Republic of Iraq

Ministry of Higher Education

and Scientific Research

University of Babylon

College of Education for Pure

Sciences

The Chaotic Properties on G-Non Autonomous Discrete Dynamical Systems

A Thesis

Submitted to the Council of the College of Education for Pure Sciences in University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics

By

Russl Abd Al-Khaleq Abd Al-Rahem Hussen

Supervised by

Prof.dr. Iftichar Mudhar Talb Al-Sharaa

2023 A.D. 1445 A.

بسم الله الرحمن الرحيم

(يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات والله بما تعملون خبير)

صدق الله العلي العظيم

Supervisor's Certification

I certify that the thesis entitled "***The chaotic properties on G-non autonomous discrete dynamical systems***" by "***Russl Abd Al-Khaleq Abd Al-Rahem Al-Joboury***" has been prepared under my supervision in Babylon University/ College of Education for Pure Sciences as a partial requirement for the degree of Master in Education / Mathematics.

Signature :

Name : Dr. Iftichar Mudhar Al-sharaa

Title: Assistant Professor

Date: / / 2023

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:

Name : Dr. Azal Jaafar Musa

Head of Mathematics Department

Title: Assistant Professor

Date: / / 2023

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis **“*The chaotic properties on G-non autonomous discrete dynamical systems*”** and I have approved this dissertation is qualified for debate

Signature:

Name: Dr.

Title:

Date:

Certification of Scientific Expert

I certify that I have read the scientific content of this thesis "***The chaotic properties on G-non autonomous discrete dynamical systems***" and I have approved this dissertation is qualified for debate

Signature:

Name: Dr.

Title:

Date:

Certification of linguistic Expert

I certify that I have read this thesis entitled **"*The chaotic properties on G-non autonomous dynamical systems* "** and corrected its grammatical mistakes; therefore, it has qualified for debate.

Signature:

Name: Dr.

Title:

Address:

Date: / / 2023

Examining Committee Certification

We certify that we have read the thesis entitled **"** ***The chaotic properties on autonomous dynamical systems*** **"** by**"** ***Russl Abd Al-Khaleq Abd Al-Rahem Al-Joboury*** **"** and as a committee examined the student in its contents and, according to our opinion, it is accepted as a thesis for the degree of Master in Education / Mathematics .

Signature:	**Signature:**
Name:	**Name:**
Title:	**Title:**
Date: / / 2023	**Date:** / / 2023
Chairman	**Member**

Signature:	**Signature:**
Name:	**Name:** ***Dr. Iftichar Mudhar Al-Al-sharaa***
Title:	**Title:**
Date: / / 2023	**Date:** / / 2023
Member /	**Member / Advisor**

Dedication To

My mother, My father,

My husband, My children, My family,

My teachers,

And my friend.

Acknowledgments

Thanks and praise to God Almighty for giving me the determination and success to complete this work

I extend my thanks and appreciation to my supervisor, Professor Iftichar Al-Sharaa (Ph.D) and for all the support, guidance she provided me to accomplish this work

I would like to thank all my professors and colleagues on the College of Education for Mathematics Sciences at the University of Babylon.

All thanks and love to my family, friends and everyone who supported and helped me.

Contents

List of Symbols

Symbol	Description
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integer numbers
$\mathbb{Z}_+$	The set of positive integer numbers
$\mathbb{Z}_-$	The set of negative integer numbers
$D(U)$	Diameter of U
$\overline{D}$	Positive upper density
$\mathbb{C}$	The cantor set
$\ell_n^m \xrightarrow{u} \ell$	ℓ_n^m uniformly convergences to a map ℓ
$\lvert x \rvert$	The absolute value of x
$B(x, \varepsilon)$	The open ball with the center x and the radius ε
X	Compact metric space
(ℓ_n)	A sequence of an uniform continuous maps from X to X, for all $n \in \mathbb{N}$

Table of Abbreviations

Abbreviation	Description
g-NDS	The general non-autonomous discrete dynamical systems
ADS	The autonomous discrete dynamical systems
DS	the discrete dynamical systems
P.O.T.P	Pseudo-orbit tracing property.
A.S.P	Average shadowing property.
A.A.S.P	Asymptotic average shadowing property
F.S.P	Fitting shadowing property.
A.F.S.P	Asymptotic fitting shadowing property
Diam$(\boldsymbol{X})$	Diameter of set X
AP($\boldsymbol{\ell_n^m}$**)**	The set of all minimal points of ℓ_n^m
CR($\boldsymbol{\ell_n^m}$**)**	The set of all chain recurrent points of ℓ_n^m

List of Publications Arising from This Thesis

1- R. A. Al-Rahim and I. Al-Sharaa," The Minimality of g-Non autonomous Discrete Dynamical Systems" Journal of Interdisciplinary Mathematics (ISSN:0972-0502 ICPS_2022_0092).

(acceptable for publication).

2- R. A. Al-Rahim and I. Al-Sharaa," The W-expansive With The Shadowing Property in g-nonautonomous Discrete Dynamical Systems" Journal of Kufa For Mathematics And Computer.

(Acceptable For Publication).`

Abstract

The aim of this work is to study some of chaotic properties on the g-non-autonomous discrete dynamical systems. Through this study, we introduced some differences between the g-non-autonomous and the autonomous of discrete dynamical systems, when they are minimal, And we explain the shadowing property on the g-non-autonomous discrete dynamical system.

Some results on the shadowing property with the chaotic dynamical systems are shown.

New concept, for example; (The w-Expansive in g-Non-Autonomous Discrete Dynamical System, The Fitting Shadowing Property on g-Non-Autonomous Discrete Dynamical System, ….) are introduced. These concepts and definitions are used to prove many new findings.

It also explains other types of shadowing properties, such as: h-shadowing property in g-non-autonomous discrete dynamical system and h-fitting shadowing property g-non-autonomous discrete dynamical system . Some definitions and findings on the uniform convergence and uniform conjugate for a series of continuous distinct mappings on a compact metric space are discussed.

The main findings proven in this study are:

Assume that (X, d) be a compact metric space (for short we write X), an g-NDS is a pair (X, ℓ_n^m) where ℓ_n be a sequence of a continuous maps $\ell_n: X \longrightarrow X$, for all $n \in \mathbb{N}$, and the composition $\ell_n^m = \ell_m \circ \ell_{m-1} \circ \cdot \cdot \cdot \cdot \circ \ell_n$, $\forall\, 0 \leq n < m \in \mathbb{N}$, then we prove:

- If the g-nonautonomous discrete dynamical system (g-NDS) is transitive then the discrete dynamical systems (DS) is transitive. And the same of (sensitive and equicontinuous properties.

- A minimum g-non autonomous discrete dynamical system exists system (g-NDS) it which has neither a sensitive nor an equicontinuous.

- If X is a compact metric space and ℓ_n: X → X, $\forall n \in \mathbb{N}$, satisfies the fitting shadowing property, then $\underbrace{\ell_n^m \times \ell_n^m \times \ell_n^m \times \ldots . \times \ell_n^m}_{k-times}$ has the fitting shadowing property.
- Assume that $\ell_n: X \to X, \forall n \in \mathbb{N}$, be an uniform continuous maps and X is a compact metric space. For each $k \in \mathbb{N}$, $(\ell_n^m)^k$ will have the fitting shadowing property (F.S.P) if ℓ_n^m for (F.S.P) of g-NDS if possesses this property $\forall\ n < m$.

- The uniform continuous maps $\ell_n: X \to X$ and $g_n: Y \to Y, \forall n \in \mathbb{N}$ where compact metric spaces are (X, d) & (Y, d'), then ℓ_n^m & g_n^m have a shadowing asymptotics fitting condition in g-NDS, $\forall\ n < m$ if and only if an asymptotic fitting shadowing characteristic in g-NDS is present in $\ell_n^m \times g_n^m$.
- If X is a compact metric space and ℓ_n: X → X, $\forall n \in \mathbb{N}$, be an uniform continuous map, then ℓ_n^m is chain transitive if it is surjective and possesses a fitting shadowing property in g-NDS $\forall\ n < m$.

Introduction

Given the importance of dynamic systems (nonautonomous and autonomous), their theories have been studied, and researched extensively in the past decades, because many natural phenomena can be modeled through a system of differential equations, or discrete dynamic systems.

For instance, Estimating species population increase, or managing the dynamics of diverse of different mechanical and electrical systems [1].

L. Snoha and S. Kolyada introduced nonautonomous discrete dynamic systems (NDS for short) in their work [2].

A. Miralles, A., M. Murillo-Arcila, M., & Sanchis, M. in [3] They explained that transitivity as well as the density for periodic points do not, in general, such as imply sensitivity; however, if uniform convergence to a sequence(ℓ_n) that induces the NDS is assumed, then sensitivity does follow. In addition, contrary to the autonomous case, they demonstrate the existence of minimal nonautonomous discrete dynamical systems that are neither equicontinuous nor sensitive.

Many chaotic properties (metrical and topological) have been studied for their importance.

In [4] Vasisht, R., & Das, R. are study some types of expansive for non-autonomous discrete dynamical systems

Additionally, because the average-shadowing feature is one for the key ideas in the qualitative theory for dynamic systems and is crucial, Niu

Yingxuan examined and explored it in [5]. Furthermore, it was demonstrated that if ℓ satisfies the average-shadowing quality and its minimum points are dense in X, ℓ is weakly mixing and completely strongly ergodic.

In [6] as well are Kulczycki, M., and Oprocha, P. talked about how the asymptotic average shadowing property (A.A.S.P) relates to other topological dynamics concepts. and shown that ℓ is completely transitive if it possesses the A.A.S.P and its minimum points are dense in X.

In [7], it was discussed addressed the dynamics of n-expansive homeomorphisms with the shadowing property specified on compact metric spaces.

In [8], explored the relationship between different expansivity, such as locally expanding, favorably expansive, and weakly locally expanding, with h-shadowing property and shadowing.

Novel concepts have been introduced in [9], including related asymptotic fitting shadowing property and its fitting shadowing property.

In [10], the general-nonautonomous discrete dynamical systems is defined by Baraa A. and Iftichar M. and they denoted by $(X, \ell_{n,\infty})$ when X is a compact topological space and $\ell_n: X \to X$, $\forall n \in \mathbb{N}$, such that $\ell_{n,\infty}$ is the sequence of the continuous maps

In our work, we will study Some chaotic properties and some types of shadowing and generalize them into the general-nonautonomous discrete dynamical systems (g-NDS).

This thesis is divided into three chapters:

In chapter one, we take two sections. **in section one**, We define and explain some concepts and properties of chaos in g-nonautonomous discrete dynamic systems. **In section two**, we define the average shadowing property and asymptotic shadowing property in g-nonautonomous discrete dynamic systems.

In chapter two, we have three sections. **in section one**, we discuss the sensitive and the equicontinuous property in g-nonautonomous discrete dynamic systems, and we prove There exists a minimal g-non autonomous discrete dynamical system which is neither equicontinuous nor sensitive,That is different for the autonomos discrete dynamic systems.

In section two, definition h-shadowing property and we discuss its relation with various expansivity in g- nonautonomous discrete dynamic systems.

In section three, we discuss the relation between w-expansive homeomorphism with shadowing property in g-nonautonomous discrete dynamic systems.

In chapter three, we take two section. **In section 1**, the asymptotic fitting shadowing property (A.F.S.P) in g-nonautonomous discrete dynamical systems (F.S.P) and fitting shadowing property (F.S.P), **in section 2**, provides a working description of the h-fitting shadowing property and a discussion of its relationship to the fitting shadowing phenomenon in g-nonautonomous discrete dynamical systems

Chapter One

Some Basic Concepts in g-Nonautonomous Discrete Systems

1.1 Preliminaries

the aim of this section is define some concept of g- non autonomous discrete dynamical systems.

Definition 1.1.1 [10]

Let (X, d) be a compact metric space , let (ℓ_n) be a sequence of an uniform continuous maps, such that $\ell_n: X \longrightarrow X$, for all $n \in \mathbb{N}$, then the composition

$$\ell_n^m = \begin{cases} \ell_m \circ \ell_{m-1} \circ \dots \dots \circ \ell_n & for\ 0 \leq n < m \\ \ell_{-m}^{-1} \circ \ell_{-(m-1)}^{-1} \circ \dots \dots \circ \ell_{-n}^{-1} & for\ m < n \leq -1 \end{cases}$$

Is called **The general-non autonomous discrete dynamical systems for short (g-NDS)**.

If (ℓ_n) is a sequence of an uniform homeomorphism map, then inverse map is given by $(\ell_n^m)^{-1} = \ell_{-m}^{-n}\ , where\ m < n \leq 0.$

In addition, we define a (***k* th –iterate of $\boldsymbol{\ell_n^m}$**)

$(\ell_n^m)^k = (g_m)_{mk}^{(m-1)k+1}$, where $0 \leq n < m \in \mathbb{N}, k > 0$ on X, where $g_m = \ell_{mk} \circ \ell_{(m-1)k+k-1} \circ \dots \circ \ell_{(m-1)k+1.}$

Thus $(\ell_n^m)^k = \ell_{(m-1)k+1,mk}$.

Definition 1.1.2 [4]

A homeomorphism $\ell_n: X \rightarrow X,$ is said to be an **uniform homeomorphism** if ℓ_n^m is an uniform continuous on X, and $(\ell_n^m)^{-1}$is an uniform continuous on $X, \forall\ 0 \leq n < m \in \mathbb{N}$.

Definition 1.1.3

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$. The sequence $\ell_n: X \to X$ is said to **uniform convergence** to $\ell: X \to X$ if every $\varepsilon > 0$, there exists $m_o \in \mathbb{N}$, $\ni, d(\ell_n^m(x), \ell(x)) \leq \varepsilon$ for every $m > m_o$ and for every $x \in$ X. we write $\ell_n \overset{u}{\to} \ell$.

Definition 1.1.4 [10]

Let (X, d) be a compact metric space and $\ell_n: X \longrightarrow X$ be an uniform continuous maps $\forall n \in \mathbb{N}$. A point $x \in X$ is called **fixed point** of $(\ell_n^m)_{0 \leq n < m}$ in g-nonautonomous discrete dynamical systems when $\ell_n^m(x) = x$, where $0 \leq n < m \in \mathbb{N}$.

Example 1.1.5

let $\ell_n: R \to R$ be a maps , $\forall n \in \mathbb{N}$, where

$\ell_1(x) = x^2$,

$\ell_2(x) = 5x^2 - 2x$,

$\ell_3(x) = \sin x$.

If $x = 0$, hence

$$\ell_1^3(x) = \ell_3 \circ \ell_2((0)^2)$$

$$= \ell_3(5(0)^2 - 2(0))$$

$$= \sin 0$$

$= 0,$

Then the point $x = 0$ is a fixed point for ℓ_1^3.

Definition 1.1.6 [10]

Let (X, d) be a compact metric space and $\ell_n: X \longrightarrow X$ be an uniform continuous map $\forall n \in \mathbb{N}$. A point $\boldsymbol{a} \in X$ is called a **periodic point** of ℓ_n^m in g-non autonomous discrete dynamical systems if there is an exists $m \in \mathbb{N}$, $\quad \ell_n^m(a) = a \; and \;\; \ell_n^k(a) \neq a, \qquad \forall\, k < m$.

Example 1.1.7

let $\ell_n: R \rightarrow R$ be a maps , $\forall n \in \mathbb{N}$, where

$\ell_1(a) = a + 3,$

$\ell_2(a) = \dfrac{a+3}{2},$

$\ell_3(a) = \sqrt[2]{a}\,.$

If $a = 2$, hence

$\ell_1^3(a) = \ell_3 \circ \ell_2(2 + 3)$

$= \ell_3(\frac{\mathbf{5+3}}{\mathbf{2}})$

$= \sqrt[2]{4}$

$= 2.$

Then the point $a = 2$ is a periodic point for ℓ_1^3 of period 3.

<u>Theorem</u> 1.1.8

Let (X, d) be a metric space without isolated point, assume that $\ell_n: X \to X$ converges pointwise to ℓ, then if $\mathcal{P}$ is a periodic point in the g-nonautonomous discrete dynamical system then $\mathcal{P}$ is a periodic point of the discrete dynamical systems.

Proof :

Assume $\mathcal{P}$ is a periodic point ,there is $0 \leq n < m$, $\forall\, n, m \in \mathbb{N}$ meaning that $\ell_n^m(\mathcal{P}) = \mathcal{P}$,

then for any $0 < i < m$, we get have

$\ell_n^{m+i}(\mathcal{P}) = \ell_{m+i} \circ \ell_{m+(i-1)} \circ \dots . . \circ \ell_{m+1}(\mathcal{P})$

Which is convergent to $\ell^i(\mathcal{P})$, in particular, for $i = m - n$, then $\ell_n^{m+i}(\mathcal{P}) = \mathcal{P}$, is convergent to $\ell^i(\mathcal{P})$ so $\ell(\mathcal{P}) = \mathcal{P}$, this mean $\mathcal{P}$ is a periodic point in (DS) .

<u>Definition</u> 1.1.9

Let (X, d) be a compact metric space and ℓ_n: X → X be an uniform continuous map $\forall n \in \mathbb{N}$ .A point $\boldsymbol{x} \in X$ is called **chain recurrent point in g-nonautonomous discrete dynamical systems** of $(\ell_n^m)_{0 \leq n < m}$, if for each $\varepsilon > 0$, there is ε -chain between x and x. **CR($\boldsymbol{\ell_n^m}$)** represents the collection of all ℓ_n^m chain repeating points.

Definition 1.1.10

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous maps, $\forall n \in \mathbb{N}$. For a real number $\eta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ in X called a **η-chain in g-nonautonomous discrete dynamical systems**:

if $d\big(\ell_j(x_m), x_{m+1}\big) \leq \eta$, for all $0 \leq n < m$.

The maps $(\boldsymbol{\ell_n^m})_{\mathbf{0 \leq n < m}}$ is called **chain-transitive in g-nonautonomous discrete dynamical systems :** if for all $x, y \in X$ and every $\eta > 0$, there is a finite η-chain in g- non autonomous discrete dynamical systems $\{x_0, x_1, \ldots x_n\}$

such that $x_0 = x$ and $x_n = y$**.**

Definition 1.1.11 [10]

Let (X, d) be a compact metric space and $\ell_n: X \longrightarrow X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, A sequence $(\ell_n^m)_{0 \leq n < m}$ is said to be **topologically transitive** of g-nonautonomous discrete dynamical systems if for any $U, V \neq \emptyset$, open sets , U & $V \subset$ X there exist, $0 \leq n < m \in \mathbb{N}$

$\ni \ell_n^m(U) \cap V \neq \emptyset$. i.e.

$N(U \cap V) = \{0 \leq n < m \in \mathbb{N}: \ell_n^m(U) \cap V \neq \emptyset\}$.

<u>Definition</u> 1.1.12 [10]

Let (X, d) be a compact metric space ,and $\ell_n: X \to X$ be an uniform continuous maps, $\forall n \in \mathbb{N}$ the sequence $(\ell_n^m)_{0\leq n<m}$ is said to be **topologically mixing in g-nonautonomous discrete dynamical systems** if whenever there are two non-empty open subsets U and V of X, there is $k \in \mathbb{Z}_+$ meaning that.

$$N(U \cap V) \supset \{k, k+1, \dots\} .$$

The $(\ell_n^m)_{0\leq n<m}$is said to **be topologically weak mixing in g-nonautonomous discrete dynamical systems:** if $\ell_n^m \times \ell_n^m$ topologically transitive in g-nonautonomous discrete dynamical systems.

$(\ell_n^m)_{0\leq n<m}$is called **chain mixing in g-nonautonomous discrete dynamical systems** : if for all $x, y \in X$ and for each $\delta > 0$, there exists $M \in \mathbb{N}$ therefore for any $m \geq M$ there is δ-chain in g-nonautonomous discrete dynamical systemsfrom x to y with length m.

<u>Proposition</u> 1.1.13

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, converges uniform to ℓ. If $(\ell_n^m)_{0\leq n<m}$ is transitive in g-nonautonomous discrete dynamical system then ℓ is transitive in the discrete dynamical systems.

Proof:

Let A, B be nonempty subsets of X, Since ℓ_n^m is transitive, then by Definition 1.1.11 there exist $m, n \in \mathbb{N}, \ni 0 \leq n < m$,

$\ni\ \ell_n^m(A) \cap B \neq \emptyset$, but $\ell_n : X \to X$ is converges uniform to ℓ, so $\ell^i\,(A) \cap B \neq \emptyset$ (where $i = m - n$),

hence ℓ is transitive. □

In order to obtain a property that is stronger than the transitive property, we must prove the set $N(A, B)$ is infinite.

Theorem 1.1.14

Let (X, d) be a metric space without isolated points and $\ell_n : X \to X$

be an uniform continuous map $\forall n \in \mathbb{N}$. If $(\ell_n^m)_{0 \le n < m}$ is topological transitive, then the set $\mathrm{N(A, B)}$ is infinite for any two non-empty open subsets $\mathrm{A, B}$ of X.

Proof:

Let $A, B\ \neq \emptyset$, Since ℓ_n^m is transitive , then $N(A, B) \neq \emptyset$,

let suppose $N(A, B)$ is finite,

$\exists\ h = \max\{0 \le n < m \in\ \mathbb{N}, \ni \ell_n^m(\mathrm{A}) \cap \mathrm{B} \neq \emptyset\}$.

Since X have no isolated points, B contain infinitely many points As a result, we may be repair a collection $B_j \subset B, j = 1,2,3, \dots, h$ of mutually disjoint open sets. since $h \in N(A, B)$,

therefore exists $x \in A\ \ni \ell_n^h(x) \in B$

Let $j_0 \in \{1,2, \dots, h + 1\}\ \ni\ \ell_n^i(x) \notin B_{j_0}$, $\forall\ i = 1,2, \dots, h$

By continuity, there is an open neighborhood $\mathrm{A}^* of\ x$ contained in A,

$\ni \ \ell_n^i(A) \cap B_{j_0} = \emptyset$ ……… (1.1)

Since ℓ_n^m is transitive, $\exists\ k \in \mathbb{N}, \ni \ \ell_n^k(A^*) \cap B_{j_0} \neq \emptyset$ …. By (1.1)

We have $k > h$ .. contradiction!

