$$

4.3- The Auxiliary Results

In this section we shall introduce results that we need in our proof of the main theorem.

Lemma4. 3.1.[4]

Let $b > 0$, $\alpha(x) = \dfrac{b^2}{(x^2 - b^2)}$, $g_m(x, b) = x^m e^{\alpha(x)+1}$, and $x \in (-b, b)$, then

$$\left| g_m^{(m)}(x, b) - m! \right| \le c(m) b^{-2} x^2, \qquad |x| < b.$$

Where c(x) always indicates a positive constant depending upon x only which takes different values in different places in this work.

Lemma 4.3. 2.[5]

Let $\delta_n \ge 0, \delta_n \to 0, n \to \infty, \ \sigma > 0$. Define

$$\bar{g}_r(x, \delta_n, \sigma) = \delta_n^{2+2\sigma} g_r(x, \delta_n) + x^{r+2} - \delta_n^{2+2\sigma} x^r, \qquad x \in (-\delta_n, \delta_n);$$

then for sufficiently large n and, $\quad x \in (-\delta_n, \ \delta_n)$

$$\bar{g}_r^{(r)}(x, \delta_n, \sigma) \ge 0.$$

We set

$$F_l(x) = \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x),$$

$$Q_l(x) = q_l(x) + n_l^{-\frac{1}{4}} \left(x^{r+2} - n_l^{-\frac{5}{2}} x^r \right),$$

$$f_n(x) = \begin{cases} x^{r+2} - n^{-\frac{5}{2}}x^r, & |x| \geq n^{-\frac{9}{8}}, \\ \overline{g_r}\left(x, n^{-\frac{9}{8}}, \frac{1}{9}\right), & |x| < n^{-\frac{9}{8}}, \end{cases}$$

where $q_l(x)$ is the algebraic polynomial of the best approximation of degree n_l to $F_{l-1}(x)$, and $\{n_l\}$ is a later result of natural numbers chosen through induction: Set n_l, to be part of natural number N,

$$n_l \geq \left\| F_{l-1}^{(2r+8)} \right\|_p.$$

Lemma 4.3.3.

For F_l and $Q_l \in L_{p[0,1]}$, we have

$$\left\| F_l - Q_l \right\|_p \sim n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}}x^r \right\|_p \sim n_l^{-\frac{9r}{8} - \frac{11}{4}},$$

$$Q_l^{(r)}(0) \leq -c(p) \, n_l^{-\frac{11}{4}}$$

satisfied.

Proof. By lemma 4.3.2

$$\left\| F_l - q_l \right\|_p = \left\| \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x) - q_l(x) \right\|_p.$$

Using the definition of $F_{l-1}(x)$ and q_l we get

$$\left\| F_l - q_l \right\|_p \leq c(p) \left\| F_{l-1}^{(2r+8)} \right\|_p n_l^{\frac{1}{2r+8}}. \tag{4.1}$$

We have

$$\left\| F_l(x) - Q_l(x) \right\|_p,$$

$$= \left\| F_l(x) - q_l(x) - n_l^{-\frac{1}{4}} \left(x^{r+2} - n_l^{-\frac{5}{2}} x^r \right) \right\|_p,$$

$$= \left\| F_{l-1}(x) - q_l(x) + n_l^{-\frac{1}{4}} \left(f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r \right) \right\|_p,$$

$$\leq c(p) \left(\left\| F_{l-1}(x) - q_l(x) \right\|_p + n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}} x^r \right\|_p \right).$$

Then

$$\left\| f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r \right\|_p = \left(\int_{-n_l^{-\frac{9}{8}}}^{n_l^{-\frac{9}{8}}} \left| e \, n_l^{-\frac{5}{2}} x^r \, e^{\left(x^2 - n_l^{-\frac{9}{4}} \right)^{\left(n_l^{-\frac{9}{4}} \right)}} \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq c(p) n_l^{\frac{1}{\frac{9}{8}r - \frac{5}{2}}} \tag{4.2}$$

$$\left\| F_l - Q_l \right\|_p \leq c(p) \left(\left\| F_{l-1}(x) - q_l(x) \right\|_p \right.$$

$$\left. + n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} - n_l^{-\frac{5}{2}} x^r \right\|_p \right).$$

