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College of Education for Pure Sciences
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The Application of Some Arithmetical Function in Graph Theory

A Thesis

Submitted to the Council of the College of Education for Pure Sciences in University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics

By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ قِيلَ ادْخُلِ الْجَنَّةَ قَالَ يَا لَيْتَ قَوْمِي يَعْلَمُونَ * بِمَا غَفَرَ لِي رَبِّي وَجَعَلَنِي مِنَ الْمُكْرَمِينَ ﴾

صدق الله العلي العظيم

سورة يس

الآية (26- 27)

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DEDICATIONS

To the first teacher of humanity and the intercessor of the nation ......
Our Prophet Muhammad.

To my great homeland……. *To the grieving Iraq*.

To the person who taught me how patience is the way to success, my support and my strength ….. *My beloved father* ,
may God prolong his life .

To whose satisfaction is my goal and ambition...She gave me a lot and did not wait for thanks .To a source of determination and will throughout my life……… *My beloved mother,*
may God prolong her life.

Companions of honest life and a pure and elegant home ....
My beloved sisters.

Researcher

Marwa

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Supervisor's Certification

I certify that the thesis entitled **" The Application of Some Arithmetical Function in Graph Theory "** *by* **" Marwa Mohammed Abed AL-ameer jawad "** has been prepared under my supervision in Babylon University / College of Education for Pure Sciences as a partial requirement for the degree of Master in Education / Mathematic.

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List of Contents

List of Symbols

Symbol	Description
PNT	Prime Number Theorem
P	Prime Number
S , i	Complex Number
$\pi\ (x)$	Counting function of prime
$N(x)$	Counting function of integer
d(a)	Counting function of divisor
σ_{α}**(a)**	Divisor function
μ(a)	The Mobius function
ϕ (a)	Euler's function
Λ(a)	The Mangoldt function
θ(a)	Chebysheve's θ -function
π(a)	Counting the number of prime
N(a)	Counting the total of natural numbers
Π(a)	**Π** - function
ψ**(a)**	Chebysheve's ψ -function
O(q(a))	Big oh notation
o(q(a))	Small oh notation
~	Asymptotic equality
Ω	**Ω** - notation and **Ω** - function
ζ(s)	Zeta function
ℜ(s)	Real part of the complex number
Δ(x)	Abundance of a
$G\ (V_{(G)}, E_{(G)})$	Graph of order n and size m

V(G)	Vertex set in a graph G
E(G)	Edge set in a graph G
$\|V\|$	Number of vertices in a graph G
|E|	Number of edges in a graph G
|A|	Cardinality of a set a
$\lceil A \rceil$	Cardinality integer $\geq a$ (ceiling)
$\lfloor A \rfloor$	Cardinality integer $\leq a$ (floor)
deg$G(v)$	Degree of vertex v in a graph G
$\delta(G)$,$\Delta(G)$	Minimum degree of vertices in G
$d(v,u)$	Shortest path joining v and u
$N\ (v)$	Open neighborhood of v in a graph G
$N\ [v]$	Closed neighborhood of v in a graph G
$< H >$ or $[H]$	Induced subgraph of a graph G
$\overline{G}$	Complement of graph G
Pn	Path graph
Nn	Null graph
γ(G)	Domination number
β(G)	Independence number
χ(G)	Chromatic number
ϖ(G)	Clique number
C_n	Cycle graph
K_n	Complete graph
D	Dominating set
N_e^n	Even number
S	Isolated vertices
g.c.d(u ,v)	greatest common divisor of u ,v

List of Figures

Abstract

The work of this thesis focuses on the use of some arithmetic functions of number theory in graph theory . Several properties of a graph theory using the **Division function** have been discussed $\boldsymbol{d(n) = \sum_{d/n} 1}$. To count the number of divisors of any natural numbers which means county the nodes of a graph. There are so many authors had been working on this subject such as ,G. Santhosh G. Singh and others. Using an arithmetical function in order to build a network graph is a little bit hard, although it gives a solid manner in graph theory. Throughout this work , some results of δ-divisor graph are presented . So, the results of these numbers of pendant vertices , the clique number, and chromatic number are used in this work. Moreover, this work proposed some properties of the complement of maximum **δ-divisor graph** as well as it discussed the Hamiltonian , semi Hamiltonian , the Gauss's function , (chromatic, independence, domination , the clique) numbers and the number of pendant vertices. From unsolved problems till now. The sequence of actual primes and the sequence of the natural numbers build from them called actual prime system. There are a several applications of the Number Theory in different areas.

Introduction

Since the last two century's the number theory is an important core of the pure mathematics subjects, where it deals either algebraically with the behaviors of the sequence of numbers and related counting functions or it deals analytically. For $\mathbb{P}$ takes an actual prime $\{2,3,5,\ldots\}$ one easly can build the sequence of related positive integers $\mathbb{N} = \{1,2,3,\ldots\}$ by using the sense of the fundamental theorem of arithmetic on $\mathbb{P}$. This tells us that every number can be represented as a product of primes with a power (i.e. $p_1^{\beta_1}.p_2^{\beta_2}\ldots p_k^{\beta_k}$ with β_k belongs to $\mathbb{N} \cup \{0\}$ and k belongs to $\mathbb{N}$). Many other authors have studied the application of the number theory in different topics see[49],[50],[52].

In all constantly evolving side of life, the Number Theory is needed to solve the problems that related to the evolution. The mathematicians create two major branches of Number Theory, Multiplicative Number Theory which deals with the distribution of the primes and Additive. And as time passes on, it became necessary to introduce Analytic Number Theory by using mathematical analysis to deal with integers. Number theory considers as one of the cores in pure mathematics. As a part of it, the study of integers in general " primes in particular" and the integer valued functions using the mathematical analysis falls under the brunch "Analytic Number Theory". Number Theory was initially one of ancient Greeks' concerns. The Pythagoras's school's mathematicians (500 BC to 300 BC) were interested to study the numbers for their mystical and numerological properties. They understood the idea of primarily and were interested to study the perfect numbers as well which are defined later. Then, in 1737, Euler presented a relation between the prime numbers and zeta function and by this relation he unleashed his inimitable mentality to present the "Analytic Number Theory" by using the techniques from mathematical analysis. After one hundred years, the German mathematician Dirichlet , in 1837 published his prime number theorem which states that: [for any two prime numbers p and q, there are infinite prime numbers that congruent with p modulo q]. He also used tools from mathematical analysis in the proof. Chebyshev and Riemann later played an important role in improving the concepts and theories about prime numbers. Riemann especially, made a breakthrough in analytic math by conjecting that all of non-trivial zeros of zeta function are on the critical line $u = 1/2$ and that is what's known now by Riemann hypothesis. It is considered one of the most important open problem because of its role in different fields of sciences. The Russian Chebyshev, played an important role in Prime Number Theorem proving and proved Bertrand's postulate which states that: [for every integer n > 1, there is at least one prime between n and 2n]. In number theory, we will deal with a number of functions, including: the DIVISOE function, and link this function with the Graph Theory [23].

Graph theory is a branch of discrete mathematics that is of great importance in solving many complex problems in various fields of science. Moreover, most fields in mathematics deal with graph theory to find a new solution or alternative solution such as fuzzy graph [26] and [27], topological graph [19], labeled graph [25], general graph [28], [29-30], and [2-31], topological indices graph [23- 32], and others. In 1736, the cornerstone of the graph theory was laid when Swiss scientist Leonard Euler published his paper on a problem “Seven Bridges of Königsberg” where he drew the first graph. In this sense, graphs are very convenient tools to convert any subject represented by vertices while the connections between them are edges. This has opened the way for introducing and publishing many graphs to solve multiple problems including, computer science and networking. The graph theory has become rich and interesting because of the problems that have grown exponentially in recent years which require much effort and time to solve, but the graph theory was present in solving such problems. Therefore, the graph theory proved to be an important mathematical tool. Graph has branched out in applications in almost all fields of science and engineering that can be used for example, electrical networks, encryption, wireless communications, image processing, computer science, specially used for study algorithms and traffic systems. Its beauty has attracted many researchers to adopt a graph theory to develop the model with other areas of applied mathematics and a variety of scientific fields including chemistry, physics and life sciences. In addition to social sciences such as business administration, sociology, economics, marketing etc…

For more information see books [24, 25]. There are many topics in the graph theory such as graphic colors, mutant theory and domination theory and numbering diagram. After more than a century of Euler's paper, a new concept was introduced called “Domination”. Where the inspiration was the nucleus of research activity in recent times due to a variety of criteria that can descend from the basic definition of domination.

horizontally and diagonally, De Jaenish, in 1862, considered the problem of finding the minimum number of queens that can be placed on a chessboard. So that each square is either 2 occupied by a queen or can be occupied by the queen. One step, It turns out that five is the minimum number of queens. There are many numbers in graph theory are been used such as independence, domination, chromatic and clique. Then:

$$\boldsymbol{d(n)} = \sum_{\boldsymbol{d/n}} \mathbf{1}$$

one of the labeled graphs is discussed, G. Santhosh and G. Singh [1] defined a δ-divisor graph . We call the divisor graph a δ-divisor graph if the vertices can be labeled with distinct integers 1,2,… , |V|. A.A. Omran in his desertion [1] discussed many properties of the δ-divisor graph. In this work, the new properties are presented in the maximum δ-divisor graph such as clique number, chromatic number, and the number of pendant vertices. Additionally, in the complete of the maximum δ-divisor graph, various numbers are determined such as the domination number, independence number, chromatic number, clique number, and the number of pendant vertices. Moreover, In this work, the relationship between graph theory and number theory is introduced. we explained that there is no inverse domination in the complete of the maximum δ-divisor graph.

The reader can find all the concepts that were not previously mentioned in [10] and [12-13]. For concrete example while shows the application of some arithmetical function such as **μ**(n) , **d**(n) the reader could see it in chapter four .

The outline of the thesis

Its beneficial to give an outline to this work as:

Chapter one : The reader could see represents some elementary needed concepts and basic definitions in Number Theory like prime number, counting and arithmetic function and some unsolved question . Also talk about weird numbers and Divisor Function with some important results about them.

Chapter two : This chapter for some needed requirements and basic definitions of graph theory that we will need later are discussed, as well as some definitions of domination in graphs and forms of domination parameters used in this work, in addition to a previous study of some domination types.

Chapter three : In this chapter are presented some properties of them are provided and results, many numbers are determined in the maximum δ-divisor graph and its complements such as (chromatic, independence, domination, the clique) numbers, the number of pendent vertices and the number of isolated vertices.

Chapter Four : In this chapter are presented some Conclusion and Future work and all References which we learned from in previous chapters.

List of Publications Arising from This Thesis

" The Complement Of The Maximum δ-Divisor Graph With New Results in δ-Divisor Graph " 4th (ICPS) , College of Education Mustansiriyah , Co – Published By (AIP Conference series) , (SCOPUS) ,ID (0202) (2023)

Chapter One

Number Theory

Chapter One

Number Theory

1. 1 Introduction

Some definitions of important concepts are needed to build a good background about this subject. Especially, prime numbers, weird numbers and some important arithmetical function example Divisor , Mobius and Euler's . This chapter contains the definition, Premlineries , examples and some theorems about them.