$N(A^*, B_{j_0}) \subset N(A, B)$

hence set of $N(A, B)$ is infinite . □

Definition 1.1.15

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, the sequence $(\ell_n^m)_{0 \le n < m}$ is said to be an **expansive** with constant expansive $e > 0$ in g-nonautonomous discrete dynamical systems when

$\forall x, y \ \in X, x \neq y, \exists\ 0 \le n < m \in \mathbb{N}, d\big(\ell_n^m(x), \ell_n^m(y)\big) > e$.

equivalently, if $x, y \ \in$ X, $d\big(\ell_n^m(x), \ell_n^m(y)\big) \le e, \forall\ 0 \le n < m \in \mathbb{N}$,

then $x = y$.

Proposition 1.1.16

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, and $\ell_n: X \to X$ converges uniformly to ℓ. then If $(\ell_n^m)_{0 \le n < m}$ is an expansive of g-nonautonomous discrete dynamical system, then ℓ is an expansive of discrete dynamical systems

Proof

Let $x \in X$ since g-NDS is an expansive with constant expansive $e > 0$,

then for every $y \in X, d(x,y) < \varepsilon,\ \exists\ 0 \le n < m\ \in \mathbb{N}$,

$d\big(\ell_n^m(x), \ell_n^m(y)\big) > e$.

Hence $\ell_n: X \to X$ converges uniformly to ℓ, there exists $\ell_n^m = \ell^i$, $(i = m - n)$

$\ni d\big(\ell_n^m(x), \ell_n^m(y)\big) = d\left(\ell^i(x), \ell^i(y)\right) > \text{e}$.

hence the DS is an expansive with constant expansive $e > 0$. □

<u>Theorem</u> 1.1.17

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform homeomorphism maps $\forall n \in \mathbb{N}$, If then the sequence $(\ell_n^m)_{0 \le n < m}$ is an expansive of g-nonautonomous discrete dynamical systems if and only if $(\ell_{-m}^{-n})_{m < n \le 0}$ is an expansive of g-nonautonomous discrete dynamical systems.

Proof :

Let $\gamma > 0$ be a constant expansive for ℓ_n^m ,

We can write $\ell_{-m}^{-n} = (\ell_n^m)^{-1}, \forall\ n, m \in \mathbb{Z}$.

Let $a, b\ \in X,\ a \ne b,\ n, m \in \mathbb{Z},\ \ni d\big(\ell_n^m(a), \ell_n^m(b)\big) > \gamma$

That is $d(((\ell_n^m)^{-1}(a), ((\ell_n^m)^{-1}(b)) > \gamma, for\ n, m \in \mathbb{Z}$.

hence $(\ell_n^m)^{-1}$ is an expansive. □

Theorem 1.1.18

Let (X, d) be a compact metric space and $(\ell_n^m)_{0 \le n < m}$ be an equicontinuous family of self-maps on X and where k is an integer greater than zero. Then map $(\ell_n^m)_{0 \le n < m}$ is expansive of g-nonautonomous discrete dynamical systems if and only if $(\ell_n^m)^k$is expansive of g-nonautonomous discrete dynamical systems.

Proof:

Let $e > 0$ be an expanding constant for ℓ_n^m.

Since ℓ_n^m is equicontinuous family, for any $m > 0$

let $mk + 1 \le j \le (m + 1)k$, $\ell_{mk+1,j}$ is uniformly continuous on X as a result, there exists a $\delta_j > 0$ meaning that $d(x, y) < \delta_j$

hence $d(\ell_{mk+1,j}(x), \ell_{mk+1,j}(y)) < e$. Note that due to equicontinuity of ℓ_n^m, δ_j does not depend on m.

Take $\delta = min\{\delta_j : mk + 1 \le j \le (m + 1)k\}$. Then for any

$0 \le n < m, d(x, y) < \delta$.

Now $(\ell_n^m)^k = (g_n^m)_{0 \le n < m}$, where $g_n^m = \ell_{(m-1)k+1,mk}$ and

$g_{n,m} = g_m \circ g_{(m-1)} \circ \cdots \circ g_n$.

So $g_{n,m} = \ell_{[n,m]k}$. Note that for any $j \ge 0$ there is an exists

$m \ge 0$ meaning that $mk \le j \le (m + 1)k$.

Now for any $m \ge 0$ and $mk \le j \le (m + 1)k$,

$(d(g_n^m(x), g_n^m(y)) < \delta$) hence $d((\ell_n^m)^k(x), (\ell_n^m)^k(y)) < \delta$

Hence $(d(\ell_{mk+1}^{j}\left((\ell_n^m)^k(x)\right), \ell_{mk+1}^{j}((\ell_n^m)^k(y)) < e$

hence $d\left((\ell_n^m)^j(x), (\ell_n^m)^j(y)\right) < e$.

Because e is the expansive constant, ℓ_n^m, $x = y$ and hence δ is the expansive constant of $(\ell_n^m)^k$.

Conversely, if $(\ell_n^m)^k$ is expansive with the expansive constant ε then of any $x, y \in X$, $x \neq y$, therefore exists $0 \leq n < m \in \mathbb{N}$ meaning that $d(g_n^m(x), g_n^m(y)) > \varepsilon$, which implies

$d((\ell_n^m)^k(x), (\ell_n^m)^k(y)) > \varepsilon$ proving that ε is an expansive constant for ℓ_n^m. □

Remark 1.1.19

Let (X, d) and (Y, d') be a compact metric space and $\ell_n: X \to X$, $g_n: Y \to Y$ be a uniform continuous maps, $\forall\, n \in \mathbb{N}$, the $d''((a, b), (a', b') = max\ \{d(a, a'), d'(b, b')\}$.

We define the map$(\ell_n^m \times g_n^m)(a, b) = (\ell_n^m(a), g_n^m(b))$, for any $a \in X, b \in Y$, for everyone $0 \leq n < m \in \mathbb{N}$. We shall demonstrate that $(X \times Y, d'')$ is a metric space.

Proof:

Let $(a, b), (a', b'), (a''b'') \in X \times Y$.

Since $d''((a, b), (a', b') = max\ \{d(a, a'), d'(b, b')\}$, also

1. $d(a, a') \geq 0, \&\ d'(b, b') \geq 0$, so that $d''((a, b), (a', b') \geq 0$.

2. To show $d''((a,b),(a',b') = 0$ iff $(a,b) = (a',b')$

If $d''((a,b),(a',b') = 0$ then $max\ \{d(a,a'),d'(b,b')\} = 0$

This implies that $d(a,a') = 0$ & $d'(b,b') = 0$,

Hence $a = a'$ & $b = b'$,

Thus $(a,b) = (a',b')$.

If $(a,b) = (a',b')$**,** then $a = a'$ & $b = b'$, also $d(a,a') = 0$

& $d'(b,b') = 0$ this implies that $max\ \{d(a,a'),d'(b,b')\} = 0$

Thus $d''((a,b),(a',b')) = 0$.

3. $d''((a,b),(a',b')) = max\ \{d(a,a'),d'(b,b')\}$

$$= max\ \{d(a',a),d'(b',b)\}$$

$$= d''((a',b'),(a,b))$$

Hence $d''((a,b),(a',b')) = d''((a',b'),(a,b))$.

4. $d''((a,b),(a',b')) = max\{d(a,a'),d'(b,b')\}$

$$\leq max\{d(a,a''),d'(b,b''),d(a'',a'),d'(b'',b')\}$$

$$\leq max\{d(a,a''),d'(b,b'')\} + max\{d(a'',a'),d'(b'',b')\}$$

$$\leq d''((a,b),(a'',b'')) + d''((a'',b''),(a',b'))$$

Hence

$$d''((a,b),(a',b')) \leq d''((a,b),(a'',b'')) + d''((a'',b''),(a',b'))$$

From **1**,**2**,**3**, and **4** thus $(X \times Y, d'')$ is a metric space. □

1.2 The Shadowing Property in The g-Nonautonomous Discrete Dynamical Systems.

in this section, we define the shadowing property, the asymptotic shadowing property, an average shadowing property in the g-nonautonomous discrete dynamical systems, some result that are related to these concepts are proved.

Definition 1.2.1

Let (X, d) be a compact metric space and $(\ell_n^m)_{0\leq n<m}$is a sequence on X. for $\delta > 0$, the sequence $\{x_i\}_{i=0}^{\infty}$ in X is say to be **an $\boldsymbol{\delta}$-pseudo orbit of $\boldsymbol{\ell_n^m}$ in g-non autonomous discrete dynamical systems** if

$$d\big(\ell_j(x_i), x_{i+1}\big) < \delta, \quad for\, n \leq j < m \ \in \mathbb{N},$$

$$d\big(\ell^{-j}(x_{m+1}), x_m\big) < \delta \quad for\, m < j \leq n.$$

Definition 1.2.2

Let (X, d) be a compact metric space, and $(\ell_n^m)_{0\leq n<m}$is a sequence on X. for $\varepsilon > 0$, a δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is say to be **$\boldsymbol{\varepsilon}$-traced in g-nonautonomous discrete dynamical systems** by $y \in X$ if $d((\ell_n^m)^i(y), x_i) < \varepsilon$ for $0 \leq n < m \ \in \mathbb{N}$.

<u>Definition</u> 1.2.3

Let (X, d) be a compact metric space , and $(\ell_n^m)_{0\leq n<m}$is a sequence on X. then ℓ_n^m is say for has **shadowing property** or **pseudo orbit tracing property** (P.O.T.P) **in g- non autonomous discrete dynamical systems** if , for each $\varepsilon > 0$ therefore exists $\delta > 0$ $\ni$ every δ-pseudo orbit is ε-traced by using some point of X.

<u>Theorem</u> 1.2.4

Let (X, d_1) and (Y, d_2) be a compact metric spaces and $(\ell_n^m)_{0\leq n<m}$, $(g_n^m)_{0\leq n<m}$are a sequence on X, Y , respectively. Define metric d on $X \times Y$ by:

$$d''\big((x_1, y_1), (x_2, y_2)\big) = \max\ \{d(x_1, x_2), d'(y_1, y_2)\},$$

(x_1, y_1) , $(x_2, y_2) \in X \times Y$. If ℓ_n^m and g_n^m have P.O.T.P of g-nonautonomous discrete dynamical systems then the sequence $\ell_n^m \times g_n^m$has P.O.T.P of g-nonautonomous discrete dynamical systems in $X \times Y$. Therefore, each finite direct product of sequence with P.O.T.P, have P.O.T.P of g-nonautonomous discrete dynamical systems.

Proof

Note that for any $0 \leq n < m$,

$(\ell_n^m \times g_n^m)(x, y) = (\ell_n^m(x), g_n^m(y))$ $(x, y) \in X \times Y$. Let $\varepsilon > 0$ be

given. Then there exist $\delta_1 > 0$ and a $\delta_2 > 0$ meaning that every δ_1-pseudo orbit of ℓ_n^m and δ_2-pseudo orbit of g_n^m can be ε-traced by some ℓ_n^m-orbit and g_n^m-orbit respectively.

Suppose that $\delta = min\{\delta_1, \delta_2\}$ and $\{(x_i, y_i)\}_{i=0}^{\infty}$be a δ-pseudo orbit of $(\ell_n^m \times g_n^m)$.

Then $d((\ell_j \times g_j)(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta$.

i.e. $d((\ell_j(x_i), g_j(y_i), (x_{i+1}, y_{i+1})) < \delta$ which is defined as d implies $d(\ell_j(x_i), x_{i+1}) < \delta \leq \delta_1$ & $d'(g_j(y_i), y_{i+1}) < \delta \leq \delta_2$. Therefore, there are $x \in X$ & $y \in Y$ meaning that $d((\ell_n^m)^i(x), x) < \varepsilon$ & $d'((g_n^m)^i(y), y_i) < \varepsilon$.

Hence $d''(((\ell_n^m)^i(x), (g_n^m)^i(y)), (x_i, y_i)) < \varepsilon$. i.e.

$d''((\ell_n^m \times g_n^m)^i(x, y), (x_i, y_i)) < \varepsilon$ which implies $\{(x_i, y_i)\}_{i=0}^{\infty}$is ε-traced by $(x, y) \in X \times Y$.

Thus $\ell_n^m \times g_n^m$ P.O.T.P is also present. According to Induction Law, any finite direct product of sequence with P.O.T.P has P.O.T.P. □

Definition 1.2.5 [4]

Let (X, d_1) and (Y, d_2) be two compact metric spaces with g-non autonomous map sequences $(\ell_n^m)_{0 \leq n < m}$ and $(g_n^m)_{0 \leq n < m}$, respectively. If there is a uniform homeomorphism $h : X \to Y$ meaning that $h \circ \ell_n^m = g_n^m \circ h$, for all$0 \leq n < m \in \mathbb{Z}$, then ℓ_n^m and g_n^m are said to be **topologically conjugate**.

Theorem 1.2.6

Let (X, d_1) and (Y, d_2) be a compact metric spaces. Let $(\ell_n^m)_{0 \leq n < m}$ and $(g_n^m)_{0 \leq n < m}$be a sequence of X and Y respectively meaning that

$(\ell_n^m)_{0\le n<m}$is uniformly conjugate to $(g_n^m)_{0\le n<m}$. If $(\ell_n^m)_{0\le n<m}$ has P.O.T.P of g-nonautonomous discrete dynamical systems. then $(g_n^m)_{0\le n<m}$ has P.O.T.P of g-nonautonomous discrete dynamical systems.

Proof:

Assume that $\varepsilon > 0$ be given. Since ℓ_n^m is uniformly conjugate to g_n^m, there exists a uniform homeomorphism $h : X \to Y$ meaning that $h \circ \ell_n^m = g_n^m \circ h$.

i.e. $\ell_n^m \circ h^{-1} = h^{-1} \circ g_n^m$, for each $0 \le n < m \in \mathbb{N}$.

As a result of uniform homeomorphism, h is now uniformly continuous., as a result, there exists an $\varepsilon_0 > 0$ meaning that $d_1(x, y) < \varepsilon_0$ implies $d_2\big(h(x), h(y)\big) < \varepsilon$. Since ℓ_n^m have P.O.T.P there is an exists a $\delta_0 > 0$ meaning that any δ_0-pseudo orbit of ℓ_n^m is ε_0-traced by ℓ_n^m orbit of some point of X. Because h being a uniform homeomorphism, $h^{-1}$is uniformly continuous map, hence for $\delta_0 > 0$ there is an exists $\delta > 0$ implying that $d_2(x, y) < \delta$ means $d_1(h^{-1}(x), h^{-1}(y)) < \delta_0$.

Suppose that $\{x_i\}_{i=0}^{\infty}$be a δ-pseudo orbit for g_n^m. i.e.

$d_2(g_j(x_i), x_{i+1}) < \delta$. i.e.

$d_1(h^{-1}\left(g_j(x_i)\right), h^{-1}(x_{i+1})) < \delta_0$. i.e

$d_1(\ell_j(h^{-1}(x_i), x_{i+1}) \le \delta_0$ meaning that $\{h^{-1}(x_i)\}_{i=0}^{\infty}$ is a δ_0-pseudo orbit for ℓ_n^m. As a result, there exists $y \in X$ meaning that $d_1((\ell_n^m)^i(y), h^{-1}(x_i)) < \varepsilon_0$ and hence $d_2(h\left((\ell_n^m)^i(y)\right), x_i) < \varepsilon$.

Now for all $0 \le n < m$,

$$h \circ \ell_n^m = h \circ \ell_m \circ \ell_{m-1} \circ \ldots \circ \ell_{n+1} \circ \ell_n$$

$$= g_m \circ h \circ \ell_{m-1} \circ \dots \circ \ell_{n+1} \circ \ell_n$$

$$\vdots$$

$$= g_m \circ g_{m-1} \circ \dots \circ g_{n+1} \circ g_n \circ h$$

$$= g_n^m \circ h$$

implies $d_2((g_n^m)^i(h(y)), x_i) < \varepsilon$.

i.e. $\{x_i\}_{i=0}^{\infty}$ is ε-traced by using $h(y) \in Y$.

Thusly g_n^m have P.O.T.P □

Theorem 1.2.7

Let (X, d) be a compact metric space and $\ell_n: X \longrightarrow X$ be an uniform continuous map, for all $n \in \mathbb{N}$ If $(\ell_n^m)_{0 \le n < m}$ has P.O.T.P of g-nonautonomous discrete dynamical systems. then $(\ell_n^m)^k$has P.O.T.P of g-nonautonomous discrete dynamical systems. for every $k > 0$.

Proof:

Suppose $k \geq 2$. And $\varepsilon > 0$ be given, since ℓ_n^m has P.O.T.P, as a result, therefore exists $\delta > 0$ meaning that each δ-pseudo orbit of ℓ_n^m is ε-traced by using some point of X.

Assume that $\{y_i\}_{i=0}^{\infty}$ be a δ-pseudo orbit of $(\ell_n^m)^k$. Then $d(\ell_n^m(y_i), y_{i+1}) < \delta$,

for all $0 \le n < m$,

For $0 \le h < k$ & $0 \le n < m \in \mathbb{N}$, put $x_{i+h} = \ell_n^m(y_i)$,

hence $\{x_i\}$ is a δ-pseudo orbit for ℓ_n^m. i.e for show :

$d(\ell_j(x_{i+h}), x_{i+h+1}) < \delta$, for $i > 0$ & $0 \leq h < k$

choose any $i > 0$, now for any j, $0 \leq h < k - 2$,

$$\ell_j(x_{i+h}) = \ell_j\big((\ell_n^m(y_i)\big)$$

$$= \ell_n^{m+j}(y_i)$$

$$= x_{i+h+1}$$

Thus $d\big(\ell_j(x_{i+h}), x_{i+h+1}\big) = 0 < \delta$, for all j, $0 \leq h \leq k - 2$.

Now for $h = k - 1$,

$$d\big(\ell_j(x_{i+k-1}), x_{i+k}\big) = d(\ell_j((\ell_n^m(y_i), x_{i+k})$$

$$= d\big((\ell_n^{m+j}(y_i), y_{i+1}\big)$$

$$< \delta$$

As a result, P.O.T.P claim of ℓ_n^m, $\{x_i\}_{i=0}^{\infty}$ is ε-traced by some $y \in X$, i.e. $d\big((\ell_n^m)^i(y), x_i\big) < \varepsilon$, for each $i \geq 0$, in particular for $i = ki$, $d((\ell_n^m)^{ki}(y), x_{ki}) < \varepsilon$.

Thus $(\ell_n^m)^k$ has P.O.T.P. □

<u>Definition</u> 1.2.8

Let (X, d) be a compact metric space , and $(\ell_n^m)_{0 \leq n < m}$is sequence on X. then the sequence $\{x_i\}_{i=0}^{\infty}$ in X is known as the **$\boldsymbol{\delta}$-average pseudo-orbit** of X in g-nonautonomous discrete dynamical systems if there exists $\delta > 0$, and a positive integer $M = M(\delta)$, as a result of which for all $m > M$, and $k \in \mathbb{N}$, we get

$$\frac{1}{m}\sum_{i=0}^{m-1} d\big(\ell_j(y_{i+k}), y_{i+k+1}\big) < \delta,\ where\ n \le j < m$$

A maps $(\ell_n^m)_{0\le n<m}$ is said to have **average shadowing property** in g-nonautonomous discrete dynamical systems if to each $\varepsilon > 0$, there is $\delta > 0$, as a result of which for all δ-average pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ in average is ε-shadowing by orbit of a point $z \in X$, its mean

$$\lim_{m\to\infty} \sup \frac{1}{m}\sum_{i=n}^{m-1} d((\ell_n^m)^i(z), x_i) < \varepsilon$$

Definition 1.2.9

A sequence $\{x_i\}_{i=0}^{\infty}$ in X is known as **the asymptotic average pseudo orbit of $(\ell_n^m)_{0\le n<m}$ in g-nonautonomous discrete dynamic systems** if

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m\_1} d\big(\ell_j(x_i), x_{i+1}\big) = 0.$$

A sequence $\{x_i\}_{i=0}^{\infty}$ in X is say to be **asymptotic shadowing in g-nonautonomous discrete dynamic systems in average** by the point $z \in X$ if

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=n}^{m\_1} d\big((\ell_n^m)^i(z), x_i\big) = 0.$$

We say that $(\ell_n^m)_{0\le n<m}$ has **the asymptotic average shadowing property in g-nonautonomous discrete dynamic systems** if any asymptotic

average pseudo orbit of $(\ell_n^m)_{0\le n<m}$ is asymptotic shadowing in g-nonautonomous discrete dynamic systems in average by the point $z \in X$.

<u>Lemma</u> 1.2.10 [9]

Let$\{c_i\}_{i=0}^{\infty}$ be a sequence for non-negative real numbers. Given a positive real number α, suppose that $C_{t,m}$ be the cardinality for $\{i < m\,, c_i \ge \alpha\}$, that is , $c_{\alpha,m} = card(\{i < m :\ c_i \ge \alpha\})$

(a) if

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} c_i \le \varepsilon\,,$$

then

$$\lim_{m\to+\infty} \sup c_{\sqrt{\varepsilon}}\ , m \le \varepsilon$$

(b) if $\{c_i\}_{i=0}^{\infty}$ is bounded by $L \in \mathbb{R}$ and

$$\lim_{m\to+\infty} \sup c_{\sqrt{\varepsilon}}\,, m \le \varepsilon$$

then

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} c_i \le (F+1)\varepsilon.$$

Proof:

(a) Assume, alternatively on the other hand that on the contrary that $c_{\sqrt{\varepsilon}}$, $m \ge \sqrt{\varepsilon}$, then which would imply the existence of a sequence $\{m_j\}_{j=0}^{\infty}$in $\mathbb{N}$ meaning that

$$\lim_{j\to+\infty} c_{\sqrt{\varepsilon}}\,, m_j = b > \sqrt{\varepsilon},$$

therefor $\delta > 0$ with $b - \delta > \sqrt{\varepsilon}$, there exists $r > 0$ meaning that

$c_{\sqrt{\varepsilon}}\,, m_j > b - \delta$, for all $j > r$.

