Republic of Iraq
Ministry of Higher Education and
Scientific Research
Babylon University
College of Education of Pure Science
Department of Mathematics

# On The Distributive Semimodules

**A Thesis**

**Submitted to the Council of the College of Education for Pure Sciences in the University of Babylon as a Partial Fulfillment of the Requirements for the Degree of Master in Education / Mathematics**

*By*

*Ali Hasan Abdoul Khaleq Khandouh*

*Supervised by*

*Prof. Asaad Mohammed Ali Alhossaini (Ph.D.)*

**2023 A.D.** **1445 A.H.**

**بِسْمِ اللَّـهِ الرَّحْمَـٰنِ الرَّحِيمِ**

« يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ»

**صَدَقَ الله العَليّ العَظِيم**

**سورة المجادلة الآية :11**

## ***Dedication***

To the martyrs of the Popular Mobilization Forces during the events of the occupation of Mosul, and to all my teachers throughout my academic journey.

# *Acknowledgment*

I deeply appreciate the invaluable guidance and support provided by my teacher and supervisor, Prof. Dr. Asaad Mohammed Ali Alhossaini. Without his unwavering commitment and mentorship, I would not have achieved the milestones I have today. As a teacher, he served as an exemplary role model, and now, as a researcher, he continues to inspire me. His dedication, wise counsel, and willingness to devote his time and effort have been instrumental in shaping my academic and professional growth. I am truly grateful to have such a remarkable individual in my life and hold him in the highest regard.

I would also like to express my heartfelt gratitude to all those who have stood by me throughout my thesis and academic journey. To my family, loved ones, faculty members, and friends, their unwavering support, encouragement, and understanding have been sources of immense strength and motivation. Your belief in my abilities and your presence in my life have made a significant impact, and I am truly grateful for your love and support.

## Abstract

Many authors have extensively studied distributive modules and their properties. However, this work focuses on the concrete development of the concept of distributive property for semimodules. In addition to provid the definition, various characterizations of this property are presented, along with illustrative examples.

To obtain interesting results, certain conditions on semirings or semimodules were required, such as subtractive, semisubtractive, cancellative, *k*-cyclic, yoked, i-regular, and *k*-regular conditions. By exploring the relationship between the distributive semimodule over a local semiring and the hollow semimodule, valuable insights were gained. Furthermore, the connections between the distributive semimodule and distributed homomorphisms over intersection processes or inverse image distributed over addition were established.

It was demonstrated that any subsemimodule and factor semimodule of a distributive semimodule retains the distributive property. Additionally, a concept of weakly distributive semimodules was introduced and investigated. It was discovered that a semimodule is distributive if and only if each of its subsemimodules is weakly distributive. Moreover, by employing the supplemented concept, various properties of distributive semimodules were derived.

Finally, the properties of summand sum and summand intersection for distributive semimodules were explored under specific conditions, yielding valid results.

# *Contents*


| | | |
|---|---|---|
| **Abstract** | | III |
| Contents | | IV |
| **List of Symbols** | | V |
| **Introduction** | | 1 |
| Chapter 1 | **Preliminaries and Examples** | |
| **1.1** | Preliminaries | 3 |
| **1.2** | Examples of Semirings and Semimodules | 9 |
| Chapter 2 | **Structure and Properties of Distributive Semimodules** | |
| **2.1** | Some Characterizations and Properties of Distributive Semimodules | 12 |
| **2.2** | The Homomorphisms of Distributive Semimodules | 24 |
| **2.3** | Some Examples | 36 |
| Chapter 3 | **On Subsemimodules of Distributive Semimodule** | |
| **3.1** | Direct Sums of Distributive Semimodule | 39 |
| **3.2** | Weakly Distributive Semimodule | 44 |
| **3.3** | More on Supplement Property in Distributive Semimodules | 47 |
| **Conclusions** | | 53 |
| **Future Works** | | 54 |
| **References** | | 55 |

## List of Symbols

| Symbol | The meaning |
|---|---|
| $R$ | The Semiring |
| $R_R$ | The Semimodule $R$ over itself |
| $\mathbb{N}$ | The set of natural numbers |
| $\mathbb{Z}_n$ | The set of integers modulo n |
| $M$ | left $R$ -semimodule |
| $\leq^e$ | Essential subsemimodule |
| $ker\pi$ | Kernel of homomorphism $\pi$ |
| $\pi(M)$ | Image of $\pi$ |
| $im\ \pi$ | Extended image of $\pi$ |
| $Ra$ | Cyclic subsemimodule generated by $a$ |
| $J(R)$ | Jacobson radical of $R$ |
| $\ll$ | Small subsemimodule |
| $HOM(M, A)$ | The set of all homomorphisms from $M$ into $A$ |
| $End(M)$ | The endomorphisms semiring of $M$ |
| $\cap$ | intersection |

| | |
|---|---|
| $\in$ | **belong to** |
| $\subseteq$ | **Subset** |
| $\leq$ | **Subsemimodule** |
| $\longrightarrow$ | **Thus, implication** |
| $\oplus$ | **Direct sum** |
| $\boldsymbol{A}/\boldsymbol{M}$ | **Factor semimodule** |
| P(X) | **The power set of a set X** |
| $\blacksquare$ | **The end proof** |

## Introduction

The semimodule $M$ is said to be distributive if, for all subsemimodules $A, B,$ and $C$ of $M,$ the following equality holds: $A \cap (B + C) = A \cap B + A \cap C.$ A semiring is termed an arithmetical semiring if it can be regarded as a distributive semimodule over itself. This concept has not been explored independently for semimodules, it has been occasionally referenced in some research studies, such as [1], [2].

It is worth noting that this property has been studied in the context of rings for ideals before modules, referred to as a "distributive ring" or "arithmetical ring." According to Fuchs [3], a commutative ring $R$ with an identity is considered arithmetical if the lattice formed by the ideals of $R$ is distributive. Several studies have been conducted on this topic, including references [4], [5], [6], and [7].

In the modules, researchers have been interested since the 1970s in the distributive feature like W. Stephenson in [8]. The purpose of this paper is to investigate the characteristics of modules whose lattice of submodules is distributive. Specifically, it aims to examine rings $R$ for which $R_R$ is a distributive module.. Erdogdu in [9] show that if a ring $R$ has a finitely generated faithful and distributive module $M$, then $R$ is arithmetical and $M$ is projective of rank one. Gave some characterizations of distributive modules in terms of the order ideals of submodules and the homomorphisms of factor modules of submodules. The author A. A. Tuganbaev studied a concept named "distributive extensions "over rings not necessarily commutative [10]. Naser Zamani found the relationship between distributive modules and primal submodules in [11], while Engin Büyükaşık in [12] gave and studied a generalization of distributive property by defining weakly distributive module. The relationship between the distributive module and the Armendariz module was gotten by Buhphang [13]. In this work, some of those concepts and results will be converted to semimodules. In addition to other research addressing our topic in semimodules, we refer to the following sources: [14] [15], [16], [17], [18], and [19].

Throughout this study, we draw upon a wealth of research contributions related to distributive modules in order to shed light on the investigation of distributive semimodules. By bridging the gap between distributive modules and distributive semimodules, we aim to enrich our understanding of the topic and contribute to the existing body of knowledge.

This work consists of three chapters. Chapter one is divided into two sections. Section one introduces the necessary definitions for the thesis, while section two provides examples of semirings and semimodules.

Chapter two comprises three sections. Section one explores the structure and properties of distributive semimodules, presenting various characterizations and properties of these modules. Section two focuses on the homomorphisms of distributive semimodules. Finally, section three presents additional examples.

Chapter three is also divided into three sections. Section one delves into the topic of direct summands and subsemimodules of distributive semimodules. Section two introduces the concept of weakly distributive semimodules. Lastly, section three explores the supplement and distributive semimodules.

# Chapter one

## Preliminaries and Examples

**Introduction**

In this chapter, we will present the most well-known fundamental results and essential facts required for the comprehensive study of this dissertation. The chapter is divided into two sections. Section 1 encompasses the preliminary concepts and theories that form the foundation of our investigation. Section 2 deals into a review of various examples of semirings and semimodules. These examples will serve as valuable reference points and learning opportunities as we progress through the upcoming chapters of this dissertation.

# 1.1 Preliminaries

In this section, we will introduce some of the definitions needed in the main results. It is to begin with the definition of a semiring**.**

**Definition1.1.1.** [20, p. 1] A **semiring** is a nonempty set $R$ on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

1. $(R, +)$ is a **commutative monoid** with identity element 0;
2. $(R, \cdot)$ is a **monoid** with identity element $1 \neq 0$ $(1 = 1_R)$;
3. Multiplication distributes over addition, i.e. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$;
4. The element 0 is the absorbing element of the multiplication, i.e. $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$.

The semiring $R$ is said to be commutative if its multiplication is commutative.

**Definition 1.1.2.** [20, p. 3] A subset $S$ of a semiring $R$ is a **subsemiring** of $R$ if it contains 0, 1, and is closed under the operations of addition and multiplication in $R$.

**Definition 1.1.3.** [20, p. 65] A non-empty subset $I$ of a semiring $R$ will be called a left (resp. right ) **ideal** of $R$ if for $a, b \in I$, and $r \in R$ imply $a + b \in I$,and $ra$ (resp. $ar$) $\in I$. $I$ is ( two- sided) ideal if it is both a left and a right ideal of $R$.

**Definition 1.1.4.** [20, p. 49] A semiring R is called **yoked (semisubtractive)** if each $x, y \in R$, there exists $z \in R$ where $x + z = y$ or $y + z = x$.

**Definition 1.1.5.** [21] If $R$ is a semiring, and $I$ ideal of $R$, if $m, m + n \in I$ implies $n \in I$, for all $m, n \in R$, then the ideal $I$ is called subtractive. The semiring is called **subtractive** if each ideal is subtractive.

**Definition 1.1.6.** [22] A nonzero ideal $(I, +, .)$ of a semiring $(R, +, .)$ is called **maximal ideal**. If $I$ is a proper ideal and there exists no proper ideal of $R$ containing $I$. (i .e) $I$ is a maximal ideal of $R$ if $I \neq R$, and if $M$ is an ideal of $R$ such that $I \subset M \subseteq R$, then $M = R$.

**Definition 1.1.7.** [23] Let $R$ be a semiring. The intersection of all maximal ideals of $R$ is called the **Jacobson radical** of $R$ and is denoted by J($R$). If $R$ has no maximal ideal then J($R$)=$R$.

**Definition 1.1.8.** [20, p. 164] Let $\mathcal{D}$ be an $R$ - semimodule, $\mathcal{F} \in L(\mathcal{D})$, then $\mathcal{F}$ induces an $R$ -congruence relation $\approx_{\mathcal{F}}$ on $\mathcal{D}$ called the Bourne relation, defined by setting $a \approx_{\mathcal{F}} b$ if and only if there exist elements $i$ and $j$ of $\mathcal{F}$ such that $a + i = b + j$. If $a \in \mathcal{D}$ then we write ${}^{a}/_{\mathcal{F}}$ instead of ${}^{a}/_{\approx_{\mathcal{F}}}$. The **factor semimodule** ${}^{\mathcal{D}}/_{\approx_{\mathcal{F}}}$ is denoted by ${}^{\mathcal{D}}/_{\mathcal{F}}$.

**Definition 1.1.9.** [24] Let $R$ be a semiring. If $R$ has only one maximal ideal, then it is called **local** semiring.

**Definition 1.1.10.** [20, p. 149] Let $R$ be a semiring, a **left *R*-semimodule** is a commutative monoid $(M, +)$ with additive identity $0$ for which we have a function $R \times M \rightarrow M$ defined by $(r, m) \mapsto rx$ (scalar multiplication), which satisfies the following conditions for all elements $r, r' \in R$ and all elements $m, m' \in M$:

1. $(rr')x = r(r'm)$;

2. $r(m + m') = rm + rm'$;

3. $(r + r')m = rm + r'm$;

4. $r0_M = 0_M = 0_R m$.

If the condition $1_R m = m$ for all $m \in M$ hold then the semimodule $M$ is said to be unitary.

**Definition 1.1.11.** [20, p. 150] A non-empty subset $N$ of a left $R$-semimodule $M$ is called a **subsemimodule** of $M$ if $N$ is closed under addition and scalar multiplication that is $N$ is an $R$-semimodule itself (denoted by $N \leq M$). The set of all subsemimodules of $M$ is denoted by $\mathrm{L}(M)$.

**Definition 1.1.12.** [20, p. 49] An element $m$ of an $R$ −semimodule $M$ is called **cancellable** if for all $m, m' \in M$ with $m + m' = m + m''$ implies that $m' = m''$. The semimodule $M$ is cancellative if and only if every element of $M$ is cancellable.

**Definition 1.1.13.** [25] Let $M$ be an $R$-semimodule. A **subtractive** subsemimodule (or $k$ −subsemimodule) $N$ is a subsemimodule of $M$ such that if $z, z + w \in N$, then $w \in N$, and the semimodule $M$ is **subtractive** if each subsemimodule is subtractive.

**Definition 1.1.14.** [26]A subsemimodule $E$ of $R$ - semimodule $M$ is **superfluous** or **small** (denoted by $E \ll M$). If any $K \in \mathrm{L}(M)$, $K + E = M$ implies $K = M$.

**Definition 1.1.15.** [26] The $R$ - semimodule $M$ is said to be **hollow** if every proper $E \in \mathrm{L}(M)$ is superfluous.

**Definition 1.1.16.** [27] A left $R$ -semimodule $M$ is **Artinian,** if and only if $M$ satisfies the DCC (Descending chain condition) on its $R$ -subsemimodules.

**Definition 1.1.17.** [27] A left $R$ -semimodule $M$ is **Noetherian**, if and only if $M$ satisfies the ACC (Ascending chain condition) on its $R$ -subsemimodules.

**Remark 1.1.18.** [28] For a semimodule $M$ over a semiring $R$ and $A$ subsemimodule of $M$, the following statements are true:

1. $M$ is Artinian if and only if every non-empty set of subsemimodules of $M$ has a minimal element.
2. The semimodule $M$ is Artinian (noetherian) if and only if both left semimodules $A$ and $M/A$ are Artinian (noetherian).
3. $M$ is noetherian if and only if every non-empty set of subsemimodules of $M$ has a maximal element.

**Definition 1.1.19.** [25] A left $R$-semimodule $M$ is called **cyclic** if $M$ can be generated by a single element i.e. $M = (x) = Rx = \{rx \mid r \in R\}$ for some $x$ in $M$.

**Definition 1.1.20.** [25] A left $R$-semimodule $M$ is called **simple** over a semiring $R$ if it has no non-zero proper subsemimodules. Equivalently, a semimodule $M$ is simple if and only if every cyclic subsemimodule generated by a non-zero element of $M$ equals $M$.