Prime number 1.1.1[41] The natural number P **is prime** if it is greater than 1 and divisible only by P and 1. The first few primes are 2,3,5,7,11,13,17,19,23,29,..

The function that counts the prime numbers less than $x \in \mathbb{R}$ is

$$\pi(x) = \sum_{p<x} 1.$$

For example, $\boldsymbol{\pi(10) = 4}$

The Greek mathematician proved that every natural number is either a prime number or a uniquely product of different powers prime numbers, this is called **The Fundamental Theorem** of Arithmetic, and the function that counts the naturals less than the integer x is :

$$N(X) = \sum_{\substack{n<x \\ n\in N}} 1$$

1. 2 Arithmetical functions

Definition 1.2.1 [41]

An arithmetical function is real or complex valued function defined on the set of natural numbers.

The following arithmetical functions are needed for our work. For all natural number n one can have :

1. **d(n)** is counting the number of **a divisors** which is defined by:

$$d(n) = \sum_{d|n} 1$$

For instance, d (10) = 4, d (101) = 2

2. $\boldsymbol{\sigma_\alpha}(\mathbf{n})$ is summing the **divisors of n with power $\boldsymbol{\alpha}$** and defined as

$$\sigma_\alpha(n) = \sum_{d^\alpha|n} d^\alpha$$

For example, $\boldsymbol{\sigma_2}$ (8) = 15

3. The **Mobius function:**

$$\boldsymbol{\mu}(\mathbf{n}) = \begin{cases} 1 \quad if\ n = 1 \\ 0\ \ if\ n > 1\ \ and\ contains\ n\ sequare\ prime\ factor \\ (-1)^n \quad if\ n > 1\ and\ \ n =\ p_1\ p_2 \ldots p_n (i.e\ sequarefree) \end{cases}$$

For example, μ(2) = -1, μ(4) = 0

4. **Euler's function ϕ (n)** is counting the numbers that coprime that is with n defined by:

$$\phi(n) = \sum_{\substack{n=1 \\ (n,a)=1 \\ n\in N}}^{a} 1$$

For example, $\boldsymbol{\phi(11) = 10}$, $\boldsymbol{\phi(20) = 8}$ $\boldsymbol{\phi\ (45) = 24}$

5. The **Von Mangoldt function**:-

$$\boldsymbol{\Lambda(n)} = \begin{cases} 0 \text{ if } \ \text{n is not a power of any prime} \\ \log(\text{p}) \text{ if } \ \text{n} = \text{p}^{\text{n}}, \text{p is a prime, n ia natural} \end{cases}$$

$\boldsymbol{\Lambda(125) = \Lambda(5^3) = \log\ (5) = 0.698}$, $\boldsymbol{\Lambda(22) = 0}$

6. Chebysheve's function $\boldsymbol{\psi(n)}$ is summing the logarithms of all primes less than or equal to a which is defined by:

$$\psi(\text{n}) = \sum_{p \le n} log(p),$$

where p runs over all primes less than or equal to n . For example if n = 9 then, **ψ(9) = 2.322.**

1. 3 The counting function of prime and integer

Important counting functions for any real number a are discussed as follows:

1) $\boldsymbol{\pi(\ a)}$ is counting the number of prime numbers less than a that is defined by:

$$\pi(a) = \sum_{p\,<\,a} 1 \quad , \ \text{where p is a prime}$$

2) N(a) is counting the total of natural numbers less than a. This function is defined by:

$$N(a) = \sum_{p<a} 1 \ ; \ \ \text{p is prime}$$

3) $\boldsymbol{\Pi(a)} = \sum_{k=1}^{\infty} \frac{1}{k}\ \pi\left(a^{\frac{1}{k}}\right) \ ; \ k \in N$

4) $\boldsymbol{\psi(a)} = \sum_{n \le a} \Lambda(n)$, where n is a natural, Λ is Von Manglodt function.

5) $\boldsymbol{\Omega(a)}$ is summing the powers of a divisors which is defined by:

$$\Omega(\mathrm{a}) = \sum_{p^k|a} \mathrm{k}$$

Some of theorems are required for our work and the proof one can see at in [41] such that :-

Theorem 1.3.1 [41]

If n ≥ 1 then

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & if\ n = 1 \\ 0 & if\ n > 1 \end{cases}$$

proof : (see [41] ,page 25)

Theorem 1.3.2 [41]

The relation between Euler's function and The Mobius function

If n ≥ 1 then

$$\boldsymbol{\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}} \quad \ldots\ldots\ldots\ldots. \quad *$$

Proof : (see [41] , page 26)

For example, **ϕ (45) = 24**

1. 4 Fundamental Theorem

There are many notations of mathematics. Where the mathematics have be needs such as big (O), small(o) , ~ and Ω which are needed for this work and they one means as following : For any two functions w and q:

Definition 1.4.1[42] For a >1, a function w is asymptotically equal to q and write w ~ q as $a \longrightarrow \infty$ to mean that $\lim_{a\to\infty}\frac{w(a)}{q(a)} = 1.$

Definition 1.4.2[42] A function w is **small o**-notation of q and write

$w(a) = o\big(q(a)\big)\ as\ a \longrightarrow \infty\ if\ \lim_{a\to\infty} \frac{w(a)}{q(a)} = 0.$

Definition 1.4. 3[42] w is **big O**-notation of q $as\ a \longrightarrow \infty$ and write

$\mathrm{w(a)} = \mathrm{O}\big(\mathrm{q(a)}\big)$ if there is a positive constant S such that $|w(a)| \le Sq(a)$.

Definition 1.4.4[42]A function w satisfies the relation $\mathrm{w(a)} = \Omega(\mathrm{q(a)})$ if there is a positive constant **S** such that $|\ w(a)| \ge Sq(a)$.

For some arbitrary large value of **a** ,then $w(a) = \Omega_+\big(q(a)\big)$ means that there is a positive constant S with $w(a) \ge Sq(a)$ and $w(a) = \Omega_-(q(a))$ means that there is a positive constant S with $w(a) = \le - Sq(a)$.

$w(a) = \Omega_+(q(a))$means that both $w(a) = \Omega_+(q(a))$ and $w(a)= \Omega_-(q(a))$ are hold.

Theorem 1.4.5[41] The Prime Number Theorem

The most prominent theorem in number theory is the prime number theorem (PNT) which states $\pi(a) \sim \frac{a}{\log(a)}$ as a goes to infinity.

The prime number theorem gives an asymptotic description of the distribution of prime numbers among natural numbers. It stated first by the French mathematician Adrien-Marie Legendre as a conjecture in 1789 when he made a study about prime numbers greater than 10^6. In 1850, the Russian Chebyshev presented an equivalent formula to Prime Number when he tried to prove it. After nine years in 1859. Bernhard Riemann wrote new ideas about Prime Number Theorem depended on the analysis of the complex zeros of Riemann zeta function.

It takes around a hundred years of trying until the theory was proved. That was independently by J. Hadamard and C. Poussin in 1896 using ideas presented by Riemann (see[41]).

Theorem 1.4.6[41]

Addressing some related theorems as follows , ∀a > 1 :

1. $\lim_{a\to\infty} \frac{\pi(a)\log(a)}{a} = 1$
2. $\lim_{a\to\infty} \frac{\theta(a)}{a} = 1$
3. $\lim_{a\to\infty} \frac{\psi(a)}{a} = 1$

Theorem 1.4.7 [41]

Let p_n be the n-th prime number, the following statement are reasonably equivalent:

1. $\lim_{a\to\infty} \frac{\pi(a)\log(a)}{a} = 1$
2. $\lim_{a\to\infty} \frac{\pi(a)\log(\pi(a))}{a} = 1$
3. $\lim_{n\to\infty} \frac{p_n}{n\log(n)} = 1$

There are a connection according between a discrete function and inferable function and this showed in the following:

1. 5 Euler's summation formula

The following theorem connects between a partial sum and an integral. It expresses a careful indication for the error made in any approximation.

Theorem 1.5.1[41]

If w is continuously differentiable on the closed interval [m, n] where 0 < m < n and w' is its derivative then:

$$\sum_{m<x\leq n} w(x) = \int_m^n w(t)dt + \int_m^n (t-[t])\, w'(\mathrm{t})\mathrm{dt}$$

$$+w(n)([n]-\mathrm{n}) - \mathrm{w(m)}([m]-\mathrm{m}).$$

1. 6 Dirichlet series

The German mathematician Johann Peter Dirichlet (1805-1859) gave an important contributions in mathematics especially, in Number Theory. He creates Analytic Number Theory by investing tools from mathematical analysis. One of the most famous theorem proved by Dirichlet is known by Dirichlet Prime Number Theorem which state that there are infinitely many prime numbers that congruent with the natural α modulo the natural ß and from the form aß + α where a is an integer and g.c,d (α, ß) = 1. The primes of the form aß + α are forming arithmetic sequence.

Before talking about Dirichlet series mathematically we need to know about the following concepts

Definition 1.6.1[43] The series $\sum_{n=1}^{\infty} \alpha_n$ an converges if the sequence of its partial sums {S_k} converges where $S_k = \sum_{n=1}^{k} \alpha_n$ and conversely, it diverges if the sequence of its partial sums diverges.

Definition 1.6.2[43] The series $\sum_{n=1}^{\infty} \alpha_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |\alpha_n|$ is convergent.

Definition 1.6.3 [43] The series $\sum_{n=1}^{\infty} \alpha_n$ is conditionally convergent if the series 1 an is convergent but the series $\sum_{n=1}^{\infty} |\alpha_n|$ is divergent.

Clearly, every absolutely convergent series and every conditionally convergent series is convergent series. Also,

Definition 1.6.4[43] The function of complex variable F(z) is called analytic if it owns a continuous derivative at each point of an open set.

One of the most beneficial tools in Analytic Number Theory is the series of the form

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$$

Which is called the Dirichlet series with coefficient α_n where α_n is arithmetical function and s is a complex number. Dirichlet series is used to generate arithmetic functions, as an example, the ordinary generating functions can be used to prove combinatorial identities, Dirichlet series can be applied to discover and prove identities among arithmetic functions. In [41] there was the statement " if $|\alpha_n|$ is bounded then $\sum_{n=1}^{\infty} \frac{\alpha_n}{n^s}$ is convergent for n > 1".

The utmost known Dirichlet series is called zeta function $\zeta(s) = \sum \frac{1}{n^s}$ which $\alpha_n = 1$ and it will be briefly explained later. Also if $\alpha_n = n$ then the Dirichlet series will equal to $\zeta(s-1) = \sum \frac{n}{n^s}$.

Definition 1.6.5 [43] The absolute convergence half plane of a Dirichlet series

The set of points {s: s= u + it) is called half plane such that u > a where a is constant. Any Dirichlet series have a half plane u > m which is the series converges and another half plane u > n which is the series is diverges. The following theorem proved this in [41].