Thus

$$\sum_{i=n}^{m_j-1} c_i \geq c_{\sqrt{\varepsilon}}\,, m_j > (b-\delta)\sqrt{\varepsilon}$$

for each $j > r$ which implies that

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} c_i \geq (b-\delta)\sqrt{\varepsilon} > \varepsilon.$$

That is a contradiction. Hence (a) is true.

(b) since

$$\sum_{i=n}^{m-1} c_i \leq \left(m - c_{\varepsilon,m}\right)\varepsilon + c_{\varepsilon,m}\, F = \varepsilon + c_{\varepsilon,m}\,(F-\varepsilon),$$

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} c_i \leq m\varepsilon + (F-\varepsilon) \lim_{m\to+\infty} \sup c_{\varepsilon,m} \leq m\varepsilon + (F-\varepsilon)\varepsilon.$$

So if m large enough then

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} c_i \leq (F+1)\varepsilon. \quad \square$$

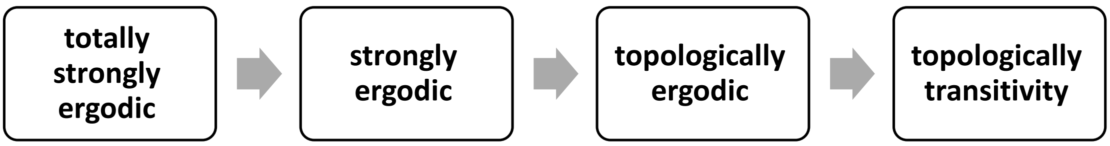
Definition 1.2.11

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous maps, we say that $(\ell_n^m)_{0 \le n < m}$ be **topological ergodic in g-nonautonomous discrete dynamical systems** if for any pair nonempty open subset $A, B \subset X$, $N(A \cap B)$ has positive upper density, that is

$$\overline{D}(N(A \cap B) = \lim_{m \to \infty} \sup \frac{card(N(A \cap B) \cap \{0,1 \dots, m-1\})}{m} > 0.$$

$J \subset \mathbb{Z}_+$is said to be **syndetic in g-nonautonomous discrete dynamical systems** if there is $M \in \mathbb{N}$ meaning that $[m, m+M] \cap J \neq \emptyset$, for every $m \in \mathbb{N}$.

$(\ell_n^m)_{0 \le n < m}$ is said to be **strongly ergodic in g-nonautonomous discrete dynamical systems** if for each pair nonempty open subset combination $A, B \subset X, N(A, B)$ is a set that is syndetic. If for each $k \in \mathbb{N}, (\ell_n^m)^k$ is referred to this as being strongly ergodic, we call $(\ell_n^m)_{0 \le n < m}$ **totally strongly ergodicin g-nonautonomous discrete dynamical systems**. It is obvious that



Definition 1.2.12

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps, $\forall n \in \mathbb{N}$. A point $x \in X$ is called **transitive** in g-nonautonomous discrete dynamical systems if when x has a dense orbit in X.

Definition 1.2.13

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous maps, $\forall n \in \mathbb{N}$, if each $x \in X$ we call this a transitive point, then we may say $(\ell_n^m)_{0 \le n < m}$ is minimal. $x \in X$ is say to be **minimal point** of g-nonautonomous discrete dynamical systems if for every neighbrhood U of y, $N(y, V)$ is syndetic, denoted by **AP(**ℓ_n^m) the set of all minimal points of $(\ell_n^m)_{0 \le n < m}$.

We generalize Proposition 2.1.13 in [9] in to g-non autonomous discrete dynamical system

Proposition 1.2.14

Let (X, d) and (Y, d') be a compacts metric spaces , $\ell_n: X \to X$ and $g_n: Y \to Y$ be an uniform continuous maps $\forall n \in \mathbb{N}$, $(\ell_n^m)_{0 \le n < m}$ and $(g_n^m)_{0 \le n < m}$ are surjective. If $(\ell_n^m)_{0 \le n < m}$ and $(g_n^m)_{0 \le n < m}$ each has dense minimal points, $\forall\ 0 \le n < m$ then so does the product $\ell_n^m \times g_n^m$.

We generalize the Proposition 3.3 in [6] in to g-non autonomous

discrete dynamical systems.

Proposition 1.2.15

Let (X, d) be a compact metric space, and suppose that $\ell_n : X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$. A point x is minimal under

$(\ell_n^m)_{0\le n<m}$ if and only if for every open neighborhood U for x the set $\{m \in \mathbb{N}: \ell_n^m(x) \in U\}$ is a syndetic

Theorem 1.2.16

Let (X, d) be a compact metric space, and suppose that $\ell_n: X \to X$ be a uniform continuous map $\forall n \in \mathbb{N}$. Suppose that there exists a point $x \in X$ that has whose orbit closure contained in some open set U that is an invariant under $(\ell_n^m)_{0\le n<m}$ (that is $\ell_n^m(U) \in U$). Suppose that $y \in X$ is a point whose orbit is metrically separated from U (that is $d(\{(\ell_n^m)^i(y)\}_{i=0}^{\infty}, U) > 0$, Then $(\ell_n^m)_{0\le n<m}$ does not have the asymptotic average shadowing property ($AASP$).

Proof.

Put $a_0 = a_1 = x$, and suppose that $\{a_i\}_{i=2}^{\infty}$be the following sequence of points:

$\{x, y, x, \ell_{n_1}^1(x), y, \ell_{n_1}^1(y), x, \ell_{n_1}^1(x), \ell_{n_2}^2(x), y, \ell_{n_1}^1(y), \ell_{n_2}^2(y), x, \ell_{n_1}^1(x), \ell_{n_2}^2(x),$

$\ell_{n_3}^3(x), y, \ell_{n_1}^1(y), \ell_{n_2}^2(y), \ell_{n_3}^3(y), x, \dots\}$

Take note that for each $j \in \mathbb{N}$ and for every integer $2^j \le k < 2^{j+1}$,we have the inequality

$$\sum_{i=0}^{k-1} d(\ell_j(a_i), a_{i+1}) \le 2j.D(X).$$

This means that

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{i=0}^{m} d(\ell_j(a_i),a_{i+1}) \le \lim_{m\to\infty}\frac{1}{2^m}2m.D(X) = 0$$

Therefore, $\{a_i\}_{i=0}^{\infty}$ is an asymptotic average pseudo-orbit of ℓ_n^m. We shall now demonstrate that no point for X may asymptotically shadow it in average, showing that ℓ_n^m lacks the A.A.S.P. Assume that $z \in X$ shadows asymptotically in average $\{a_i\}_{i=0}^{\infty}$. The orbit of z must therefore reach U at some point, else for every $m \in \mathbb{N}$ we get

$$\frac{1}{2^m}\sum_{i=n}^{2^m-1} d(a_i,(\ell_n^m)^i(z)) \ge \frac{1}{2}d(\{(\ell_n^m)^i(x)\}_{i=0}^{\infty},\{(\ell_n^m)^i(z)\}_{i=0}^{\infty}) > 0$$

There cannot be any asymptotic shadowing within the mean. Hence, $\ell_n^M(z) \in U$ for some $M \in \mathbb{N}$. Then, by the invariance of U for each $m \ge M$, we have $\ell_n^M(z) \in U$and for m large enough

$$\frac{1}{2^m}\sum_{i=n}^{2^m-1} d\left(a_i,(\ell_n^m)^i(z)\right) \ge \frac{1}{3}.d(\{(\ell_n^m)^i(y)\}_{\boldsymbol{i=0}}^{\infty},U) > 0.$$

This also avoids $\boldsymbol{z}$ forming asymptotically shadowing for average$\{a_i\}_{i=2}^{\infty}$.

<u>Theorem</u> 1.2.17

Let (X,d) be a compact metric space, and $\ell_n : X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$. Assume that $(\ell_n^m)_{0\le n<m}$ has the asymptotic average shadowing property ($AASP$) of g-nonautonomous discrete dynamical systems, and minimal points of ℓ_n^m are dense in X.

then $(\ell_n^m)_{0\le n<m}$ is transitive.

Proof.

To prove that ℓ_n^m is transitive. Fix any nonempty open sets U, V and choose a pair of minimal points $x \in U$ and $y \in V$. set two open sets U' and V' together with a number $\varepsilon > 0$, so that the following conditions are satisfied:$x \in U' \subset U$, $y \in V' \subset V$, $d(U', X/U) > \varepsilon$ and $d(V', X/V) > \varepsilon$.

Put $a_0 = a_1 = x$ and let $\{a_i\}_{\boldsymbol{i=2}}^{\infty}$ be as follows sequence of points:

$\{x, y, x, \ell_{n_1}^1(x), y, \ell_{n_1}^1(y), x, \ell_{n_1}^1(x), \ell_{n_2}^2(x), y, \ell_{n_1}^1(y), \ell_{n_2}^2(y), x, \ell_{n_1}^1(x),$

$\ell_{n_2}^2(x), \ell_{n_3}^3(x), y, \ell_{n_1}^1(y), \ell_{n_2}^2(y), \ell_{n_3}^3(y), x, \dots\}$

In the proof of Theorem 1.2.16, we have shown that $\{a_i\}_{\boldsymbol{i=0}}^{\infty}$ is an asymptotic average pseudo-orbit of ℓ_n^m. Using the A.A.S.P we fix a point $z \in X$ that asymptotically shadows it in average. By Proposition 1.2.15 the sets

$\{i \in \mathbb{N}: (\ell_n^m)^i(x) \in U'\}$, and $\{i \in \mathbb{N} : (\ell_n^m)^i(x) \in V'\}$ are syndetic. Choose a number M greater than zero to avoid having sequences of consecutive integers longer than M in the complements of these two sets in $\mathbb{N}$. Use uniform continuity of ℓ_n^m to pick a number $\delta > 0$, meaning that for any $a, b \in X$ the condition $d(a, b) < \delta$ implies that $d\left(\ell_n^i(a), \ell_n^i(b)\right) < \varepsilon$, for every $i \in \{0, \dots, M-1\}$ and $n < i$.

By an argument similar to that used to prove Theorem 1.2.16, there exists $M_1 \in \mathbb{N}$ meaning that $d\left(a_{M_1}, (\ell_n^m)^{M_1}(z)\right) < \delta$ and we get $a_{M_1+i} = \ell_n^{k+i}(x)$ for $i \in \{0, \dots, M-1\}$ and some $k \in \mathbb{N}$. Take note that this, in addition to picking δ and M, ensures that for some $M_2 \in \{M_1, \dots, M_1 + M - 1\}$, we have $(\ell_n^m)^{M_2}(z) \in U$.

In a similar way, we can pick $M_3 > M_2$ meaning that $d(a_{M_3}, (\ell_n^m)^{M_3}(z)) < \delta$, and we get $a_{M_3+i} = \ell_n^{\iota+i}(y)$, for and some $i \in \{0, \dots, M-1\}$, where $\iota \in \mathbb{N}$.

Once more, this, along with the options of δ and M, ensures that for some $M_4 \in \{M_3, \dots, M_3 + M - 1\}$, we have $(\ell_n^m)^{M_4}(z) \in V$. Thus the orbit of $(\ell_n^m)^{M_2}(z)$, which is a point from U, intersects the set V, i.e. $\ell_n^m(U) \cap V \neq \emptyset$,

hence ℓ_n^m is transitive. □

We show that, there is exists $(\ell_n^m)_{0 \leq n < m}$ be open, expansive and transitive but does not have the average shadowing property of g-nonautonomous discrete dynamical systems

<u>Example</u> 1.2.18

Let ℓ_n^m be an open, expansive and transitive $\forall 0 \leq n < m, \ni n, m \in \mathbb{N}_0$. and $X = \{a, b\}$ be any two points set with the discrete metric d, and

let ℓ_n^m be the cyclic permutation of X, that is, $\ell_n^m(a) = b, \ell_n^m(b) = a$. We will prove that ℓ_n^m does not have the average shadowing property of g-nonautonomous discrete dynamical systems.

Fix $\varepsilon = 1/4$ and take any $\delta > 0$

and let m be meaning that $1/m < \delta$. Put $M = 2m$ and consider the sequence

$$x_i = \begin{cases} (\ell_n^m)^j(a), & if\ i = 2sM + j\ for\ some\ s \geq 0\ \&\ 0 \leq j < M \\ (\ell_n^m)^j(b), & if\ i = (2s+1)M + j\ for\ some\ s \geq 0\ \&\ 0 \leq j < M \end{cases}$$

In other words,

$$x_0, x_1, \dots . = \underbrace{a, b, a, b, \dots, a, b}_{2m}, \underbrace{b, a, b, a, \dots, b, a}_{2m}, \dots$$

We demonstrate that the sequence $\{x_i\}_{\boldsymbol{i=o}}^{\infty}$ is a δ-average-pseudo-orbit, with $M\delta = M$ given above. Let $m > M$ be given, say $m = lM + j$ for some $l \geq 1$, and $0 \leq j < M$. If we fix any $k \geq 0$, then the set $\{i: 0 \leq i < m\ and\ d(\ell_j(x_{i+k}), x_{i+k+1}) = 0\}$ has at most $l + 1$ elements. Since $d(\ell_j(x_i), x_{i+1}) \neq 0$ implies $d\big(\ell_j(x_i), x_{i+1}\big) = 1$, we have

$$\frac{1}{m}\sum_{i=0}^{m-1} d\big(\ell_j(x_{i+k}), x_{i+k+1}\big) \leq \frac{1}{lM}(l+1) \leq \frac{2}{M} = \frac{1}{m} < \delta$$

which shows that the sequence $\{x_i\}_{\boldsymbol{i=o}}^{\infty}$ is indeed a δ-average-pseudo-orbit. For every $s \geq 1$ the following holds

$$\frac{1}{2Ms}\sum_{i=0}^{2Ms-1} d(\big(\ell_n^m)^i(a), x_i\big)$$

$$= \frac{1}{2Ms}\sum_{i=0}^{s-1}\sum_{j=0}^{M-1} d((\ell_n^m)^{M(2i+1)+j}(a), (\ell_n^m)^j(b)$$

$$= \frac{1}{2Ms}\sum_{i=0}^{s-1}\sum_{j=0}^{M-1} d((\ell_n^m)^j(a), (\ell_n^m)^j(b)$$

$$= \tfrac{1}{2} > \varepsilon = \tfrac{1}{4},$$

which implies

$$\lim_{m\to\infty} \sup \frac{1}{m}\sum_{i=n}^{m-1} d((\ell_n^m)^i(a), x_i) \geq \varepsilon$$

And we also get

$$\lim_{m\to\infty} \sup \frac{1}{m}\sum_{i=n}^{m} d((\ell_n^m)^i(b), x_i) \geq \varepsilon$$

To put it another way, for $\varepsilon = \frac{1}{4}$, the δ-average pseudo orbit $\{x_i\}_{i=0}^{\infty}$ constructed above cannot be ε-shadowed in average by any point of the space X □

Lemma 1.2.19

Let (X, d) be a compact metric space, and $\ell_n : X \to X, \forall n \in \mathbb{N}$ is an uniform continuous surjective with an average shadowing and shadowing properties of g-nonautonomous discrete dynamical systems, then $(\ell_n^m)_{0\leq n<m}$ is transitive

Proof

We can assume, without losing generality, that diam $X \leq 1$. Fix any nonempty open sets $U, V \subset X$ and choose points $p \in U$, and $q \in V$. Let $\varepsilon > 0$ meaning that $B(p, \varepsilon) \subset U$ and $B(q, \varepsilon) \subset V$. For that ε take $\delta > 0$ provided by the shadowing property. Through uniform continuity, we can discover $\eta < \delta$ such that $d(x, y) < \eta$ implies that $d\big(\ell_n^m(x), \ell_n^m(y)\big) < \delta$, for any $x, y \in X$. For $\eta/3$ we take $\xi > 0$ as in the definition for average shadowing property 1.2.8. Let M be an integer such that $2/\xi < M$, and suppose that $r \in X$ be meaning that$(\ell_n^m)^N(r) = q$. Consider the sequence

$$x_i = \begin{cases} (\ell_n^m)^j(p), if\ i = 2s.M + j\ for\ som\ s \geq 0\ \&\ 0 \leq j < M \\ (\ell_n^m)^j(r)\ , if\ i = (2s+1).M + j\ for\ some\ s \geq 0\ \&\ 0 \leq j < M. \end{cases}$$

In other words,

$$x_0, x_1, \dots, x_{2M-1} = \underbrace{p, \ell_n^m(p), \dots, (\ell_n^m)^{M-1}(P)}_{M} \underbrace{, r, \ell_n^m(r), \dots, (\ell_n^m)^{M-1}(r)}_{M}$$

and repeats ad infinitum this initial sequence. Similarly to Example 1.2.18, one may simply meaning that $\{x_i\}_{\boldsymbol{i=0}}^{\infty}$ constructed above is a periodic ξ-average-pseudo-orbit. By an average shadowing property 1.2.9, we can find a point $z \in X$, which $\eta/3$-traces on average our ξ- average-pseudo-orbit. Specifically,

we **assert** that: for every $K > 0$, there is an integer σ meaning that $2\sigma\, M > K$, and $d((\ell_n^m)^j(z), (\ell_n^m)^l(p)) < \eta$, for some $j \in [2\sigma\, M, (2\sigma+1)M)$, and $l = i - 2\sigma\, M$.

Suppose that our claim does not hold, that is, of some σ_0 and each $s \geq \sigma_0$

we have $d((\ell_n^m)^i(z), (\ell_n^m)^l(p)) \geq \eta$, for all $i \in [2sM\,, (2s+1)M)$, and

$l = i - 2sM$. It follows that for all $s \geq \sigma_0$ we have

$$\sum_{i=2sM}^{(2s+1)M-1} d((\ell_n^m)^i(z), x_i) \geq M\eta$$

which implies

$$\lim_{m\to\infty}\sup\frac{1}{m}\sum_{i=0}^{m-1} d((\ell_n^m)^i(z),x_i) \geq {}^{\eta}/_{2} > {}^{\eta}/_{3},$$

But this contradicting the reality that $z \in X$ is $\eta/3$-tracing on average our ξ-pseudo-orbit $\{x_i\}$.

By our claim, there are $i > 0$ and $0 \leq l < M$, such that $d((\ell_n^m)^i\,(z),(\ell_n^m)^l(p)) < \eta$.

In a similar manner, we establish that there exist integers j, and γ, where $j > i$, and $0 \leq \gamma < M$ meaning that $d((\ell_n^m)^j\,(z),(\ell_n^m)^\gamma(r)) < \eta$. By the choice of η, and $i,j,l,\gamma,$ the sequence

$p,\ell_n^m(p),\dots,(\ell_n^m)^l(p),(\ell_n^m)^{i+1}(z),(\ell_n^m)^{i+2}(z),\dots,(\ell_n^m)^j(z),(\ell_n^m)^{\gamma+1}(r),$
$\dots,(\ell_n^m)^M(r),(\ell_n^m)^{M+1}(r),\dots$

is a δ-pseudo-orbit of ℓ, and thus by using shadowing there is a point y which ε-traces it. We see that $y \in B(p,\varepsilon) \subset U$, and $(\ell_n^m)^{j-i+l+M-\gamma-2}(y) \in B((\ell_n^m)^M(r),\varepsilon) \subset V$ hence $\ell_n^m$is transitive. □

Chapter Two

Some Metric Chaotic Properties in g-Nonautonomous Discrete Dynamical Systems

2.1 Some Metric Chaotic Propertys on g-Nonautonomous Discrete Dynamical Systems

In this section, We discuss the sensitive and the equicontinuous properties in g-nonautonomous discrete dynamical systems, and we notice that there exists a minimal g-nonautonomous discrete dynamical systems which is neither sensitive nor equicontinuous

<u>Definition</u> 2.1.1 [10]

Let (X,d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, the sequence $(\ell_n^m)_{0 \le n < m}$is said to has **sensitive dependence on initial Condition in g-nonautonomous discrete dynamical systems,** if

$\forall\ x \in X$ exists a constant $\delta > 0$,$\forall\ \varepsilon > 0$, such that $\exists\ y \in X$

where $d(x,y) < \varepsilon$,$\exists\ d\big(\ell_n^m(x), \ell_n^m(y)\big) > \delta$, for $0 \le n < m \in \mathbb{N}$.

<u>Proposition</u> 2.1.2

Let (X,d) be a compact metric space, if ℓ is transitive map and non-sensitive of autonomous discrete dynamical system (ADS), then there exists $(\ell_n^m)_{0 \le n < m}$which is transitive of g-nonautonomous discrete dynamical systems (g-NDS), and all its points are periodic but non sensitive dependence on initial condition .

Proof:-

Suppose that ℓ is injective & surjective map, denoted by ℓ^{-i} the i-th iterate $\overbrace{\ell^{-1} \circ \ell^{-1} \circ \ell^{-1} \circ \cdots\cdots \circ \ell^{-1}}^{i-_{time}}$ and define the (g-NDS) where The sequence (G_n) is gevin by :

$(\ell_1, \ell_2, \ell_3, \dots \ell_n, \dots \dots) = (\ell, \ell^1, \ell^{-1}, \ell^2, \ell^{-2}, \ell^3, \ell^{-3} \dots \dots \ell^n, \ell^{-n}, \dots)$

So that the sequence

$\ell_n^n = \ell_n = \ell^n$

$\ell_n^{n+1} = \ell^{-n} \circ \ell^n = \mathrm{I}d_X$

$\ell_n^{n+2} = \ell^{n+1} \circ (\ell^{-n} \circ \ell^n) = \ell_{n+1} = \ell^{n+1}$

$(\ell_n^n, \ell_n^{n+1}, \ell_n^{n+2}, \dots \dots) = (\ell^n, \mathrm{I}d_X, \ell^{n+1}, \mathrm{I}d_X, \dots \dots)$

So, every $x \in X$ is periodic since $\ell_n^{2k} = \mathrm{I}d_X$ for any $k \in \mathbb{N}$, $\ni 2k > n$

$(\ell_n^{even} = \mathrm{I}d_X)$, $(\ell_n^{odd} = \ell^{n+j}, j = 0,1,2, ..)$

Now, let $U, V \neq \emptyset$ be open sets of X.

since (ADS) is transitive, hence $\exists\ i \in \mathbb{N} \ni \ell^i(U) \cap V \neq \emptyset$.

there exists $m, n \in \mathbb{N}$ & m is odd, $\ni \ell_n^m = \ell^i$, & $0 \leq n < m$

then $\ell_n^m(U) \cap V \neq \emptyset$, hence the (g-NDS) is transitive .