From (4.1) and (4.2), we get

$$\left\| F_l - Q_l \right\|_p \leq c(p) \left(\left\| F_{l-1}^{(2r+8)} \right\|_p n_l^{\frac{1}{2r+8}} + n_l^{-\frac{1}{2}} \, n_l^{\frac{1}{\frac{9}{8}r - \frac{5}{2}}} \right).$$

From our hypothesis lemma 4.3.2

$$\left\| F_l - Q_l \right\|_p \leq c(p) \left(n_l \, n_l^{\frac{1}{2r+8}} + n_l^{-\frac{1}{2}} \, n_l^{\frac{1}{\frac{9}{8}r - \frac{5}{2}}} \right)$$

$$\le c(p)\, n^{-\frac{9r}{8}-\frac{1}{4}}. \tag{4.3}$$

We have

$$\left\| q_l \right\|_p \le c(p)\,\left| F_l \right| n_l^{-r-8}.$$

$$\left\| F_l - Q_l \right\|_p = \left\| \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x) - q_l(x) + n_l^{-\frac{1}{4}}\left(x^{r+2} - n_l^{-\frac{5}{2}} x^r \right) \right\|_p$$

$$\le c(p) \sum_{i=1}^{l} \left\| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}} x^r \right\|_p \, n_l^{-\frac{1}{4}}$$

$$\le c(p)\, n_l^{-\frac{1}{4}} \left\| f_{n_l}(x) - x^{r+2} + n_l^{-\frac{5}{2}} x^r \right\|_p.$$

Lemma 4.3. 4.

For any $p(x) \in \pi_{n_l} \cap \Delta^r$ and large enough l, we have

$$\left\| F_l - p(x) \right\|_p \ge c(p) n_l^{-r-\frac{11}{4}}.$$

Proof.

$$\left\| Q_l^{(r)} - p(x)^{(r)} \right\|_p \le c(p)\, n_l^{r} \left\| Q_l - p(x) \right\|_p$$

$$= c(p)\, n_l^{r} \left\| Q_l - F_l + F_l - p(x) \right\|_p$$

$$\le c(p)\, n_l^{r} \left(\left\| Q_l - F_l \right\|_p + \left\| F_l - p(x) \right\|_p \right)$$

Thus

$$\left\| Q_l^{(r)} - p^{(r)}(x) \right\|_p \le c(p)\, n_l^{r} \left(\left\| Q_l - F_l \right\|_p + \left\| F_l - p(x) \right\|_p \right).$$

From (4.3), we get

$$\left\| F_l - Q_l \right\|_p \le c(p) n^{-\frac{9}{8}r - \frac{1}{4}}$$

$$\left\| Q_l^{(r)} - p^{(r)}(x) \right\|_p \le c(p) \, n_l^r \left(n^{-\frac{9}{8}r - \frac{1}{4}} + \left\| F_l - p(x) \right\|_p \right)$$

$$\left\| F_l - p(x) \right\|_p \ge n_l^{-r} \left\| Q_l^{(r)} - p^{(r)}(x) \right\|_p - n^{-r - \frac{9}{8}r - \frac{1}{4}}.$$

Proof of the main result Theorem 4.2.1.

From Lemma 4. 3.2, we see that there exists $M > 0$ such that $n \ge M$,

$$(\Delta_h^r f_n) \ge 0.$$

Now select $\{n_l\}$ by induction. Set $n_l = M$,

$$n_{l+1} = 2 \left(n_l^{4(r+3)} + \left[\left\| F_l^{(2r+8)} \right\| \right] + \left[\left\| F_l^{(r+3)} \right\| \right]^5 + 1 \right) \text{ for } l = 1,2,3, \dots.$$