Theorem 1.6.6[41]

Suppose the series $\sum \left|\frac{\alpha_n}{n^s}\right|$ is divergent for any s. Then there is a constant K, named the abscissa of absolute convergence that is $\sum \frac{\alpha_n}{n^s}$ is absolutely convergent if u > K is not absolutely convergent if u < K.

We conclude from the above information, there three possibilities about Dirichlet series convergence one of which can come true:

1. It does not converge for any s
2. It converges for all s.
3. There exist K such that the series converges for s> K and diverges for s < K.

One can take $\alpha_n = 2^n$ as an example of the first case where$2^n n^{-2} \to \infty$ for any s. Also, $\alpha_n = 0$ for any n is a suitable example of the second case where in the third case, one can take c = ∞ and c = - ∞ and apply it in the first and second case respectively.

Theorem 1.6.7[41]

The sum function $A(s) = \sum \frac{\mathrm{a}(a)}{n^s}$ of a Dirichlet series is analytic in its half an plane of convergence u > c and its derivative A' (s) is represented in this haf plane by the Dirichlet series

$$A'(s) = -\sum_{n=1}^{\infty} \frac{a(n)\log(n)}{n^s}$$

We now need actually to define the convolution cause we need it in solving some examples .

Definition 1.6.8 [41][42] For any two arithmetical functions w, q ∈ S and natural numbers a,b and t, the convolution is defined by

$$(w * q)(a) = \int_{-1}^{a} w\left(\frac{a}{t}\right) dq(t)$$

And in term of summation convolution can be defined as

$$(w * q)(a) = \sum_{b \leq a} w(b) q\left(\frac{a}{t}\right)$$

Definition 1.6.9 [41] If w and q are arithmetical function then Dirichlet convolution for w and q is known as

$$h(a) = \sum_{b|a} w(b) q\left(\frac{a}{t}\right)$$

Theorem 1.6.10 [41] Multiplication of Dirichlet series

Let w and q be two Dirichlet series defined as:

$$w(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \qquad \text{for } u > a$$

$$q(s) = \sum_{n=1}^{\infty} \frac{q(n)}{n^s} \qquad \text{for } u > b$$

Then in the half plane that both series are converges absolutely we have

$$w(s)q(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

Where h(s) is Dirichlet convolution of w(s) and q(s)

Theorem 1.6.11 [41] Uniqueness Theorem

For any two absolutely convergent Dirichlet series for $u > u_\alpha$

$$w(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \qquad and \qquad q(s) = \sum_{n=1}^{\infty} \frac{q(n)}{n^s}$$

If w(s) = q(s) for any s in an infinite sequence $\{S_k\}$ such that $S_K \to \infty$

as K→ ∞ , then w(n) = q(n) for any n.

Theorem 1.6.12[41] Mobius inversion formula

The following

$$f(a) = \sum_{b|a} g(b)$$

Indicates that

$$g(a) = \sum_{b|a} f(b)\, \mu\left(\frac{a}{b}\right)$$

besides the converse is true.

1. 7 The Dirichlet product of arithmetic functions

The two obvious operations on the set of arithmetic functions are point wise addition and multiplication. The constant functions $f = 0$ and $f = 1$ are neutral elements with respect to these operations, and the additive and multiplicative inverses of a function f are given by $-f$ and $1/f$, respectively. While these operations are sometimes useful, by far the most important operation among arithmetic functions is the so-called Dirichlet product, an operation that, at first glance, appears mysterious and unmotivated, but which has proved to be an extremely useful tool in the theory of arithmetic functions.

Definition 1.7.1[48] Given two arithmetic functions f and g , **the Dirichlet product** (or **Dirichlet convolution**) of f and g, denoted by $f * g$, is the arithmetic function defined by:

$$(f * g)(n) = \sum_{d|n} f(d)\, g(n/d)$$

In particular, we have $(f * g)(1) = f(1)\, g(1)$, $(f * g)(p) = f(1)\, g(p) + f(p)\, g(1)$ for any prime p and $(f * g)(p^m) = \sum_{k=0}^{m} f(p^k)\, g(p^{m-k})$ for any prime power p^m .

It is sometimes useful to write the Dirichlet product in the symmetric form

$$(f * g)(n) = \sum_x f(a)\, g(b)$$

where the summation runs over all pairs (a , b) of positive integers whose product equals n. The equivalence of the two definitions follows immediately. from the fact that the pairs (d, n/d), where d runs over all divisors of n, are exactly the pairs (a, b) of the above form.

One motivation for introducing this product is the fact that the definitions of many common arithmetic functions have the form of a Dirichlet product, and that many identities among arithmetic functions can be written concisely as identities involving Dirichlet products.

Maximal order of the divisor function 1.7.2[48] Dirichlet's theorem gives an estimate for the "average order" of the divisor function, but the divisor function can take on values that are significantly smaller or significantly larger than this average. Regarding lower bounds, we have $d(n) = 2$ whenever n is prime, and this bound is obviously best possible. The problem of obtaining a similarly optimal upper bound is harder. It is easy to prove that d(n) grows at a rate slower than any power of n, in the sense that, for any given $\epsilon > 0$ and all sufficiently large n, we have $d(n) \leq n^{\epsilon}$ This can be improved to

$d(n) \leq \exp\{(1+\epsilon)(\log 2)(\log n) / (\log \log n)\}$, for any $\epsilon > 0$ and $n \geq n_0(e)$, a bound that is best-possible, in the sense that, if $1 + \epsilon$ is replaced by $1 - \epsilon$, it becomes false.

The Dirichlet divisor problem 1.7.3[48] Let Δ(x) denote the error term in Dirichlet's theorem, i.e,,$\Delta(x) = \sum_{n \le x} d(n) - x\ logx - (2x-1)\ x$ Thus $\Delta(x) = O(\sqrt{x})$ by Dirichlet's theorem. The problem of estimating $\Delta(x)$is known as the Dirichlet divisor problem and attained considerable notoriety. The problem is of interest, partly because it is a difficult problem that is still largely unsolved, but mainly because in trying to approach this problem one is led to other deep problems (involving so-called "exponential sums") which have connections with other problems in number theory, including the Riemann Hypothesis. Thus, significant progress on this problem will likely have ramifications on a host of other

problems. Most of the known results are estimates of the form(*) $\Delta(x)= O(x^{\theta})$ with a certain constant θ. Dirichlet's theorem shows that one can take θ=1⁄2. In the other direction, G.H. Hardy proved in the early part of the 20th century that the estimate does not hold with a value of θ that is less than 1/4. It is conjectured that 1/4 is, in fact, the "correct" exponent, but this is still open. Nearly 100 years ago, G.F. Voronoii proved that one can take θ = 1/3= 0.333.... but despite enormous efforts by many authors not much progress has been made: the current record for θ is near 0.31.

1. 8 Zeta function

Zeta function ζ(s) is a function defined as a series by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ It was studied firstly by Euler where in 1737, he provided a linking between zeta function and prime numbers by proving the identity

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p$ is prime $\frac{1}{1-p^{-s}}$ That is sometimes, zeta function named by Euler zeta function.

After then, Riemann entered the complex numbers in the study of zeta function by taking complex values of s and now, the zeta function is widespread as Riemann zeta function. Zeta function is analytically continues and converges if $\Re(s) > 1$. Thus for s = u + it, u and t are reals, we have

$$\zeta(\mathrm{s}) = \sum_{n=1}^{\infty} n^{-(u+it)} = \sum_{n=1}^{\infty} n^{-(u+it)} \log(n)$$

$$= \sum_{\mathrm{n}=1}^{\infty} \mathrm{n}^{-\mathrm{u}} \mathrm{e}^{-\mathrm{it}\log(\mathrm{n})}, \text{hence } |\mathrm{n}^{\mathrm{s}}| = \mathrm{n}^{\mathrm{u}}$$

Actually, zeta function ζ(s) is a Mellin transform of f(x) = [x] (see[41]). In fact, Riemann zeta function is analytic at all the points of the complex plane except at the simple pole s = 1 with residue 1 and this is proved in [41]. In particular, if s is near 1 then

$$\lim_{s\to 1} \left|\zeta(\mathrm{s}) - \frac{1}{s-1}\right| = \gamma$$

Where γ is Euler-Mascheroni constant and equal approximately to 0.57721665 (see[44]). If t = 0 then zeta function is divergent for u = 1 and absolutely convergent for u > 1. This was proved in elementary calculus.
Many mathematicians tried to dilate the concept of the zeta function, in 1901 Ernest Barnes defined zeta function and named Shintani zeta function as:

$$\zeta_N(\mathrm{s}, \mathrm{w}|, \dots \dots a_N) = \sum_{n_{1,\dots\dots} n_N} \frac{1}{(w + a_1 n_1 + \cdots + a_N a_N)^2}$$

As a generalization, Adolf Hurwits defined zeta function with respect to two complex numbers s and r with real parts greater than one and zero respectively as

$$\zeta(\mathrm{s},\mathrm{r}) = \sum_{n=0}^{\infty} \frac{1}{(n+r)^s}$$

This series is absolutely convergent for all s and r in the half-plane u > 1 and analytic in the same half-plane where Riemann zeta function is $\zeta(\mathrm{s},1)$.

In 1996 Crandall defined zeta function with respect zeros of Airy function as

$$\zeta A_i(\mathrm{s}) = \sum_{i=1}^{\infty} \frac{1}{|a_i|^s}$$

where $\Re(s) > \frac{2}{3}$ and A_i is Airy function which is defined as

$$A_i(\mathrm{x}) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

and a_i are the zeros of Airy function such that |a1| < |a2| < …

One of the most important usages of zeta function is when Euler proved in 1737 Euclid's theorem (which is talking about the infiniteness of prime numbers) by showing that the series $\sum \frac{1}{p}$ is divergent (where p run through all prime numbers). He inferred this result from the fact that zeta function tend to infinity when s = 1.

Another formula of zeta function can be written if $s \in (0,1)$ as

$$\zeta(\mathrm{s}) = \lim_{a\to\infty}\left(\sum_{n\le a} \frac{1}{n^s} - \frac{a^{1-s}}{1-s}\right)$$

And by this formula one can get the following theorem

Theorem 1.8.1[41]

If a ≥ 1 and s > 0, s ≠ 1 then

$$\sum_{n\le a} \frac{1}{n^s} = \frac{a^{1-s}}{1-s} + \zeta(\mathrm{s}) + 0\left(\frac{1}{a^s}\right)$$

1. 9 Perfect numbers

The sum-of-divisors function is important because of its connection to so-called perfect numbers, that is, positive integers n that are equal to the sum of all their pro per divisors, i.e., all positive divisors except n itself. Since the divisor $d=n$ *is* counted in the definition of *σ(n)* , the sum of proper divisors of n is *σ(n)-n*. Thus, an integer n is perfect if and only if *σ(n)=2n.* For example, 6 is perfect, since 6 = 1+ 2+ 3. It is an unsolved problem whether there exist infinitely many perfect numbers. However, a result of Euler states:

Theorem 1.9.1 [48] (Euler)

An even integer n is perfect if and only if *n* is of the form *n* = $2^{p-1}(2^p - 1)$ where p is a prime and $2^p - 1$is also prime.