Now to prove that (g-NDS) is not sensitive, since (ADS) is not

sensitive dependence on initial condition for any $k \in \mathbb{N}$, $\exists\ x_k \in X$ &

$0 < \varepsilon_k < \frac{1}{K}$, $\ni$ $\sup d(\ell^i(x_k), \ell^i(y)) < \frac{1}{K}$,for all $y \in$ X, where

$d(x_k, y) < \varepsilon_k$, for any $i \in \mathbb{N}$, $\exists m, n \in \mathbb{N}$, $\ni 0 \leq n < m$, $\ni$ $\ell_n^m = \ell^i$, so for any $k \in \mathbb{N}$ we have

$sup(d(\ell^i(x_k), \ell^i(y)) = sup(d(\ell_n^m(x_k), \ell_n^m(y)) < \frac{1}{k}$,

for any $y \in X \ni d(x_k, y) < \varepsilon_k$, so (g-NDS) is not sensitive

dependence on initial condition. □

Proposition 2.1.3

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, and $\ell_n^m \xrightarrow{u} \ell$. If $(\ell_n^m)_{0 \le n < m}$ has sensitive dependence on initial condition of g-nonautonomous discrete dynamical system, then the ℓ has sensitive dependence on initial condition of discrete dynamical systems.

Proof :

Let $\boldsymbol{x} \in X$, since the (g-NDS) is sensitive dependence on initial condition,

$\exists\, \varepsilon > 0, \forall\, \delta > 0\,, \exists y \in B_{(X)}$, there exist $m, n \in \mathbb{N}, \ni 0 \le n < m$,
$\ni\ d(x, y) <\ \delta\,, \quad d\big(\ell_n^m(x), \ell_n^m(y)\big) > \varepsilon$

(since $\ell_n^m \xrightarrow{u} \ell$)

There exists $\ell_n^m = \ell^i, (i = m - n)$

$\ni d(\ell_n^m(x), \ell_n^m(y)) = d(\ell^i(x), \ell^i(y)) > \varepsilon$
then the DS is sensitive dependence on initial condition. □

Definition 2.1.4 [10]

Let (X, d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, the sequence $(\ell_n^m)_{0\leq n<m}$ is said to be an **equicontinuous in g-nonautonomous discrete dynamical systems** when for every $\varepsilon > 0, \exists\ \delta > 0, \ni\ d\big(\ell_n^m(x), \ell_n^m(y)\big) < \varepsilon$, for all $m, n \in \mathbb{N}$,

$\ni\ 0 \leq n < m$ when $d(x, y) < \delta$.

Propostion 2.1.5

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, and $\ell_n^m \xrightarrow{u} \ell$. If $(\ell_n^m)_{0\leq n<m}$ is an equicontinuous of g-nonautonomous discrete dynamical system then ℓ is an equicontinuous of discrete dynamical system

Proof :

Let $x \epsilon X$, since (g-NDS) is an equicontinuous then

$\forall\varepsilon > 0, \exists\ \delta > 0, d\big(\ell_n^m(x), \ell_n^m(y)\big) < \varepsilon, \forall\ y \in X$

When $d(x, y) < \delta$,for all $0 \leq$ n $< m, \&\ n, m \in \mathbb{N}$

We can found $\ell_n^m = \ell^i,\ \ni\ i = m - n$ (when $\ell_n^m \xrightarrow{u} \ell$)

hence $d\big(\ell_n^m(x), \ell_n^m(y)\big) = d\Big(\ell^i(x), \ell^i(y)\Big) < \varepsilon$

hence the DS is an equicontinuous map. □

<u>Theorem</u> 2.1.6

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, where the sequence $(\ell_n^m)_{0 \le n < m}$ be topologically transitive g-nonautonomous discrete dynamical system, if the sequence $(\ell_n^m)_{0 \le n < m}$ is an equicontinuous at x_0, then x_0 is a transitive point.

Proof:-

Suppose that $y \in X$ & $\varepsilon > 0$, since $(X, \ell_{n,m})$ is an equicontinuous at x_0, then $d(\ell_n^m(x_0), \ell_n^m(z)) < \varepsilon$,

when $z \in B_\delta(x_0), \forall\, \varepsilon > 0, m > n \ge 0, \forall n, m \in \mathbb{N}$,

since $(X, \ell_{n,m})$ is transitive, hence $\exists\, B_\delta(x_0), B_\varepsilon(y) \subset X, \ni$

$\ell_n^m(B_\delta(x_0)) \cap B_\varepsilon(y) \ne \emptyset$.

Let $\ell_n^m(z) \in \ell_n^m(B_\delta(x_0)) \cap B_\varepsilon(y)$ hence $d(\ell_n^m(z), y) < \varepsilon$,

We get $d(\ell_n^m(x_0), y) \le d\left(\ell_n^m(x_0), \ell_n^m(z)\right) + d\,(\ell_n^m(z), y)$

$$\le \varepsilon + \varepsilon$$

$$\le 2\varepsilon$$

hence $\ell_n^m(x_0) \in B_{2\varepsilon}(y)$

hence x_0 is transitive point. □

<u>Definition</u> 2.1.7 [10]

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, the sequence $(\ell_n^m)_{0 \le n < m}$ is called **minimal of g-**

nonautonomous discrete dynamical systems, if all the points of X are transitive.

<u>Definition</u> 2.1.8

Let (X, d) be an autonomous discrete dynamical systems (ADS for short), a subset a $A \subseteq X, \ni \ell_i^{-1}(A) \subseteq A$, is said to be an **inversely invariant.** and the set of all equicontinuous points in X is denoted by $\boldsymbol{\zeta_\varepsilon}$.

<u>Remark</u> 2.1.9

Let (X, d) is an autonomous discrete dynamical systems, $\zeta_\varepsilon$is a set of equicontinuous points, then the following is hold; If $\zeta_\varepsilon = X$, implies the autonomous discrete dynamical system is equicontinuous. ……….(2.1)

If $\zeta_\varepsilon \neq \emptyset$ hence

$\zeta_\varepsilon = \left\{x \in X, \forall\, z, y \in B_\delta(x), \forall i \geq 0, d\left(\ell^i(z), \ell^i(y)\right) < \varepsilon\right\}$ hence ζ_ε is inversely invariant open & $\zeta = \cap\ \zeta_{1/m}, m > 0$,then the autonomous discrete dynamical system is almost equicontinuous . …… (2.2)

If $\zeta_\varepsilon = \emptyset$, implies

$$\exists\, x \in X, \ni \forall z, y \in B_\delta(x), \exists n \in \mathbb{N}\,, d\left(\ell^i(z), \ell^i(y)\right) > \varepsilon$$

hence either $d(\ell^i(x), \ell^i(y)) > \frac{\varepsilon}{2}$ or $d(\ell^i(x), \ell^i(z)) > \frac{\varepsilon}{2}$

Then the autonomous discrete dynamical system is a sensitive with sensitive constant $\frac{\varepsilon}{2}$. ………(2.3)

Remark 2.1.10

If the autonomous discrete dynamical system is equicontinuous then be almost equicontinuous.

Remark 2.1.11

If the autonomous discrete dynamical system minimal, then it's transitive (by definition), but the conversely not necessary.

Theorem 2.1.12 [15]

Let (X, d) be a compact metric space and $\ell: X \to X$ be a continuous map $\forall n \in \mathbb{N}$, such that for each $\varepsilon > 0$, ζ_ε is inversely invariant, If ℓ is an a minimal then it′s either sensitive or almost equicontinuous.

Proof:

since ζ_ε is inversely invariant, since ℓ is minimal then it's transitive hence there exists $U = X \setminus \zeta_\varepsilon$, $\exists\, i > 0, i \in \mathbb{N}, \ni\ \ell^i(\zeta_\varepsilon) \cap U \neq \emptyset$,

or $\ell_i^{-1}(\zeta_\varepsilon) \cap U \neq \emptyset$

By Definition 2.1.8 that $\ell_i^{-1}(\zeta_\varepsilon) \subseteq \zeta_\varepsilon$

hence $\ell_i^{-1}(\zeta_\varepsilon) \cap U \subseteq \zeta_\varepsilon \cap U$, but $\zeta_\varepsilon \cap U = \emptyset$, that contradiction!

hence either $\zeta_\varepsilon = \emptyset$ or $\zeta_\varepsilon \neq \emptyset$,

By Remark 2.1.9

If $\zeta_\varepsilon = \emptyset$

$\zeta_\varepsilon = \{x \in X, \exists\, z, y \in\ B_\delta(x), \exists\, \varepsilon > 0\ \&\ \varepsilon \in \mathbb{N}\ \ni d(\ell^i(z), \ell^i(y)) >$

$\varepsilon\ , \forall\ i \in\ \mathbb{N}\ \}$ hence $d(\ell^i(x), \ell^i(y)) > \frac{\varepsilon}{2}\ or\ d(\ell^i(x), \ell^i(z)) > \frac{\varepsilon}{2}$

hence the (ADS) is a sensitive with sensitive constant $\frac{\varepsilon}{2}$.
Or $\zeta_\varepsilon \neq \emptyset$ hence $\zeta = \cap\ \zeta_{1/m}$,

Hence ℓ is an almost equicontinuous. □

<u>Theorem</u> 2.1.13

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, there exists the sequence $(\ell_n^m)_{0 \le n < m}$ is a minimal which is neither equicontinuous nor sensitive .

Proof :

Suppose that $\{M_1, M_2\}$ be a paitition for the cantor set with M_i a clopen subset for all $i = 1,2$, and consider a minimal equicontinuous isometry Ton $\mathbb{C}$.

Now , give the clopen sets $C_i = T(M_i)$ for $i = 1,2$, our initial step is to identify a sequence of homeomorphisms $\{g_n, n \in\ \mathbb{N}\ \}$ on $\mathbb{C}$ that will allow us to construct a non-equicontinuous g-NDS for this choose a point x_1 in $C_2$and consider a sequence $\ell_n^m, 0 \le n < m \in \mathbb{N}$ which satisfies for each $n \in \mathbb{N}$ the following:

K_n is a clopen subset of $\mathbb{C}$ whose dimeter is $1/_m$,

$K_{n+1} \subsetneq K_n$,

$x_1 \in K_{n+1}$

Now, choose clopen subsest S_1, S_2 of $C_2 \ni d(S_1, S_2) = \varepsilon_0 > 0$, select for each $m \in \mathbb{N}$ homeomorphisms

$$w_m : C_2 \backslash K_n \rightarrow C_2 \setminus (S_1 \cup S_2),$$

$$h_m : K_n \backslash K_{n+1} \rightarrow S_1,$$

$$q_m : K_{n+1} \rightarrow S_2.$$

We define G_m as follow

$$G_m = \begin{cases} x & if \quad x \in C_1, \\ w_m(x) & if \quad x \in C_2 \setminus K_n, \\ h_m(x) & if \quad x \in K_n \setminus K_{n+1}, \\ q_m(x) & if \quad x \in K_{n+1}. \end{cases}$$

Now suppose that $(\mathbb{C}, \ell_n^m)$ be a (g-NDS) defined by

$\{T, \ell_1, \ell_1^{-1}, T, T^{-1}, \ell_2, \ell_2^{-1}, T^2, T^{-2}, \ell_3, \ell_3^{-1}, T^3, T^{-3}, \ldots\ldots\}$

we get

$\ell_n^n = T \qquad , \ell_n^{n+1} = \ell_n + T$

$\ell_n^{n+2} = \ell_n^{-1} \circ \ell_n \circ T = T$

$\ell_n^{n+3} = T^n \circ \ell_n^{-1} \circ \ell_n \circ T = T^{n+1}$

$\ell_n^{n+4} = T^{-n} \circ T^n \circ \ell_n^{-1} \circ \ell_n \circ T = T$

Notice that the sequence $\{T^m, m \in \mathbb{N}\}$ is subsequence for $\{\ell^m, m \in \mathbb{N}\}$ thus transitivity and minimality of $(\mathbb{C}, \ell_n^m)$, follow from transitivity and minimality of $(\mathbb{C}, T)$. Moreover, since the restriction of

$G_m to\ C1$ is the identity for every $m \in \mathbb{N}$ the g-NDS $(\mathbb{C}, \ell_n^m)$ is an equicontinuous at every $x \in C_1$ hence $(\mathbb{C}, \ell_n^m)$ is not sensitive. We will prove that $(\mathbb{C}, \ell_n^m)$ is non equicontinuous for fixed $\delta > 0$. We may choose $m \in \mathbb{N} \ni {}^{1}/_{m} < \delta, if\ x \in \ell_n^m \backslash \ell_n^{m+1}$, then $G_m(x) = h_m(x) \in S_1$, since $G_m(x_1) = q_m(x_1) \in S_2$ we have $d((\ell_m \circ T^n)\, T^{-n}(x), (\ell_m \circ T^n)\, T^{-n}(x_1)) \geq \varepsilon_0$

since T^{-n} is an isometry, this proves that $(\mathbb{C}, \ell_n^m)$ is not equicontinuous at $T^{-n}(x_1)$. □

2.2 The Relation between Expansivity and h-Shadowing in g-Nonautonomus Discrete Dynamical Systems

In this section, we study and definition h-shadowing property and discuss its relation with various expansivity such as locally expanding, weakly locally expanding, expansive, weakly expanding small distances in g-nonautonomous discrete dynamical systems.

Definition 2.2.1

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, the sequence $(\ell_n^m)_{0 \le n < m}$has **h-shadowing property in g- non autonomous discrete dynamical systems** if for every $\varepsilon > 0$ as a result, there exists $\delta > 0$, meaning that for all δ-pseudo-orbit $\{x_0, x_1, \dots x_{\tau}\} \subseteq X$, there is $y \in X$ with

$$d(y, x_0) < \varepsilon, d\big((\ell_n^m)^i(y), x_i\big) < \varepsilon, \forall\, n \le i < m, and\ \ (\ell_n^m)^{\tau}(y) = x_{\tau}\ .$$

Definition 2.2.2

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, then the sequence $(\ell_n^m)_{0 \le n < m}$ is **weakly expanding small distances in g- non autonomous discrete dynamical systems** if there exists $\gamma > 0$, meaning that for all $x, y \in X$, $d(x, y) < \gamma\ then\ d(\ell_j(x), \ell_j(y)) > d(x, y)$, for every $j \in \mathbb{Z}$.

<u>Definition</u> 2.2.3

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, then the sequence $(\ell_n^m)_{0 \le n < m}$ called **inverse equicontinuous in g- non autonomous discrete dynamical systems** if for every $x \in X$, and for each $\varepsilon > 0$, as a result, there exists $\delta(x) > 0$,meaning that $B_{\delta(x)}(\ell_i(x)) \subseteq \ell_i(B_\varepsilon(x))$ for all $i \in \mathbb{Z}$, in which $B_\varepsilon(x)$ is the open ball with a center x and radius ε.

<u>Definition</u> 2.2.4

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, then the sequence $(\ell_n^m)_{0 \le n < m}$is **locally expanding in g- non autonomous discrete dynamical systems** when there exists $\lambda > 1$, meaning that for each $x \in X, i \ge 1$ & $\varepsilon > 0$,

$B_{\lambda\varepsilon}(\ell_i(x)) \subseteq \ell_i(B_\varepsilon(x))$.

<u>Definition</u> 2.2.5

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, then the sequence $(\ell_n^m)_{0 \le n < m}$ is **weakly locally expanding in g- non autonomous discrete dynamical systems** when there exists $\gamma > 0$, meaning that :for every $x \in X$, $i \ge 1$ & $\varepsilon > \gamma$,

$$B_\varepsilon\big(\ell_i(x)\big) \subseteq \ell_i\big(B_\varepsilon(x)\big) .$$

<u>Lemma</u> 2.2.6

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, Assume that the sequence $(\ell_n^m)_{0 \le n < m}$ is a small distance that is weakly growing in an inverse equicontinuous fashion in g-nonautonomous discrete dynamical systems then it has weakly locally expanding property in g-nonautonomous discrete dynamical systems.

Proof.

Suppose thst $\gamma > 0$ be constant, as in the definition of distances that expand weakly over short distances.. Since ℓ_n^m is an inverse equicontinuous, for all $x \in X$, there exists $\lambda(x) > 0$, meaning that: $B_{\lambda(x)}(\ell_i(x)) \subseteq \ell_i(B_{\frac{\lambda}{2}}(x))$ for all $i \ge 1$ …… (2.4)

We denote $B_{\frac{\gamma}{2}}(x) \cap \ell_i^{-1}(B_{\frac{\lambda(x)}{2}}(\ell_i(x))$ by $\mathcal{U}_i$, then for $x \in \mathcal{U}_i$, there exists $\eta = \eta(x) < \frac{\lambda(x)}{2}$ such that $B_\eta(x) \subseteq \mathcal{U}_i$, .

Now we have $\ell_i(\mathcal{U}_i) = B_{\frac{\lambda(x)}{2}}(\ell_i(x))$ in fact if $y \in B_{\frac{\lambda(x)}{2}}(\ell_i(x))$ then using (2.4) we have $y = \ell_i(t)$ with $t \in B_{\frac{\gamma}{2}}(x)$ and

$y \in B_{\frac{\lambda(x)}{2}}(\ell_i(x))$ implies $t \in \ell_i^{-1}(B_{\frac{\lambda(x)}{2}}(\ell_i(x)),$

hence $B_{\frac{\lambda(x)}{2}}(\ell_i(x)) \subseteq \ell_i(\mathcal{U}_i)$.

Let $z \in B_\eta(x), \rho < \eta, \ni B_\rho(z) \subseteq B_\eta(x)$

Then $\ell_i(z) \in \ell_i(\mathcal{U}_i) = B_{\frac{\lambda(x)}{2}}(\ell_i(x)),$

thus $B_\rho(\ell_i(z)) \subseteq B_{\lambda(x)}(\ell_i(x)) \subseteq \ell_i\left(B_{\frac{\gamma}{2}}\right)$. …….(2.5)

We denote $B_{\frac{\lambda}{2}}(x) \cap \ell_i^{-n}(B_\rho(\ell_i(\mathrm{z}))$ by $\mathcal{V}_i$, as demonstrated by the proof of an analogous result for $\mathcal{U}_i$ we see that$\ell_i(\mathcal{V}_i) = B_\rho(\ell_i(z))$.

We claim $\mathcal{V}_i \subseteq B_\rho(z)$

Suppose that $\mathcal{V}_i \nsubseteq B_\rho(z)$, then there is $y \in \mathcal{V}_i - B_\rho(z)$.

$$z \in B_\eta(x) \subseteq \mathcal{U}_i \subseteq B_{\frac{\gamma}{2}}(x), \qquad so\ \ \rho < d(y,z) \leq d(y,x), d(x,z) < \gamma.$$

Now, based on this relationship and the fact that ℓ_n^m is faintly expanding over short distances, we can conclude that $d(\ell_i(y), \ell_i(z)) \geq d(y,z) > \rho$ which is in contradiction with $y \in \mathcal{V}_i$. Therefore we get $\mathcal{V}_i \subseteq B_\rho(z)$ which produces

$B_\rho\big(\ell_i(z)\big) = \ell_i(\mathcal{V}_i) \subseteq \ell_i(B_\rho(z))$.

Now X is compact, and there is

$x_1, x_2, \dots, x_m$ in X meaning that $X \subseteq \bigcup_{i=1}^m B_{\frac{\eta(x_i)}{2}}(x_i)$

Define $r = \min\{\frac{\eta(x_i)}{2}\}$ and considering $x \in X$ and $\rho < r$,

so there is $1 \leq i \leq m$

meaning that $x \in B_{\frac{\eta(x_i)}{2}}(x_i)$ which suggests $B_\rho(x) \subseteq B_{\eta(x_i)}(x_i)$, and therefore $B_\rho\big(\ell_i(x)\big) \subseteq \ell_i\left(B_\rho(x)\right)$,

hence ℓ_n^m has weakly locally expanding property. □

Theorem 2.2.7

Let (X, d) be a compact metric space, and $\ell_n : X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, If the system $(\ell_n^m)_{0 \le n < m}$ is expanding locally, then it has h-shadowing property in g-nonautonomous discrete dynamical systems.