**Definition 1.1.21.** [29] An $R$ -semimodule $M$ is called **uniserial** if for any two subsemimodules $H$ and $L$ of $M$, either $H \subseteq L$ or $L \subseteq H$.

**Definition 1.1.22.** [30] If $R$ is a semiring. The $R$-semimodule $M$ is **semisubtractive** if each $x, y \in M$, there exists $z \in M$ where $x + z = y$ or $y + z = x$.

**Definition 1.1.23.** [20, p. 3] An element $e$ of a semiring $(R, +, .)$ is called an **multiplicatively idempotent** element if $e^2 = e$.( In this work, we will use "**idempotent**" instead of "**multiplicatively idempotent**" for the sake of brevity and simplicity.)

**Definition 1.1.24.** [31] The $R$ - subsemimodule $V$ of $R$ - semimodule $D$ is called **large** or **essential** if $V \cap G = 0$, implies $G = 0$, for all $G \in L(D)$, denoted by $V \leq^e D$. In this case، we say that $D$ is an **essential extension** of $V$.

**Definition 1.1.25.** [32] A nonzero semimodule $M$ is called a **uniform** semimodule if the intersection of any two nonzero subsemimodules is nonzero.

This is equivalent to saying that every nonzero subsemimodule of $M$ is an essential subsemimodule.

**Definition 1.1.26.** [30] The $R$ - subsemimodule $V$ of $R$ - semimodule $M$ is called the **relative complement** of $F \in L(\mathcal{D})$, if $V \cap F = 0$ and $V$ is maximal with this property.

**Definition 1.1.27.** [20, p. 150] Let $M$ be $R$ - semimodule, and $h, k \in M$ we define :

1- $(Rh: k) = \{r \in R: rk \in Rh\}$.
2- $(0: h) = l(\mathrm{h}) = \{\mathrm{r} \in \mathrm{R}: \mathrm{rh} = 0\}$.

**Definition 1.1.28.** [20, p. 156] Let $M$ and $D$ be $R$ - semimodules the map $f: M \to D$ is called a homomorphism if for each $e, n \in M, s \in R$

1- $f(e+n) = f(e) + f(n)$.
2- $f(s\,n) = sf(n)$.

**Definition 1.1.29.** [33] For a homomorphism of $R$ - semimodules ɥ: $\mathcal{B} \to \mathcal{D}$ we define

1. ɥ$(\mathcal{B}) = \{$ ɥ$(e) \mid e \in \mathcal{B}\}$.
2. ker(ɥ) $= \{ e \in \mathcal{B} \mid$ ɥ$(e) = 0\}$.
3. $Im$(ɥ) $= \{k \in \mathcal{D} \mid k +$ ɥ$(e) =$ ɥ$(\acute{e})$ for some $e, \acute{e} \in \mathcal{B}\}$.
4. ɥ is ***i*-regular** if ɥ$(\mathcal{B}) = Im(\mathcal{B})$.
5. ɥ is ***k*-regular**, if ɥ$(e) =$ ɥ$(\acute{e})$ implies $e + h = \acute{e} + \acute{h}$ for some $h, \acute{h} \in$ ker(ɥ).
6. ɥ is **regular**, if ɥ is $i$-regular and $i$-regular.
7. $Hom\,(\mathcal{B}, \mathcal{D}) = \{$ɥ$: \mathcal{B} \to \mathcal{D} \mid$ ɥ is a homomorphism$\}$.
8. $End\,(\mathcal{B}) = \{$ɥ$: \mathcal{B} \to \mathcal{B} \mid$ ɥ is a homomorphis$\}$.
9. ɥ is a **monomorphism**, if and only if it is injective. [20, p. 169]
10. ɥ is an **epimorphism**, if and only if it is a surjective and ɥ$(\mathcal{B})$ is subtractive [20, p. 169]
11. ɥ is an **isomorphism** if ɥ is monomorphism and epimorphism. [20]

**Remark1.1.30.** [33] Let ɥ: $\mathcal{B} \to \mathcal{D}$ be homomorphism of $R$ – semimodules , then

1. $\ker(\text{ɳ})$, and $Im(\text{ɳ})$are subtractive subsemimodule of $\mathcal{B}$ and $\mathcal{D}$ respectively.
2. $\text{ɳ}(\mathcal{B})$ is a subtractive subsemimodule of $\mathcal{D}$ if and only if ɳ is $i$-regular.
3. ɳ is a monomorphism if $\ker(\text{ɳ}) = 0$, the converse is true if $\mathcal{B}$ is semisubtractive and $\mathcal{D}$ is cancellative.
4. ɳ is an epimorphism if ɳ is $i$-regular.
5. ɳ is a monomorphism, then ɳ is $k$-regular.
6. ɳ is a monomorphism if and only if ɳ is $k$-regular and $ker(\text{ɳ}) = 0$.

**Definition1.1.31.** Let $\alpha: A \to B$ and $\beta: B \to A$ be homomorphisms between left R-semimodules $A$ and $B$ have the property that $\alpha\beta$ is the identity map on $A$, we say that each of the maps is a splitting map for the other.

**Definition1.1.32.** [34] A subsemimodule $U$ of $M$ is called a **fully invariant** subsemimodule if $f(U) \subseteq U$ for every f $\in End\ (M)$.

**Definition1.1.33.** [34]A semimodule $U$ is said to be **duo** if each subsemimodule of $U$ is fully invariant.

**Definition1.1.34.** [35] If $\mathcal{D}$ is an $R-$semimodule, the set $\mathcal{D}\coprod\mathcal{D}' = \{(e, y): e \in \mathcal{D}, \in \mathcal{D}'\}$ is an $R$–semimodule with two operations of addition and scalar multiplication on $\mathcal{D}\coprod\mathcal{D}'$ setting $(e_1, y_1) + (e_2, y_2) = (e_1 + e_2, y_1 + y_2)$, and for all $r \in R$, $\text{r}(e, \text{y}) = (\text{r}e, \text{ry})$.

**Definition 1.1.35.** [36] Let $\mathcal{D}$ be an $R-$semimodule, and $\mathcal{J}, \mathcal{S} \in L(\mathcal{D})$. The subsemimodule $\mathcal{J}$ is called **supplement** of $\mathcal{S}$ in $\mathcal{D}$ if $\mathcal{J} + \mathcal{S} = \mathcal{D}$, and $\mathcal{J}$ is minimal with this property. That is, if $\mathcal{F} + \mathcal{S} = \mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{J}$, then $\mathcal{F} = \mathcal{J}$. A semimodule $\mathcal{D}$ is called supplemented if each $\mathcal{S} \in L(\mathcal{D})$ has a supplement in $\mathcal{D}$.

**Definition 1.1.36.** [36] A semimodule $\mathcal{D}$ is called an **amply supplemented** if whenever $\mathcal{F} + \mathcal{H} = \mathcal{D}$, then $\mathcal{F}$ has a supplement in $\mathcal{D}$ contained in $\mathcal{H}$.

**Definition 1.1.37.** [20, p. 184] Let $\mathcal{D}$ be an $R-$semimodule, and $\mathcal{L}$, $\mathcal{L}_* \in L(\mathcal{D})$. $\mathcal{D}$ is called a **direct sum** of $\mathcal{L}$and $\mathcal{L}_*$ , if each $u \in \mathcal{D}$ can be represented uniquely as $u = k + h$, where $k \in \mathcal{L}$ and $h \in \mathcal{L}_*$, then we can say that $\mathcal{L}$ (similarly $\mathcal{L}_*$) is a **direct summand** of $\mathcal{D}$, and denoted by $\mathcal{D} = \mathcal{L}\oplus\mathcal{L}_*$. It's clear that $\mathcal{D}$ and 0 are always direct summand of $\mathcal{D}$. If the only direct summand $\mathcal{D}$ and 0, then $\mathcal{D}$ is called **indecomposable.**

**Definition 1.1.38.** [30] A subsemimodule $\mathcal{K}$ of semimodule $\mathcal{D}$ is called **closed** if $\mathcal{K}$ has no proper essential extension in $\mathcal{D}$ (denoted by $\mathcal{K} \leq^{c} \mathcal{D}$).

**Definition 1.1.39.** [30] A subsemimodule $\mathcal{K}$ of a semimodule $\mathcal{D}$ is said to be a **closure** of a subsemimodule $\mathcal{N}$ in $\mathcal{D}$, if $\mathcal{K}$ is closed and $\mathcal{N}$ essential in $\mathcal{K}$.

## 1.2 Examples of Semirings and Semimodules

Rings with identity are clearly semirings, but there are many other interesting examples of semirings. We show some of them in this section.

Modules are clearly semimodules. Also, any semiring over itself is a semimodule. If $R$ is considered as a left $R$-semimodule it will be denoted by ${}_{R}R$. But there are many other interesting examples of semimodules.

**Example 1.2.1.** Let $B = \{0, 1\}$, where $1 + 1 = 1$ in $B$. Then $B$ is a semiring and $B$ is called the **boolean semiring**.

**Example 1.2.2.** [20, p. 5] With the usual operations of addition and multiplication of integers, the set $\mathbb{N}$ of nonnegative integers is a semiring. The same is true for the set $\mathbb{Q}^{+}$ of all nonnegative rational numbers, for the set $\mathbb{R}^{+}$ of all nonnegative real numbers, and, in general, for $S^{+} = S \cap \mathbb{R}^{+}$, where S is any subring of $\mathbb{R}$. Clearly $\mathbb{N}$ is a subsemiring of $\mathbb{Q}^{+}$ and $\mathbb{Q}^{+}$ is a subsemiring of $\mathbb{R}^{+}$. Note that $\{0, 1, 2, 3\} \cup \{x \in Q | x \geq 4\}$ is an example of a subsemiring of $\mathbb{R}^{+}$ which is not of the form $S^{+}$ for some subring S of $\mathbb{R}$.

**Remark 1.2.3.** [37] The only subtractive ideals of the semiring $(\mathbb{N}, +, \cdot)$ are the cyclic ideals.

**Example 1.2.4.** [31] Let $R = \{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$ and $n \geq 2$ and define $x \oplus y = min\{x, y\}$ and $x \odot y = max\{x, y\}$ for all $x, y \in R$. $(R, \oplus, \odot)$ is a semiring with $0_R = n, 1_R = 0$.

**Example 1.2.5.** [20, p. 60] If $T$ is a topology on a nonempty set $Y$ then $(T, \cup, \cap)$ is a semiring with additive identity $\emptyset$ and multiplicative identity $Y$.

**Example 1.2.6.** [31] Let $R = \{0, h, 1\}$ define addition and multiplication operations on R as follows.

$i$) $0_R = 0, 1_R = 1;$

ii) $1 + 1 = 1 + h = 1, h + h = h;$

iii) $0.0 = 0 . h = 0, h . h = h.$

Then $(R, +, .)$ is a commutative semiring.

Let $\mathcal{D} = \{0, 1, h, k\}$ with the same operations defined in R and

1) $0_{\mathcal{D}} = 0_R = 0;$

2) $k + k = k, k + 1 = k + h = h;$

3) $0 . k = h. k = 0, 1. k = k.$

It is easy to see that $(\mathcal{D},+)$ is an $R-$semimodule.

**Example 1.2.7.** [20] The monoid $M = (\mathbb{Z}_n, +_n)$ is $\mathbb{N}$-semimodule, $\mathbb{N}$ is the semiring of nonnegative integers with the usual operations of addition and multiplication.

**Example 1.2.8.** [20] Let $\mathbb{N}$ be a semiring of non-negative integers with the usual operations of addition and multiplication. Let $h, k \in \mathbb{N}$, and let $h \sqcup k$ be the greatest common divisor of $h$ and $k$ in $\mathbb{N}$. The monoid $M = (\mathbb{N}, \sqcup)$ is $\mathbb{N}$ - semimodule.

**Example 1.2.9.** [20] The set $\mathbb{N}$ of all nonnegative integers with addition and multiplication given by $n + m = max\{n, m\}, n \cdot m = nm$, for all $n, m \in \mathbb{N}$. It's clear that $R = (\mathbb{N}, +, \cdot)$ is a semiring. Hence, $R$ is a left $R$-semimodule.

**Example 1.2.10.** [20, p. 159] If $M$ and $N$ are left $R$-semimodules then we will denote the set of all $R$-homomorphisms from $M$ to N by $Hom_R(M, N)$. If α and β belong to $Hom_R(M,N)$ then so does the map $\alpha + \beta$ from $M$ to $N$ which is defined by $(\alpha + \beta)m = \alpha m + \beta m$. It is easy to check that $(Hom_R(M,N),+)$ is an $R$-semimodule. Hence we have also $(End_R(M),+)$ is an $R$-semimodule.

**Example 1.2.11.** If $(Z_2,+_2,._2)$ is a semiring. Let $(X,+)$be commutative monoid, where $X = \{0,1,2,\ldots\ldots,k\}$, $k \in \mathbb{N}$, and $h + k = \max\{h,k\}$ for all $h,k \in X$. Define multiplication operation as follows: $0_{Z_2}.k$=$0_X$ and $1_{Z_2}.k = k$, for all $k \in X$. Then $X$ is a $Z_2$–semimodule.

**Example 1.2.12.** Let R = $\{0,1\}$ be boolean semiring, recall that $(a) = \{an: n \in \mathbb{N}\}$. If $\mathcal{D} = \{(0),(2),(4),(8)\}$, where $(\mathcal{D},\cup)$ is a commutative monoid. Define multiplication operation as follows: $0.a = 0, 1.a = a$, for all $a \in \mathcal{D}$. Then $\mathcal{D}$ is a R–semimodule.

# Chapter two

## Structure and Properties of Distributive Semimodules

## Introduction

This chapter, reviewed the definition of a distributive semimodule, along with some characterizations, properties, and examples. Certain results obtained in homomorphisms of distributive modules were generalized under specific conditions. We aimed to minimize the conditions required for generalization. To avoid the need for a subtractive semimodule condition, we introduced the concept of a *k*-cyclic semimodule. Various distributive semimodules can be easily observed, including simple semimodules, uniserial semimodules, and subsemimodules of distributive semimodules. The chapter also explores the relationship between the distributive semimodule and otherconcepts such as the Noetherian( Artinian, hollow) semimodules, and local semiring.

Also, the chapter begins with characterizations and properties of distributive semimodules, followed by the study of homomorphisms. Finally, several examples are provided for illustration.

## 2.1 Some characterizations and properties of distributive semimodules

**Definition 2.1.1.** [1] Let $R$ be a semiring. A left $R$-semimodule $\mathcal{D}$ is called a distributive semimodule if, for all subsemimodules $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ of $\mathcal{D}$, the following equality holds: $\mathcal{A} \cap (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$.