This result is not hard to prove, using the multiplicity of **σ** (n).The problem with this characterization is that it is not known whether there exist infinitely many primes p such that $2^p - 1$ is also prime. (Primes of this form are called Mersenne primes, and whether there exist infinitely many of these is another famous open problem.)

There is no analogous characterization of odd perfect numbers; in fact, no single odd perfect number has been found, and it is an open problem whether odd perfect numbers exist.

1. 10 The Weird numbers

As a part of our purpose of this work, moving our attention to go through some relevant concepts such that ***"studying the weird numbers"***. The aim of studying this topic is to link it with the applications of the graph Till now, no one knows if there are an odd weird numbers at least less than 10^{28}. Benkoski defined the weird numbers in 1972 (see[45]) and he proved with Erdos that weird numbers have positive density. Factually, the weird numbers are infinite because weird number multiplied by a prime larger than its sum of divisors is again weird and this was proved in [42]. Now some definitions related to the main subject "weird numbers" will be stated as:

Definition 1.10.1[45] If x is a positive integer and $\alpha(\mathrm{x}) = 2\mathrm{x}$ then x is named perfect . where α(x) is the sum of divisor function . The first perfect numbers are 6, 28, 496,...

Definition 1.10.2 [45] If x is a positive integer and $\alpha(\mathrm{x}) \geq 2\mathrm{x}$ then x is named an abundant. The first few abundant numbers are 12, 18, 20,…
$\Delta(\mathrm{x}) = \alpha(\mathrm{x}) - 2\mathrm{x}$ is named the abundance of x.

Definition 1.10.3 [45] If x is a positive integer and $\alpha(\mathrm{x}) < 2\mathrm{x}$ then x is named deficient. The first deficient numbers are 1, 2, 3, 4, 5, 7....

Definition 1.10.4 [45] If x is a positive integer and equal to sum of all or some of its proper divisors then x is the pseudo perfect number. The first few pseudo perfect numbers are 6, 12, 18,…

Definition 1.10.5 [45] A positive integer x is weird if it is abundant but not pseudo perfect. The first few weird numbers are 70, 836, 4030, 5830, 7192, 7912, 9272...
Kravitz in 1976 showed in [46] that for a natural number n and prime number p. if

$$q = \frac{2^n p - (p+1)}{(p+1) - 2^n}$$

is a prime then w = $2^{n-1}pq$ is weird. Kravitz used Mersenne prime (the prime number of the form $(2^k - 1)2^{16} - 1$ as p and 57 as n and get a large weird number w=2.55 x 10^{57}.

The following remarks that proved in [47] can be noted

1. The proper divisors of the deficient number are deficient.
2. There are abundant numbers but not weirds like 18.
3. The abundant number is weird iff its abundance cannot be expressible as a summation of distinct proper divisors of it.

1 . 11 The divisor and sum-of-divisors functions

Theorem 1.11.1[48] The divisor function d(n) and the sum-of-divisors function o(n) are multiplicative. Their values at prime powers are given by

$$d(p^m) = m + 1, \sigma(p^m) = \frac{p^{m+1} - 1}{P - 1}$$

Proof: (see [48] . pag.30)

1 . 12 The Division Algorithm and Greatest Common Divisor

Given $a, b \in \mathbb{N}$ with $a \geq b \geq 2$, there exists a unique pair of integers q, r such that

$$a = bq + r \quad with \quad 0 \leq r < b$$

We say q is the quotient and r is the remainder when a is divided by b. This is called the Division algorithm.

Proposition 1.12.1[48]

Let G be a subgroup of ℤ under addition (i.e. G C ℤ, G contains 0, is closed under addition, and if $a \in G$ then so is- a). Then there exists a unique integer d ≥ 0 such that G = d ℤ; i.e.

$$G = \{0, \pm d, \pm 2d, \dots\}.$$

Theorem 1.12.2[48]

Let $m, n \in \mathbb{N}$. Then there exists $k, l \in \mathbb{Z}$ such that

$(m, n) = km + In.$

In particular, if m and n are coprime, then there exist integers k, l such that

$$km + ln = 1 .$$

Theorem (Euclid's Lemma)1.12.3[48]

Let n, a, b be integers. If $n|ab$ and $(n, a) = 1$, then $n|b$.

In particular, for p prime; $p|ab$ implies $p|a$ or $p|b$.

1 . 13 Theorem (Fundamental Theorem of Arithmetic)[48]

Every natural number n greater than 1 can be written uniquely in the form

$$n = p_1^{a1} p_2^{a2} \dots p_k^{ak}$$

1 . 14 The Euclidean Algorithm

The *Division algorithm* can be used repeatedly to compute the gcd of two numbers .

Theorem (Euclidean Algorithm) 1.14.1[41]

Let $a > b \geq 2$, with b $\nmid$ a. Let $ro = a \; and \; r_1 = b$. Apply the division algorithm repeatedly to obtain a set of remainders $r_2, r_3, \dots, r_{n+1}$:

$$r_0 = r_1 q_1 + r_2 \qquad 0 < r_2 < r_1$$

$$r\_1 = r\_2\ q\_2 + r\_3 \qquad 0 < r\_3 < r\_2$$

$$\vdots$$

$$r_{n-2} = r_{n-1} q_{n-1} + r_n \qquad 0 < r_n < r_{n-1}$$

$$r_{n-1} = r_n q_n + r_{n+1}\ ,$$

$with\ r_{n+1} = 0. Then\ \ r_{n-2} = (a, b)$. Hence r_n = (a,b) Example Find (210, 1001) = 7 . We have:

1001= 210.4 + 161

210 = 161.1 + 49

161= 49.3 +14

49 = 14.3 +7

14 = 7.2 + 0

Chapter Two

Graph Theory

Chapter Two

Graph Theory

2.1 Inttoduction

In this chapter moving our attention to some application of some arithmetical function in the reality .As known there are so many applications of the number theory in different areas . Here the work consists the application of some Arithmetical functions on graph. So, for the mention purpose some of the following graphical concepts are needed:

2.2 Basic Definitions

The city's people were wondering whether it was possible to pass on the seven bridges. Only once and return to the starting point without repeating any bridge . In 1736, Leonard Euler (1707-1783) came up with an answer to this question, proving that it was not possible to pass on the seven bridges at exactly the same time.Use Euler to answer the question a simple way by eliminating all the unnecessary features of the city. He drew a picture of the city where he represented the land b vertices (or points) and bridges by edges (or lines) as shown in Fig.2.1(c). This mathematical structure is called the graph. From these simple assets, the theory of graph has grown into a strong mathematical theory in mathematics and solved many complex life problems.

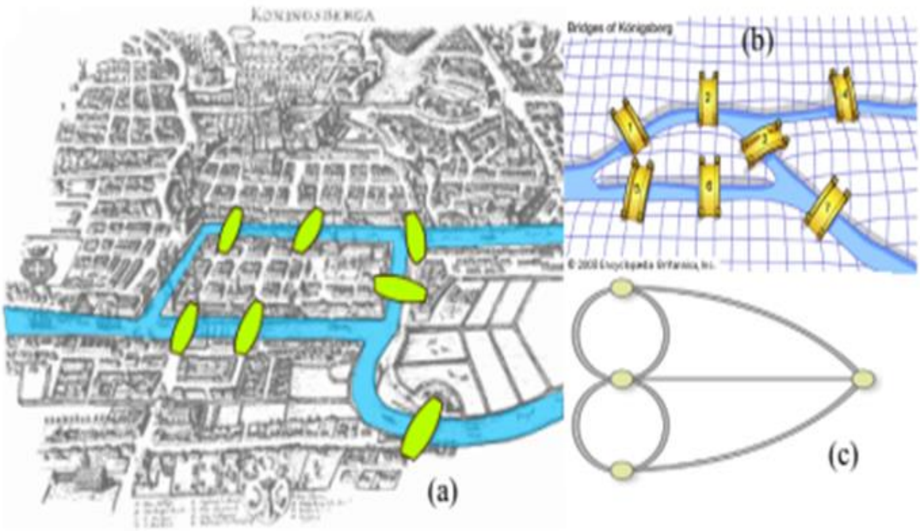


Figure 2.1: Seven bridges in Kaliningrad (Konigsberg), Russia

In view of the above, there is a need to know the basic concepts of the graph on which we depend on in thesis.

Definition 2.2.1[39] A graph G = (V(G), E(G)) or G = (V, E) consists of two sets V(G) or V, the vertex set of the graph, which is a non-empty set of elements called vertices (or points) and E(G) or E the edge set of the graph, which is a possibly empty set of elements called edges (or lines), such that each edge e in E is assigned as an unordered pair of vertices called the end vertices of e. A graph G with n vertices and m edges is called a (n, m)-graph.

Definition 2.2.2[39] A graph G is called **undirect graph** when the pair of vertices representing any edge is unordered.

Example 2.2.3: A graph G of 6 vertices and 7 edges such that

$V(G) = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ a set of vertices and

$E(G) = \{V_IV_2, V_IV_5, V_1V_6, V_2V_3, V_2V_4, V_3V_4, V_5V_6\}$ a set of edge of a graph C or $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

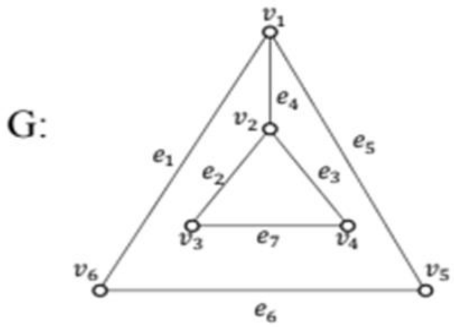


Figure 2.2 : Show graph G has 6 vertices and 7 edges.

Definition 2.2.4[34] The **order** of G is n = | V(G) | and the size of G is m = | E(G) |. In Fig 2.2, order and **size** of G is 6 and 7 respectively.

Definition 2.2.5[34] A graph G is **trivial** if a vertex set of G is a singleton and its contains no edges.

Definition 2.2.6[39] A **finite graph** is a graph that has finite number of vertices and finite number of edges. Otherwise, a graph is called infinite graph.

Definition 2.2.7 [36] If two vertices of a graph are joined by an edge then these vertices are called **adjacent vertices**.

Definition 2.2.8 [36] If two or more edges of a graph have a common vertex then these edges are called **adjacent edges**. Note that if e = vu is an edge of G, then e is incident to u and we also say that u and v are the endpoints of e.

Definition 2.2.9 [37] The **open neighborhood** N(v) of the vertex v consists of the set vertices adjacent to v, that is, $N(v) = \{u \in V: vu \in E\}$, and the **closed neighborhood** of v is N[v] = N(v) U {v}.

In Fig. 2.2, N (v_1) = { V_2, V_5, V_6} and N[V1] = { V_1, V_2, V_5, V_6}.