Proof

Assume that ℓ_n^m is an expanding locally, there exists $\lambda > 1$ meaning that for each $i \geq 1$ and $\varepsilon > 0$, we get $B_{\lambda\varepsilon}(\ell_i(x)) \subseteq \ell_i(B_\varepsilon(x))$. For a fixed $\varepsilon > 0$, we have set $\delta = (\lambda - 1)\varepsilon$, therefore for every $x \in X$ and $i \geq 1$ $B_{\varepsilon+\delta}(\ell_i(x)) \subseteq B_{\lambda\varepsilon}(\ell_i(x)) \subseteq \ell_i(B_\varepsilon(x))$

Let $\{x_0, x_1, x_2 \ldots, x_\tau\} \subseteq X$ be a δ-pseudo-orbit for ℓ_n^m Then $d(\ell_\tau(x_{\tau-1}), x_\tau) < \delta$ implies that $x_\tau \in B_{\varepsilon+\delta}(\ell_\tau(x_{\tau-1}))$, hence there is a point $y_{\tau-1} \in B_\varepsilon(x_{\tau-1})$ such that $\ell_m(y_{\boldsymbol{\tau}-1}) = x_\tau$ and so we have:

$$d(\ell_{\tau-1}(x_{\tau-2}), y_{\tau-1}) \leq d(\ell_{\tau-1}(x_{\tau-2}), x_{\tau-1}) + d(x_{\tau-1}, y_{\tau-1})$$
$$< \delta + \varepsilon$$

In other word, $y_{\tau-1} \in B_{\varepsilon+\delta}\big(\ell_{\tau-1}(x_{\tau-2})\big)$,

so there exists $y_{\tau-2} \in B_\varepsilon(x_{\tau-2})$

such that $\ell_{\boldsymbol{\tau}-1}(y_{\boldsymbol{\tau}-2}) = y_{\boldsymbol{\tau}-1}$. By recycling this line of reasoning, we get at

$y_{\boldsymbol{\tau}-2}, y_{\boldsymbol{\tau}-3}, \ldots, y_0$ in X such that for all $0 \le i \le m - 1$,

$\ell_{i+1}(y_i) = y_{i+1}$

and $d(y_i, x_i) < \varepsilon$ which proves the h-shadowing property of ℓ_n^m in g-NDS. □

Theorem 2.2.8

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, Assume that there is a continuous map ℓ meaning that $\ell_n^i \to \ell$ pointwise. In case the sequence $(\ell_n^m)_{0 \le n < m}$ is an inverse equicontinuous and ℓ is weakly faintly expanding over short distances, then $(\ell_n^m)_{0 \le n < m}$ has the h-shadowing property in g-nonautonomous discrete dynamical systems if it weakly expands over short distances and has a g-nonautonomous discrete dynamical system.

Proof:

There exists $\gamma > 0$, meaning that $d(x, y) < \gamma$ indicates that $d(x, y) < d\big(\ell(x), \ell(y)\big)$ & $d(x, y) < d\left(\ell_j(x), \ell_j(y)\right)$, for all $j \ge 1$ By using Lemma 2.2.6, there is $r > 0$, meaning that$\forall\, \rho < r, \forall\, j \ge 1$, we get

$B_\rho(\ell_j(x)) \subseteq \ell_j(B_\rho(x))$.

Suppose that $\varepsilon > 0$, we set $0 < \varepsilon' < \min\{\gamma, r, \varepsilon\}$, and define: $\eta(\varepsilon') = \sup\{\, d(x, y) : d(\ell_j(x), \ell_j(y)) < \varepsilon', i \ge 1\}$.

Hence $\eta(\varepsilon') \le \varepsilon'$.

We claim that $\eta(\varepsilon') < \varepsilon'$. Indeed , if $\eta(\varepsilon') = \varepsilon'$, there exist sequences $\{d(x_j, y_j)\}_{j=0}^{\infty}$& $\{k(j)\}_{j=0}^{\infty} \subseteq \mathbb{N}$ meaning that $d\left(\ell_{k(j)}(x_j), \ell_{k(j)}(y_j)\right) < \varepsilon'$ & $\lim\limits_{j \to \tau} d(x_j, y_j) = \eta(\varepsilon') = \varepsilon'$.

Since X is compact, a subsequence exists. $\{c_j\}_{j=0}^{\infty} \subseteq \mathbb{N}$ meaning that $x_{c_j} \to x_0$ & $y_{c_j} \to y_0$.

$\varepsilon' = \eta(\varepsilon') = \lim\limits_{j \to \tau} d(x_{c_j}, y_{c_j}) = d(x_0, y_0) < d(\ell(x_0), \ell(y_0))$

$$= \lim_{j \to \tau} d\left(\ell_{k(c_j)}\left(x_{c_j}\right), \ell_{k(c_j)}\left(y_{c_j}\right)\right) \leq \varepsilon',$$

That is not feasible.

considering now $0 < \delta < \min\{r, \gamma, \varepsilon' - \eta(\varepsilon')\}$.

Let $\{x_0, x_1, \dots, x_\tau\}$ be a δ-pseudo –orbit for ℓ_n^m then $d(\ell_\tau(x_{\tau-1}), x_\tau) < \delta$, which implies that there is $y_{\tau-1} \in B_\delta(x_{\tau-1})$ such that $\ell_\tau(y_{\tau-1}) = x_\tau$. Since $d(\ell_\tau(x_{\tau-1}), x_\tau) < \delta \leq \varepsilon'$, We have $d(x_{\tau-1}, y_{\tau-1}) < \eta(\varepsilon') \leq \varepsilon'$.

And

$$d(\ell_{\tau-1}(x_{\tau-2}), y_{\tau-1}) < d(\ell_{\tau-1}(x_{\tau-2}), x_{\tau-1}) + d(x_{\tau-1}, y_{\tau-1})$$

$$< \delta + \eta(\varepsilon') < \varepsilon' < r.$$

Therefore there is $y_{\tau-2} \in B_{\varepsilon'}(x_{\tau-2}) \subseteq B_\gamma(x_{\tau-2})$ such that $\ell_{\tau-1}(y_{\tau-2}) = y_{\tau-1}$.

Hence

$d(x_{\tau-2}, y_{\tau-2}) < d(\ell_{\tau-1}(x_{\tau-2}), y_{\tau-1}) < \delta + \eta(\varepsilon') < \varepsilon' \leq \varepsilon$.

By Repeating, we can find points $y_{\tau-1}, y_{\tau-2}, \dots, y_0$ in X such that for all $0 \leq j \leq m-1$, $\ell_{j+1}(y_j) = y_{j+1}$ & $d(y_j, x_j) < \varepsilon$.

Further more $(\ell_n^m)^\tau(y_0) = x_\tau$, hence ℓ_n^m has h-shadowing property. □

<u>Theorem</u> 2.2.9

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, the sequence $(\ell_n^m)_{0 \leq n < m}$ be an expansive and has shadowing property in g-nonautonomous discrete dynamical systems then it has h-shadowing property in g-nonautonomous discrete dynamical systems.

Proof:

Let $e > 0$ be an expansive constant, $\delta > 0$ & $\varepsilon > \delta$ is provided by the shadowing of ℓ_n^m.

Let $\{x_0, x_1, x_2, \dots \dots, x_\tau\}$ is a δ-pseudo-orbit. The sequence:

$\{x_0, x_1, x_2, \dots \dots, x_{\boldsymbol{\tau}}, \ell_{\boldsymbol{\tau}+1}(x_{\boldsymbol{\tau}}), \ell_{\boldsymbol{\tau}+1}^{\boldsymbol{\tau}+2}(x_{\boldsymbol{\tau}}), \ell_{\boldsymbol{\tau}+1}^{\boldsymbol{\tau}+3}(x_{\boldsymbol{\tau}}), \dots .\}$

is an infinite δ-pseudo-orbit .since ℓ_n^m has the shadowing property, there is $y \in X$,$\ni$ $\forall\, i \geq 1, d((\ell_n^m)^{\tau+i}(y), \ell_{\boldsymbol{\tau}+1}^{\boldsymbol{\tau}+i}(x_{\boldsymbol{\tau}})) < \varepsilon$

when $(\ell_n^m)^{\boldsymbol{\tau}}(y) = x_{\boldsymbol{\tau}}$. Thus the sequence ℓ_n^m has h-shadowing property in g-NDS. ◻

2.3 The Relation Between The w-Expansive and The Shadowing Property in g-Nonautonomous Discrete Dynamical Systems.

In this section, we discuss the dynamics of w-expansive homeomorphism with shadowing property defined on compact metric space

Definition 2.3.1

Let (X, d) be a compact metric space, and $\ell_n: X \longrightarrow X$ be an uniform continuous map $\forall\, n \in \mathbb{N}$. The sequence $(\ell_n^m)_{0 \le n < m}$ is said to be ***w*-expansive in g-nonautonomous discrete dynamical systems** : if exists a constant $e > 0$ (called an w-expansive constant) meaning that for each $x \in X$. and let the set

$$A(x, e) = \{\, y \in X : d(\ell_n^m(x), \ell_n^m(y) \le e, 0 \le n < m \in \mathbb{N}\,\}$$

It's has w different points (where $w = m - n$),

i.e. for every $x \in X$, such that $\exists\ e > 0, \ni$

$$A(x, e) = X - \{\, y \in X : d(\ell_n^m(x), \ell_n^m(y) > e, 0 \le n < m\ \in \mathbb{N}\,\}$$

which be a finite set of w different point.

Theorem 2.3.2

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform homeomorphism maps $\forall n \in \mathbb{N}$, then the sequence $(\ell_n^m)_{0 \le n < m}$ is w-expansive in g-nonautonomous discrete dynamical systems if and only if $(\ell_n^m)^{-1}, \forall\, m < n \leq 0$ is w-expansive in g-nonautonomous discrete dynamical systems.

Proof

Suppose that $e > 0$ be an w-expansive constant for ℓ_n^m $\forall\, 0 \leq n < m$, for a fixed $x \in X$;

the set $\{y \in X: d(\ell_n^i(x), \ell_n^i((y)) \leq e, 0 \leq n < i \in \mathbb{N}\}$ has at most w elements, i.e. the set

$\{y \in X: d((\ell_n^i)^{-1}(x), (\ell_n^i)^{-1}(y)) \leq e, 0 \leq n < i \in \mathbb{N}\}$ has at most w-elements. Thus, $(\ell_n^m)^{-1}$ is w-expansive. The converse can be proved similarly. □

<u>Theorem</u> 2.3.3

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ of an equicontinuous uniformly continuous maps $\forall n \in \mathbb{N}$, for any positive integer k, the sequence $(\ell_n^m)_{0 \leq n < m}$is an w-expansive if and only if $(\ell_n^m)^k$ is an w-expansive, $\forall\, 0 \leq n < m \in \mathbb{N}$.

Proof

Suppose that $e > 0$ be an w-expansive constant for $\ell_{n,m}$ Since ℓ_n^m is an equicontinuous family of maps. For each $h \geq 0$, $hk + 1 \leq j \leq (h+1)k$, ℓ_n^j is uniformly continuous on X and therefor there exists $e_j > 0$ meaning that $d(x, y) \leq e_j$ implies $d\left(\ell_{hk+1}^j(x), \ell_{hk+1}^j(y)\right) \leq e$.

Since $\ell_n: X \to X,\ n \in \mathbb{N}$ are equicontinuous maps, e_j does not depend on h. take

$e' = \min\{e_j: hk+1 \leq j \leq (h+1)k\}$. Therefore for any $h \geq 0$, $d(x,y) \leq e'$ implies $d\left(\ell_{hk+1}^{j}(x), \ell_{hk+1}^{j}(y)\right) \leq e$.

Let $(\ell_n^m)^k = g_n^m, \forall k \in \mathbb{N}$, where $g_m = \ell_{(m-1)k+1}^{mk}$ and $g_n^m = g_m \circ g_{m-1} \circ \ldots \circ g_n$. Not that $\ell_n^{mk} = g_n^m$. Thus , for any $h \geq 0$ and $hk \leq j \leq (h+1)k$, $d(g_n^h(x), g_n^h(y) \leq e$ which implies $d(\ell_n^{hk}(x), \ell_n^{hk}(y)) \leq e'$ and hence we get that $d(\ell_{hk+1}^{j}\left(\ell_n^{hk}(x)\right), \ell_{hk+1}^{j}\left(\ell_n^{hk}(y)\right)) \leq e$ which implies $d\left(\ell_n^{j}(x), \ell_n^{j}(y)\right) \leq e, \exists\, n < j$. Since e is an w-expansive constant for ℓ_n^m, the set $A(x,e)$ has at most w element (where $w = m - n$). Therefore $\{y \in X: d(g_n^i(x), g_n^i(y)) \leq e'; 0 \leq n < i\}$ has at most w element, and hence $(\ell_n^m)^k$ is w-expansive with a constant of w- expansive $e' \geq 0$.

Conversely, if $(\ell_n^m)^k$ is w-expansive which constant of m- expansiveness $e > 0$, then for any $x \in X$, the set $A(x,e)$ has at most w element, where $g_m = \ell_{(m-1)k+1}^{mk}$. thus,

the set $\{y \in X: d(\ell_n^{ik}(x), \ell_n^{ik}(y)) \leq e, for\ 0 \leq n < i\}$

$= \{d(\ell_n^{j}(x), \ell_n^{j}(y)) < e, 0 \leq n < j\}$ has at most w element.

Hence, $\ell_n^m$is w-expansive with a constant of w-expansiveness $e > 0$ in g-nonautonomous discrete dynamic systems. □

Theorem 2.3.4

Let (X, d) and (Y, d') be a compact metric spaces, $\ell_n: X \to X$ and $g_n: Y \to Y$ be a maps $\forall n \in \mathbb{N}$, such that $(\ell_n^m)_{0 \leq n < m}$ is uniformly conjugate to$(g_n^m)_{0 \leq n < m}$. If $(\ell_n^m)_{0 \leq n < m}$ is an w-expansive of g- nonautonomous discrete dynamical systems, then so is $(g_n^m)_{0 \leq n < m}$.

Proof

Suppose that c > 0 be an w-expansiveness constant to ℓ_n^m

Since ℓ_n^m is uniformly conjugate for g_n^m As a result, uniform homeomorphism exists.

$h : X \to X$ meaning that $h \circ \ell_n^m = g_n^m \circ h,$

for all $0 \leq n < m \in \mathbb{N}$. Thus, $\ell_m^n \circ h^{-1} = h^{-1} \circ g_m^n$,

for all $0 \leq n < m \in \mathbb{N}$.

therefore for every $e > 0$, there is $e' > 0$ such that for $\boldsymbol{x}, y \in Y$,

$d'(x, y) \leq e' \; implies \; d(h^{-1}(x), h^{-1}(y)) \leq e.$

For a fixed $x \in Y$, the set

$S = \{y \in Y : d'\left(g_n^m(x), g_n^m(y)\right) \leq e', \forall\, 0 \leq n < m \in \mathbb{N}\} \subseteq$
$\{y \in Y : d\,(h^{-1}(g_n^m(x)), h^{-1}(g_n^m(y))) \leq e', \forall\, 0 \leq n < m \in \mathbb{N}\}$

$= \{y \in Y : d(\ell_n^m(h^{-1}(x)), \ell_n^m(h^{-1}(y))) \leq e, 0 \leq n < m \in \mathbb{N}\}.$

Since ℓ_n^m is w-expansive with w-expansiveness constant $e > 0$, therefore, S has w of different points (where $w = m - n$), and hence g_n^m is an w-expansive of g-NDS with w-expansiveness constant $e' > 0$. □

Theorem 2.3.5

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, for all $w \in \mathbb{N}$, there is an w-expansive homeomorphism, defined in g-nonautonomous discrete dynamical systems , that is not $(w - 1)$-expansive, has the shadowing property in g-nonautonomous discrete dynamical systems and admits an infinite number of chain recurrent classes.

Proof:

Considered the expansive homeomorphism g_n^m defined in a compact metric space (M, d_0) and satisfying the shadowing property. Further, assume that it have the infinite number of periodic points $\{p_k\}_{k\in\mathbb{N}}$, which we as can assume have different orbits. Define X as the set $M \cup E$, where E is an infinite enumerable set. In this instance, there is a bijection $r : N \to E$.

Suppose that

$$Q = \{(a, b, c) : a \in \{1, \dots, m - 1\};\ m \in \mathbb{N},\ b \in \{k\}_{k\in\mathbb{N}}\ ,$$

$$c \in \{0, \dots, \pi(pk) - 1\}; k \in \mathbb{N}\}$$

And note that there exists a bijection $s : Q \to N$. Consider the bijection $q : Q \to E$ defined by $q(a, b, c) = r \circ s(a, c, b)$.

Thus, any point $u \in E$ has the form $u = q(a, b, c)$ for some

$(a, b, c) \in Q$.

Define a map $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v) = \begin{cases} 0, & u = v, \\ d_0(u,v), & u, v \in M, \\ \frac{1}{k} + d_0\big(u, g_n^c(p_b)\big), & u \in M, v = q(a,b,c), \\ \frac{1}{k} + d_0(g_n^c(p_b), v), & u = q(a,b,c), v \in M, \\ \frac{1}{k} + \frac{1}{w} + d_0\big(g_n^c(p_b), g_n^r(p_h)\big), & u = q(a,b,c), v = q(e,h,r), \\ & b \neq h \; or \; c \neq r, \\ \frac{1}{k}. & u = q(a,b,c), v = q(e,b,c) \\ & , a \neq e. \end{cases}$$

Then to prove that ℓ_n^m is w-expansive where ($w = m - n$, $\forall\, n, m \in \mathbb{Z}$), The expansiveness of g_n^m assures the existence of a number $\delta > 0$ meaning that if $d(g_n^k(u), g_n^k(v)) \leq \delta$, for every $0 \leq n < k \in \mathbb{Z}$, then $u = v$. Suppose that $\{u_n, u_{n+1} \ldots, u_m, u_{m+1}, \ldots, u_{m+n}\}$ are $w + 1$ different points of X satisfyin:

$$d\left(\ell_n^k(u_i), \ell_n^k(u_j)\right) \leq \delta, \qquad 0 \leq n < k \in \mathbb{N},$$

For every pair $(i, j) \in \{n, \ldots, m+n\} \times \{n, \ldots, m+n\}$. And most one of these points belong to M, and at least m of them belong to E, by

$$d(\ell_n^s(u), \ell_n^s\big(q(a,b,c)\big) \leq \delta, \forall \;\; 0 \leq n < \; s \in \mathbb{Z} .$$

We claim that at least two for these points are of the form $q(a, b, c)$ and $q(a, h, r)$ with $b \neq h$. Moreover, if this is not the case, then two of them are of the form $q(a, b, c)$ and $q(a, b, r)$ with $c \neq r$. It follows that for every $0 \leq n < s \in \mathbb{Z}$ we get

$$d(g_n^s(g_n^c(p_b)), g_n^s(g_n^r(p_b)) = d(\ell_n^s(q(a,b,c)), \ell_n^s(q(a,b,r)) - \frac{2}{k}$$

$$< d(\ell_n^s(q(a,b,c)), \ell_n^s(q(a,b,r))$$

$$\leq \delta.$$

This implies that $g_n^c\,(p_b) = g_n^r(p_b)$, Consequently, this implies that $c = r$, resulting in a contradiction.

For each $0 \leq n < s \in \mathbb{Z}$ the following holds

$$d(g_n^s(g_n^c(p_b)), g_n^s(g_n^r(p_h)) = d(\ell_n^s(q(a,b,c)), \ell_n^s(q(e,h,r)) - \frac{2}{k}$$

$$< d(\ell_n^s(q(a,b,c)), \ell_n^s(q(e,h,r))$$

$$\leq \delta.$$

Therefore, $g_n^c(p_b) = g_n^r(p_h)$ and $p_b = p_h$, which is a contradiction with the fact that $b \neq h$. Noting that M has an infinite number of periodic points $\{p_k\}_{k\in\mathbb{N}}$ is essential to understanding that this is possible.

Now to prove that ℓ_n^m is non $(w-1)$-expansive: For every $\delta > 0$, choose $k \in \mathbb{N}, \ni \frac{1}{k} < \delta$, and the set A$\left(p_k, \frac{1}{k}\right)$ contains w different points ($w = m - n$) hence, for each $a \in \{1, \dots, m-1\}$ the point $q(a,b,0)$ Belongs to A$\left(p_k, \frac{1}{k}\right)$ that's implies the A(p_k, δ) contains at least w different points and that ℓ_n^m is non $(w-1)$-expansive. To prove ℓ_n^m has the shadowing property in g-nonautonomous discrete dynamical systems,

Since g_n^m have the shadowing property , for each $\varepsilon > 0$ we can consider $\delta_g > 0$ meaning that every δ_g-pseudo orbit of g_n^m is $\frac{\varepsilon}{2}$-shadowed. Choose L belongs to $\mathbb{N}$ such that $\frac{1}{L} < \min\{\frac{\varepsilon}{2}, \frac{\delta_g}{3}\}$ and let $\delta = \frac{1}{L}$.

If $\{u_i\}_{i\in\mathbb{N}} \subset X$ is a δ-pseudo orbit of ℓ then either $\{u_i\}_{i\in\mathbb{N}}$, is one of the orbits $\{q(a,b,c);\ c \in \{0,\dots,\pi(p_k)-1\},\ a \in \{1,\dots,m-1\},\ b \in \{1,\dots,L-1\}\}$, or $\{u_i\}_{i=0}^{\infty}$ does not contain any point of these orbits. In the first case $\{u_i\}_{i=0}^{\infty}$ is obviously shadowed, therefore, we will concentrate on the second case. Thus if $u_i = q(a,b,c)$ then $b \geq L$. Define a sequence $\{v_i\}_{\boldsymbol{i=n}}^{\boldsymbol{m}} \subset X$ by

$$v_i = \begin{cases} u_i, & u_i \in M, \\ g_n^c(p_b), & u_i = q(a,b,c). \end{cases}$$

The sequence $\{v_i\}_{i=0}^{\infty}$ is a δ_g-pseudo orbit for g_n^m since, for every $i \in \mathbb{N}$ the following holds:

$$d\big((g_n^m)^{i+1}(v), v_{i+1}\big) = d\big((\ell_n^m)^{i+1}(v), v_{i+1}\big)$$

$$\leq d\Big((\ell_n^m)^{i+1}(v), (\ell_n^m)^{i+1}(u)\Big) + d\big((\ell_n^m)^{i+1}(u), u_{i+1}\big) + d(u_{i+1}, v_{i+1}),$$

$$\leq \frac{1}{L} + \frac{1}{L} + \frac{1}{L}$$

$$\leq \delta_g .$$

Then there exists $u \in M$ such that

$$d((g_n^m)^i(u), v_i) < \frac{\varepsilon}{2}, i \in Z.$$

It follows that $\{u_i\}_{i=0}^{\infty}$ is ε-shadowed by u, since for each $i \in Z$ the following holds:

$$d\big((\ell_n^m)^i\,(u),u_i\big) \le d\big(\,(\ell_n^m)^i(u),v_i\big) + d(v_i,u_i)$$

$$\le \frac{\boldsymbol{\varepsilon}}{\mathbf{2}} + \frac{\mathbf{1}}{\boldsymbol{L}}$$

$$\le \varepsilon\,.$$

Given that this is applicable to any $\varepsilon > 0$, we obtain that ℓ_n^m have an shadowing property in g-nonautonomous discrete dynamical systems.