**Remark 2.1.2.** Consider an $R$ -semimodule $\mathcal{D}$ and subsemimodules $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ of $\mathcal{D}$. We note that $\mathcal{A} \cap (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$ is equivalent to $\mathcal{A} + (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} + \mathcal{B}) \cap (\mathcal{A} + \mathcal{C})$. To see this, assume $\mathcal{A} \cap (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$. Then by hypothesis, we have $(\mathcal{A} + \mathcal{B}) \cap (\mathcal{A} + \mathcal{C}) = [(\mathcal{A} + \mathcal{B}) \cap \mathcal{A}] + [(\mathcal{A} + \mathcal{B}) \cap \mathcal{C}] = \mathcal{A} + (\mathcal{A} \cap \mathcal{C}) + (\mathcal{B} \cap \mathcal{C}) = \mathcal{A} + (\mathcal{B} \cap \mathcal{C})$. Conversely, if $\mathcal{A} + (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} + \mathcal{B}) \cap (\mathcal{A} + \mathcal{C})$, then we have $(\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C}) = [(\mathcal{A} \cap \mathcal{B}) + \mathcal{A}] \cap [(\mathcal{A} \cap \mathcal{B}) + \mathcal{C}] = \mathcal{A} \cap [(\mathcal{A} \cap \mathcal{B}) + \mathcal{C}] = \mathcal{A} \cap [(\mathcal{C} + \mathcal{B}) \cap (\mathcal{C} + \mathcal{A})] = \mathcal{A} \cap (\mathcal{B} + \mathcal{C})$. Therefore, proving either one of the equivalencies above suffices to show that the semimodule is distributive.

**Lemma 2.1. 3.** [38] For any $k$-regular homomorphism $f : M \rightarrow N$ from a subtractive semimodule $M$ to a semimodule $N$ , and any subsemimodule $A$ of $M$, then $f^{-1} f(A) = A + Ker(f)$.

**Lemma 2.1. 4.** Let $U$ and $A$ be $R$ – semimodules and μ∈ $Hom\ (U, A\ )$, and $M \in L(A)$,

then $\mu\left(\mu^{-1}(M)\right) = M \cap \mu(U)$.

**Proof.** The same as in the modules (see [39, p. 44]**).** ■

**Remark 2.1.5:** If μ∈ $Hom\ (U, A\ )$, where $U$ and $A$ are $R$ – semimodules, then:

i) For $W, N \in L(U)$, $\mu(W + N) = \mu(W) + \mu(N)$,

ii) for $Q, S \in L(A)$, $\mu^{-1}(Q \cap S) = \mu^{-1}(Q) \cap \mu^{-1}(S)$,

iii) $\mu^{-1}\left(\mu(U)\right) = U$,

**Proof.** it's clear. ■

**Proposition 2.1. 6.** Let $U$ and $A$ be $R$-semimodules, μ a $k$-regular homomorphism from $U$ to $A$

i. If $A$ is distributive, $W$ and $N$ subsemimodules of $A$ with $\mu^{-1}(W) + \mu^{-1}(N)$ subtractive, then $\mu^{-1}(W + N) = \mu^{-1}(W) + \mu^{-1}(N)$.

ii. If $U$ is distributive, $T$ and $S$ subsemimodules of $U$, then $\mu(T \cap S) = \mu(T) \cap \mu(S)$.

**Proof.**

(i) It is clear that $\mu^{-1}(W) + \mu^{-1}(N) \subseteq \mu^{-1}(W + N)$.

Let $x \in \mu^{-1}(W + N)$, then $\mu(x) \in W + N$. So $\mu(x) \in (W + N) \cap \mu(U)$.

Since $A$ is distributive, $\mu(x) \in [W \cap \mu(U) + N \cap \mu(U)]$, by Lemma 2.1.4 then $\mu(x) \in [\mu(\mu^{-1}(W)) + \mu(\mu^{-1}(N))]$ this mean $\mu(x) = \mu(x_1) + \mu(x_2)$ with $x_1 \in \mu^{-1}(W)$, $x_2 \in \mu^{-1}(N)$ by Remark 2.1.5 $\mu(x) = \mu(x_1 + x_2)$.

By hypothesis, μ a $k$-regular, hence $x + k_1 = x_1 + x_2 + k_2$ for some $k_1, k_2 \in ker\mu$. Now, $x_1 + x_2 + k_2 \in \mu^{-1}(W) + \mu^{-1}(N)$ (since $ker\mu \leq \mu^{-1}(N)$ ), so $x + k_1 \in \mu^{-1}(W) + \mu^{-1}(N)$. But $k_1 \in ker\mu \subseteq \mu^{-1}(W) + \mu^{-1}(N)$, then $x \in \mu^{-1}(W) + \mu^{-1}(N)$(by subtractive property) .

(ii) It is clear that $\mu(T \cap S) \subseteq \mu(T) \cap \mu(S)$. Let $y \in \mu(T) \cap \mu(S)$, then $y = \mu(\mathrm{t}) = \mu(\mathrm{s})$, $\mathrm{t} \in T$, $s \in S$. Since , μ is $k$-regular, $t + k_1 = s + k_2$ for some $k_1, k_2 \in ker\mu$. Then $t + k_1 \in (T + ker\mu) \cap (S + ker\mu)$, since $U$ is distributive, so $t + k_1 \in T \cap S + ker\mu$ hence $t + k_1 = x + k_3$ where $x \in T \cap S$, $k_3 \in ker\mu$. Then $\mu(\mathrm{t}) = \mu(x)$implies $y = \mu(x) \in \mu(T \cap S)$. Therefore $\mu(T \cap S) = \mu(T) \cap \mu(S)$

**Proposition 2.1. 7.** Let $\mathcal{D}$ and $\mathcal{M}$ be $R$- semimodules, and $\mathcal{N}, \mathcal{H} \in L(\mathcal{D})$.
Consider the following statements:

i. $\mathcal{D}$ is a distributive semimodule.
ii. For any homomorphism $f: \mathcal{D} \to \mathcal{M}$ from a subtractive semimodule $\mathcal{D}$ to a semimodule $\mathcal{M}$ satisfies that $f(N \cap H) = f(N) \cap f(H)$
iii. For any $g \in Hom(\mathcal{M}, \mathcal{D})$ , then $g^{-1}(\mathcal{N} + \mathcal{H}) = g^{-1}(\mathcal{N}) + g^{-1}(\mathcal{H})$ ).
iv. $\mathcal{D}$ and $\mathcal{M}$ are modules and $\mathcal{D}$ is distributive.

Then the following implications: iii → i, ii → i, iv → ii, and iv → iii hold.

**Proof.** (iii→i)

If $\mathcal{B}, \mathcal{N}, \mathcal{H} \in L(\mathcal{D})$ and $\sigma: \mathcal{B} \to \mathcal{D}$ be the inclusion map, so

$\mathcal{B} \cap (\mathcal{N} + \mathcal{H}) = \sigma^{-1}(\mathcal{N} + \mathcal{H}) = \sigma^{-1}(\mathcal{N}) + \sigma^{-1}(\mathcal{H}) = \mathcal{B} \cap \mathcal{N} + \mathcal{B} \cap \mathcal{H}$.

(ii→i)

Let $\mathcal{B}, \mathcal{N}, \mathcal{H} \in L(\mathcal{D})$ and let $\pi: \mathcal{D} \to \mathcal{D}/_{\mathcal{N}}$ be the natural epimorphism

Since $\pi$ is $k$-regular and $\mathcal{D}$ subtractive by Lemma 2.1.3 we have:

$\pi^{-1}\big(\pi(\mathcal{B} \cap \mathcal{H})\big) = \mathcal{B} \cap \mathcal{H} + ker\pi = \mathcal{B} \cap \mathcal{H} + \mathcal{N}$ ….*

Also, by hypotheses $\pi(\mathcal{B} \cap \mathcal{H}) = \pi(\mathcal{B}) \cap \pi(\mathcal{H})$

$\pi^{-1}\big(\pi(\mathcal{B} \cap \mathcal{H})\big) = \pi^{-1}\big(\pi(\mathcal{B}) \cap \pi(\mathcal{H})\big) = \pi^{-1}(\pi(\mathcal{B})) \cap \pi^{-1}(\pi(\mathcal{H})$

$= (\mathcal{B} + ker\pi) \cap (\mathcal{H} + ker\pi) = (\mathcal{B} + \mathcal{N}) \cap (\mathcal{H} + \mathcal{N})$ ….**

Therefore, the distributivity of $\mathcal{D}$ follows from the equivalence of the two expressions (*) and (**).

(iv → ii )

Let $f \in Hom(\mathcal{D}, \mathcal{M})$, such that $f(m) = f(n)$, where $m, n \in \mathcal{D}$

Since $\mathcal{D}$ and $\mathcal{M}$ are modules we have,

$f(m) - f(n) = 0$ which implies that

$f(m - n) = 0$. Therefore, $m - n \in kerf$

Let $k = m - n$, so $f$ is $k$-regular. By the Proposition2.1.6, we have

$f(\mathcal{N} \cap \mathcal{H}) = f(\mathcal{N}) \cap f(\mathcal{H})$ for any $\mathcal{N}, \mathcal{H} \in L(\mathcal{D})$.

(iv → iii )

Now, since $g$ is $k$-regular, Proposition 2.1.6 implies that $g^{-1}(\mathcal{N} + \mathcal{H}) = g^{-1}(\mathcal{N}) + g^{-1}(\mathcal{H})$. ■

**Proposition 2.1. 8:** Let $\mathcal{D}$ and $M$ be $R$ – semimodule, and q $\in Hom\ (\mathcal{D}, M)$, then.

i) If $\mathcal{D}$ is a distributive semimodule and $q$ is $k-$ regular , then q($\mathcal{D}$) is a distributive semimodule.

ii) If $M$ is a distributive semimodule, $\mathcal{D}$ is subtractive semimodule , and $q$ is a monomorphism, then $\mathcal{D}$ is distributive.

**Proof:**

i) Let $A, B,$ and $C \in L(q(\mathcal{D}))$, since

$$A \cap (B + C) = A \cap q(D) \cap \ [B \cap q(D) + C \cap q(D)]$$

$$= q\left(q^{-1}(A)\right) \cap \left[q\left(q^{-1}(B)\right) + q\left(q^{-1}(C)\right)\right] \text{ ( by Lemma 2.1.4)}$$

by Proposition 2.1.6

$$A \cap (B + C) \ = q[q^{-1}(A) \cap \left(q^{-1}(B) + q^{-1}(C)\right)]$$

$$= q[q^{-1}(A) \cap q^{-1}(B) + q^{-1}(A) \cap q^{-1}(C)] \text{(since } \mathcal{D} \text{ is a distributive)}$$

$$= q[q^{-1}(A \cap B) + q^{-1}(A \cap C)] \text{( by Remmark 2.1.5)}$$

$$= q(q^{-1}(A \cap B)) + q(q^{-1}(A \cap C))$$

$$= (A \cap B) \cap q(\mathcal{D}) + (A \cap C) \cap q(\mathcal{D}) \text{(by Lemma 2.1.4)}$$

$$= (A \cap B) + (A \cap C)$$

ii) Let $W, N,$ and $Z \in L(\mathcal{D})$, put $E = q(W), T = q(N)$ and $V = q(Z) \in L(M)$, by monomorphism then $q^{-1}(E) = W, q^{-1}(T) = N,$ and $q^{-1}(V) = Z$, and $q$ is $k-$ regular by Remark 1.1.30 then

$$W \cap (N + Z) = q^{-1}(E) \cap \left(q^{-1}(T) + q^{-1}(V)\right) = q^{-1}(E) \cap \ q^{-1}(T + V)$$

by Proposition 2.1.6

$$W \cap (N + Z) = q^{-1}(E \cap \ (T + V)) = q^{-1}(E \cap T + E \cap V)$$

$$= \mathrm{q}^{-1}(E \cap T) + \mathrm{q}^{-1}(E \cap V)$$

$$= \mathrm{q}^{-1}(E) \cap \mathrm{q}^{-1}(T) + \mathrm{q}^{-1}(E) \cap \mathrm{q}^{-1}(V) = W \cap N + W \cap Z. \quad \blacksquare$$

**Corollary 2.1. 9.** Let $\mathcal{D}$ and $M$ be subtractive $R$ −semimodules with $M$ isomorphic to $\mathcal{D}$, then $\mathcal{D}$ is distributive if and only if $M$ is distributive.

**Proof.** It is clear by using Proposition 2.1.8. ■

**Proposition 2.1. 10.** Let $\mathcal{D}$ be an $R$ -semimodule. If for any two elements $x, y \in \mathcal{D}$ there are $t, v, r, s \in$ R such that $v + t = 1$, $tx = ry$, and $vy = sx$, then $\mathcal{D}$ is a distributive semimodule.

**Proof.** Assume that $\mathcal{A}, \mathcal{B}, C \in L(\mathcal{D})$ and $u \in \mathcal{A} \cap (\mathcal{B} + \mathcal{C})$. We need to show that $u \in (\mathcal{A} \cap \mathcal{B} + \mathcal{A} \cap \mathcal{C})$. Let $u = x + y$, where $x \in \mathcal{B}$, $y \in \mathcal{C}$. By hypothesis, there are $t, v \in$ R such that $v + t = 1$, $tx = ry$ so $tx \in$ Ry, $vy = sx$ so $vy \in$ R$x$, implies that $vu = vx + vy \in$ R$u \cap$ R$x \subseteq \mathcal{A} \cap \mathcal{B}$ and $tu = tx + ty \in$ R$u \cap$ R$y \subseteq \mathcal{A} \cap \mathcal{C}$, and $vu + tu = u \in \mathcal{A} \cap \mathcal{B} + \mathcal{A} \cap \mathcal{C}$. ■

**Proposition 2.1. 11.** If $\mathcal{D}$ is a distributive $R$− semimodule, and $\mathcal{O} \in L(\mathcal{D})$, then

i) $\mathcal{O}$ is distributive.
ii) $\mathcal{D}/_{\mathcal{O}}$ distributive if $\mathcal{O}$ is subtractive.