Define

$$f(x) = \sum_{i=1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x).$$

It is clear that $f \in L_{p[0,1]}^r \cap \Delta^r$. For any $p(x) \in \pi_{n_l} \cap \Delta^r$,

$$\left\| f(x) - p(x) \right\|_p = \left\| f(x) - F_l + F_l - p(x) \right\|_p$$

$$\ge -\left\| p(x) - F_l \right\|_p + \left\| F_l - f \right\|_p$$

$$= \left| \, \left\| F_l - f \right\|_p - \left\| p(x) - F_l \right\|_p \right|$$

$$= \left| \, \left\| \sum_{i=1}^{l} n_i^{-\frac{1}{4}} f_{n_i}(x) - \sum_{i=l}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x) \right\|_p - \left\| p(x) - F_l \right\|_p \right|$$

$$= \Big| \, \Big\| \sum_{i=1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x) \, \Big\|_p - \| p(x) - F_l \|_p \, \Big|.$$

$$> \| F_l - p(x) \|_p - \Big\| \sum_{i=l+1}^{\infty} n_i^{-\frac{1}{4}} f_{n_i}(x) \Big\|_p.$$

Then using Lemma 4.3. 4, we get

$$\| f(x) - p(x) \|_p > c(p) n_l^{-r-\frac{11}{4}} - n_{n+1}^{-\frac{1}{4}}$$

$$> c(p) n_l^{-r-\frac{11}{4}}.$$

Since p is r-monotone best approximation to f. So

$$E_{n_j}^{(r)}(f)_p > c(p) n_l^{-r-\frac{11}{4}} > c(p) n_l^{-r}.$$

On the other hand using Lemma4.3.3, we get

$$E_{n_l}^{(r)}(f)_p \geq c(p) \omega_{r+3} \left(f, \frac{1}{n} \right)_p.$$

Conclusion

Many papers were introduced for the approximation of monotone functions using monotone polynomials. But we sometimes need to approximate monotone function by rational monotone polynomials. In this thesis, we fill this gap, and estimate the degree of the best constrained approximation by using rational monotone polynomials and rational r-monotone polynomials.

Future Work

In the future we intend to define rational neural network, i.e neural networks with rational activation function. Then we study the approximation of function in the quasi normed spaces in terms of this rational neural network.

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الخلاصة

تم تقديم العديد من البحوث حول التقريب الرتيب باستخدام متعددات الحدود الرتيبة. لكن في بعض الاحيان تحتاج الى ان يكون تقريب الدوال الرتيبة متعدد حدود رتيبة ونسبيه.

في هذه الرسالة سنعالج تلك المشكلة بدراسة التقريب المقيد الرتيب باستخدام متعددات الحدود النسبية والرتيبة من النوع-r.

كذلك درس الباحثون العلاقة بين درجه التقريب بقيود ودرجه التقريب بغير قيود وبرهنوا المتراجعة(*)

$$E_n(f) \leq C \, E_n^{(r)}(f) \qquad (*)$$

حيث C هو ثابت موجب.

وظل عكس المتراجحة(*) غير مبرهن وخاصة للدوال التي تنتمي الى فضاء معياري كاذب.

سنقوم ببرهان العلاقة بين درجه تقريب الرتيب لدوال في الفضاءات $L_{p[0,1]}$ ودرجه التقريب الغير رتيب.

قام الباحثون مؤخرا بتقديم العديد من النظريات حول التقريب الرتيب والمحدد ال 3-رتيب و r-رتيب لدوال في الفضاء $L_{p[0,1]}$.

وهنا تولد السؤال هل بالإمكان تقويه نظرياتهم بدلالة معاير نعومه برتب اعلى؟ الجواب كلا وهذا ما تم برهانه في نظريتنا السالبة لدوال r-رتيب في الفضاء الكاذبة $L_{p[0,1]}$ عندما $0 < p < 1$.

التقريب النسبي الرتيب في الفضاءات $p < 1, L_p$

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل كجزء من متطلبات نيل

درجة الماجستير في التربية / الرياضيات

من قبل

وهلاء عباس عاجل شناوة

بإشراف

أ.د. ايمان سمير بهيه