Now to prove ℓ_n^m admits an infinite number of chain-recurrent classes, the different periodic orbits in E belong to different chain-recurrent classes. Indeed, every point $q(a,b,c\,) \in E$ satisfies :

$$d(q(a,b,c\,),u) \ge \frac{1}{k}\,, u \in X - \{q(a,b,c\,)\}.$$

This means that if $0 < \varepsilon < \frac{1}{k}$ then the orbit of $q(a,b,c\,)$ cannot be connected by ε-pseudo orbits with any other point of X. This proves that the chain recurrent class of $q(a,b,c)$ contains only its orbit. □

Chapter Three

The Fitting Shadowing Property on g-Nonautonomous Discrete Dynamical Systems

3.1 The Fitting Shadowing Property on g-Nonautonomous Discrete Dynamical Systems

In this section, we define and discuss the relation along the three concepts, which are an fitting shadowing property (F.S.P), the asymptotic fitting shadowing property (A.F.S.P), and the relation between an fitting shadowing and the average shadowing property in g-nonautonomous discrete dynamical systems. Additionally, we study the relationship between fitting shadowing and the average shadowing property in g-nonautonomous discrete dynamical systems. We investigate the concept of the fitting shadowing property as well as the asymptotic fitting shadowing property in g-nonautonomous discrete dynamical systems with chaos.

Definition 3.1.1

Let (X, d) be a compact metric space, and $\ell_n: X \longrightarrow X$ be an uniform continuous maps $\forall n \in \mathbb{N}$. A sequence $\{x_i\}_{i=0}^{\infty}$ in X is called a ***α*-fitting pseudo-orbit** of $(\ell_n^m)_{0 \le n < m}$ **in g- non autonomous discrete dynamical systems** if there exists $\alpha > 0$ and a positive integer $M = M(\alpha)$, such that for every $w \geq M$ and $k \in \mathbb{N}$, we get $\sum_{i=0}^{w-1} d\big(\ell_j(x_{i+k}), x_{i+k+1}\big) < \alpha,\ \ where\, n \le j < m$

A sequence $(\ell_n^m)_{0 \le n < m}$ are say to have the **fitting shadowing property of g-non autonomous discrete dynamical systems :** if for all $\varepsilon > 0$ there is $\alpha > 0$ meaning that every α-fitting pseudo-orbit is ε-shadowed in fitting by the orbit of some point $y \in X$, that is

$$\lim_{w\to\infty} sup \sum_{i=n}^{w-1} d\big((\ell_n^m)^i(y), x_i\big) < \varepsilon.$$

Definition 3.1.2

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous maps $\forall n \in \mathbb{N}$, a sequence$\{x_i\}_{i=0}^{\infty}$ in X is called **asymptotic fitting pseudo-orbit** in g-NDS of $(\ell_n^m)_{0\le n<m}$ if

$$\lim_{m\to\infty} \sum_{i=0}^{m-1} d\big(\ell_j(x_i), x_{i+1}\big) = 0\,,\ \ where\, n \le j < m$$

A sequence $\{x_i\}_{i=0}^{\infty}$ in X is say to be **asymptotically shadowing in fitting** in g-NDS by using point $w \in$ X, if

$$\lim_{m\to\infty} \sum_{i=n}^{m-1} d((\ell_n^m)^i(w)\,, \boldsymbol{x}_i) = 0$$

We are saying that. $(\ell_n^m)_{0\le n<m}$ has asymptotic fitting shadowing property in g-NDS if any asymptotic fitting pseudo-orbit of $(\ell_n^m)_{0\le n<m}$ in g-NDS is asymptotically shadowing in fitting using the point $w \in$ X.

Proposition 3.1.3

Let (X, d) and (Y, d') be two compact metric space, $\ell_n: X \to X$, & $g_n: Y \to Y$ be map, $\forall n \in \mathbb{N}_0$. if $(\ell_n^m)_{0\le n<m}$ and $(g_n^m)_{0\le n<m}$ have the fitting shadowing Property of g-nonautonomous discrete dynamical systems, then the product $\ell_n^m \times g_n^m$ have the fitting shadowing property of g-nonautonomous discrete dynamical systems, $\forall\, 0 \le n < m \in \mathbb{N}$.

Proof:

Let (X,d) and (Y,d') be two compact metric space, $\ell_n: X \to X$, & $g_n: Y \to Y$ be maps, then By Remark 1.1.19 Let $F = \max\{\, \check{d}(x,y): x,y \in X \times Y\}$ $and\ \gamma = (\frac{\varepsilon}{2(F+1)})^2$.

Since $\ell_n^m$and g_n^m has an fitting shadowing property, for $\gamma > 0, \exists\, \alpha > 0$ Such that all α-fitting pseudo-orbit ℓ_n^mor every α-fitting pseudo- orbit g_n^m is γ-shadowing in fitting by some point $s \in X$ $(t \in Y)$. Assume that $\{(x_i, y_i)\}_{i=0}^{\infty}$ is α-fitting pseudo-orbit $\ell_n^m \times g_n^m$**,** from the Definition of the fitting pseudo-orbit 3.1.1, we evidently have ℓ_n^m and g_n^m are α-fitting pseudo-orbits of ℓ_n^m and g_n^m respectively. Thus there are $s \in X, t \in Y$ such that :

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} d\big((\ell_n^m)^i(s), x_i\big) \le \gamma, \ldots\ldots (3.1)$$

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} d'((g_n^m)^i(t), y_i) \le \gamma, \ldots\ldots (3.2)$$

We write $c_i = d\big((\ell_n^m)^i(s), x_i\big)$, $\quad b_i = d'\big((g_n^m)^i(t), y_i\big)$,

$w_i = \max\{(c_i, b_i)\}$, $\quad B_{\iota,m} = card(\{i < m: b_i \ge \iota\})$,

$W_{\iota,m} = card(\{i < m: w_i \ge \iota\})$,

hence that $W_{\iota,m} \le C_{\iota,m} + B_{\iota,m}$. By Lemma 1.2.10 (a) & (3.1) , (3.2),

It follows that

$$\lim_{m\to+\infty} \sup C_{\sqrt{\gamma}}, m \le \sqrt{\gamma} \quad \& \quad \lim_{m\to+\infty} \sup B_{\sqrt{\gamma}}, m \le \sqrt{\gamma}\,.$$

so

$$\lim_{\boldsymbol{m}\to+\infty} \sup M_{\sqrt{\gamma}}, m \leq \sqrt{\gamma}\,.$$

by Lemma 1.2.10 (b), we get that

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} c_i \leq 2(\mathrm{F}+1)\sqrt{\gamma} = \varepsilon.$$

That is

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} d^*(\,((\ell_n^m)^i \times (g_n^m)^i)(s,t), (\,x_i, y_i))$$

$$= \lim_{\boldsymbol{m}\to+\infty} \sup \sum_{\boldsymbol{m}\to+\infty}^{\boldsymbol{m}-\boldsymbol{1}} \max\{d\big((\ell_n^m)^i(s), x_i\big),\ d'\big((g_n^m)^i(t), y_i\big)\}$$

$$= \lim_{\boldsymbol{m}\to+\infty} \sup \sum_{i=n}^{m-1} w_i < \varepsilon$$

therefore ; $\ell_n^m \times g_n^m$ possesses the fitting shadowing property. □

<u>Corollary</u> 3.1.4

Let (X, d) be a compact metric space, if $\ell_n: X \to X$ has the fitting shadowing property of g-nonautonomous discrete dynamical systems, then the $\underbrace{\ell_n^m \times \ell_n^m \times \ell_n^m \times \ldots . \times \ell_n^m}_{m-times}$ has the fitting shadowing property of g-nonautonomous discrete dynamical systems.

Proof:

We can show the proof by Induction Law and Proposition 3.1.3. □

<u>Theorem</u> 3.1.5

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$. if $(\ell_n^m)_{0\le n<m}$ has the fitting shadowing property (F.S.P) of g-nonautonomous discrete dynamical systems, then $(\ell_n^m)^k, \forall\, 0 \le n < m \in \mathbb{N}$, has the fitting shadowing property (F.S.P) of g-nonautonomous discrete dynamical systems, for each $k \in \mathbb{N}$.

Proof:

Since $k \in \mathbb{N}$, and since ℓ_n^m has F.S.P for any $\frac{\varepsilon}{k} > 0$, there is $\delta > 0, \ni$ for each δ-fitting pseudo -orbit is $\frac{\varepsilon}{k}$ –shadowing in fitting by some point in X. suppose $\{y_i\}_{i=0}^{\infty}$; $0 \le n < m \in \mathbb{N}$, is δ-fitting pseudo –orbit of $(\ell_n^m)^k$,that is, $\exists\, M = M(\delta) > 0, \forall\, m \ge M$.

$$\sum_{i=n}^{m-1} d(\ell_n^m(y_{i+h}), y_{i+h+1}) < \delta, \qquad \forall\, m > M \text{ and } h \in \mathbb{Z}_+.$$

We write $x_{ik+t} = (\ell_n^m)^k(y_i),\ \forall\, 0 \le t < k,\ i \in \mathbb{Z}_+$, that is

$$\{x_i\}_{i=0}^{\infty} = \{y_n, \ell_n^m(y_n), \ldots, (\ell_n^m)^{k-1}(y_n), y_{n+1}, \ldots, (\ell_n^m)^{k-1}(y_{n+1}), \ldots\}$$

We have

$$\sum_{i=n}^{m-1} d\big(\ell_j(x_{i+h}), x_{i+h+1}\big) < \delta, \forall\, m \geq M, and\ h \in \mathbb{Z}_+ .$$

then $\{x_i\}_{i=0}^{\infty}$ is δ-fitting pseudo-orbit of ℓ_n^m. So there is $z \in X, \ni$

$$\lim_{m\to+\infty} \sup \sum_{i=n}^{m-1} d((\ell_n^m)^i(z), x_i) < \frac{\varepsilon}{k} \ldots\ldots (3.3)$$

Claim : there are infinite $s \in \mathbb{N}$, meaning that

$$\sum_{i=n}^{s-1} d\big((\ell_n^m)^{ki}(z), x_i\big) < \varepsilon.$$

To Proof the claim. Contrariwise, assume that there exists $k_0 \in \mathbb{N}$ meaning that

$$\sum_{i=n}^{s-1} d(\,(\ell_n^m)^{ki}(z), x_i) \geq \varepsilon$$

For all $s \geq k_0$, then

$$\lim_{m\to+\infty} \sum_{i=n}^{m-1} d((\ell_n^m)^i(z), x_i) > \frac{\varepsilon}{k}$$

which corresponds to (3.3). The substantiation of the claim has been finalized. By the claim, we get

$$\lim_{m\to+\infty} \sum_{i=n}^{m-1} d\big((\ell_n^m)^{ki}(z), x_{ki}\big) < \varepsilon,$$

since $x_{ki} = y_i$,

$$\lim_{m\to+\infty}\sum_{i=n}^{m-1} d(((\ell_n^m)^k)^i(z), y_i) < \varepsilon \,.$$

Hence, $(\ell_n^m)^k$has the fitting shadowing property (F.S.P) in g-NDS . □

<u>Remark</u> 3.1.6

By definition of fitting shadowing property in g-non autonomous discrete dynamical system, we get that, if $(\ell_n^m)_{\mathbf{0}\le n<m}$ has fitting shadowing property of g-nonautonomous discrete dynamical systems then $(\ell_n^m)_{0\le n<m}$ has average shadowing property of g-nonautonomous discrete dynamical systems.

<u>Lemma</u> 3.1.7

If $\{a_i\}_{i=n}^{m}$ be a constrained sequence of positive real numbers, then the following condition has an equivalent:

(1) $\lim_{m\to\infty}\sum_{i=n}^{m-1} a_i = 0,$

(2) There exists a subset $J\subset\mathbb{N}$ of zero density, where $d(J) = 0$ and

$$\lim_{m\to\infty}\frac{card(J\cap\{0,1,2,\dots,m-1\}}{m} = 0$$

such that

$$\lim_{m\notin J} a_i = 0$$

Proposition 3.1.8

Let (X, d) and (Y, d') be a compact metric spaces, $\ell_n: X \to X$ and $g_n: Y \to Y$ be a maps $\forall n \in \mathbb{N}$, then $(\ell_n^m)_{0 \le n < m}$ and $(g_n^m)_{0 \le n < m}$ have the asymptotic fitting shadowing property of g-nonautonomous discrete dynamical systems if and only if $\ell_n^m \times g_n^m, \forall\, 0 \le n < m$ has the asymptotic fitting shadowing property of g-nonautonomous discrete dynamical systemscharacteristic.

Proof

Assume that ℓ_n^m and g_n^m,$\forall 0 \le n < m$, possess the asymptotic shadowing fitting property in g-NDS. Suppose that $\{(x_i, y_i)\}_{i=0}^{\infty}$ be asymptotic fitting pseudo-orbit of $\ell_n^m \times g_n^m$ that is

$$\lim_{m \to \infty} \sum_{i=0}^{m-1} d''((\ell \times g)_j(x_i, y_i), (x_{i+1}, y_{i+1})) = 0 \ldots (3.4)$$

This implies that

$$\lim_{m \to \infty} \sum_{i=0}^{m-1} d(\ell_j(x_i), x_{i+1}) = 0 \ldots (3.5)$$

$$\lim_{m \to \infty} \sum_{i=0}^{m-1} d'(g_j(y_i), y_{i+1}) = 0 \ldots (3.6)$$

So $\{x_i\}_{\boldsymbol{i=0}}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are fitting asymptotic pseudo-orbit of ℓ_n^m and g_n^m, $\forall\, n < m$. Respective. Consequently, there are two points $z_1, z_2 \in X$ meaning that

$$\lim_{m\to\infty} \sum_{i=n}^{m-1} d(\,(\ell_n^m)^i(z_1), x_i) = 0 \ldots (3.7)$$

$$\lim_{m\to\infty} \sum_{i=n}^{m-1} d'(\,\left(g_n^m\right)^i(z_2), y_i\right) = 0 \ldots (3.8)$$

By Lemma 3.1.7 and (3.7) there is a set $J_0 \subset \mathbb{Z}_+$of zero density such that

$$\lim_{j\to\infty} d(\,(\ell_n^m)^i(z_1), x_j) = 0$$

where $j \notin J_0$. Likewise, a set exists $J_1 \subset \mathbb{Z}_+$ for zero density meaning that

$$\lim_{j\to\infty} d'(\,(g_n^m)^i(z_2), y_j) = 0$$

where $j \notin J_1$. Let $J = J_0 \cap J_1$, then J subset of zero density and

$$\lim_{j\to\infty} d''((\,\ell_n^m \times g_n^m)^i(z_1, z_2), \left(x_j, y_j\right) = 0$$

where $j \notin J$. So By Lemma 3.1.7 we have

$$\lim_{m\to\infty} \sum_{i=n}^{m-1} d''((\ell_n^m \times g_n^m)^i\,(z_1, z_2), \left(x_j, y_j\right) = 0$$

Thus $\ell_n^m \times g_n^m$ possesses the shadowing property of asymptotic fitting in g-NDS.

Conversely, Likewise, we can proof that if $\ell_n^m \times g_n^m$ possesses the shadowing property of asymptotic fitting in g-NDS then ℓ_n^m and g_n^m possesses the shadowing property of asymptotic fitting in g-NDS. □

<u>Proposition</u> 3.1.9

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, if $(\ell_n^m)_{0\le n<m}$ have the asymptotic fitting shadowing property of g-nonautonomous discrete dynamical systems, then $(\ell_n^m)^k, \forall\, k > 0$, has the asymptotic fitting shadowing property of g-nonautonomous discrete dynamical systems.

Proof

Assume that ℓ_n^m has the asymptotic fitting shadowing property and $k > 0$. Suppose that $\{x_i\}_{i=0}^{\infty}$be an asymptotic fitting pseudo-orbit of $(\ell_n^m)^k$, where $k = 1,2,..$, that is

$$\lim_{m\to\infty} \sum_{i=0}^{m-1} d(\,\ell_n^m(x_i), x_{i+1}) = 0 \;.... (3.9)$$

Let $z_{tk+h} = \ell_n^m(x_i)$, for all $0 \le h < k$ and every $t \ge 0$

Since

$$\lim_{m\to\infty} \sum_{i=0}^{m-1} d(\ell_j(z_i), z_{i+1}) \le \frac{m}{tk+h} \lim_{m\to\infty} \sum_{i=0}^{t} d(\ell_n^m(x_i), x_{i+1})$$

We get from (3.9) that

$$\lim_{m\to\infty} \sum_{i=0}^{m-1} d\big(\ell_j(z_i), z_{i+1}\big) = 0$$

That is the sequence $\{z_i\}_{i=0}^{\infty}$ is an asymptotic fitting pseudo-orbit of ℓ_n^m. So there is a point $w \in X$ such that

$$\lim_{m\to\infty} \sum_{i=n}^{m} d\big((\ell_n^m)^i(w), z_i\big) = 0 \qquad (3.10)$$

Note that

$$\lim_{m\to\infty}\sum_{i=n}^{t-1} d((\ell_n^m)^{ki}(w),x_i) \le \lim_{m\to\infty}\sum_{s=n}^{t-1}\sum_{h=n}^{k-1} d((\ell_n^m)^{sk+h}\,(w),z_{tk+h}$$

$$= \lim_{m\to\infty}\sum_{i=n}^{tk-1} d((\ell_n^m)^{i}(w),z_i)$$

From (3.10) that

$$\lim_{m\to\infty}\sum_{i=n}^{t-1} d\big((\ell_n^m)^{ki}(w),x_i\big) = 0$$

This showing $(\ell_n^m)^k$ has the asymptotic fitting shadowing property of g-NDS. □

Definition 3.1.10

Let (X,d) be a compact metric space and $\ell_n: X \to X$ be an uniform continuous map ,$\forall n \in \mathbb{N}$, if each $x \in X$ transitive point, then we say $(\ell_n^m)_{0\le n<m}$ is minimal in g-nonautonomous discrete dynamical systems. $x \in X$ is said to be **minimal point** of g-nonautonomous discrete dynamical systems if for every neig-hbrhood U of y, $N(y,V)$ is syndetic, denoted by **AP** (ℓ_n^m) the set of all minimal points of $(\ell_n^m)_{0\le n<m}$ in g-nonautonomous discrete dynamical systems

we generalize the Proposition 2.1.13 in [9] for g-nonautonomous discrete dynamical system

Proposition 3.1.11

Let (X, d) and (Y, d') be a compacts metric spaces , $\ell_n: X \to X$ and $g_n: Y \to Y$,be a continuous map ,$\forall n \in \mathbb{N}$, and $(\ell_n^m)_{0\leq n<m}$ and $(g_n^m)_{0\leq n<m}$are surjective maps. If $(\ell_n^m)_{0\leq n<m}$ and $(g_n^m)_{0\leq n<m}$each has dense minimal points then so does the product $\ell_n^m \times g_n^m$ of g-nonautonomous discrete dynamical systems,$\forall\ 0 \leq n < m$.

Theorem 3.1.12

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map $\forall n \in \mathbb{N}$, if $(\ell_n^m)_{0\leq n<m}$ has the fitting shadowing property of g-nonautonomous discrete dynamical systems and the set of all minimal points for $(\ell_n^m)_{0\leq n<m}$ are dence in X, then $(\ell_n^m)_{0\leq n<m}$ is strongly ergodic. of g-nonautonomous discrete dynamical systems.

Proof

Suppose that S_1 and S_2 are two non-empty open subsets of X. Since the minimal points of ℓ_n^m are dense in $X\ \forall\ n < m$, we can pick $s \in S_1 \cap AP(\ell_n^m)$ and $t \in S_2 \cap AP(\ell_n^m)$, $\varepsilon > 0$ meaning that $B(s, \varepsilon) \subset S_1$, $B(t, \varepsilon) \subset S_2$

where $B(a, \varepsilon) = \{b \in X: d(a, b) < \varepsilon\}$. by $s, t \in AP(\ell_n^m)$, we have that $J_s = \{m \in \mathbb{Z}_+: \ell_n^m(s) \in B\left(s, \frac{\varepsilon}{2}\right)\}$ and $J_t = \{m \in \mathbb{Z}_+: \ell_n^m(t) \in B\left(t, \frac{\varepsilon}{2}\right)\}$ are Syndetic, which implies that there are $W_1, W_2 \in \mathbb{N}$ meaning that $[w, w + W_1] \cap J_s \neq \emptyset$and $[w, w + W_2] \cap J_t \neq \emptyset$ for each $w \in \mathbb{N}$. Let $W = \max\{W_1, W_2\}$. There exists $\gamma > 0$ such that $d(s, t) < \gamma$ implies $d((\ell_n^m)^i(s), (\ell_n^m)^i(t)) < \frac{\varepsilon}{2}$, for $i = 0,1,2, \dots, W$, by continuity and compactness. since ℓ_n^m has the fitting shadowing property, for $\frac{\gamma}{2} > 0$, there are γ_1, γ_2 such that $0 < \gamma_1 < \gamma_2 < \frac{\gamma}{2}$ meaning that for each γ_1-fitting pseudo-orbit is $\frac{\gamma}{2}$- shadowing in fitting in g-NDS by using some point in X.

pick $W_0 \in \mathbb{N}$ meaning that if $m \in \mathbb{Z}$ is a large enough number then $\frac{3D}{m\,W_0} < \gamma_1$ and $\frac{3D}{m^2 W_0} < \gamma_2$, where $D = diam\,(x)$, that is, $D = \max\{d(s,t): s,t \in X\}$, we define the $2W_0$ periodic sequence $\{z_i\}_{i=0}^{\infty}$ with

$$z_0 = s, \quad z_1 = \ell_n^m(s), .. \quad , z_{W_0-1} = (\ell_n^m)^{W_0-1}(s),$$
$$z_{W_0} = t, \quad z_{W_0+1} = \ell_n^m(t), .. \quad , ..., z_{2W_0-1} = (\ell_n^m)^{W_0-1}(t).$$

For $m \geq W_0$ and $0 \leq r < +\infty$

$$\sum_{i=0}^{m-1} d(\ell_j(z_{i+r}), z_{i+r+1}) < \frac{[\frac{m}{W_0}] \times \frac{3D}{m^2}}{W_0} < \frac{3D}{mW_0} \leq \gamma_1.$$

Therefore, this means that $\{z_i\}_{\boldsymbol{i=0}}^{\infty}$ is a periodic γ_1-fitting pseudo orbit of ℓ, it is possible that it is γ_2-shadowing in fitting by some point $z \in X$, which means

$$\lim_{m\to\infty} \sup \sum_{\boldsymbol{i=n}}^{m} d((\ell_n^m)^i(z), z_i) < \frac{\gamma}{2} ..... (3.11)$$

Claim

(1) there is an infinite $\boldsymbol{i} \in \mathbb{N}$ meaning that

$z_i \in \{s, \ell_n^m(s), \dots, (\ell_n^m)^{W_0-1}(s)\}$ and $d((\ell_n^m)^i(z), z_i) < \gamma$.