**Proof:**

i) It is clear since subsemimodules of $\mathcal{O}$ are subsemimodules of $\mathcal{D}$.
ii) Let $\pi: \mathcal{D} \to \mathcal{D}/_{\mathcal{O}}$ be the natural epimorphism. Since $\mathcal{D}$ is a distributive semimodule and π is $k$-reguler, then by Proposition 2.1.8 $\pi(\mathcal{D}) = \mathcal{D}/_{\mathcal{O}}$ is a distributive semimodule. ■

**Proposition 2.1. 12.** If $\mathcal{D}$ is a uniserial $R$− semimodule, then $\mathcal{D}$ is distributive.

**Proof.** Let $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \in L(\mathcal{D})$, we have to show that $\mathcal{J}_1 \cap (\mathcal{J}_2 + \mathcal{J}_3) \subseteq (\mathcal{J}_1 \cap \mathcal{J}_2) + (\mathcal{J}_1 \cap \mathcal{J}_3)\ldots *$( since the other direction is satisfied). Since $\mathcal{D}$ is uniserial, it is enough to discuss only two cases :

**Cas1:** $\mathcal{J}_2 \subseteq \mathcal{J}_3$ or $\mathcal{J}_3 \subseteq \mathcal{J}_2$, then $\mathcal{J}_2 + \mathcal{J}_3 = \mathcal{J}_3$ or $\mathcal{J}_2 + \mathcal{J}_3 = \mathcal{J}_2$, hence $\mathcal{J}_1 \cap (\mathcal{J}_2 + \mathcal{J}_3) = \mathcal{J}_1 \cap \mathcal{J}_3$ or $\mathcal{J}_1 \cap \mathcal{J}_2 \subseteq (\mathcal{J}_1 \cap \mathcal{J}_2) + (\mathcal{J}_1 \cap \mathcal{J}_3)$.

**Cas2:** $\mathcal{J}_1 \subseteq \mathcal{J}_2$ or $\mathcal{J}_2 \subseteq \mathcal{J}_1$,

if $\mathcal{J}_1 \subseteq \mathcal{J}_2$ implies $\mathcal{J}_1 \cap (\mathcal{J}_2 + \mathcal{J}_3) = \mathcal{J}_1$ and $(\mathcal{J}_1 \cap \mathcal{J}_2) + (\mathcal{J}_1 \cap \mathcal{J}_3) = \mathcal{J}_1 + \mathcal{J}_1 \cap \mathcal{J}_3$, hence $\mathcal{J}_1 \cap (\mathcal{J}_2 + \mathcal{J}_3) \subseteq (\mathcal{J}_1 \cap \mathcal{J}_2) + (\mathcal{J}_1 \cap \mathcal{J}_3)$,

if $\mathcal{J}_2 \subseteq \mathcal{J}_1$, so either $\mathcal{J}_3 \subseteq \mathcal{J}_2$, then $\mathcal{J}_1 \cap (\mathcal{J}_2 + \mathcal{J}_3) = \mathcal{J}_2$ and $(\mathcal{J}_1 \cap \mathcal{J}_2) + (\mathcal{J}_1 \cap \mathcal{J}_3) = \mathcal{J}_2 + \mathcal{J}_3 = \mathcal{J}_2$ or $\mathcal{J}_2 \subseteq \mathcal{J}_3$, then $\mathcal{J}_1 \cap (\mathcal{J}_2 + \mathcal{J}_3) = \mathcal{J}_1 \cap \mathcal{J}_3$ and $(\mathcal{J}_1 \cap \mathcal{J}_2) + (\mathcal{J}_1 \cap \mathcal{J}_3) = \mathcal{J}_2 + \mathcal{J}_1 \cap \mathcal{J}_3$. Therefore, in all case * is satisfied. ■

**Proposition 2.1. 13:** If $\mathcal{D}$ is a simple $R-$ semimodule, then $\mathcal{D}$ is distributive.

**Proof:** It is clear. ■

**Lemma 2.1. 14.** [20, p. 156] Let $\mathcal{D}$ be an $R$ -semimodule, $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3 \in L(\mathcal{D})$ satisfying the condition that $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and $\mathcal{H}_2$ is subtractive, then $\mathcal{H}_2 \cap (\mathcal{H}_1 + \mathcal{H}_3) = \mathcal{H}_1 + (\mathcal{H}_2 \cap \mathcal{H}_3)$.

**Lemma 2.1. 15:** [26] Let $R$ be a semiring. Then $J(R)$ is small in $R$.

**Corollary 2.1. 16:** For a semiring $R$, every ideal subset of $J(R)$ is small in $R$.

**Proof:** By Lemma 2.1.15 ■

In order to get some results, a condition weaker than subtractive semimodule is needed, that is, only some kind of subsemimodules to be subtractive. In the following such condition will be introduced. A semimodule is *k*- cyclic if any cyclic subsemimodule of it is subtractive. For example the ℕ-semimodule ℕ is k-cyclic but is not subtractive.

**Lemma 2.1. 17.** Let $\mathcal{D}$ be an $R$ -semimodule, then

i) $l(h+k) \subseteq (Rh{:}\,k) \cap (Rk{:}\,h)$, for all $h,\ k \in U,$ if $\mathcal{D}$ is a $k-$ cyclic.
ii) $l(h) \subseteq (Rk{:}\,h)$, for all $h,\ k \in \mathcal{D}$.

**Proof:**

i) Assume $r \in l(h+k), then\ \ r(h+k) = 0 = rh + rk$

as $rh, 0 \in Rh$ , by subtractive then $rk \in Rh$ so $r \in (Rh\colon k)$, and in the same way, we obtain $r \in (Rk\colon h)$, then $r \in (Rh\colon k) \cap (Rk\colon h)$.

ii) Assume $r \in l(h)$, then $rh = 0$, so $rh \in Rk$ then $r \in (Rk\colon h)$. ■

**Lemma 2.1. 18.** Let $R$ be a yoked cancellative semiring, and Î a subtractive ideal of $R$. If $l(c) \subseteq$ Î,$c \in R$,and $Rc =$ Î$c$,then $R =$ Î.

**Proof.** If $R \neq$ Î, then there exists $a \in R$ and $a \notin$ Î, but $ac = sc$ for some $s \in$ Î,

since $a, s \in R$ there exists $h \in R \ni a + h = s \ or \ s + h = a$, if $a + h = s$ then $ac + hc = sc$, hence $hc = 0$, so h $\in l(c) \subseteq$ Î, by subtractive $a \in$ Î, similarly if s $+$ h $= a$. This is a contradiction, therefore $R =$ Î. ■

**Lemma 2.1. 19.** Let $R$ be a yoked, subtractive , cancellative semiring, and $\mathcal{D}$ a $k-$ cyclic $R$ -semimodule, if $Rh + Rk = R(h+k) + (Rh \cap Rk)$ for all $h,\ k \in \mathcal{D}$, then $R = (Rh\colon k) + (Rk\colon h)$.

**Proof.** Since $Rh = Rh \cap (\,Rh + Rk)$, and by hypothesis $Rh = Rh \cap [R(h+k) + (Rh \cap Rk)]$. By Lemma 2.1.14 we get

$$Rh = [\,Rh \cap R(h+k)]\ + (Rh \cap Rk)$$

Now,

$$[\,Rh \cap R(h+k)]\ + (Rh \cap Rk) = [(Rh\colon (h+k))(h+k)] + (Rh \cap Rk)$$

$$= (Rh\colon k)(h+k)] + (Rh \cap Rk) = (Rh\colon k)(h+k)] + (Rk\colon h)h$$

$$= [(Rh\colon k) + (Rk\colon h)]h$$

By Lemma 2.1.16 and Lemma 2.1.17 we get $R = (Rh\colon k) + (Rk\colon h)$. ■

In the following theorem we give a characterization of distributive R– semimodule.

**Theorem 2.1. 20.** Let $R$ be a semiring, then $\mathcal{D}$ is a distributive $R$ – semimodule, if and only if $R(h+k) = \big(Rh \cap R(h+k)\big) + (Rk \cap R(h+k))$, for all $h,\ k \in \mathcal{D}$.

**Proof.** Suppose that $\mathcal{D}$ is a distributive $R$ – semimodule, and $h,\ k \in \mathcal{D}$, then

$$R(h+k) = (Rh + Rk)\ \cap R(h+k) = (Rh \cap R(h+k)) + (Rk \cap R(h+k)).$$

Conversely, let $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in L(\mathcal{D})$. It is clear $(\mathcal{W}_3 \cap \mathcal{W}_1) + (\mathcal{W}_3 \cap \mathcal{W}_2) \subseteq \mathcal{W}_3 \cap (\mathcal{W}_1 + \mathcal{W}_2)$, so we need to show the other direction $\mathcal{W}_3 \cap (\mathcal{W}_1 + \mathcal{W}_2) \subseteq (\mathcal{W}_3 \cap \mathcal{W}_1) + (\mathcal{W}_3 \cap \mathcal{W}_2)$. Assume that $z = h + k \in \mathcal{W}_3 \cap (\mathcal{W}_1 + \mathcal{W}_2)$ where $z \in \mathcal{W}_3, h \in \mathcal{W}_1,$ and $k \in \mathcal{W}_2$. $Rz = R(h+k)$ by hypothesis $Rz = (Rh \cap Rz) + (Rk \cap Rz)$,then $z \in (\mathcal{W}_3 \cap \mathcal{W}_1) + (\mathcal{W}_3 \cap \mathcal{W}_2)$. ■

**Lemma 2.1. 21.** Let $U$ and $A$ be $R$ – semimodules. If $\omega,$ and $\mu \in Hom\ (U, A\ )$ where $\omega$ is *i*-regular, then $(\omega + \mu)^{-1}\big(\omega\ (U)\big) = \ \mu^{-1}\big(\omega(U)\big)$.

**Proof.** $(\omega + \mu)^{-1}\big(\omega\ (U)\big) = \{\ e\ \in U : (\omega + \mu)(e) \in \omega(U)\}$

$= \{\ e\ \in U : \omega(e) + \mu(e) \in \omega(U)\}$

$= \{\ e\ \in U : \mu(e) \in \omega(U)\}$ (By Remark 1.1.30)

$= \ \mu^{-1}\big(\omega(U)\big)$. ■

**Proposition 2.1. 22.** Let $U$ be a subtractive $R$ - semimodule, $A$ is a distributive $R$ – semimodule. If $\mu, \omega\ \in Hom\ (U, A)$ such that $\mu +\ \omega$ is *k*-regular. Then,

i) $U = \ \omega^{-1}\ \mu(U) + \ \mu^{-1}\ \omega(U)$.

ii) $C = C \cap\ \ \omega^{-1}\big(\mu(C)\big) + \ C \cap\ \mu^{-1}(\ \omega(C))$, for any $C \in L(U)$.

**proof.**

i) since $(\mu+\ \omega)(U)=\mu(U)+\ \omega\ (U)$ , by Remark 2.1.5 we get

$$(\mu+\ \omega)^{-1}[\ (\mu+\ \omega)(U)]=\ (\mu+\ \omega)^{-1}[\mu(U)+\ \omega\ (U)],\ \text{so}$$

$$U=(\mu+\ \omega)^{-1}\left(\ \mu(U)+\ \omega(U)\right)=(\mu+\ \omega)^{-1}\mu(U)+\ (\mu+\ \omega)^{-1}\ \omega(U).$$

By Lemma (2.1.21) we get $U=\ \ \omega^{-1}\ \mu(U)+\ \mu^{-1}\ \omega(U).$

ii) Let $\mu i$ and $\omega i$ be homomorphisms such that $i: C\to U$ inclusion map.

Now, apply (i) we get

$$C=\ (\omega i)^{-1}\left(\mu i(C)\right)+\ (\mu i)^{-1}\left(\omega i(C)\right)$$

$$=C\ \cap\ \omega^{-1}\left(\mu(C)\right)+C\ \cap\ \mu^{-1}\left(\omega(C)\right).\ ■$$

**Corollary 2.1. 23.** If $U$ is a distributive $R$-semimodule, $\omega\in End\ (U)$ is $i$ −regular. Then $B=\left(B\cap\omega\ (B)\right)+\left(\ B\ \cap\omega^{-1}(B)\right)$.

**Proof.** By using Proposition 2.1.22 with $\mu=i$ the identity on $U$. ■

**Proposition 2.1. 24**. If $\mathcal{D}$ is a $k-$ cyclic distributive $R-$ semimodule, then $R=(Rh: k)+(Rk: h)$, for some $h,\ k\in\mathcal{D}$.

**Proof.** Let $\xi,£\in\mathrm{Hom}(R,\mathcal{D})\ni\xi(1)=\mathrm{h}$, and $£(1)=k$.

$$£^{-1}\xi(R)=\{r\in R: £(r)\in Rh\}=\{r\in R: rk\in Rh\}=(Rh: k),$$

similarly

$$\xi^{-1}£(R)=(Rk: h).$$

Since $\mathcal{D}$ is a $k-$ cyclic, then $\xi,£$ are $i-$ regular. By Proposition 2.1.22 we get $R=\ \ £^{-1}\ \xi(R)+\ \xi^{-1}\ £(R)$, implies that $R=(Rh: k)+(Rk: h)$. ■

**Theorem 2.1. 25.** Let $R$ be a yoked, subtractive, and cancellative semiring, if $\mathcal{D}$ is a subtractive $R$ -semimodule, then the following are equivalent:

i) $R=(Rh: k)+(Rk: h)$, for some $h,\ k\in\mathcal{D}$.

ii) $R(h+k) = \big(Rh \cap R(h+k)\big) + (Rk \cap R(h+k))$, for all $h,\ k \in \mathcal{D}$.

iii) $\mathcal{D}$ is a distributive $R-$ semimodule.

**Proof :** i ↔ii, to prove that

$R(h+k) = \big(Rh \cap R(h+k)\big) + (Rk \cap R(h+k))$

$\leftrightarrow R(h+k) = \big(Rh:(h+k)\big)(h+k) + (Rk:(h+k))(h+k)$

$\leftrightarrow R(h+k) = [\big(Rh:(h+k)\big) + (Rk:(h+k))](h+k)$

$\leftrightarrow R = [(Rh:k) + (Rk:h)](h+k)$ by Lemma 2.1.17 and Lemma 2.1.18

then $R = [(Rh:k) + (Rk:h)]$.

ii→ iii by Theorem 2.1.20 and iii→ i by Proposition 2.1.24. ■

**Corollary 2.1. 26.** Let $R$ be a yoked, subtractive , and cancellative semiring. If $\mathcal{D}$ is a subtractive $R$ –semimodule, then,

i) If $Rh + Rk = R(h+k) + (Rh \cap Rk)$for all $h,\ k \in \mathcal{D}$, then $\mathcal{D}$ is distributive $R-$ semimodule.

ii) If $\mathcal{D}$ is a distributive $R-$ semimodule, and $Rh \cap Rk = 0$, then $l(h) + l(k) = R$.

iii) If $\mathcal{D}$ is a distributive $R-$ semimodule, and $l(h) \subseteq J$(R), then Rh is large in $\mathcal{D}$.