Definition 2.2.10 [36] The **degree** of a vertex v in a graph G, denoted by deg(v) is the number of edges incident with v. (For example, in Fig.2.2, deg (v_1) = 3 , i.e. deg(v) = | N(v)|)

Definition 2.2.11 [38] The **maximum degree** of a graph G, denoted by $\Delta(G)$ is the maximum value among the degrees of all the vertices of G, i.e , $\Delta(G) = max_{v\in V(G)} \deg(v)$.

Similarly, we define the **minimum degree** of a graph G and denote it by $\delta(G)$, i.e., $\delta(G) = \min_{v\in V(G)} \deg(v)$. In Fig. 2.2, $\Delta(G) = 3$ and $\delta(G) = 2$.

Definition 2.2.12[6] A vertex of degree 0 is an **isolated vertex** of G, A vertex of degree 1 is called a **pendant vertex** of G (or end -vertex or leaf), and the unique edge of G incident to such a vertex of G is a **pendant edge** of G.

Definition 2.2.13 [39] If for some positive integer k, deg(v) = k for every vertex v of the graph G, then G is called **k-regular.**

Definition 2.2.14 [38] A loop is an edge whose end-vertices are the same.

Definition 2.2.15 [38] Multiple edges are edges with the same pair of end vertices.

Example 2.2.16 In Fig 2.3 show loops and multiple edge where the edges e_1, and e_2 are loops, while the edges e3 and e4 are multiple edges.

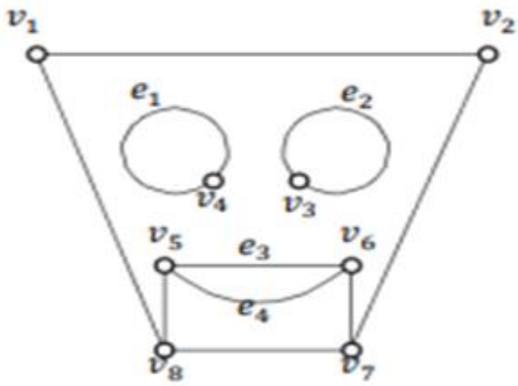


Figure 2.3: A graph has loops and multiple edges

Definition 2.2.17 [39] Simple graph is **undirected** graph without loops and multiple edges. In other words, simple graph is a pair G = (V, E) where V is an arbitrary set and E is a set of unordered pairs of distinct elements from V.

Definition 2.2.18 [35] A **u-v walk** W in G is a sequence of vertices in G beginning with u and ending at v such that consecutive vertices in the sequence are adjacent.

Definition 2.2.19 [35] A u-v walk in a graph in which no vertices are repeated is a **u-v path** and denoted by P_n.

Definition 2.2.20 [38] A **cycle** graph is one that is obtained by joining the two end-vertices of a path graph. Thus, the degree of each vertex of a cycle graph is two. A cycle graph with n vertices is often denoted by C_n

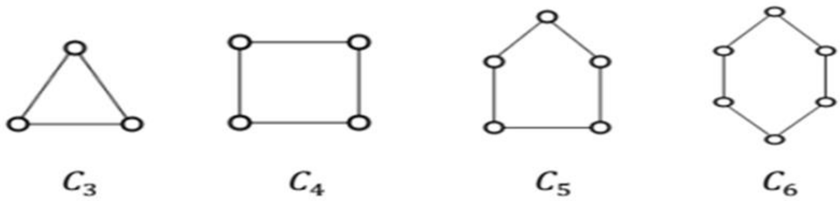


Figure 2.4: Cycle Graphs G_3 , G_4 , G_5 and G_6

Definition 2.2.21[38] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs The **union** of G_1 and G_2, denoted by $G_1 \cup G_2$, is another graph $G = (V, E)$ whose vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. (See Fig. 2.5).

Definition 2.2.22 [37] The **join** $G = G_1 + G_2$, where $G1$ and $G2$ are disjoint graphs is a graph which has a vertex set $V(G)= V(G_1) \cup V(G2)$ and edge set

$E(G) = E(G_{\ 1}) \cup\ E(G_{\ 2}) \cup \{uv: u\ \in\ V(G_1)\ , v\ \in\ V(G_2)\}$ (See Fig. 2.5).

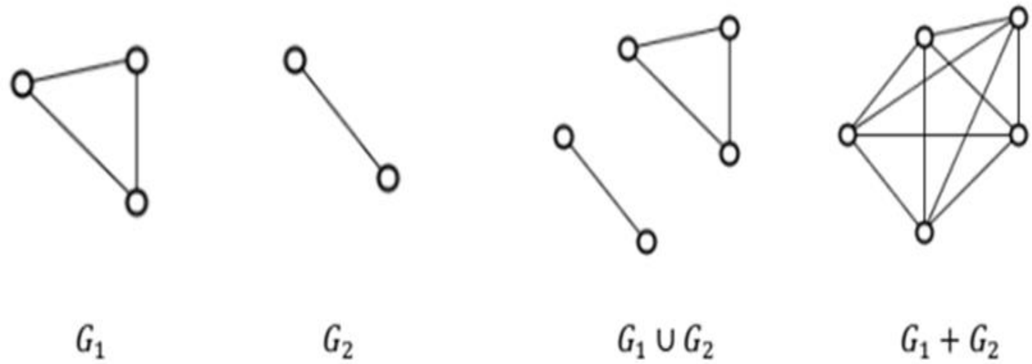


Figure 2.5: The union and Join two graphs G_1 and G_2

Definition 2.2.23 [35] A graph G is **complete** if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. (Fig.2.6 (show complete graph where n = 1, ... ,5).denoted by $\bar{G}$

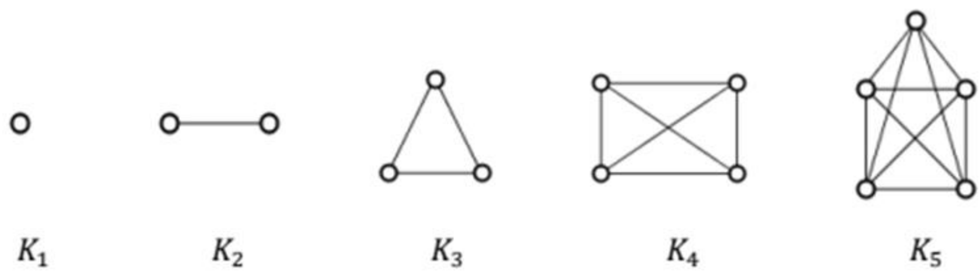


Figure 2.6: Complete Graphs K_1 , K_2 , K_3 , K_4 and K_s

Definition 2.2.24 [37] A graph G is **connected** if for every pair u, v of vertices there exist a u - v path, otherwise, G is disconnected.

Definition 2.2.25 [39] A graph H is a **subgraph** of G if every vertex of H is a vertex of G, and every edges of H is an edge of G. In other words, $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$.

Definition 2.2.26 [36] A **spanning subgraph** is a subgraph containing all the vertices of G.

Definition 2.2.27 [34] A subgraph H of G is said to be an i**nduced subgraph** of G if each edge of G having its ends in V(H) is also an edge of H. Such H is denoted by G[H] also denoted by $\langle H \rangle$ G or simply by $\boldsymbol{\langle H \rangle}$**.**

Example 2.2.28: In Fig.2.7 we show types of **subgraphs** of a graph G such that G1 and G2 are induced subgraph, while G_3 and G_4 are spanning subgraph and G_5 is subgraph but neither induced nor spanning subgraph.

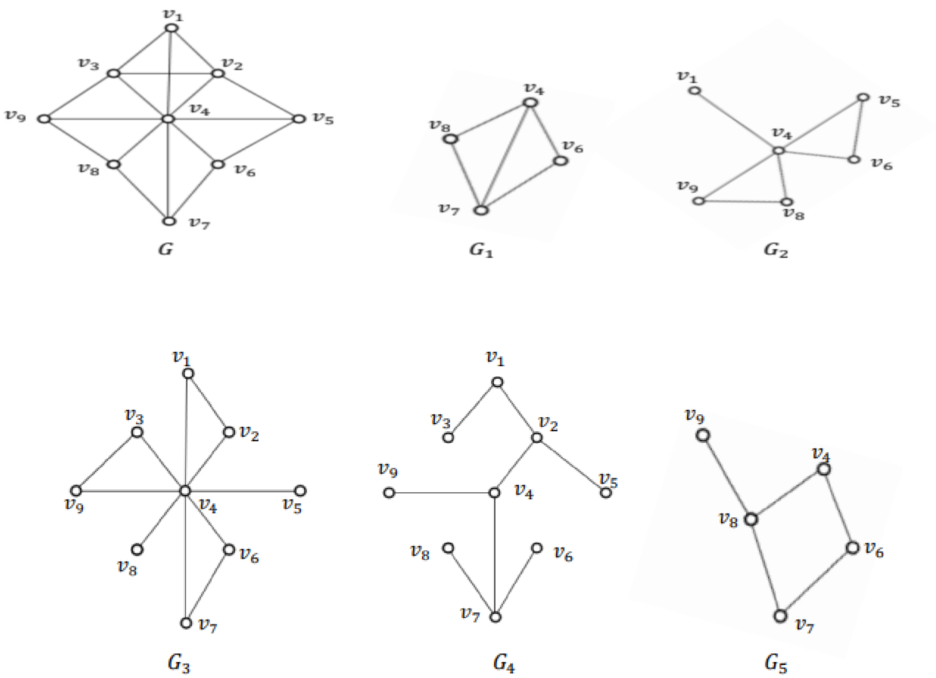


Figure 2.7: G1 Sub graph G2,G_5 induced sub graph and G_3, G_4, spanning sub graph

Definition 2.2.29 [36] The **complement** G^C of a graph G also has V(G) as its vertex set, but two vertices are adjacent in G^C if and only if they are not adjacent in G. (For example, see Fig.2.8)

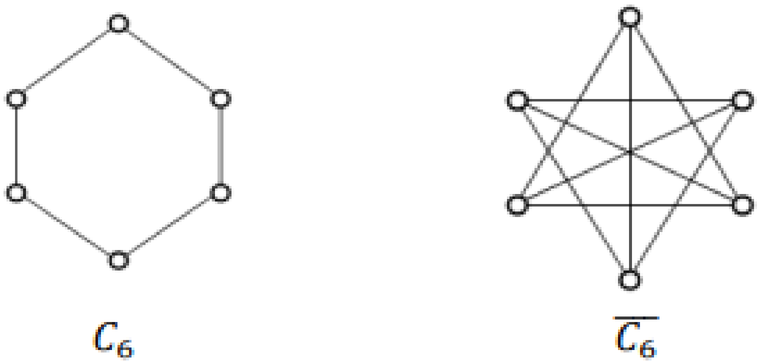


Figure 2.8: A cycle graph and its complement**.**

Definition 2.2.30 [39] If E= ∅، in a graph G (V, E) then such a graph without any edges is called **a null graph** and denoted by N_n with n vertices. (An example can be seen in Fig.2.9)

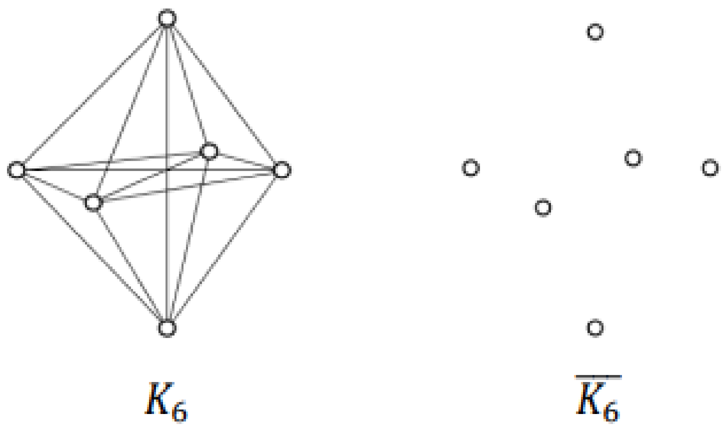


Figure 2.9: Null graph of K6.