(2) there is an infinite $i \in \mathbb{N}$meaning that

$z_i \in \{t, \ell_n^m(t), \ldots, (\ell_n^m)^{W_0-1}(t)\}$ and $d((\ell_n^m)^i(z), z_i) < \gamma$.

To Proof (1): On the contrary, suppose there is $L \in \mathbb{N}$, meaning that if $z_i \in \{s, \ell_n^m(s), \ldots, (\ell_n^m)^{W_0-1}(s)\}$, then $d((\ell_n^m)^i(z), z_i) \geq \gamma$. For every $i \geq L$, we have

$$\lim_{m\to\infty} \sup \sum_{i=n}^{m} d((\ell_n^m)^i(z), z_i) \geq \frac{\gamma}{2},$$

Which contracts (3.11), as a result, the assertion is correct.

The Proof (2) is equivalent to proof (1).

By using a claim, there are $i_0 > W_0$, $\quad 0 \leq J_0 \leq W_0 - 1$

$$k_0 > i_0 + W, \quad 0 \leq I_0 \leq W_0 - 1$$

Therefore

$$z_{i_0} = (\ell_n^m)^{\boldsymbol{J_0}}(s), \quad d((\ell_n^m)^{i_0}(z), z_{i_0}) < \gamma \quad (3.12)$$
$$z_{k_0} = (\ell_n^m)^{\boldsymbol{I_0}}(t), \quad d((\ell_n^m)^{k_0}(z), z_{k_0}) < \gamma \quad (3.13)$$

Since $[J_0, J_0 + W] \cap J_s \neq \emptyset$. There are $0 \leq i,\ I \leq W$ such that

$$(\ell_n^m)^{J_0+J}(s) \in B\left(s, \frac{\varepsilon}{2}\right), \ (\ell_n^m)^{I_0+I}(t) \in B\left(t, \frac{\varepsilon}{2}\right).$$

By the Formula (3.12) and (3.13), we get

$$d((\ell_n^m)^{i_0+J}(z), (\ell_n^m)^{J_0+J}(s)) < \frac{\varepsilon}{2}$$

$$d((\ell_n^m)^{k_0+I}(z), (\ell_n^m)^{I_0+I}(t)) < \frac{\varepsilon}{2}$$

Thus $\quad (\ell_n^m)^{i_0+J}(z) \in B(s, \varepsilon) \subset S_1, \ d((\ell_n^m)^{k_0+I}(z) \in B(t, \varepsilon) \subset S_2$.

Let $w_0 = k_0 + I - i_0 - J > 0$, it following that

$S_1 \cap (\ell_n^m)^{-w_0}(S_2) \neq \emptyset$, since S_1, S_2 are arbitrary, ℓ_n^m is topological transitive. we write $F = S_1 \cap (\ell_n^m)^{-w_0}(S_2) \neq \emptyset$,

consequently, there is $q \in AP(\ell_n^m) \cap F$.

Let $k = \{m \in \mathbb{Z}_+ : (\ell_n^m)(q) \in W\}$, then k is syndetic.

When $c \in k, F \cap (\ell_n^m)^{-c}(F) \neq \emptyset$,

since $\emptyset \neq S_1 \cap (\ell_n^m)^{-w_0}(S_2) \cap (\ell_n^m)^{-c}(S_1 \cap (\ell_n^m)^{-w_0}(S_2)) \subset$

$S_1 \cap (\ell_n^m)^{-(w_0+c)}(S_2)$.

$N(S_1, S_2) \supset \{w_0 + c : c \in k\}$. Therefore $N(S_1, S_2)$ is syndetic, since S_1, S_2 are arbitrary, ℓ_n^m is strongly ergodic. □

<u>Theorem</u> 3.1.13

Let (X, d) be a compact metric space, and $\ell_n : X \to X$ be an uniform continuous map,$\forall n \in \mathbb{N}$. if $(\ell_n^m)_{0 \le n < m}$ have the fitting shadowing property of g-nonautonomous discrete dynamical systems and minimal points of $(\ell_n^m)_{0 \le n < m}$are dence in X, then $(\ell_n^m)_{0 \le n < m}$ is totally strongly ergodic of g-nonautonomous discrete dynamical systems.

Proof

For any $k \in \mathbb{N}$, By Theorem 3.1.5 $(\ell_n^m)^k, \forall\, k > 0$. have an fitting shadowing property. It is well-common knowledge that $AP(\ell_n^m) = AP((\ell_n^m)^k)$. By using Theorem 3.1.12 for $(\ell_n^m)^k$, we are able to determine that the function is strongly ergodic. Because of this $(\ell_n^m)^k$is totally strongly ergodic. □

we generalize Lemma 2.1.3 in [9] for g-nonautonomous discrete dynamical system

Lemma 3.1.14

Let (X, d) be a compact metric space, and $\ell_n: X \rightarrow X$ is an uniform continuous map ,$\forall n \in \mathbb{N}$. The map $(\ell_n^m)_{0 \leq n < m}$If and only if it is chain transitive, it is completely chain mixing.

Lemma 3.1.15

Let (X, d) be a compact metric space, and $\ell_n: X \rightarrow X$ is an uniform continuous map ,$\forall n \in \mathbb{N}$. If $(\ell_n^m)_{0 \leq n < m}$is surjective and has the fitting shadowing property in g-nonautonomous discrete dynamical systems $\forall\, n < m$, then $(\ell_n^m)_{0 \leq n < m}$ is chain mixing.

Proof

It was proved in Theorem 3.1.5 that if ℓ_n^m have the fitting shadowing property, then so does $(\ell_n^m)^k, \forall\, k > 0$, By using Lemma 3.1.14 if $(\ell_n^m)^k$ is chain transitive for $\forall\, k > 0$, then there's chain mixing. As a result, proving that ℓ_n^m is chain transitive is sufficient. It does not affect the rule of the theory if we suppose that diam $(X) \leq 1$.

Fix any $\varepsilon > 0$ and $x, y \in X$. Let δ be provided by an fitting shadowing property for $\varepsilon/2$. Suppose that $k_0 \geq 2$ be meaning that $2/k_0 < \delta$ and suppose that $k_i = 2^i k_0$, for $i \geq 1$. For $h = 0, 1, 2, \ldots$ we use surjectivity to fix a point $y_h \in (\ell_n^m)^{k_{2h+1}+1}(x)$ and define a sequence

$$\varphi = \{x, \ell_n^m(x), \ldots, (l_n^m)^{k_0-1}(x), y_0, \ell_n^m(y_0), \ldots, (\ell_n^m)^{k_1-1}(y_0), y, \ell_n^m(x),$$

$$\ldots, (\ell_n^m)^{k_2-1}(x), y_1, \ell_n^m(y_1), \ldots, (\ell_n^m)^{k_3-1}(y_1), \ldots, x, \ell_n^m(x), \ldots$$

$$, (\ell_n^m)^{k_{2h}-1}(x), x_h, \ell_n^m(y_h), \ldots, (\ell_n^m)^{k_{2h+}-1}(y_h), \ldots.\}$$

Let $l(h) = (2^h - 1)k_0$. According to its definition of φ, we get, $\varphi_{l(\mathbf{2i+2})-1} = y$ and $\varphi_{l(2i+1)} = y_i$, for each $i = 0, 1, 2, ..$ Therefore $\varphi_{l(i)}, \ldots, \varphi_{l(\mathbf{i+1})-\mathbf{1}}$ is the initial segment of length m_h for the orbit of x if h is even, and of y_w, where $w = (h-1)/2$, if h is an odd. Note that if we correct any $i \geq 0$ then in the sequence $\varphi_i, \ldots, \varphi_{i+k_0}$ There is only one possible position h with $d((\ell_n^m)^i(\varphi_h), \varphi_{h+1}) > 0$. Therefore for every $w > k_0$ we have

$$\sum_{h=i}^{i+w-1} d((\ell_n^m)^i(\varphi_h), \varphi_{h+1}) \leq \left(\frac{w}{k_0} + 1\right) diam(X) \leq \frac{w + k_0}{k_0} \leq \frac{2}{k_0} < \delta,$$

which shows that φ is an δ-fitting-pseudo-orbit. Let $z \in X$ be a point which ε-shadows φ on fitting. This implies that there exist are $p, q \in N$ meaning that $d\big((\ell_n^m)^p(y), (\ell_n^m)^q(z)\big) < \varepsilon$ and $r, s, t \in N$ such that $q < l(2t) \leq r$, $s < m_{2t+1}$, and $d((\ell_n^m)^r(x), (\ell_n^m)^s(y_w)) < \varepsilon$, for otherwise we would have

$$\sum_{i=n}^{l(h)-1} d((\ell_n^m)^i(z),\varphi_i) \geq \frac{1}{l(h)} \sum_{i=n}^{l(h)-1} d\big((\ell_n^m)^i(z),\varphi_i\big) \geq \frac{2^{h-1}k_0}{(2^h-1)k_0}\varepsilon \geq \frac{\varepsilon}{2}$$

for each adequately large $h \in N$ of some fixed parity (even or odd).

We come to the conclusion that for some p, q, r, s, t with $q < r-1$ and $s < m_{2t+1}$the sequence

$x, \ell_n^m(x), \dots, (\ell_n^m)^{p-1}(x), (\ell_n^m)^q(z), \dots, (\ell_n^m)^{r-1}(z), (\ell_n^m)^s(y_i), \dots, y$

is a finite ε-pseudo-orbit from x to y □.

Theorem 3.1.16

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous map,$\forall n \in \mathbb{N}$. If $(\ell_n^m)_{0\leq n<m}$ has the asymptotic fitting shadowing property of g-nonautonomous discrete dynamical systems $\forall\, n < m$, then $(\ell_n^m)_{0\leq n<m}$is chain transitive map.

Proof.

Suppose that x, y are two distinct points in X and δ is a positive number. It is sufficient to demonstrate that x leads to y via δ - -chain. First, we construct a sequence $\{w_i\}_{i=0}^{\infty}$ as following. Suppose that $w_0 = x,\;\; w_1 = y,\;\; w_2 = x,\;\; w_3 = y,\;\; w_4 = x,\;\; \ell_n^1(x), y_{-1},\;\; w_5 = y, ...$

$$.., w_{2^k} = x, \ell_n^1(x), \dots, \ell_n^{2^{k-1}-1}(x), y_{-2^{k-1}+1}, \dots., y_{-1}, \dots, w_{2^{k+1}-1} = y.$$

For all $0 \leq n < m \;\; \in \mathbb{N}$,

where $\ell_{-i+1}\big(y_{-j}\big) = y_{-j+1}$, for each $j > 0$ and $y_0 = y$.

for $2^k \leq n < 2^{k+1}$,

$$\lim_{m\to\infty}\sum_{i=0}^{m-1} d\big(\ell_j(w_i), w_{i+1}\big) < \frac{2(k+1)D}{2^k},$$

where D is the diameter of X, that is, $D = max\{d(x,y): x, y \in X\}$.

Hence

$$\lim_{\boldsymbol{m}\to\infty}\sum_{\boldsymbol{i=0}}^{\boldsymbol{m-1}} d\big(\ell_j(w_i), w_{i+1}\big) = 0$$

That is, $\{x_i\}_{\boldsymbol{i=0}}^{\infty}$ is an asymptotic-average pseudo orbit of ℓ_n^m. Since ℓ_n^m have the A.A.S.P, there is a point z in X meaning that

$$\lim_{m\to\infty}\sum_{i=n}^{m-1} d\big((\ell_n^m)^i(z), w_i\big) = 0 \qquad (3.14)$$

For the positive number δ above, by the continuity of ℓ_n^m, there is $\eta \in (0,\delta)$ meaning that $d(u,v) < \eta$ implies $d\big(\ell_n^m(u), \ell_n^m(v)\big) < \delta$, for all $u, v \in X$. We now hase the following claim.

Claim. (1) There are an infinite number of positive integers j meaning that

$w_{m_j} \in \{x, \ell_n^1(x), \ldots, \ell_n^{2j-1}(x)\}$ and $d(\ell_n^{mj}(z), w_{mj}) < \eta$;

(2) There are an infinite number of positive integers l meaning that $w_{m_l} \in \{y_{-2^l+1}, \ldots, y_{-1}, y\}$ and $d\big(\ell_n^{ml}(z), w_{m_l}\big) < \eta$.

To Prove Claim We prove merely the conclusion and do so without loss of generality.

(1) Assume on the contrary that for all integers $k > M$ meaning that there is a positive integer M, whenever $w_i \in \{x, \ell_n^1(x), \ldots, \ell_n^{2^k-1}(x)\}$,

it is obtained that $d((\ell_n^m)^i(z), w_i) \geq \eta$. Then it would be obtained that

$$\lim_{m\to\infty} \inf \sum_{i=0}^{m-1} d((\ell_n^m)^i(z), w_i) \geq \frac{\eta}{2}$$

Which is inconsistent with (3.14). This demonstrates that the result (1) is correct.

According to the claim, we can select two positive integers j_0 and l_0 meaning that

$m_{j_0} < m_{l_0}$ and $w_{m_{j_0}} \in \{x, \ell_n^1(x), \dots, \ell_n^{2^{j_0-1}}(x)\}$ and

$d((\ell_n^m)^{m_{j_0}}(z), w_{m_{j_0}}) < \eta$;

$w_{m_{l_0}} \in \{y_{-2^{l_0}+1}, \dots, y_{-1}, y\}$ and $d((\ell_n^m)^{m_{l_0}}(z), w_{m_{l_0}}) < \eta$. It may be

assumed $w_{m_{j_0}} = \ell_n^{j_1}(x)$ for some $j_1 > 0$; $w_{m_{l_0}} = y_{-l_1}$ for some $l_1 > 0$

This gives a δ-chain from x to y: $x, \ell_n^1(x), \dots, \ell_n^{j_1}(x) = w_{m_{j_0}}, \ell_n^{m_{j_0+1}}(z), \ell_n^{m_{j_0+2}}(z), \dots, \ell_n^{m_{l_0-1}}(z),\ w_{m_{l_0}} = y_{-l_1}, y_{-l_1+1}, \dots, y.$

Thus, ℓ_n^m is chain transitive. □

<u>Lemma</u> 3.1.17

Assume that (X, d) and (Y, d') be a compact metric spaces, $\ell_n: X \to X$ and $g_n: Y \to Y$ be uniform continuous maps. If $(\ell_n^m)_{0\leq n<m}$ and $(g_n^m)_{0\leq n<m}$ have the fitting shadowing property in g-nonautonomous discrete dynamical systems and dense minimal points, then

(1) $\ell_n^m \times g_n^m$ has the fitting shadowing property in g-nonautonomous

discrete dynamical systems, $\forall\ 0 \leq n < m$.

(2) each of $\ell_n : X \to X$ and $g_n : Y \to Y$ that are weakly disjoint, that is $\ell_n^m \times g_n^m$ is topological transitive, $\forall\ 0 \leq n < m$.

(3) $(\ell_n^m)_{0\leq n<m}$ is very weakly mixing and totally strongly ergodic.

Proof

Since ℓ_n^m and g_n^m every has a fitting shadowing property and dense minimal points, by Theorem 3.1.12 : each of ℓ_n^m and g_n^m are strongly ergodic, and By Definition 1.2.11 so that ℓ_n^m and g_n^m each are topological transitive,

To prove (1) : by Proposition 3.1.3 and Proposition 3.1.11 ,$\ell_n^m \times g_n^m$has the fitting shadowing property and dence minimal points.

To prove (2) : by Theorem 3.1.12 and (1), $\ell_n^m \times g_n^m$ is topological ergodic, hence it's a topological transitive .

To prove (3) : frome (2), that $\ell_n^m$is weakly mixing. by Theorem 3.1.13 ℓ_n^m is totally strongly ergodic. □

3.2 The h-Fitting Shadowing Property in g-Nonautonomus Discrete Dynamical Systems

In this section, we present the concept of the fitting shadowing property for homeomorphism maps in g-nonautonomous discrete dynamical systems (h-fitting shadowing property), also we define a new concept which are the h-fitting Lyapunov stable points, the h- fitting sensitive points in g-nonautonomous discrete dynamical systems.

<u>Definition</u> 3.2.1

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous homeomorphism map $\forall n \in \mathbb{N}$. For $\delta > 0$ a sequence $\{x_i\}_{i\in\mathbb{Z}}$ in X is called a **$\boldsymbol{\delta}$-h-fitting pseudo-orbit in g-nonautonomous discrete dynamical systems** of ℓ, if

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d\big(\ell_j(x_i), x_{i+1}\big) < \delta.$$

A maps $(\ell_n^m)_{0\le n<m}$is say to has the **h-fitting shadowing property in g-nonautonomous discrete dynamical systems** if for each $\boldsymbol{\varepsilon} > 0$, there is $\delta > 0$, meaning that all δ-h- fitting pseudo-orbit $\{x_i\}_{\boldsymbol{i}\in\mathbb{Z}}$ is ε-shadowing in fitting of some point $z \in X$, that is

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d((\ell_n^m)^i(z), x_i) \le \varepsilon.$$

<u>Lemma</u> 3.2.2

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous homeomorphism map $\forall n \in \mathbb{N}$. For each $\alpha > 0$ there exists $\beta > 0$ meaning that if $\{w_i\}_{i\in\mathbb{Z}}$ is a β-h-fitting pseudo-orbit of ℓ then $\{u_i\}_{i\in\mathbb{Z}}$ is a α-h- fitting pseudo-orbit for ℓ^{-1}, such that $u_i = w_{-i}$, for every $i \in \mathbb{Z}$.

Proof:

Assume that $\alpha > 0$, since ℓ^{-1} is continuous map then there exists $0 < \alpha' < \alpha$ meaning that if $d(u, w) < \alpha'$ then $d(\ell^{-1}(u), \ell^{-1}(w)) < \alpha$

for all $u, w \in X$.

Suppose that $\{w_i\}_{i\in\mathbb{Z}}$ is an $\frac{(\alpha')^2}{D}$ –h-fitting pseudo-orbit for ℓ, where D is diameter of X.

Suppose that $S = \{i \in \mathbb{Z} \mid d(\ell_j(w_i), w_{i+1}) \geq \alpha' \}$,

$S_m = S \cap \{-m+1, \dots, 0, \dots, m-1\}$,

$K_m = S^c \cap \{-m+1, \dots, 0, \dots, m-1\}$,

And $\lim_{m\to\infty} \sup card\,(S_m) = \text{I}$.

Since

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d(\ell_j(w_i), w_{i+1}) =$$

$$\lim_{m\to\infty} \sup(\sum_{i=0}^{m-1} d(\ell_j(w_i), w_{i+1}) + \sum_{i=1-m}^{0} d(\ell_j(w_i), w_{i+1})$$

$$< \frac{(\alpha')^2}{D}$$

Then,

$$\lim_{m\to\infty} \sup \sum_{i\in S_m} \alpha' \leq \lim_{m\to\infty} \sup \sum_{i\in S_m} d(\ell_j(w_i), w_{i+1}) < \frac{(\alpha')^2}{D}.$$

Also Ɩ $\alpha' < \frac{(\alpha')^2}{D}$ and is implying that Ɩ $< \frac{\alpha'}{D}$. Consider the following sequence: $\{u_i\}_{i\in\mathbb{Z}}$, in which by case hypothesis $u_i = w_{-i}$. Then:

$$\sum_{i=1-m}^{m-1} d\big((\ell_j)^{-1}(u_i), u_{i+1}\big) = \sum_{i\in S_m} d\big((\ell_j)^{-1}(u_i), u_{i+1}\big) +$$

$$\sum_{i\in K_m} d\big((\ell_j)^{-1}(u_i), u_{i+1}\big)$$

Note that :

$u_{i+1} = (\ell_j)^{-1}\left(\ell_j(u_{i+1})\right) = (\ell_j)^{-1}(\ell_j(w_{-i-1}))$ and $(\ell_j)^{-1}(u_i) = \ell^{-j}(w_{-i})$.