**Proof:**

i) It is direct by using Lemma 2.1.19 and Theorem 2.1.25.

ii) If $\mathcal{D}$ is distributive $R-$ semimodule, then by Theorem 2.1.24, $R = (Rh:k) + (Rk:h)$. Since $(Rh:k) = \{r \in R : rk \in Rh\} = \{r \in R : rk = 0\} = l(k)$, because $Rh \cap Rk = 0$, similarly $(Rk:h) = l(h)$.

iii) Assume that $l(h) \subseteq J(R)$, and $Rh \cap U = 0$, where $U \ll \mathcal{D}$ if $k \in U$, then $Rh \cap Rk = 0$. By (ii) then $l(h) + l(k) = R$, but $l(h)$ is small in $R$ by

Corollary 2.1.16, so $l(k) = R$ and $k = 0$, hence $U = 0$, therefore, Rh $<^e \mathcal{D}$. ■

**Proposition 2.1. 27.** If $\mathcal{D}$ is a distributive $R-$ semimodule, and $\mathcal{B} \in L(\mathcal{D})$. Then $\mathcal{B}$ has a unique relative complement.

**Proof:** Assume that $\mathcal{B}_1, \mathcal{B}_2 \in L(\mathcal{D})$ are relative complements of $\mathcal{B}$ , then $\mathcal{B} \cap \mathcal{B}_1 = 0 \wedge \mathcal{B} \cap \mathcal{B}_2 = 0$. So, $(\mathcal{B} \cap \mathcal{B}_1) + (\mathcal{B} \cap \mathcal{B}_2) = 0 = \mathcal{B} \cap (\mathcal{B}_1 + \mathcal{B}_2)$, but $\mathcal{B}_1 \subseteq (\mathcal{B}_1 + \mathcal{B}_2)$ , $\mathcal{B}_2 \subseteq (\mathcal{B}_1 + \mathcal{B}_2)$, and $\mathcal{B}_1, \mathcal{B}_2$ are maximal with the property, then $\mathcal{B}_1 = \mathcal{B}_2$. ■

**Lemma 2.1.28:** Let $C$ be a fully invariant subsemimodule of $R$-semimodule $M$, and $A, B$ be subsemimodules of $M$ such that $A \cap B = 0$ . If each idempotent endomorphism of every 2-generated subsemimodule of $M$ is extended to an endomorphism of $M$. Then $C \cap (A + B) = C \cap A + C \cap B$.

**Proof:** Assume that $H = (Re_1 \oplus Re_2)$ where $e_1 \in A$, $e_2 \in B$. Let $f_i : H \to Re_i$ be natural projections. Idempotent endomorphisms $f_i$ of $H$ are extended to endomorphism $g_i \in End(M)$. Hence $g_1(e_1) = e_1$, $g_2(e_2) = e_2$ and $g_1(e_2) = g_2(e_1) = 0$. If $h = e_1 + e_2 \in C \cap (A + B)$, then $e_i = g_i(e_1 + e_2) = g_i(h)$. By hypostasis $g_i(h) \in C$, therefore $h = g_1(e_1) + g_2(e_2) \in (C \cap A + C \cap B)$. ■

**Proposition 2.1.29**. Assume that each idempotent endomorphism of any 2-generated left ideal of any arbitrary factor semiring $B$ of a commutative semiring $R$ is extended to an endomorphism of $B$ (as a semimodule over itself). If $R$ is subtractive, then it is distributive.

**Proof.** It is clear that a commutative semiring $R$ is duo as ( a left or right) semimodule over itself, for if $f \in End({}_RR)$ and $I$ is a left ideal of $R$, then for each $x \in I$, $f(x) = xf(1) = f(1)x \in I$, that is $f(I) \subseteq I$. This property hold for any factor of ${}_RR$. Let $F, G,$ and $H$ be left ideals of $R$, and let $h: R \to R/(G \cap H)$ be a natural epimorphism.

Not that, $h$(G) and $h$(H) are subsemimodules of the factor semiring $R/(G \cap H)$, with $h$(F) $\cap$ $h$(G) $= 0$.

Applying Lemma 2.1.28 to $R/(G \cap H)$, we have

$$h(\mathrm{F}) \cap \left(h(\mathrm{G}) + h(\mathrm{H})\right) = h(\mathrm{F}) \cap h(\mathrm{G}) + h(\mathrm{F}) \cap h(\mathrm{H})$$

the left hand side

$$F/_{G \cap H} \cap (G/_{G \cap H} + H/_{G \cap H}) = {F \cap (G + H)}/_{(G \cap H)}$$

the right hand side

$$(F/_{G \cap H} \cap G/_{G \cap H}) + (G/_{G \cap H} \cap H/_{G \cap H}) = {(F \cap G + F \cap H)}/_{(G \cap H)}$$

Therefore $F \cap (G + H) = (F \cap G + F \cap H)$, then is , ${}_R R$ is distributive. ■

**Proposition 2.1.30:** If $M$ is a subtractive cyclic semimodule over a uniserial semiring $R$, then $M$ is a distributive semimodule.

**Proof.** Assume that $M = Rm$, for some $m$ in $M$, and let $\gamma: R \to M$ be a homomorphism map such that $\gamma(r) = rm$, then $\gamma$ is an epimorphism with $\ker(\gamma) = l(m)$, hence $M \cong R/_{l(m)}$. By Proposition 2.1.12, $R$-semimodule $R$ is a distributive semimodule. So, $R/_{l(m)}$ is a distributive semimodule, by Corollary 2..1.9 $M$ is a distributive semimodule. ■

## 2.2 The homomorphisms of distributive semimodules

**Lemma 2.2.1.** [28]Let $B$ be a semisubtractive cancellative semimodule, and $\{B_i\}_{i \in I}$ is a family of subsemimodules of $B$, such that $\sum_{i \in J} B_i$ is subtractive for each $J \subseteq I$. Then $\sum_{i \in I} B_i = \oplus_{i \in I} B_i$ if and only if $B_j \cap \sum_{i \in I/\{j\}} B_i = 0$ for each $i \in J$.

**Lemma 2.2.2.** If $A, B \in L(M)$, $A \cap B = 0$, $A + B$ is **cancellative** and **semisubtractive**, then there is a bijection between the complements of $B$ in $A \oplus B$ and the split of maps the projection of $A \oplus B$ onto A.

**proof.** Let $p: A \oplus B \to A$ be a natural projection then $p$ splits. So there exists $q: A \to A \oplus B$ such that $pq = 1_A$, hence $A \oplus B = Imq \oplus kerp = Imq \oplus B$. It is clear that $Imq$ is a complement of $B$ in $A \oplus B$. Now if $q \neq \acute{q}$ are distinct split maps of $p$ then $q(a) \neq \acute{q}(a)$ for some $a \in A$.

Claim: $q(a) \notin Im\acute{q}$, and then $Im\ q \neq Im\ \acute{q}$.

For, if $q(a) \in Im\acute{q}$, then $q(a) =$ q'( $\acute{a}$) for some $\acute{a} \in A$. Hence $p\big(q(a)\big) =$ $p\big($q'($\acute{a}$)$\big)$, so, $a = \acute{a}$ (since $pq = pq' = 1_A$) that is, $q(a) = q(\acute{a}) = \acute{q}(\acute{a})$ $C$!, therefore $Im\ q \neq Im\ \acute{q}$.

Hence there is a bijective between the split maps of p and the complements of $B$ in $A \oplus B$. ■

**Lemma 2.2.3.** If $A, B \in L(M)$, $A \cap B = 0$, $A + B$ is **cancellative** and **semisubtractive**, then there is a bijection between $Hom\ (A, B)$ and the split maps of the projection p of A⊕B onto $A$ .

**proof.** Let α ∈ Hom $(A, B)$ and $p: A \oplus B \to A$ be the natural projection

Define $q: A \to A \oplus B$ by $a \mapsto a + \alpha(a)$, then

$$(pq)(a) = p(a + \alpha(a)) = a$$

$q$ is a right inverse of $p$, that is, a split map of $p$

It is clear that if $\alpha \neq \alpha' \in Hom(A, B)$, then for some $a \in A$.

$$\alpha(a) \neq \acute{\alpha}(a), \qquad q(a) = a + \alpha(a) \neq a + \acute{\alpha}(a) = \acute{q}\ (a).$$

On the other hand, if $q$ is a split map of $p$, then it is 1-1, and it induces an $\alpha \in Hom\ (A, B)$, by $\alpha = \pi_B\ q$ where $\pi_B$ is the natural projection of $A \oplus B$ onto $B$. Therefore, there is a bijection between $Hom\ (A, B)$ and the split maps $p$. ■

**Corollary 2.2.4:** If $A, B \in L(M)$,$A \cap B = 0$, $A + B$ is cancellative and semisubtractive, then there is a bijective between $Hom\ (A, B)$ and the set of complements of B in $A \oplus B$.

**Proposition2.2.5**. A lattice $L$ is distributive if and only if for each interval $I$ of $L$ any two elements of $I$ which are related in $I$ are equal. (See [40, p. 68] )

**Proposition 2.2.6.** Suppose that $M$ is a cancellative and semisubtractive semimodule, with $A, B \in L(M)$ such that $A \cap B = 0$. Then $M$ is a distributive semimodule if and only if $Hom\ (A, B) = 0$.

**proof.** Since $A \cap B = 0$, by Corollary 2.2.4, there is a bijection between $Hom\ (A, B)$ and the set of complements of $B$ in $A \oplus B$, and by Proposition2.2.5 a lattice is distributive if and only if relative complements are unique. Now, we get that $Hom\ (A, B) = 0$ if and only if $M$ is a distributive semimodule. ■

**Corollary 2.2.7.** Suppose that $M$ is a cancellative and semisubtractive semimodule. If $A, B \in L(M)$ , with $A \cap B$ subtractive subsemimodule, then ${}^{M}/_{(A \cap B)}$ is a distributive semimodule if and only if $Hom\ \left({}^{A}/_{(A \cap B)}, {}^{B}/_{(A \cap B)}\right) = 0$.

**Corollary 2.2.8.** If $M$ is a cancellative, semisubtractive, and distributive semimodule, such that $M = A + B$, where $A, B$ are subtractive subsemimodules of $M$, then $Hom\ \left({}^{M}/_{A}, {}^{M}/_{B}\right) = 0$.

**Corollary 2.2.9.** Let $M$ be distributive, cancellative, and semisubtractive semimodule over semiring R. If $l(m) = 0, m \in M$, then $Rm$ is an essential subsemimodule of $M$.

**Proof.** Let $f$: R → Rm be an epimorphism such that $f(r) = rm$ for all $r \in R$. Since $kerf = l(m) = 0$, then $f$ is a monomorphism, hence R ≅ Rm. If Rm $\cap\, A = 0$, for some $A \in L(M)$, if $0 \neq a \in A$, then there exist $g: R \to A$ defined by $g(r) = ra$ for all $r \in R$. But we have $h: Rm \to R$ ( inverse of $f$ ) which is not zero hence $0 \neq gh \in Hom(\mathrm{Rm}, A)$ C!, hence $A = 0$ .■

**Proposition 2.2.10.** Let $U$ be a distributive $R$ -semimodule and $B$ be an $R$ – semimodule, and $\omega, g \in Hom\ (U, B)$ such that $\omega + g$ is k- regular. Then,

i) $0 = g(\ker \omega) \cap \omega(ker\ g)$.

ii) If $U$ is a subtractive semimodule, then for any $C \in L(B)$, $C = ( C +$

$\omega(g^{-1}(C)) \cap \Big( C + g(\omega^{-1}(C))\Big)$.

**Proof.**

i) Since $(\omega + g)( ker\omega \cap \ker g) = 0$, then by Proposition 2.1.6 we get that $(\omega + g)(ker\omega) \cap (\omega + g)(kerg) = 0$ , so

$$[\omega\ (\ker \omega) + g\ (\ker \omega)] \cap [\ \omega\ (kerg) + g\ (\ker g)] = 0, \text{ then}$$

$$g(\ker \omega) \cap \omega\ (\ker g) = 0$$

ii) If $\Pi$ is the natural epimorphism of $B$ onto $B/C$ , and $\Pi\omega$, $\Pi g \in Hom\ (U, B/C)$Now apply (i), we get $C = \Pi g\ (\ker \Pi\omega) \cap \Pi\omega\ (\ker \Pi g) = \Pi g\ (\omega^{-1}(C)) \cap \Pi\omega\ (g^{-1}(C))$, since $\ker \Pi\omega = \omega^{-1}(C)$, and $\ker \Pi\, g = g^{-1}(C)$, then $C = ( C + \omega(g^{-1}(C)) \cap \Big( C + g(\omega^{-1}(C))\Big)$. ■

**Corollary 2.2.11.** Let $U$ be a distributive $R$-semimodule and $\omega \in End\ (U)$ with

$\omega + i$ is $k$- regular where $i$ is the identity on $U$. If $B \in L\ (U)$, then $B = \big(B + \omega^{-1}(B)\big) \cap \big( B + \omega\ (B)\big)$.

**Proof.** By using Proposition 2.2.10 we get that

$$B = \big(B + \omega(i^{-1}(B)\big) \cap ( B + i(\omega^{-1}\ (B))).$$

Since $i^{-1}(B) = B$, and $i(\omega^{-1}\ (B)) = \omega^{-1}\ (B)$, then

$$B = \big(B + \omega^{-1}(B)\big) \cap \big( B + \omega\ (B)\big). \blacksquare$$

**Proposition 2.2.12.** Let $U$ be subtractive and distributive $R$ -semimodule, $B \in L\ (U)$ and $\omega \in End\ (U)$ is $k$ – regular.

i) If $\omega$ is $i - regular$ then $B = \big(B \cap \omega^{-1}(B)\big) + \omega\ (B \cap \omega^{-1}(B))$.

ii) If $\omega + i$ is $k$ – regular where $i$ is identity then $B = \big(B + \omega(B)\big) \cap \omega^{-1}\ (\omega(B) + B)$.

**Proof.**

i) By Corollary 2.1.23, we get

$$B = \left(B \cap \omega^{-1}(B)\right) + (B \cap \omega(B))$$

$$= \left(B \cap \omega^{-1}(B)\right) + (B \cap [\omega(U) \cap \omega(B)]\ )$$

$$= \left(B \cap \omega^{-1}(B)\right) + (B \cap \omega(U) \cap \omega(B)\ )$$

$$= \left(B \cap \omega^{-1}(B)\right) + \left(\omega\left(\omega^{-1}(B)\right) \cap \omega(B)\ \right),$$

by Proposition 2.1.6, we get

$$\mathrm{B} = B \cap \omega^{-1}(B) + \omega(\omega^{-1}(B) \cap B\ )$$

ii) By Corollary 2.2.11, we get

$$B = \left(B + \omega(B)\right) \cap \left(B + \omega^{-1}(B)\right)$$

then

$$\mathrm{B}\ = \left(B + \omega(B)\right) \cap \left(B + \omega^{-1}(0 + B)\right)$$

Now, by using Proposition 2.1.6, we have

$$\mathrm{B} = \left(B + \omega(B)\right) \cap \left(B + \omega^{-1}(0) + \omega^{-1}(B)\right)$$

$$= \left(B + \omega(B)\right) \cap [(B + \ker \omega) + \omega^{-1}(B)]$$

$$= \left(B + \omega(B)\right) \cap \left(\omega^{-1}\omega(B) + \omega^{-1}(B)\right) = \left(B + \omega(B)\right) \cap \omega^{-1}(\omega(B) + B). \blacksquare$$

**Corollary 2.2.13.** Let $U$ be a subtractive and distributive $R$ -semimodule, $B \in L(U)$, suppose that $\omega \in End\ (U)$ and $\omega$ is regular, then

i) $\omega^{-1}(B) \cap \omega(B) \subseteq B \subseteq \omega^{-1}(B) + \omega(B)$.

ii) $B \cap \omega^2(B) \subseteq \omega(B) \subseteq B + \omega^2(B)$.