Definition 2.2.31 [34] A connected graph without cycles is defined as a **tree.**

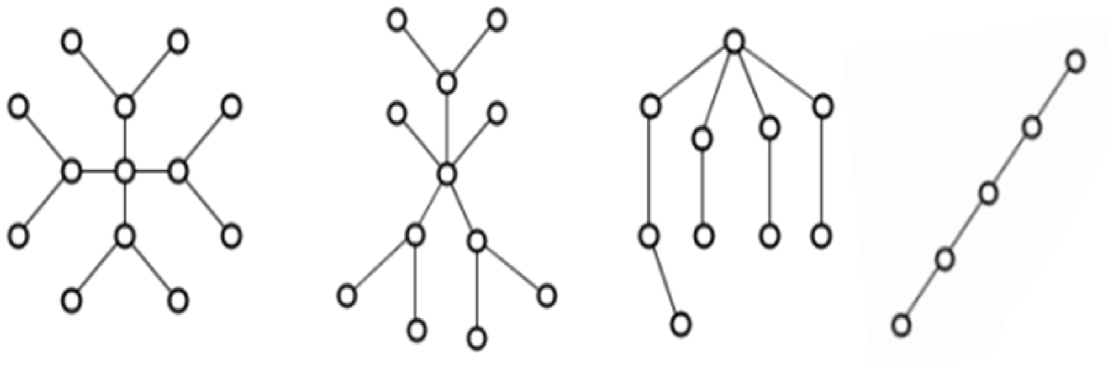


Figure 2.10: Some tree graphs

Definition 2.2.32 [37] For a graph G = (V, E), a set S ⊆V is **independent** if no two vertices in S are adjacent.

Definition 2.2.33 [39] A graph $G = (V_{(G)}, E_{(G)})$ is said to be **connected** if there is a path between every pair of its vertices. A graph which is not a connected is called a disconnected graph and every part is called component (as an example. See Figure 2.11).

Figure 2.11: Disconnected graph with two components

Definition 2.2.34 [39] A graph $G_1 = (V_1, E_1)$ is said to be **isomorphic** to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1, is an edge with end vertices u_1, and u_2 in G_1 then the corresponding edge e_2 in G_2 has its end vertices v_1 and v_2 in G_2 which corresponds to u_1 and u_2, respectively. Such a pair of correspondence is called a graph isomorphism and denoted by ($G_1 \cong G_2$). (see Fig.2.12).

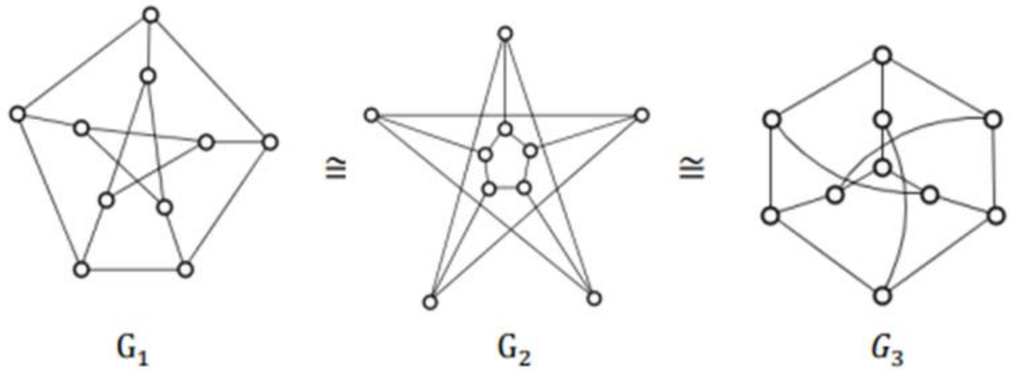


Figure 2.12: Three isomorphic graphs

Definition 2.2.35 [35] For a connected graph G, we define the **distance** between two vertices u and v is the smallest length of any u - v path in G and is denoted by d (u, v).

Definition 2.2.36 [27] A graph G is called **Hamiltonian graph** if it has a Hamiltonian cycle.

Definition 2.2.37 [27] A **Hamiltonian cycle** in graph G is cycle which includes every vertex in G.

Definition 2.2.38 [27] A non – Hamiltonian graph G is **semi – Hamiltonian** if there exists a path passing through every vertex

2.3 Domination in Graph Theory

The origin of the idea of domination in the graph started from the problem of the five queens in the chessboard game. In 1850, several chess players were interested in the minimum number of queen or is attacked by a describes how to determine the minimum number of queen.

This problem queens placed on the chessboard so that all chess boxes are controlled by the dominant queens recall that a queen can move any number of squares horizontally, vertically, or diagonally on the chessboard. (as shown Fig.2.13).

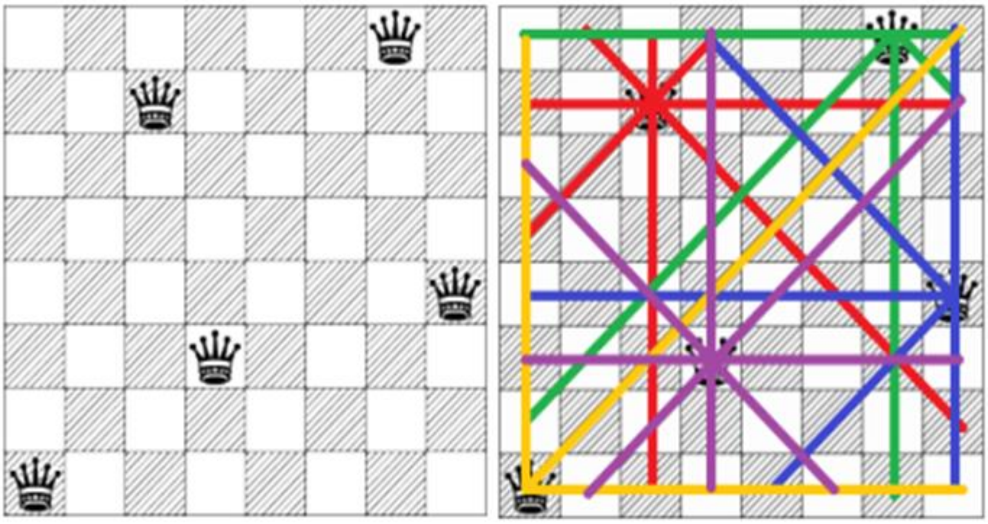


Figure 2.13: Five Queen problem

Away from the chess game, in our lives there are many other applications for the concept of domination in the graph such as networks, radio stations, bus routes schools, social media, camera locations in buildings, power grids, coding theory. Information and details can be found in [40,30].

The concept of the domination number of a graph was first defined in 1958 by Berge [26] where he called this number “coefficient of external stability". However, it was until 1962 when Ore [25] gave this number the name "dominating set" and "domination number" , the way to dominate a graph either as few as possible of vertices or edges.

Definition 2.3.1 [37] A set D ⊆ V of vertices in a graph G = (V, E) is called a **dominating set** if every vertex v ∈ V is either an element of D or is adjacent to an element of D.

Definition 2.3.2 [37] A dominating set D is called a **minimal dominating** set if has no proper subset $\overline{D} \subset D$is dominating set.

Definition 2.3.3 [37] The **domination number $\boldsymbol{\gamma}$(G)** of a graph G equals the minimum cardinality of a set of minimal dominating set of G. Such a set is called the γ −set of G.

Example 2.3.4. In the following Figure

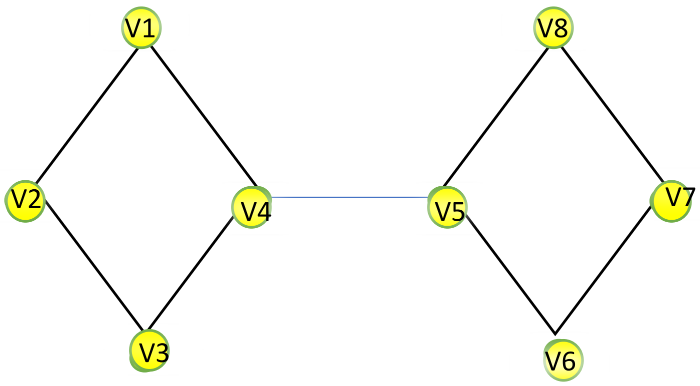


Figure 2.14: Domination number of graph G.

D 1 = {V_4, V_5, V_7} is minimal dominating set of G.

D 2= {V_1, V_3, V_8, V_6} is minimal dominating set of G.

D 3 = {V_2‘V_4, V_5, V_7} is not minimal dominating set of G.

$\gamma\ (G) =\ 4$

Definition 2.3.5 [37] The maximum cardinality of all independent sets is called the **independence number** of the graph G and denoted by $\boldsymbol{\beta(G)}$ (Haynes et al., 1998).

Definition 2.3.6[36] The **chromatic number** of a vertex- coloring of G is an assignment of colors to all its vertices such that all pairs of adjacent vertices are assigned different colors., denoted by $\boldsymbol{\chi(G)}$ is the smallest number colors necessary for coloring G (Brooks, 1941).

Definition 2.3.7 [37] The order of largest complete (each vertex in it is adjacent to all other vertices in it) subgraph of a graph G is called the **clique number**, denoted by $\boldsymbol{\varpi(G)}$ **.**

Definition 2.3.8 [24] Let n be a positive integer. if a and b are integers, then a is said to be **congruent to b modulo n**, which is written $a \equiv b$ (mod n), if n divides$(a - b)$. we call n the modulus of the congruence .

Chapter Three

The Complement of the

Maximum δ-Divisor Graph

Chapter Three

The Complement of the Maximum δ-Divisor Graph with New Results Graph in δ-Divisor

3.1 Introduction

This is chapter shows the application of some arithmetical function in graph theory. As the reader know that addressing any application of some number theory concepts are not always doable, this work concentrates on using: $\boldsymbol{d(n) = \sum_{d/n} 1}$ or **(deg Vn) = dn – 1**

To count the number of any divisors of any natural number that is; county the nodes by using $d(n)$ .So this needs to go through some needed define:

Definition 3.1.1[1] A δ-divisor graph if V is labeled by a set of integer and for each edge $uv \in E$ either $f(u)/ f(v)$ or vice versa .