So, $$d\big((\ell_j)^{-1}(u_i), u_{i+1}\big) = d((\ell_j)^{-1}(w_{-i}), (\ell_j)^{-1}(\ell_j(w_{-i-1}))$$

$$= d((\ell_j)^{-1}\left(\ell_j(w_{-i-1})\right)(\ell_j)^{-1}(w_{-i})).$$

Therefore;

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d((\ell_j)^{-1}(u_i), u_{i+1})$$

$$\leq \lim_{m\to\infty} \sup \sum_{i\in S_m} d\big((\ell_j)^{-1}(u_i), u_{i+1}\big) +$$

$$\lim_{m\to\infty} \sup \sum_{i\in K_m} d((\ell_j)^{-1}(u_i), u_{i+1})$$

$$\leq \text{I}D + \lim_{m\to\infty} \sup \sum_{i\in K_m} d((\ell_j)^{-1}\left(\ell_j(w_{-i-1}), (\ell_j)^{-1}(w_{-i})\right).$$

$$\leq \alpha' + \alpha < 2\alpha$$

So, $\{u_i\}_{i\in\mathbb{Z}}$ is 2α-h-pseudo-orbit for ℓ^{-1}. □

Proposition 3.2.3

Let (X, d) be a compact metric spacem and $\ell_n: X \to X$ be an uniform continuous homeomorphism map $\forall n \in \mathbb{N}$, if $(\ell_n^m)_{0\leq n<m}$ has the h-fitting shadowing property of g- nonautonomous discrete dynamical systems, then $(\ell_n^m)^{-1}$has the h-fitting shadowing property of g- nonautonomous discrete dynamical systems, $\forall\, m < n \leq 0$.

Proof

Given $\varepsilon > 0$, since ℓ_n^m have the h-fitting shadowing property there is also $\theta > 0$ meaning that for any θ-h-fitting pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}}$, we may be findz $z \in X$ which

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d((\ell_n^m)^i(z), x_i) < \varepsilon$$

By Lemma 3.2.2 there exists $\theta' > 0$, meaning that if $\{z_i\}_{i\in\mathbb{Z}}$ is θ'-h-fitting pseudo-orbit for ℓ^{-1}, then $\{x_i\}_{i\in\mathbb{Z}}$ is a θ'-h-fitting pseudo-orbit for ℓ Where $x_i = z_{-i}$, for every $i \in \mathbb{Z}$. Therefore if $\{z_i\}_{i\in\mathbb{Z}}$ is a θ'-h-fitting pseudo-orbit for ℓ^{-1}, then there exists $z \in X$, such that

$$\lim_{m\to\infty}\sup\sum_{\boldsymbol{i=1-m}}^{m-1} d\big((\ell_n^m)^i(z), x_i\big) < \varepsilon$$

This infers the following

$$\lim_{m\to\infty}\sup\sum_{i=1-m}^{m-1} d\big(((\ell_n^m)^{-1})^{-i}(z), z_{-i}\big) < \varepsilon$$

So, $(\ell_n^m)^{-1}$has h-fitting shadowing property. □

Proposition 3.2.4

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous homeomorphism map $\forall n \in \mathbb{N}$, If $(\ell_n^m)_{0\le n<m}$ have the h-fitting shadowing property in g- nonautonomous discrete dynamical systems, then it has fitting shadowing property in g- nonautonomous discrete dynamical systems.

Proof:

Assume that be an arbitrary positive number and $\gamma > 0$ be an ε-modulus of h- fitting shadowing property. Let $\{x_i\}_{i=-\infty}^{\infty}$ is an γ-fitting pseudo orbit, As a result, there is a natural number $M = M(\gamma) > 0$ meaning that for every $m \ge M$ and $k \in \mathbb{Z}$

$$\sum_{i=0}^{m-1} d\big(\ell_j(x_{i+k}), x_{i+k+1}\big) < \frac{\gamma}{2}$$

This implies that

$$\sum_{\boldsymbol{i=0}}^{\boldsymbol{2m-1}} d\big(\ell_j(\boldsymbol{x}_{i-m}), \boldsymbol{x}_{i-m+1}\big) < \frac{\gamma}{2} < \gamma$$

for each $m \geq M$. Then,

$$\lim_{m\to\infty} \sup \sum_{\boldsymbol{\iota}=-\boldsymbol{m}}^{\boldsymbol{m}} d(\ell_j(\boldsymbol{x}_\iota), \boldsymbol{x}_{\iota+1}) < \gamma$$

Therefore; $\{\boldsymbol{x}_i\}_{\boldsymbol{i}=-\infty}^{\infty}$ is an γ-h-fitting pseudo-orbit in g-NDS. By h-fitting shadowing property there exists $z \in X$ meaning that

$$\lim_{m\to\infty} \sup \sum_{i=-m}^{m} d((\ell_n^m)^i\,(z), x_i) < \frac{\varepsilon}{2}$$

Then

$$\lim_{m\to\infty} \sup \sum_{i=0}^{m-1} d((\ell_n^m)^i(z), x_i) \leq 2 \lim_{m\to\infty} \sup \sum_{i=-m}^{m} d((\ell_n^m)^i(z), x_i)$$

$$< 2 \times \frac{\varepsilon}{2}$$

$$< \varepsilon.$$

Hence ℓ_n^m has the fitting shadowing property. □

Definition 3.2.5

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous homeomorphism $\forall n \in \mathbb{N}$, a point $x \in A \subset$ X is said to be **h-Lyapunov stable point for $(\boldsymbol{\ell_n^m})_{\boldsymbol{0 \leq n < m}}$in $\boldsymbol{A}$ in g-nonautonomous discrete dynamical systems,** if for any $\varepsilon > 0$, such that, $\exists\ \delta > 0$, meaning that for every $z \in A$, satisfying $d(x, z) < \delta$, we have

$d\big(\ell_n^m(x), \ell_n^m(z)\big) < \varepsilon$, for all $0 \leq n < m \in \mathbb{Z}_+$.

<u>Definition</u> 3.2.6

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous homeomorphism $\forall n \in \mathbb{N}$, a point $x \in A \subset X$ is said to be **h-fitting Lyapunov stable point for $(\boldsymbol{\ell_n^m})_{0 \leq n < m}$ in A in g-nonautonomous discrete dynamical systems**, if for any $\varepsilon > 0$, such that, $\exists\ \delta > 0$, meaning that for every $z \in A$ satisfying $d(x, z) < \delta$, we have

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d((\ell_n^m)^i(x), (\ell_n^m)^i(z)) < \varepsilon.$$

Otherwise, we refer to this point as **h-fitting sensitive in A in g-nonautonomous discrete dynamical systems**.

<u>Proposition</u> 3.2.7

Let (X, d) be a compact metric space, and $\ell_n: X \to X$ be an uniform continuous homeomorphism $\forall n \in \mathbb{N}$, and B be a subset of X. every Lyapunov stable point for $(\ell_n^m)_{0 \leq n < m}$ in B of g- nonautonomous discrete dynamical systems $\forall\ n < m$, is an h-fitting Lyapunov stable point for $(\ell_n^m)_{0 \leq n < m}$ in B of g- nonautonomous discrete dynamical systems .

Proof:

Suppose $x \in B$ is Lyapunov stable point for ℓ_n^m in B, and $\varepsilon > 0$. Then there exists a number $\gamma > 0$, meaning that for each $z \in B$ satisfying $d(x, z) < \gamma$, implies $d((\ell_n^m)^i(x), (\ell_n^m)^i(z)) < \frac{\varepsilon}{2m}$, where m is a large integer number and for all $i \in \mathbb{Z}$

Therefore ;

$$\sum_{i=1-m}^{m-1} d((\ell_n^m)^i(x),(\ell_n^m)^i(z)) < 2m \times \frac{\varepsilon}{2m} < \varepsilon$$

This means that

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d(\,(\ell_n^m)^i(x),(\ell_n^m)^i(z)) < \varepsilon.$$

Thus $\boldsymbol{x}$ is an h-fitting Lyapunov stable point for ℓ_n^m in B in g-NDS. □

Proposition 3.2.8

Let (X,d) be a compact metric space, and $\ell_n\colon X \to X$ be an uniform continuous homeomorphism map $\forall n \in \mathbb{N}$, and A a subset of X, if u is an h-fitting sensitive point for $(\ell_n^m)^k$ in A in g-nonautonomous discrete dynamical systems for every a nonzero integer number k, and $\forall\ 0 \le n < m$, then u is an h-fitting sensitive point for $(\ell_n^m)_{0\le n<m}$ in A in g- nonautonomous discrete dynamical systems.

Proof:

By Definition 3.2.6 , we can find $\varepsilon > 0$ such that for any $\gamma > 0$ there exists $v \in X$ such that $d(u,v) < \gamma$ and

$$\lim_{m\to\infty} \sup \sum_{i=1-m}^{m-1} d(((\ell_n^m)^k)^i(u),((\ell_n^m)^k)^i(v)) < \varepsilon$$

Obviously, we have

$$\frac{1}{|k|}\sum_{i=1-m}^{m-1} d((\ell_n^m)^{ki}(u),(\ell_n^m)^{ki}(v)) \le \frac{1}{|k|}\sum_{i=1-m}^{km-1} d((\ell_n^m)^i(u),((\ell_n^m)^i(v)).$$

Hence,

$$\frac{1}{|k|}\left[\lim_{m\to\infty}\sup\sum_{i=1-m}^{m-1} d(((\ell_n^m)^k)^i(u),((\ell_n^m)^k)^i(v))\right]$$

$$\leq \lim_{m\to\infty}\sup\frac{1}{|k|}\sum_{i=m-1}^{km-1} d((\ell_n^m)^i(u),(\ell_n^m)^i(v))$$

So,

$$\lim_{m\to\infty}\sup\frac{1}{|k|}\sum_{i=1-km}^{km-1} d((\ell_n^m)^i(u),(\ell_n^m)^i(v)) < \frac{\varepsilon}{k}$$

Thus, u is an h-fitting sensitive point for $\ell_n^{\boldsymbol{m}}$ in A in g-NDS. □

Conclusion and future work

This chapter is divided into two section , the first section is the Conclusion and the second section about the future work

4.1Conclusion

In this work, we prove the following results :

- $\forall n \in \mathbb{N},\ \ell_n: X \to X$ is transitive map and $\ell_n \overset{u}{\to} \ell$, then $\ell: X \to X$ is transitive map.

- if $(\ell_n^m)_{0 \le n < m}$ is an expansive map if and only if $(\ell_n^m)^{-1}$ is an expansive map.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ and $g_n: Y \to Y$, have P.O.T.O then $(\ell_n^m \times g_n^m)$ has P.O.T.P.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$, has P.O.T.P and uniform conjugate to $g_n: Y \to Y$ then the last has P.O.T.P.

- if$\forall n \in \mathbb{N}\ \ell_n: X \to X$ and $g_n: Y \to Y$ are surjective and have dence minimal points then so does the product $(\ell_n^m \times g_n^m)$.

- if $\forall n \in \mathbb{N}$, $\ell: X \to X$ is transitive map non sensitive dependence on initial condition then there exists $\ell_n: X \to X$ is transitive and all it is points are periodic but non sensitive dependence on initial condition.

- there exists $\ell_n: X \to X$ is minimal then it is neither sensitive nor equicontinuous.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ is an inverse equicontinuous and weakly expanding small distance then it has weakly locally expanding.

- if $\forall n \in \mathbb{N}$, $\ell_n: X \to X$,is locally expanding then it has h-shadowing property.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ is w-expansive if and only if $(\ell_n^m)^k, \forall k > 0$

  is w- expansive $\forall\, n < m$.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ and $g_n: Y \to Y$ are possesses fitting shadowing then $(\ell_n^m \times g_n^m)$ has the fitting shadowing property $\forall\, n < m$.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ has fitting shadowing then $(\ell_n^m)^k$ has fitting shadowing $\forall\, n < m$.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ and $g_n: Y \to Y$ are possesses asymptotic fitting shadowing then $(\ell_n^m \times g_n^m)$ has an asymptotic fitting shadowing property $\forall\, n < m$.

- If $\forall n \in \mathbb{N}$, $\ell_n: X \to X$ has fitting shadowing and AP(ℓ_n^m) is dense then ℓ_n^m is strongly $\forall\, n < m$.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ has fitting shadowing then it is chain mixing.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ has h- if it possesses fitting shadowing, then it possesses the property of fitting shadowing $\forall\ n < m$.

- if $\forall n \in \mathbb{N}\ \ell_n: X \to X$ unified homeomorphism, with A constituting a subset of X, if x is h-fitting sensitive point for $(\ell_n^m)^k, \forall\ k > 0$. then x is h-fitting sensitive point for ℓ_n^m , $\forall\ n < m$.

- if $\forall n \in \mathbb{N}, \ell_n: X \to X$ only when uniform homeomorphism each point of Lyapunov stability of ℓ_n^m in B is h-fitting point of Lyapunov stability of ℓ_n^min B.

4.2 Future Work

In future work, we will try to research and work on the following**:**

1-We will study other chaotic properties and generalize them to g-nonautonomous discrete dynamical systems, and study how it is transmitted from g-nonautonomous discrete dynamical systems to autonomous discrete dynamical systems.

2-We study the difference between some chaotic properties and concepts between g-nonautonomous discrete dynamical systems And autonomous discrete dynamical systems.

3-We study other types of shadowing and generalize them in the g-nonautonomous discrete dynamical systems.

4-We study stability and try to generalize it in g-nonautonomous discrete dynamical systems.

Reference

References

[1] Caraballo, T., & Han, X. (2017). *Applied nonautonomous and random dynamical systems: applied dynamical systems*. Springer.

[2] Kolyada, S., & Snoha, L. (1996). Topological entropy of nonautonomous dynamical systems. *Random and Computational Dynamics*, *4*(2), pp 205-233

[3] Miralles, A., Murillo-Arcila, M., & Sanchis, M. (2018). Sensitive dependence for nonautonomous discrete dynamical systems. *Journal of Mathematical Analysis and Applications*, *463*(1), pp 268-275.

[4] Vasisht, R., & Das, R. (2022). Generalizations of expansiveness in non-autonomous discrete systems. *Bulletin of the Iranian Mathematical Society*, *48*(2), pp 417-433.

[5] Niu, Y. (2011). The average-shadowing property and strong ergodicity. *Journal of Mathematical Analysis and Applications*, *376*(2), pp 528-534.

[6] Kulczycki, M., & Oprocha, P. (2010). Exploring the asymptotic average shadowing property. *Journal of Difference Equations and Applications*, *16*(10), pp1131-1140.

[7] Carvalho, B., & Cordeiro, W. (2016). N-expansive homeomorphisms with the shadowing property. *Journal of Differential Equations*, *261*(6), pp3734-3755.

[8] Rasuli, H., & Memarbashi, R. (2017). On the relation of shadowing and expansivity in nonautonomous discrete systems. *Analysis in Theory and Applications*, *33*(1), pp11-19.

[9] AL-Sharaa, I. M., & AL-Joboury, R. S. (2016). Fitting Shadowing Property. *Journal of Babylon University*, 24.(5)

[10] Hussein, B. A., & Al-Sharaa, I. (2022). Some chaotic properties for g-non autonomous systems. *Journal of Positive School Psychology*, 6 (4) pp 4641-4647.

[11] Thakkar, D., & Das, R. (2014). On nonautonomous discrete dynamical systems. *International Journal of Analysis*, *2014 (538691), pp 6*-12.

[12] Vasisht, R., & Das, R. (2021). Specification and shadowing properties for non-autonomous systems. *Journal of Dynamical and Control Systems*, 28, pp 481-492

[13] Thakkar, D., & Das, R. (2014). Topological stability of a sequence of maps on a compact metric space. *Bulletin of Mathematical Sciences*, *4*, pp 99-111.

[14] Kwietniak, D., & Oprocha, P. (2012). A note on the average shadowing property for expansive maps. *Topology and its Applications*, *159*(1), pp 19-27.

[15] Mangang, K. B. (2017). Equicontinuity and Sensitivity of Sequence Dynamical Systems. *Global Journal of Pure and Applied Mathematics*, *13*(8), pp 4223-4228.

[16] AL-Sharaa, I. M., & AL-Joboury, R. S. (2018). Asymptotic fitting shadowing property. *Albahir journal*, *7*, pp13-14

[17] Glasner, E., & Weiss, B. (2000). Locally equicontinuous dynamical systems. In *Colloquium Mathematicum* Polska Akademia Nauk. Instytut Matematyczny PAN..84 (2), pp. 345-361.

[18] Sakai, K. (2010). Positively expansive differentiable maps. *Acta Mathematica Sinica, English Series*, *26*(10), pp1839-1846.

[19] Li, N., & Wang, L. (2020). Sensitivity and chaoticity on nonautonomous dynamical systems. *International Journal of Bifurcation and Chaos*, *30(10*), 2050146.

[20] Huang, Q., Shi, Y., & Zhang, L. (2015). Sensitivity of non-autonomous discrete dynamical systems. *Applied Mathematics Letters*, *39*, pp 31-34.

[21] Lan, Y., & Peris, A. (2018). Weak stability of non-autonomous discrete dynamical systems. *Topology and its Applications*, *250*, pp 53-60.

[22] Salman, M., & Das, R. (2020). Multi-transitivity in non-autonomous discrete systems. *Topology and its Applications*, *278*, 107237.

[23] Shao, H., Shi, Y., & Zhu, H. (2018). On distributional chaos in non-autonomous discrete systems. *Chaos, Solitons & Fractals*, *107*, pp 234-243.

[24] Fatehi Nia, M. (2013). h–Average-Shadowing Property and Chaos. *Journal of Dynamical Systems and Geometric Theories*, *11*(1-2), pp39-49.

[25] Sakai, K. (2003). Various shadowing properties for positively expansive maps. *Topology and its Applications*, *131*(1),pp 15-31.

[26] Shi, Y., & Chen, G. (2009). Chaos of time-varying discrete dynamical systems. *Journal of Difference Equations and applications*, *15*(5), pp 429-449.

[27] Mañé, R. (1979). Expansive homeomorphisms and topological dimension. *Transactions of the American Mathematical Society*, *252*,

pp 313-319.

[28] Jiang, Y., Lu, T., Pi, J., & Anwar, W. (2022). The retentivity of four kinds of shadowing properties in non-autonomous discrete dynamical systems. *Entropy*, *24*(3), 397, pp 26-32 .

[29] Kulczycki, M., Kwietniak, D., & Oprocha, P. (2013). On almost specification and average shadowing properties. *Math.224(2014),pp 241-278*

[30] Jiao, L., Wang, L., & Li, F. (2019). Average shadowing property and asymptotic average shadowing property of linear dynamical systems.

International Journal of Bifurcation and Chaos, *29*(12), 1950170.

[31] Richeson, D., & Wiseman, J. (2008). Chain recurrence rates and topological entropy. *Topology and its Applications*, *156*(2), pp251-261

الخلاصة

الهدف من هذا العمل هو دراسة بعض الخواص الفوضوية في الأنظمة المنفصلة المستقلة الديناميكية -g. خلال هذه الدراسة قدمت بعض الاختلاف بين النظام المنفصل غير المستقل الديناميكي –g والنظام المنفصل المستقل الديناميكي عندما يكونان في الحد الادنى. ووضحنا خاصية الظل في النظام المنفصل غير المستقل –g, وظهرت بعض النتائج لخاصية الظل مع الأنظمة الديناميكية الفوضوية. ومفاهيم جديدة عل سبيل المثال (التوسعة –w في النظام -g NDS, خاصية الظل الملائم في g-NDS,...) قدمت. استخدمت هذه المفاهيم والتعاريف لإثبات العديد من النتائج الجديدة.

وكذلك شرحت انواع الظل اخرى مثل: خاصية الظل -h في g-NDS و خاصية ا الظل الملائم –h في g-NDS . تمت مناقشة بعض النتائج والتعاريف حول التقارب المنتظم لمتتابعة دوال مختلفة ومستمرة في فضاء متري مرصوص.

النتائج الرئيسية التي تم اثباتها في هذه الدراسة:

ليكن (X, d) فضاء متري مرصوص, (للاختصار نكتب X) فان النظام g-NDS عبارة عن زوج حيث (X, ℓ_n^m)

ان ℓ_n^m هي متتابعة من الدوال المستمره $\ell_n: X \to X$ لكل $n \in \mathbb{N}$, والتركيب $\forall\, 0 \leq n < m \in \mathbb{N}, \ell_n^m = \ell_m \circ \ell_{m-1} \circ \cdots \circ \ell_n$, , فانه اثبت :

- اذا كان النظام الديناميكي المنفصل وغير مستقل –g متعديا فانه النظام المنفصل ومستقل متعديا وكذلك بالنسبة (خصائص الحساسية وعدم التساوي و التوسع).
- يوجد نظام الديناميكي المنفصل وغير المستقل –g حد ادنى والذي يكون لا يملك حساسيه ولا خاصية عدم التساوي.
- اذا كان X فضاء متري مرصوص , فاذا كان $\ell_n: X \to X$, $\forall n \in \mathbb{N}$, يملك خاصية الظل الملائم في g-NDS فانه $\underbrace{\ell_n^m \times \ell_n^m \times \ell_n^m \times \ldots \times \ell_n^m}_{k-times}$ يملك خاصية الظل الملائم في .g-NDS

- ليكنX فضاء متري مرصوص و $\ell_n: X \to X$, $\forall n \in \mathbb{N}$, دوال مستمرة بانتظام. فاذا كان لكل $k \in \mathbb{N}$, ل$(\ell_n^m)^k$, ستملك خاصية الظل اذا كان ℓ_n^m لها خاصية الظل في g-NDS متحققه $\forall n \in \mathbb{N}$.
- الدوال المستمره بانتظام $\ell_n: X \to X$ & $g_n: Y \to Y$ $\forall n \in \mathbb{N}$, عندما (X, d) و (Y, d') فضاءات مترية مرصوصة ,فانه ℓ_n^m و g_n^m تملكان خاصية الظل الملائم التقريبي في g-NDS $\forall\, n < m$, اذا وفقط اذا كان خاصية الظل الملائم التقريبي في g-NDS , متحققه ل $\ell_n^m \times g_n^m$
- ليكن (X, d) فضاء متري مرصوص و $\ell_n: X \to X$ دوال مستمرة $\forall n \in \mathbb{N}$.فان ℓ_n^m سلسة متعدية اذا كانت شامله وتملك خاصية الظل الملائم في g-NDS, $\forall\, n < m$.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

الخواص الفوضوية في الانظمة الدينمية المتقطعه غير المستقله G

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة/جامعة بابل كجزء من متطلبات نيل

درجة الماجستير في التربية/الرياضيات

من قبل

رسل عبد الخالق عبد الرحيم حسين

بإشراف

ا. د. افتخار مضر طالب الشرع

1445 هـ 2023 م