**Proof.** (i)

Let $B \in L(U)$ and $\omega \in End(U)$, since $B \cap \omega^{-1}(B) \subseteq B,$ hence

$$\omega\left(B \cap \omega^{-1}(B)\right) \subseteq \omega(B),$$

then, by Proposition(2.2.12)(i) we get

$$B = \left(B \cap \omega^{-1}(B)\right) + \omega\left(B \cap \omega^{-1}(B)\right),$$

so $B \subseteq \left(B \cap \omega^{-1}(B)\right) + \omega(B)$ hence $B \subseteq \omega^{-1}(B) + \omega(B)$. Also $\omega^{-1}(B) \cap \omega(B) \subseteq B$ when $\omega$ is $k$- regular, since $\omega^{-1}(B) \subseteq \omega^{-1}\left(B + \omega(B)\right)$ and $\omega(B) \subseteq B + \omega(B)$ then $\omega^{-1}(B) \cap \omega(B) \subseteq \omega^{-1}\left(B + \omega(B)\right) \cap B + \omega(B)$. Now by Proposition (2.2.12)(ii) we get $\omega^{-1}(B) \cap \omega(B) \subseteq B$.

(ii)

By (i) we get $\omega^{-1}(B) \cap \omega(B) \subseteq B \subseteq \omega^{-1}(B) + \omega(B)$

$\omega(\omega^{-1}(B) \cap \omega(B)) \subseteq \omega(B) \subseteq \omega(\omega^{-1}(B) + \omega(B))$

$\omega(\omega^{-1}(B)) \cap \omega^2(B) \subseteq \omega(B) \subseteq \omega(\omega^{-1}(B)) + \omega^2(B)$

$B \cap \omega^2(B) \subseteq \omega(B) \subseteq B \cap \omega(B) + \omega^2(B)$

$B \cap \omega^2(B) \subseteq \omega(B) \subseteq B \cap \omega(B) + \omega^2(B) \subseteq B + \omega^2(B)$

$B \cap \omega^2(B) \subseteq \omega(B) \subseteq B + \omega^2(B)$. ■

If $U$ is a subtractive and distributive semimodule and $\mu \in End\,(U)$ is regular then by Proposition (2.2.12) any subsemimodule $B \in L(U)$, can be written in the form $B = T \cap \mu^{-1}(T)$ and if $\mu, \omega + i$ are $k$- regular then by Proposition (2.2.12) $B$ can be written in form $B = W + \mu(W)$ for some $T, W \in L(U)$. These representations are unique as in the following sense.

**Corollary 2.2.14.** Suppose that $W, T$ are subsemimodules of a distributive and subtractive semimodule $U$ and $\mu \in End\,(U)$.

i) If $\mu$ and $\mu + i$ are $k$- regular and $W + \mu(W) = T + \mu(T)$ then $W = T$.

ii) If $\mu$ is regular and $W \cap \mu^{-1}(W) = T \cap \mu^{-1}(T)$ then $W = T$.

**Proof.**

i) By Proposition 2.2.12, we get

$$W = (W + \mu(W)) \cap \mu^{-1}(W + \mu(W)) = (T + \mu(T)) \cap \mu^{-1}(T + \mu(T)) = T.$$

ii) Follows similarly. ■

**Corollary 2.2.15.** Let $U$ be a distributive semimodule and $N \in L(U)$ and $ȡ \in End(U)$.

i) If $ȡ, ȡ + i$ are $k$- regular and $ȡ^m(N) \subseteq \sum_{j=0}^{m-1} ȡ^j(N)$ for some $m \geq 1$, then $ȡ(N) \subseteq N$.

ii) If $ȡ$ is regular and $\bigcap_{i=0}^{m-1} ȡ^{-j}(N) \subseteq ȡ^{-m}(N)$ for some $m \geq 1$, then $ȡ(N) \subseteq N$.

**Proof.**

i) if $m = 1$ is trivial, we assume that the case is true when $m$, by induction, we prove that is true when $m + 1$, suppose that

$ȡ^{m+1}(N) \subseteq \sum_{j=0}^{m} ȡ^j(N)$ and define $N_m = \sum_{j=0}^{m} ȡ^j(N)$ .

Then

$$ȡ(N_{m-1}) = ȡ(N) + ȡ^2(N) + \cdots + ȡ^m(N),$$

$$N_{m-1} + ȡ(N_{m-1}) = N + ȡ(N) + ȡ^2(N) + \cdots + ȡ^m(N) = N_m,$$

$$N_{m-1} + ȡ(N_{m-1}) = N_m \ldots\ldots (1)$$

Now

$$ȡ^{m+1}(N) \subseteq \sum_{j=0}^{m} ȡ^j(N) = N_m .$$

$$ȡ(N_m) = N_m + ȡ^{m+1}(N) \subseteq N_m + N_m$$

$$then\ ȡ(N_m) \subseteq N_m$$

So, $N_m = N_m + ȡ(N_m) \ldots\ldots (2)$ . Then $N_m = N_{m-1}$ By Corollary2.2.14(i)

$$\text{hence } ȡ^m(N) \subseteq N_m \subseteq N_{m-1} = \sum_{j=0}^{m-1} ȡ^j(N)$$

$$ȡ^m(N) \subseteq \sum_{j=0}^{m-1} ȡ^{-j}(N) \text{ by hyopesis then } ȡ\,(N) \subseteq N.$$

ii) Also we can use induction on $m$.

Define $N_m = \bigcap_{j=0}^{m} ȡ^{-j}(N)$ and suppose that $\bigcap_{j=0}^{m} ȡ^{-j}(N) \subseteq ȡ^{-m-1}(N)$

now

$$ȡ^{-1}(N_{m-1}) = ȡ^{-1}(N) \cap ȡ^{-2}(N) \cap \dots \cap ȡ^{-m}(N)$$

$$N_{m-1} \cap ȡ^{-1}(N_{m-1}) = N \cap ȡ^{-1}(N) \cap ȡ^{-2}(N) \cap \dots \cap ȡ^{-m}(N) = N_m.$$

$$\text{then } N_m = N_{m-1} \cap ȡ^{-1}(N_{m-1}) \dots .*$$

Since $N_m = \bigcap_{j=0}^{m} ȡ^{-j}(N) \subseteq ȡ^{-m-1}(N)$,

$$N_m = N_m \cap N_m \subseteq ȡ^{-m-1}(N) \cap N_m$$

$$N_m \subseteq N \cap [\, ȡ^{-1}(N) \cap ȡ^{-2}(N) \cap \dots \cap ȡ^{-m}(N) \cap ȡ^{-m-1}(N)]$$

$N_m \subseteq N \cap ȡ^{-1}(N_m)$ implies that $N_m = N \cap ȡ^{-1}(N_m) \dots **$

using Corollary 2.2.14. on * and ** we get $N_m = N_{m-1}$so $\bigcap_{i=0}^{m-1} ȡ^{-1}(N) = N_{m-1} = N_m \subset ȡ^{-N_m}(N)$ . By induction, then $ȡ\,(N) \subseteq N$. ■

**Proposition 2.2.16.** If $U$ and $B$ are semimodules and $ȡ, g \in Hom\,(U,B)$, such that $ȡ$ and $g$ are $i-regular$. If $U$ is a hollow semimodule and $B$ a distributive semimodule then either $ȡ(U) \subseteq g(U)$ or $g(U) \subseteq ȡ(U)$.

**Proof.** By Proposition 2.1.22 we get $U = g^{-1}ȡ(U) + ȡ^{-1}g(U)$. Also, by hypothesis we have $U$ is hollow, hence $g^{-1}ȡ(U) = U$ or $ȡ^{-1}g(U) = U$. If $g^{-1}ȡ(U) = U$ implies that $g(\, g^{-1}ȡ(U)) = g(\,U)$, since $g\, g^{-1}ȡ(U) \subseteq ȡ(\,U)$

therefore $g(U) \subseteq ȡ(U)$. Similarly when $ȡ^{-1}g(U) = U$ we get $ȡ(U) \subset g(U)$. ■

**Corollary 2.2.17.** If $R$ is a local semiring and $B$ is a $k$-cyclic distributive $R$ – semimodule $R$, then $B$ is uniserial semimodule.

**Proof.** Since $R$ is local then $R$ is a hollow $R$-semimodule , putting $U = R$ in Proposition (2.2.16) then $ȡ(R) \subset g(R)$ or $g(R) \subset ȡ(R)$ for any $ȡ, g \in Hom\ (R, B)$.

Now let $I, W$ any two subsemimodules of $B$ and $I \nsubseteq W$, this mean $\exists\ x \in I$ and $x \notin W$ and define ȡ and $g$ such that $ȡ(\mathrm{r}) = rx, \forall r \in R$ and $\forall\ y \in W$, $g(\mathrm{r}) = ry, \forall r \in R$. By Remark 1.1.30 we get ȡ and $g$ are $i -$ regular and by Proposition (2.2.16) we get

$Ry \subset Rx, \forall\ y \in W$ implies that $W \subseteq Rx$ hence $W \subseteq I$. ■

**Corollary 2.2.18.** Let $R$ be a local semiring and $R$ a $k$-cyclic semimodule over $R$, then $R$ is a distributive semimodule if and only if $R$ is uniserial.

**Proof.** The first direction is verified by Corollary 2.2.17. The other direction is clear since any uniserial semimodule is distributive. ■

**Theorem 2.2.19.** Let $U$ be a subtractive and Artinian semimodules, $B$ is a distributive semimodule, $ȡ, g \in Hom\ (U, B)$ such that $ȡ + g$ is $k$-regular. If $ker\ ȡ \subseteq ker\ g$ , then $g\ (U) \subseteq ȡ\ (U)$.

**Proof.** Assume that $ker\ ȡ \subseteq ker\ g$ and $g\ (U) \nsubseteq ȡ\ (U)$, let $\Omega = \{\ N \in L\ (U) : g(N) \subseteq ȡ\ (N)\}$ define $M = \sum N\ , s.t\ \ N \in \Omega$ ,

$$g\ (M) = g\ (\sum N) = \sum g(N) \subseteq \sum ȡ(N) = ȡ\ (M)$$

then $M$ the largest subsemimodule of $U$ in $\Omega$, and $M$ proper in $U$ because of

$g\ (U) \subseteq ȡ\ (U)$ , Now $0 = g\ (ker\ g\ ) \subseteq ȡ\ (kerg\ )$,then $ker\ g\ \in \Omega\ , \Omega \neq \varnothing$ .

By hypothesis $U$ is Artinian then $U/_M$ is Artinian, so $U/_M$ has a minimal subsemimodule say $C/_M$ implies that $C/_M \neq 0$ and $C/_M$ is simple . Since $M \subseteq C \subseteq U$, then $g(M) \subseteq ȡ\ (M) \subseteq ȡ\ (C)$, and $M \subseteq g^{-1}\big(g\ (M)\big) \subseteq g^{-1}\big(ȡ\ (M)\big) \subseteq g^{-1}\big(ȡ\ (C)\big)$ , implies that $M \subseteq g^{-1}\big(ȡ\ (C)\big)$ and $M \subseteq C$ , then $M \subseteq C\ \cap\ g^{-1}\big(ȡ\ (C)\big) \subseteq C$ and $[C\ \cap\ g^{-1}\big(ȡ\ (C)\big)]/M\ \subseteq C/_M$.

Since $C/_M$ is simple, then $[C\ \cap\ g^{-1}\big(ȡ\ (C)\big)]/M = \bar{0}$ , hence $C\ \cap\ g^{-1}\big(ȡ\ (C)\big) = M$. By Proposition (2.2.10) we get $C = (C\ \cap\ g^{-1}ȡ\ (C)) + (C\ \cap\ ȡ^{-1}g\ (C)$. Now,

$$ȡ(C) = ȡ(C\ \cap\ g^{-1}ȡ\ (C)) + ȡ(C\ \cap\ ȡ^{-1}g\ (C))$$

$$\subseteq ȡ\ (M) + \big(\ ȡ(\ C) \cap ȡ\ ȡ^{-1}g\ (C)\big)$$

$$= ȡ\ (M) + \big[ȡ(\ C) \cap \big(g\ (C) \cap\ ȡ\ (A)\big)\big]$$

$$= ȡ\ (M) + [ȡ(\ C) \cap\ ȡ\ (A) \cap g\ (C)]$$

$$= ȡ\ (M) + [ȡ(\ C) \cap g\ (C)]$$

$$= ȡ\ (M) + [ȡ(\ C) \cap\ g\ (C) \cap g\ (A)]$$

$$= ȡ\ (M) + g\ g^{-1}[ȡ(\ C) \cap g\ (C)]$$

$$= ȡ\ (M) + g\ [g^{-1}ȡ(\ C) \cap g^{-1}g\ (C)]$$

$$= ȡ\ (M) + g\ [g^{-1}ȡ(\ C) \cap (C + \ker g\ )]$$

$$= ȡ\ (M) + g\ [g^{-1}ȡ(\ C) \cap C\ ] = ȡ(M) + g\ (\ M) = ȡ(M)$$

then $ȡ\ (C) \subseteq ȡ\ (M)$ so $ȡ\ (C) = ȡ\ (M)$

implies that $ȡ^{-1}ȡ\ (C) = ȡ^{-1}ȡ\ (M)$ hence $C + ker\ ȡ = M + ker\ ȡ$

and C = M [since $ker\ ȡ\ \subseteq M\ \subseteq C$ ] this is a contradiction then our assumption is false, hence $g\ (U) \subseteq ȡ\ (U)$. ■

In addition if $U$ and $B$ are semimodules and ȡ$, g \in Hom\ (U, B)$ when $U$ is Artinian, $B$ is distributive, and ȡ $+\ g$ is $k-$ regular if $ker$ ȡ $= ker\ g$ then $g\ (U) =$ ȡ $(U)$.