Definition 3.1.2[11] Consider G of order n be a δ-divisor graph and if add any edge to this graph yields a non- δ-divisor graph, then this graph is called **the maximum δ-divisor graph**.(For example , see Fig. (3.1))

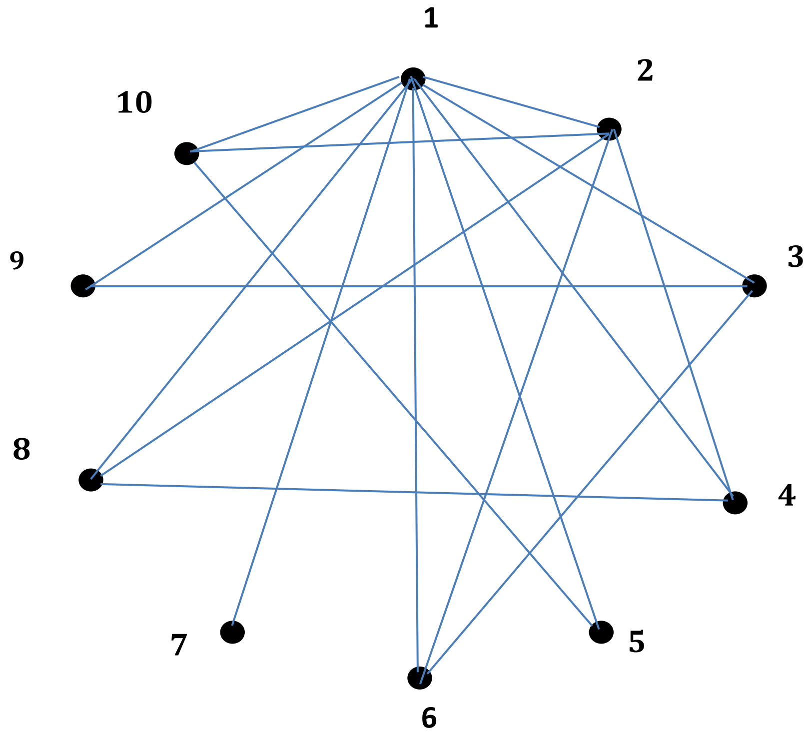


Figure 3.1 : The maximum δ-divisor graph of order 10.

Definition 3.1.3 [11] The number of prime number not exceeding x is called the **Gauss's function π(x)** and can be written as $\pi(x) = |\{p : p \text{ is prime}, 2 \leq p \leq x\}|$.

Note 3.1.4 For all definitions in this chapter we define the labeling function by: $f(v_i) = i \ , i = 1, \ldots, n.$

Theorem 3.1.5 [1] Let G be the maximum δ-divisor graph. Then,

(i) $\beta(G) = \left\lceil \frac{n}{2} \right\rceil$.

(ii) $\gamma(G) = 1$.

Theorem 3.1.7 [1] Let G be the maximum δ-divisor graph, then, $\gamma^{-1}(G) = \pi(n)$.

3. 2 Main results

Theorem 3.2.1. The clique number of the maximum δ-divisor graph of order n is equal to m+1, where m is the maximum number such that $2^m \leq n$.

Proof. Let

$$S = \left\{ v\,; f(v) = 2^i\,; 1 \leq i \leq m\,;\ here\ m\ is\ the\ maximum\ number\ such\ that\quad 2^m \leq n \right\}$$

since for every pair of vertices in the set S say u and w, these vertices are adjacent since u divides w or w divides u. Thus, the set S is a clique subgraph in G. the vertex v_1 is adjacent to other vertices, since $f(v_1) = 1$, and 1 divides all integer numbers. Therefore, the set $S \cup [v_1]$ is a clique in G. For each vertex not in the set $S \cup [v_1]$, there exists at least one prime number say $q \neq 2$ divides the labeled of this vertex so this vertex does not divide any vertex in the set S and vice versa. Thus, the set $S \cup [v_1]$ is the maximal clique in G. The other maximal clique subset in being following form:

$$\left\{ v\,; f(v) = r^i;\ 1 \leq i \leq m \quad and \quad 3 \leq r \leq m\,;\ \text{m is the maximum number such that } i^m \leq n \right\}$$

and the number of vertices in each subset of this form is less than of the set S (for an example, see fig. 3.2). Therefore, the set $S \cup [v_1]$ is the maximum clique subset, so the clique number is equal to

$|\, S \cup [v_1]\ | = m + 1$ ■

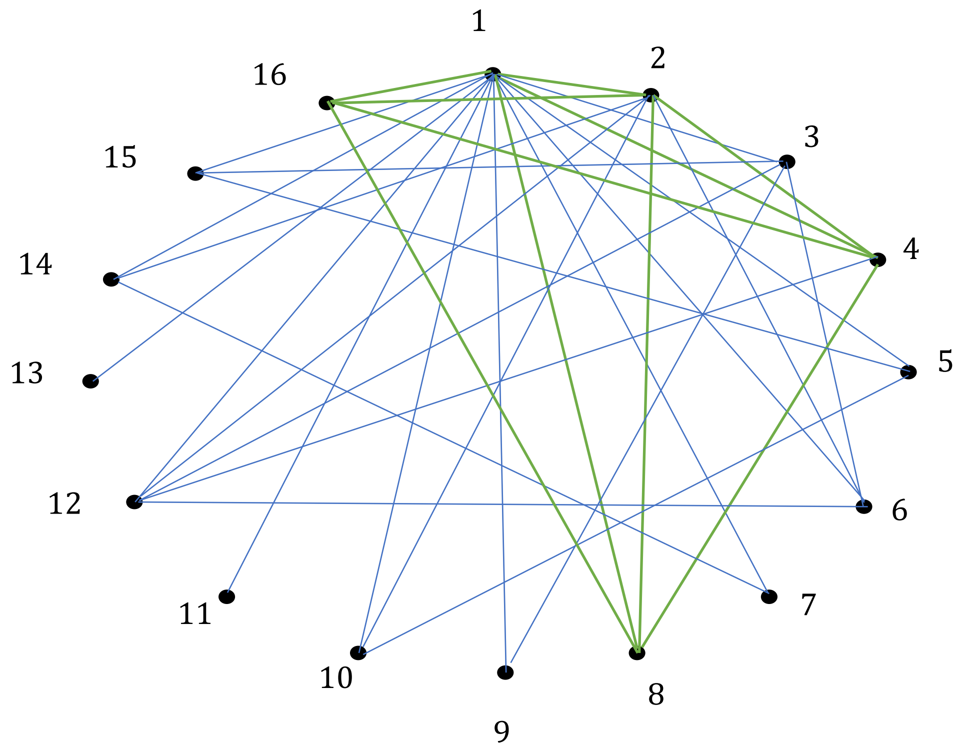


Figure 3.2: The maximum clique subgraph of maximum δ-divisor graph of order 16.

Corollary 3.2.2. $\chi\,\overline{(G)} = \mathrm{m} + 1$

Proposition 3.2.3. Let G be the maximum δ-divisor graph of order n, then the set of the pendants vertices is:

$$\left\{\begin{array}{ll} \{v_1, v_2\}, & \text{if } n = 2 \\ \{v_1, v_3\}, & \text{if } n = 3 \\ \left\{\begin{array}{c} p_i,\ \ p_i \text{ is prime number such that } p_i > \left\lceil \frac{n}{2} \right\rceil, \text{if } \left\lceil \frac{n}{2} \right\rceil \text{ is prime} \\ \text{and } \ p_i \geq \left\lceil \frac{n}{2} \right\rceil, \text{if } \left\lceil \frac{n}{2} \right\rceil \text{ is not prime} \end{array}\right\}, & \text{if } n > 3 \end{array}\right\}$$

Proof.

Four cases are discussed below:

Case 1. If $n = 1$, then $G \equiv K_1$, so there is no pendant vertex.

Case 2. If $n = 2$, then $G \equiv K_2$, so there are two pendant vertices.

Case 3. If $n = 3$, then $G \equiv P_3$, so again there are two pendant vertices.

Case 4. If $n > 3$, then there are four subcases:

Subcase 1. If $f(v_i) < \left\lceil \frac{n}{2} \right\rceil$, then the neighborhood of the vertex v_i contains at least two vertices one of them is v_1 and the other of labeled $2f(v_i)$. Thus, the vertex v_i is not a pendant .(see Fig. 3.2)

Subcase 2. If $f(v_i) = \left\lceil \frac{n}{2} \right\rceil$, then there are two subcases:

I)If n is even, then the vertex of labeled $\frac{n}{2}$ is adjacent to at least two vertices one of them is v_1 and the other of labeled n. .(see Fig. 3.2)

II) If n is odd, then there are two subcases:

a) If $\left\lceil \frac{n}{2} \right\rceil$ is not prime then there are at least two primes number say p and q less than $\left\lceil \frac{n}{2} \right\rceil$. Therefore, the two vertices that have labeled p and q are adjacent to the vertex of labeled $\left\lceil \frac{n}{2} \right\rceil$. Thus, the vertex of labeled $\left\lceil \frac{n}{2} \right\rceil$ is not a pendant.

b) If $\left\lceil \frac{n}{2} \right\rceil$ is prime then the vertex of labeled $\left\lceil \frac{n}{2} \right\rceil$ is adjacent to only one vertex v_1, since $2\left\lceil \frac{n}{2} \right\rceil > n$.

Subcase 3. If $f(v_i) > \left\lceil \frac{n}{2} \right\rceil$, then there are two subcases:

I) If $\left\lceil \frac{n}{2} \right\rceil$ is not prime, then by the same manner in case 4(subcase 2 (II(a))), the vertex of labeled $\left\lceil \frac{n}{2} \right\rceil$ is not pendant.

II) If $\left\lceil \frac{n}{2} \right\rceil$ is prime then the vertex of labeled $\left\lceil \frac{n}{2} \right\rceil$ is adjacent to only one vertex v_1, since $2\left\lceil \frac{n}{2} \right\rceil > n$■

Note 3.2.4. There is no isolated vertex in the maximum δ-divisor graph.

3.3 The complement of the maximum δ-divisor graph of order n.

Theorem 3. 3.1. Consider G be the maximum δ-divisor graph of order n, then $\overline{G}$ has the following properties:

1. The set of isolated vertices contains two vertices if $n = 2$, otherwise contains one vertex.

2. The graph $\overline{G} - v_1$ is Hamiltonian if n is odd and semi - Hamiltonian if n is even.

Proof. 1) Two cases are discussed:

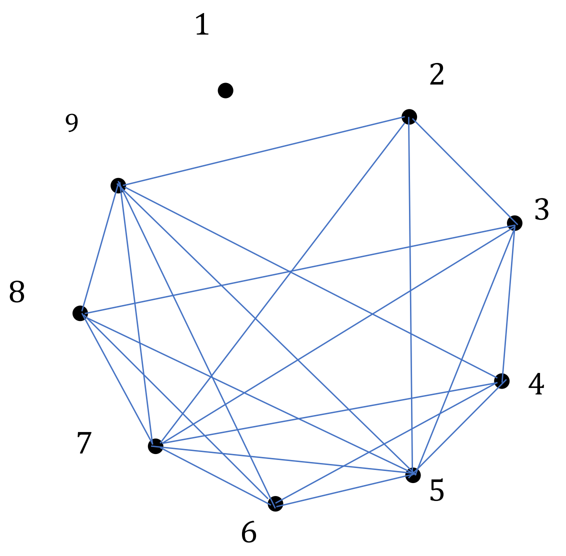
Case 1. If $n = 2$, then $\overline{G} \equiv N_2$, so $\overline{G}$ has two isolated vertices.

Case 2. If $n \neq 2$, then the vertex v_1 is the only isolated vertex since for each vertex $v_i \neq v_1$, the vertex v_i is adjacent to at least one vertex v_{i-1} or v_{i+1}.

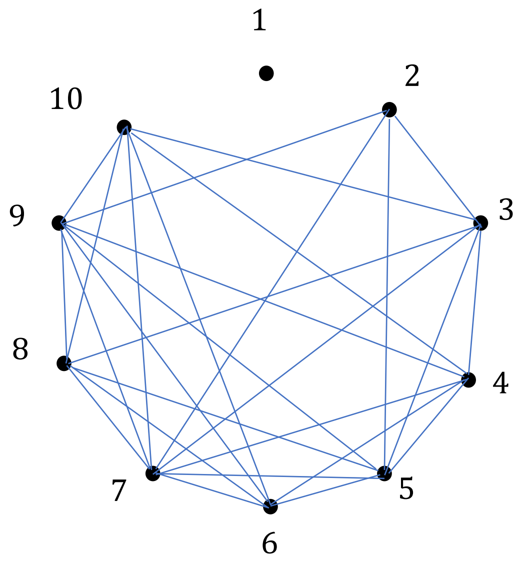
2) The vertex v_2 is adjacent to v_3 and the vertex v_n is adjacent to the vertex v_{n-1}. For each vertex $v_2 < v_i < v_n$, the vertex v_i is adjacent to the two vertices v_{i-1} *and* v_{i+1}. Thus, there is a path passing all vertices in the graph $\bar{G} - v_1$ and there are two cases as follows:

Subcase 1. If n is odd , then the vertex v_n is adjacent to the vertex v_2. Thus, there is a cycle passing all vertices in the graph $\overline{G} - v_1$, so this graph is Hamiltonian (as shown in Fig.3.3 (B))

Subcase 2. If n is even , then the vertex v_n is not adjacent to the vertex v_2. Thus, there is no cycle passing all vertices in the graph $\overline{G} - v_1$, so this graph is semi Hamiltonian (as shown in Fig. 3.3 (A))■



A



B

Figure 3.3: Complement of maximum δ-divisor graph of order 9 and 10.

Theorem 3.3.2. Let $\overline{G}$ be the complement of the maximum δ-divisor graph G of order n , then

$$\deg(v_2) = \pi(n) - 1 + \left|\{m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}\}; 2 \text{ dont divided } m\right|.$$

Proof. In graph $\overline{G}$ it is obvious that the labeled vertex v_2 is adjacent to all vertices which have labeled prime numbers, so $\deg(v_2) \geq \pi(n) - 1$. Now, the other vertices that mean the vertices of labeled not prime is written by the form $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, so there are two subcases to discuss this case:

Subcase 1. If 2 *divided* m, then the vertex v_2 is not adjacent to these vertices.

Subcase 2. If 2 *not divided* m, then the vertex v_2 is adjacent to these vertices.

Thus, depending on two subcases above, $deg(v_2) = \pi(n) - 1 + \left|\{m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}\}; 2\ dont\ divided\ m\right|$. ■

Theorem 3.3.3. Let $\overline{\mathrm{G}}$ be the complement of the maximum δ-divisor graph of order n, then, the clique number of the maximum δ-divisor graph is π(n).

Proof. Let S be the set of all vertices of labeled prime numbers, then every two vertices in the set S are adjacent in the graph $\overline{\mathrm{G}}$. Thus, the set S is a clique subgraph of $\overline{\mathrm{G}}$. Therefore, the clique number greater than or equal to the cardinal of the set S. For each vertex say u not belong to the set S, there are at least two vertices have different prime labeled say p and q, so the vertex u not adjacent to at least two vertices which have the labeled p and q. Therefore, the clique number is equal to $|S| = \pi(n)$. ■

Corollary 3.3.4 $\chi(\overline{\mathrm{G}}) = \pi(\mathrm{n})$.

Theorem 3.3.5 Let $\overline{\mathrm{G}}$ be the complement of the maximum δ-divisor graph of order n, then, $\gamma(\overline{\mathrm{G}}) = \begin{cases} 1, \text{if } \mathrm{n} = 1 \\ 2, \text{if } \mathrm{n} = \mathrm{p} \\ 3, \text{if } \mathrm{n} \neq \mathrm{p} \end{cases}$, where p is a prime number.

Proof. There are three cases will be as follows:

Case 1. If n = 1, then it is obvious that $\gamma(\overline{\mathrm{G}}) = 1$.

Case 2. If $n = p$, then the vertex v_n is adjacent to all other vertices except the isolated vertex v_1. Thus, $\gamma(\overline{\mathrm{G}}) = 2$

Case 3. If $n \neq p$, then there is no vertex adjacent to all vertices in $\overline{\mathrm{G}} - \{v_1\}$, therefore, $\gamma(\overline{\mathrm{G}}) \geq 3$. Let $D = \{v_1, v_{n-1}, v_n\}$, so for each vertex in $\overline{G} - \{v_1\}$ this vertex is ad to at least one vertex of the two vertices v_{n-1} *or* v_n . Thus, the set D is the minimum dominating set and $\gamma(\overline{G}) = 3$. ■

Theorem 3.3.6 Let $\overline{G}$ be the complement of the maximum δ-divisor graph of order n, then, $\beta(\overline{G}) = m$, where m is the maximum number such that $2^m \leq n$.

Proof. From Theorem 3.2.1, the maximum clique subgraph in G isomorphic to the induced subgraph spanning by the set $S = \{v; f(v) = 2^i; 1 \leq i \leq$
$m;\ where\ m\ is\ the\ maximum\ number\ such\ that\ 2^m \leq n\}$. Thus, in $\bar{G}$, the set S is independent. Now, if the vertex u does not belong to the set S, there is at least one prime not equal to two divides the labeled of the vertex u. Thus, the vertex u is adjacent to some vertices in the set S. Therefore, the set S is the maximum independent set and $\beta(\bar{G}) = m$, *where m is the maximum number such that* $2^m \leq n$. ■

Note 3. 3.7 There is an inverse dominating set for all complements of the *maximum δ-divisor graph*.

Proposition 3.3.8 Let $\overline{G}$ be the complement of the maximum δ-divisor graph of order n, then, the pendant set contains two vertices if $n = 3,4$, otherwise there are no pendant vertices.

Proof. There are two cases as follows.

Case 1. If $n = 1,2$, then $\overline{G} \equiv N_n$ where N_n is null graph, so the is no pendant vertex.

Case 2. If $n = 3,4$, then $\overline{\mathrm{G}} - \{v_1\} \equiv N_i; i = 2,3$. Thus, the number of pendant set is two.

Case 3. $n > 4$, then each vertex $v_i; 2 < i < n$ is adjacent to at least two vertices v_{i-1} and v_{i+1}, so all these vertices are not pendant. The remaining vertices are v_2 and v_n, the vertex v_2 is adjacent to two vertices at least v_3 and v_5, since $n > 4$. The vertex v_n is adjacent to two vertices at least one of them is v_{n-1} and the other is v_{n-2}. Therefore, there is no pendant vertex (as an example, see Figure3.3) ■

Chapter Four

Conclusion and Future work

4 . 1 Conclusions

The Number Theory and Graph theory are a broad research field, and each time new and distinct ideas can be added from this work, the following conclusions can be drawn:

- The new results of δ-divisor graph are presented. These results are the number of pendant vertices, the clique number, and the chromatic number.
- Moreover, many properties in the complement of the maximum δ -divisor graph are discussed, like Hamiltonian and semi-Hamiltonian, the Gauss's function, and the number of pendant vertices.
- A new graph is constructed in this work, it is called The clique number of the maximum δ-divisor graph of order n is equal to m+1.
- A new graph is constructed in this work, The set of isolated vertices contains two vertices if n=2 , otherwise contains one vertex.
- The graph is $\overline{G} - \boldsymbol{v_1}$ Hamiltonian if n is odd and semi - Hamiltonian if n is even.

4 . 2 Future work

Future work necessary to :

1. Find more properties for divisor function graph and its inverse properties as well as for the domination of the edges and its inverse.
2. Create a new definition of the Divisor function graph by introducing more conditions and finding all the properties that are related to it and its inverse.
3. The search for the possibility of modified Divisor function graph by inserting the terms of known domination parameters such as the definition of connected domination, total domination and independent domination, etc
4. Study Divisor function in digraph.
5. Apply the definition of inverse Divisor function to special graphs and find their properties.

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الخلاصة

يركز عمل هذه الرسالة على استخدام بعض الوظائف الحسابية لنظرية الاعداد في نظرية الرسم البياني. من الواضح أن معالجة أي تطبيق لبعض مفاهيم نظرية الأعداد ليست دائمًا قابلة للتنفيذ .لقد تم مناقشة العديد من خصائص الرسم البياني بأستخدام دالة القسمة: $d(n) = \sum_{d/n} 1$ **(the Divisor Function)** لحساب عدد قواسم أي أعداد طبيعية مما يعني حساب عقد الرسم البياني . كان هناك الكثير من المؤلفين الذين عملوا على هذا الموضوع مثل G. Santhosh and G. Singh وآخرين. يعد استخدام دالة حسابية لإنشاء رسم بياني للشبكة أمرًا صعبًا بعض الشيء ، على الرغم من أنه يعطي طريقة قوية في نظرية الرسم البياني. لذلك تم استخدام دالة δ - divisor خلال هذا العمل ، تم تقديم بعض نتائج الرسم البياني للمقسوم عليه. لذلك تم استخدام نتائج هذه الأعداد من الرؤوس المعلقة ، وعدد الزمرة ، والرقم اللوني في هذا العمل . علاوة على ذلك ، اقترح هذا العمل بعض خصائص تكملة الرسم البياني الأقصى للمقسوم كما ناقش دالة Hamiltonian ، وشبه Hamiltonian ، ودالة Gauss، ، وأرقام (لونية ، واستقلالية ، وهيمنة ، وزمرة) وعدد الرؤوس المعلقة. من المشاكل التي لم تحل حتى الآن . يسمى تسلسل الأعداد الأولية الفعلية وتسلسل الأعداد الطبيعية المبنية منها بالنظام الأولي الفعلي . و هناك عدة تطبيقات لنظرية الأعداد في مجالات مختلفة .

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كلية التربية للعلوم الصرفة
قسم الرياضيات

تطبيقات بعض الدوال الحسابية في نظرية البيان

رسالة

مقدمة الى مجلس كلية التربية للعلوم الصرفة / جامعة بابل
كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات

من قبل

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بإشراف

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