**Corollary 2.2. 20.** Let $U$ be an Artinian distributive and subtractive semimodule and $H \in L(U)$. If $f: H \to U$ is a monomorphism then $f\ (H) = H$.

**Proof.** Since $U$ is Artinian then $H$ is Artinian, and if $i$: $H \to U$ is the inclusion map, then $f + i$ is a monomorphism .By Remark 1.1.30 we get $f + i\ is\ k-$ regular and by monomorphism $ker\ i\ = 0$, also $ker\ f\ = 0$. By Theorem 2.2.19, we get $f\ (H) = i(H) =\ H.$ ■

**Theorem 2.2.21.** Let $M$ be a subtractive and distributive semimodule, $N$ a Noetherain semimodule, $f, g\ \in Hom\ (M, N)$ such that $f + g$ is $k$-regular, if $g\ (M) \subseteq f\ (M)$ then $ker\, f \subseteq kerg.$

**Proof.** Suppose that $g\ (M) \subseteq f\ (M)$ but $ker\, f \nsubseteq kerg$. Let $A = \cap \{B \in L(N): f^{-1}(B) \subseteq g^{-1}\ (B)\}$

note that $f^{-1}(N) = g^{-1}\ (N) = M$, that is, the above set is not empty. Clearly, $A$ is the smallest subsemimodule of $N$ such that $f^{-1}(A) \subseteq g^{-1}\ (A)$. In particular, we see that $A \subseteq g\ (M) \subseteq f\ (M)$…** [since $f^{-1}(\ g\ (M)) \subseteq g^{-1}\left(\ g\ (M)\right)$].

Since ker $f \nsubseteq kerg$, $A \neq 0$ and so, by hypothesis, we can find a subsemimodule $X$ such that $0 \leq X \leq A$ and $A/_X$ is simple. But $g\ (f^{-1}(X)) \subseteq g\ (f^{-1}(A)) \subseteq$ A, and so $A = X + g\ (f^{-1}(X))$..∗[ since $f^{-1}(X) \nsubseteq g^{-1}\ (X)$ and $X$ is maximal in $A$].

By Proposition 2.2.10

$$X = (\ X + g\ (f^{-1}(X))) \cap (X + f(g^{-1}\ (X))$$

and so

$$X = A \cap (X + f(g^{-1}\ (X))$$

leads to

$$f^{-1}(X) = f^{-1}(A) \cap [f^{-1}(X) + f^{-1}\big(f(g^{-1}(X)\big)]$$

$$f^{-1}(X) = f^{-1}(A) \cap [f^{-1}(X) + g^{-1}(X)]$$

[from * $g^{-1}(A) = g^{-1}(X) + f^{-1}(X)$]

$$f^{-1}(X) = f^{-1}(A) \cap g^{-1}(A) = f^{-1}(A)$$

$$X \cap f(M) = A \cap f(A)$$

, but $X \subseteq$ A $\subseteq f(M)$ by **

so $X = A$ a contradiction, hence $A = 0$, that is $ker\, f = f^{-1}(A) \subseteq g^{-1}(A) = ker g.$

**Proposition 2.2.22.** Let $\mathcal{D}$, and $N$ be modules over a semiring $R$. Where $\mathcal{D}$ be distributive , $N$ is a Noetherain and $A, B$ are submodules of $\mathcal{D}$. If $\mathcal{D}/_A \cong \mathcal{D}/_B$, then $A = B$.

**Proof.** Assume that $A, B \in L(\mathcal{D})$, such that $\mathcal{D}/_A \cong \mathcal{D}/_B$ by $\alpha$. Let $\beta_C: \mathcal{D} \to \mathcal{D}/_C$ be the natural epimorphism for $C \in L(\mathcal{D})$. Then $\alpha\beta_A: \mathcal{D} \to \mathcal{D}/_B$ is an epimorphism and $\alpha\beta_A + \beta_B$ is $k$-regular ( since $\mathcal{D}, N$ are modules). Thus $\alpha\beta_A(\mathcal{D}) = \beta_B(\mathcal{D})$, so by using Proposition 2.2.21 $\ker(\alpha\beta_A) = \ker(\beta_B)$, hence $A = B$. ■

As in modules, if $M$ is an $R$-semimodule and $l(M) = I$, then $M$ is an $R/_I$-semimodule. Define $(r/_I)m = rm$, then $r_1/_I = r_2/_I$ implies that $r_1 + i = r_2 + j$ for some $i, j \in I$, thus if $(r_1 + i)m = (r_2 + j)m \to r_1m + im = r_2m + jm$, but $im = jm = 0$ then $r_1m = r_2m$. So, this multiplication is well-defined. The rest of the conditions of multiplication are easy to prove. We will use the foregoing note in the following lemma.

**Lemma 2.2.23.** If $M$ is an $R$-semimodule, then $M/_{IM}$ is an $R/_I$-semimodule, where $I$ is an ideal of R.

**Proof.** Since $M$ is an $R$-semimodule, then by definition of factor semimodule $M/_{IM}$ is an $R$-semimodule. Now, $l(M/_{IM}) = I$, so $M/_{IM}$ is an $R/_{I}$-semimodule. ■

**Proposition 2.2.24.** Let $R$ be a local semiring with a unique maximal ideal $I$ and $M$ a $k$-cyclic $R$-semimodule with $M \neq IM$. Then $M$ is cyclic R-semimodule if $M$ is distributive.

**Proof.** If $I$ is the maximal ideal and $M \neq IM$ by Lemma2.2.23 $M/_{IM}$ is a simple $R/_{I}$-semimodule, then there exists $x \in M$ whose image in $M/_{IM}$ generates the semimodule. Then clearly $x \in IM$ and $IM + Rx = M$. By Corollary 2.2.17 then $M$ is a uniserial semimodule and since $Rx \nsubseteq IM$, we have $IM \subseteq Rx$. Therefore it follows that $Rx = M$. ■

## 2.3 Some examples

Initially, we will present some examples in the modules, after which we will proceed to showcase some examples in the semimodules.

**Remark 2.3.1.** Any cyclic $\mathbb{Z}$ −module is a distributive module ( see [41 ،p. 198]). For example, $\mathbb{Z}$ −module $\mathbb{Z}$ is a distributive module.

**Example 2.3.2.** Let $N$ be a field, and $R$ be a 5 −dimensional algebra over the field $N$ generated by all $(3 * 3)$ −matrices of the form

$\begin{pmatrix} n_{11} & n_{12} & n_{13} \\ 0 & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}$, where $n_{ij} \in N$, then $D = Re_{11} = Ne_{11} + Ne_{12} + Ne_{13}$ is distributive $R$ −module.( see [42, p. x])

**Example 2.3.3.** The $\mathbb{Z}$ −module $\mathbb{Z}_{p^n}$, where $p$ is a prime number, and $\in \mathbb{N}$. Since $\mathbb{Z}_{p^n}$ is a uniserial $\mathbb{Z}$ −module, then $\mathbb{Z}_{p^n}$ is a distributive module.

**Example 2.3.4.** The $(\mathbb{Z}_2, +_2, ._2)$ is the yoked semiring, cancellative, and subtractive. Let $(X, +)$ be a commutative monoid, where $X = \{0,1,2, \dots \dots, k\}$, $k \in$

$\mathbb{N}$, and $h+k=\max\{h,k\}$ for all $h,k\in X$. Define multiplication operation as follows: $0_{\mathbb{Z}_2}.\,k$=$0_X$ and $1_{Z_2}.\,k=k$, for all $k\in X$. Then the $\mathbb{Z}_2-$semimodule $X$ is subtractive, distributive, and satisfies Theorem 2.1.25.

**Example 2.3.5.** Let $(\mathbb{N},+,.)$be the semiring, with usual operations, It is easy to see that $(\mathbb{Z}_4,+_4)$ is a distributive $\mathbb{N}-$semimodule.

**Example 2.3.6.** Let R $=\{0,1\}$ be a semiring, and $\mathcal{D}=\{(0),(2),(4),(8)\}$ , as discussed in Example 1.2.12. Then $\mathcal{D}$ is a distributive R$-$semimodule because

$\mathcal{M}_1\cap(\mathcal{M}_2+\mathcal{M}_3)=\mathcal{M}_1\cap(\mathcal{M}_2\cup\mathcal{M}_3)=(\mathcal{M}_1\cap\mathcal{M}_2)\cup(\mathcal{M}_1\cap\mathcal{M}_3)=$
$(\mathcal{M}_1\cap\mathcal{M}_2)+(\mathcal{M}_1\cap\mathcal{M}_3)$, for all $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3\in L(\mathcal{D})$.

**Example 2.3.7.** Let $\mathcal{D}=\{0,1,h,k\}$ be a semimodule over semiring R $=\{0,h,1\}$, , as discussed in Example 1.2.6. $\mathcal{D}$ is distributive R$-$semimodule because

$\mathrm{R}(c+v)=\big(\mathrm{R}c\cap\mathrm{R}(c+v)\big)+(\mathrm{R}v\cap\mathrm{R}(c+v))$, for all $c,v\in\mathcal{D}$.

Since $\mathrm{R}0=\{0\}$, $\mathrm{R}1=\{0,1,\mathrm{h}\}$, $\mathrm{Rh}=\{0,\mathrm{h}\}$, $\mathrm{Rk}=\{0,\mathrm{k}\}$, then

If $c=1,v=h$ , then $\mathrm{R}(1+h)=\mathrm{R}1=\mathrm{R}1+\mathrm{Rh}=(\mathrm{R}1\cap\mathrm{R}1)+(\mathrm{R}h\cap\mathrm{R}1)$.

If $c=1,v=k$, then$\mathrm{R}(1+k)=\mathrm{R}h=\mathrm{R}h+\{0\}=(\mathrm{R}1\cap\mathrm{R}h)+(\mathrm{R}k\cap\mathrm{R}h)$.

If $c=h,v=k$, then$\mathrm{R}(h+k)=\mathrm{R}h=\mathrm{R}h+\{0\}=(\mathrm{R}h\cap\mathrm{R}h)+(\mathrm{R}k\cap\mathrm{R}h)$.

**Example 2.3.8.** Let $X$ be any nonempty set, then $(P(X),\cup,\cap)$ is a semiring. Since the first operation of semiring is union, and the intersection is distributive over the union in set theory, then $P(X)$ is arithmetical semiring, thus $P(X)$ is distributive semimodule over itself.

**Example 2.3.9.** The $\mathbb{Z}_2-$semimodule $X$, as discussed in Example 1.2.11 is a distributive semimodule, since it satisfies Theorem 2.1.25.

**Example 2.3.10.** Let $(R,+,.)$ be a semiring, where $R=\{0,1,2\}$,and for all $h,k\in R$,

$h + k = \max\{h, k\}$ ,$h.k = \min\{h, k\}$.

If $h = 0, k = 1,$ so $R0 = \{0\}, R1 = \{0,1\}, R(0 + 1) = R1$, then

$R(h + k) = \big(Rh \cap R(h + k)\big) + (Rk \cap R(h + k)) \to R1 = R0 \cap R1 + R1 \cap R1$.

If $h = 0, k = 2,$ so $R0 = \{0\}, R2 = R$, $R(0 + 2) = R$, then $R = R0 \cap R + R2 \cap R$.

If $h = 1, k = 2,$ so $R1 = \{0,1\}, R2 = R$, $R(1 + 2) = R2$, then $R = R2 = R1 \cap R + R \cap R$. By Theorem 2.1.20, R is a distributive R – semimodule.

**Remark 2.3.11.** If $\mathcal{D} \neq \{0\}$ is any R – semimodule, then $\mathcal{B} = \mathcal{D}\coprod\mathcal{D}$ is not a distributive R – semimodule.

# Chapter three

## Distributive Property and other Concepts in Semimodules

**Introduction**

This chapter consists of three sections. In the first section, we generalized some of the properties obtained by A. A. Tuganbaev [17], and [19]. As for the second section, we discussed the concept of weakly distributive semimodules, which Büyükaşık and Demirci discussed in [12]. As for the last section, there were generalizations for some associated definitions to the supplement and finding relationships between those concepts and related concepts of distributive semimodules.

## 3.1 Direct Sums of Distributive Semimodule

**Proposition 3.1.1.** Let $\{\mathcal{A}\}_{i\in I}$ be a family of subsemimodules of a distributive semimodule $\mathcal{D}$. Then

i. $\mathcal{B} \cap \sum_{i\in I} \mathcal{A}_i = \sum_{i\in I}(\mathcal{B} \cap \mathcal{A}_i)$ for any subsemimodule $\mathcal{B}$ of $\mathcal{D}$. However, the dual statement $\mathcal{B} + (\cap \mathcal{A}_i) = \cap_{i\in I} (\mathcal{B} + \mathcal{A}_i)$ does not hold in general.

ii. If $\mathcal{B}$ is a subsemimodule of $\mathcal{D}$ such that $\mathcal{B} \cap \sum_{i\in I} \mathcal{A}_i \neq 0$, then $\mathcal{B} \cap \mathcal{A}_i \neq 0$ for some $i \in I$.

iii. If $\{\mathcal{B}\}_{i\in I}$ is the set of subsemimodules of $\mathcal{D}$ such that $\mathcal{B}_i$ is an essential subsemimodule of $\mathcal{A}_i$, then $\sum_{i\in I} \mathcal{B}_i$ is an essential subsemimodule of $\sum_{i\in I} \mathcal{A}_i$.

iv. If $\mathcal{B}$ is an essential subsemimodule of $\mathcal{A}_i$, $\forall\, i \in I$ ,then $\mathcal{B}$ is essential in $\sum_{i\in I} \mathcal{A}_i$

**Proof.**

(i) Let $x \in \mathcal{B} \cap \sum_{i\in I} \mathcal{A}_i$ **,** then there are finite set $K \subseteq I$, such that $x = \sum_{i\in K} g_i \in \mathcal{B} \cap \sum_{i\in K} \mathcal{A}_i$, since $\mathcal{D}$ is distributive then $\mathcal{B} \cap \sum_{i\in K} \mathcal{A}_i = \sum_{i\in K}(\mathcal{B} \cap \mathcal{A}_i) \subseteq \sum_{i\in I}(\mathcal{B} \cap \mathcal{A}_i)$. For the other direction, by this consider, let $\mathbb{Z}$-semimodule $\mathbb{Z}$ and $\mathcal{B} = (3)$ and $\mathcal{A}_i = (2^i)$ for all natural number $i$. Clearly, by Remark 2.3.1 $\mathbb{Z}$ is a distributive semimodule. However, $\mathcal{B} + (\cap \mathcal{A}_i) = (3)$ whereas $\cap_{i\in I} (\mathcal{B} + \mathcal{A}_i) = \mathbb{Z}$.

(ii) Follow from (i).

(iii) Let $\mathcal{C}$ be a nonzero subsemimodule of $\sum_{i\in I}\mathcal{A}_i$, then $0 \neq \mathcal{C} = \mathcal{C} \cap \sum_{i\in I}\mathcal{A}_i = \sum_{i\in I}\mathcal{C} \cap \mathcal{A}_i$ by (ii) $\mathcal{C} \cap \mathcal{A}_j \neq 0$ for some $j \in I$, in addition, since $\mathcal{B}_j$ is an essential $\mathcal{A}_i$ then $(\mathcal{C} \cap \mathcal{A}_j) \cap \mathcal{B}_j \neq 0$ hence $\mathcal{C} \cap \mathcal{B}_j \neq 0$ implies that $\sum_{i\in I}\mathcal{C} \cap \mathcal{B}_i \neq 0$ (since $\mathcal{C} \cap \mathcal{B}_j \neq 0$ subsemimodule of $\sum_{i\in I}\mathcal{C} \cap \mathcal{B}_i$ by distributive we get that $\mathcal{C} \cap \sum_{i\in I}\mathcal{B}_i \neq 0$ and $\sum_{i\in I}\mathcal{B}_i$ is an essential of $\sum_{i\in I}\mathcal{A}_i$.

(iv) Follows from (iii). ■

**Proposition 3.1.2.** Let $\mathcal{D}$ be a distributive semimodule, and $\mathcal{H} \in L(\mathcal{D})$, then

i. $\mathcal{H}$ has a unique relative complement $\mathcal{B}$ in $\mathcal{D}$, and $\mathcal{B}$ coincides with the sum of all subsemimodules of $\mathcal{D}$ which have zero intersection with $\mathcal{H}$.

ii. $\mathcal{H}$ has a unique closure $\mathcal{A}$ in $\mathcal{D}$, and $\mathcal{A}$ coincides with the sum of all essential extensions of $\mathcal{H}$ in $\mathcal{D}$.

**Proof.**

(i) Let $\Omega = \{B_i \in L(\mathcal{D}): B_i \cap \mathcal{H} = 0\}$ and let $\mathcal{B} = \sum_{B_i\in\Omega} B_i$, by Proposition (3.3.1)(ii) then $(\sum_{B_i\in\Omega} B_i) \cap \mathcal{H} = 0$, hence $\mathcal{B}$ the largest subsemimodule of $\mathcal{D}$ which has zero intersection with $\mathcal{H}$, and by Proposition 2.1.27 $\mathcal{B}$ is a unique complement.

(ii) Indeed, taking $\mathcal{A} = \sum_{i\in I}\mathcal{A}_i$, where $\mathcal{A}_i$ is an essential extension of $\mathcal{H}$. By Proposition (3.1.1)(iv), $\mathcal{A}$ is an essential extension of $\mathcal{H}$. If we take $\mathcal{A}'$ as an essential extension of $\mathcal{A}$, then $\mathcal{A}'$ is an essential extension of $\mathcal{H}$, by the hypothesis $\mathcal{A}' = \mathcal{A}$ hence $\mathcal{A}$ is closed in $\mathcal{D}$. Therefore, $\mathcal{A}$ is a unique closure of $\mathcal{H}$ in $\mathcal{D}$. ■

**Proposition 3.1.3.** Let $\mathcal{D}$ be a distributive semimodule, and $N, G \in L(\mathcal{D})$, then

i. If $G \cap N \leq^e N$, then $G \leq^e G + N$.

ii. If $G$ is closed in $\mathcal{D}$ and $G \cap N \leq^e N$, then $N \subseteq G$.

iii. If $G$ is closed in $\mathcal{D}$ and $N$ is uniform, then either $N \subseteq G$ or $G \cap N = 0$.

iv. The intersection of any two distinct closed and uniform subsemimodules of $\mathcal{D}$ is equal to zero.

**Proof.**

(i) If we assume $G \cap N \leq^e N$, and it's clear $G \leq^e G$. By Proposition (3.1.1)(iii) we get $G + G \cap N = G \leq^e G + N$.

(ii) Since $G$ is closed and $G \cap N \leq^e N$, then by (i) $G \leq^e G + N \leq \mathcal{D}$, implies that $G = G + N$, hence $N \subseteq G$.

(iii) If $G \cap N \neq 0$, then $G \cap N \leq^e N$ since $N$ is uniform. By (ii) then $N \subseteq G$.

(iv) Follows from (iii) ■

**Proposition 3.1.4.** Let $\mathcal{D}$ be a semisubtractive, cancellative, and distributive semimodule and $\{\mathcal{A}_i\}_{i \in I}$ is the set of all distinct closed uniform subsemimodules of $\mathcal{D}$ such that $\sum_{i \in J} \mathcal{A}_i$ is subtractive for any $J \subseteq I$ .then $\sum_{i \in I} \mathcal{A}_i, = \bigoplus_{i \in I} \mathcal{A}_i$.

**Proof.** By Proposition (3.1.3)(iv) the intersection of any two distinct closed and uniform subsemimodules is equal to zero and by Proposition (3.1.1)(ii) and Lemma 2.2.1, then $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$. ■

**Proposition 3.1.5.** Let $\mathcal{D}$ be a distributive semimodule and $\mathcal{D} = \bigoplus_{i \in I} \mathcal{A}_i = \bigoplus_{j \in J} \mathcal{B}_j$. where all $\mathcal{A}_i$ and $\mathcal{B}_j$ are nonzero indecomposable semimodules, then there is a bijection $\sigma: I \to J$ such that $\mathcal{A}_i$ coincides with $\mathcal{B}_{\sigma(i)}$.

**Proof.** For any $i \in I$, then $\mathcal{A}_i = \mathcal{A}_i \cap \bigoplus_{j \in J} \mathcal{B}_j$ we get by Proposition (3.3.1)(i) $\mathcal{A}_i = \bigoplus_{j \in J} (\mathcal{A}_i \cap \mathcal{B}_j)$, but $\mathcal{A}_i$ is indecomposable semimodules, then exist $k \in J$ where $\mathcal{A}_i = \mathcal{A}_i \cap \mathcal{B}_k$ and $\mathcal{A}_i \cap \bigoplus_{j \neq k \in J} \mathcal{B}_j = 0$, hence $\mathcal{A}_i = \mathcal{B}_k$. ■

If $\mathcal{N}$ is a subsemimodule of a left $R$ - semimodule $\mathcal{D}$, then $e\mathcal{N}$ is a subsemimodule of left $eR$e - semimodule $\mathcal{D}$, where e is a non-zero idempotent element of the semiring $R$. This can be easily verified by assuming $en \in e\mathcal{N}$ and $ere \in eR$e to imply $ere(en) = e(ren) \in e\mathcal{N}$, and if $en_1, en_2 \in e\mathcal{N}$, then $e(n_1 + n_2) \in e\mathcal{N}$. Now, if we assume that $\mathcal{D}$ is cancellative, and semisubtractive left $R-$ semimodule and $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$, such that $\mathcal{D}_{i=\{1,2\}}$ subtractive and $\mathcal{D}_1 \cap \mathcal{D}_2 = 0$, it can be directly proven by Lemma 2.2.1 the left $eR$e$-$ semimodule $\mathcal{D}e$ is a direct sum of $e\mathcal{D}_1$ and $e\mathcal{D}_2$, where $e$ is a non-zero idempotent of the semiring $R$. Generalizing this result, we obtain the following property.

**Proposition 3.1.7.** Let $t$ be a non-zero idempotent element of the semiring $R$, then

i. Assume that $\mathcal{D}$ is a cancellative, and semisubtractive left $R-$semimodule and $\mathcal{H} \in L(\mathcal{D})$. The rule $f(\mathcal{H}) = t\mathcal{H}$ gives a surjective lattice homomorphism $f: L(\mathcal{D}) \rightarrow L(t\mathcal{D})$, where $t\mathcal{D}$ is a left $eRe$ – semimodule, and if $\mathcal{D} = \oplus_{i\in I}\mathcal{D}_i$, then $t\mathcal{D} = \oplus_{i\in I} t\mathcal{D}_i$.

ii. Let $\mathcal{D}$ be a cancellative, and semisubtractive left $R-$semimodule, if $\mathcal{D}$ is distributive ( direct sum of distributive semimodules) then the left $tRt-$semimodule $t\mathcal{D}$ is distributive (direct sum of distributive semimodules).

**Proof.** The part (i) can be verified directly.

(ii) If $\mathcal{D}$ is distributive, and $A, B, C$ are subsemimodules of $tRt-$semimodule $t\mathcal{D}$, then $A = tA', B = tB', C = tC'$ for some $A', B', C'$ subsemimodules of $\mathcal{D}$. Now $A \cap (B + C) = tA' \cap (tB' + tC')$, but $tA', tB', tC'$ are subsemimodules of $\mathcal{D}$, too so, $A \cap (B + C) = (tA' \cap tB') + (tA' \cap tC') = (A \cap B) + (A \cap C)$. ∎

**Proposition 3.1.8.** Let $R$ be semiring with the identity $1 \neq 0$, such that $1 = \sum_{i=1}^{n} e_i$ , $e_i$ is non-zero idempotent, and $e_i e_j = 0$ when $i \neq j$. Assume that $\mathcal{D}$ is a cancellative, and semisubtractive left $R-$semimodule, then

i. The left $R-$semimodule $\mathcal{D}$ is the direct sum of the left $e_iRe_i-$semimodule $e_i\mathcal{D}$.

ii. If $\mathcal{N}$and $\mathcal{H}$ are subsemimodules of the semimodule $\mathcal{D}$, then $\mathcal{N} + \mathcal{H} = \sum_{i=1}^{n}(e_i\mathcal{N} + e_i\mathcal{H})$ and $\mathcal{N} \cap \mathcal{H} = \sum_{i=1}^{n} e_i\mathcal{N} \cap e_i\mathcal{H}$.

iii. A left $R-$semimodule $\mathcal{D}$ is distributive if and only if all the left $e_iRe_i-$semimodules $e_i\mathcal{D}$ are distributive.

**Proof.**

(i) Since $\mathcal{D} = 1.\mathcal{D} = (\sum_{i=1}^{n} e_i)\mathcal{D}$ and if $x \in e_i\mathcal{D} \cap e_j\mathcal{D}$ ,then $x = e_i d_i = e_j d_j$ implies that $e_i e_i d_i = e_i e_j d_j \rightarrow e_i d_i = 0$, hence $x = 0$, now by Lemma 2.2.1 $\mathcal{D} = \oplus_{i\in I} e_i\mathcal{D}$.

(ii) $\mathcal{N} + \mathcal{H} = \sum_{i=1}^{n} e_i \mathcal{N} + \sum_{i=1}^{n} e_i\mathcal{H} = \sum_{i=1}^{n}(e_i\mathcal{N} + e_i\mathcal{H})$ , similarly $\mathcal{N} \cap \mathcal{H} = \sum_{i=1}^{n} e_i\mathcal{N} \cap \sum_{i=1}^{n} e_i\mathcal{H} = \sum_{i=1}^{n} e_i\mathcal{N} \cap e_i\mathcal{H}$.

(iii) Follows from (ii). ■

**Proposition 3.1.9.** Let $\mathcal{D}$ be a cancellative, semisubtractive, and distributive R−semimodule. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are direct summands of $\mathcal{D}$, then $\mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 + \mathcal{A}_2$ are direct summands of $\mathcal{D}$.

**Proof.** By hypothesis $\mathcal{D} = \mathcal{A}_1 \oplus \mathcal{H}_1 = \mathcal{A}_2 \oplus \mathcal{H}_2$ for some $\mathcal{H}_1, \mathcal{H}_2 \in L(\mathcal{D})$.Hence $\mathcal{A}_1 = \mathcal{A}_1 \cap \mathcal{D} = \mathcal{A}_1 \cap (\mathcal{A}_2 \oplus \mathcal{H}_2) = \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{H}_2$, since $\mathcal{D} = \mathcal{A}_1 \oplus \mathcal{H}_1$, then $\mathcal{D} = (\mathcal{A}_1 \cap \mathcal{A}_2) + (\mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{H}_1)$, since $\mathcal{A}_1 \cap \mathcal{A}_2$ and$\mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{H}_1$ are subtractive ( [43], Lemma2) and $(\mathcal{A}_1 \cap \mathcal{A}_2) \cap (\mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{H}_1) = 0$, hence $\mathcal{A}_1 \cap \mathcal{A}_2$ is direct summand by Lemma 2.2.1.

Now to prove that $\mathcal{A}_1 + \mathcal{A}_2$ is a direct summand of $\mathcal{D}$, at first

$$\begin{aligned}
\mathcal{A}_1 + \mathcal{A}_2 = (\mathcal{A}_1 + \mathcal{A}_2) \cap \mathcal{D} &= \mathcal{A}_1 \cap \mathcal{D} + \mathcal{A}_2 \cap \mathcal{D} \\
&= \mathcal{A}_1 \cap (\mathcal{A}_2 + \mathcal{H}_2) + \mathcal{A}_2 \cap (\mathcal{A}_1 + \mathcal{H}_1) \\
&= \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{A}_2 \cap \mathcal{A}_1 + \mathcal{A}_2 \cap \mathcal{H}_1 \\
&= \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_2 \cap \mathcal{H}_1 + \mathcal{A}_1 \cap \mathcal{H}_2 \\
&= \mathcal{A}_2 \cap (\mathcal{A}_1 + \mathcal{H}_1) + \mathcal{A}_1 \cap \mathcal{H}_2 = \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{H}_2.
\end{aligned}$$

.

Now we get $\mathcal{D} = (\mathcal{A}_2 + \mathcal{H}_2) \cap (\mathcal{A}_1 + \mathcal{H}_1) = \mathcal{A}_2 + \mathcal{H}_2 \cap \mathcal{A}_1 + \mathcal{H}_2 \cap \mathcal{H}_1 = (\mathcal{A}_1 + \mathcal{A}_2) + (\mathcal{H}_2 \cap \mathcal{H}_1)$, since $(\mathcal{A}_1 + \mathcal{A}_2), (\mathcal{H}_2 \cap \mathcal{H}_1)$are subtractive ( [43], Lemma2), and $(\mathcal{A}_1 + \mathcal{A}_2) \cap (\mathcal{H}_2 \cap \mathcal{H}_1) = 0$ , then $\mathcal{A}_1 + \mathcal{A}_2$ is direct summand of $\mathcal{D}$ by Lemma 2.2.1. ■

Following Taganbaev in [19] we use a concept arithmetical module. This means a semimodule is arithmetical if the lattice of its fully invariant subsemimodules is distributive.

**Proposition 3. 1.10.** Let $M$ be a direct sum of distributive semimodules $M_{i=\{1,2\}}$, then $M$ is an arithmetical semimodule.

**Proof.** Assume that $N, K,$ and $L$ are fully invariant subsemimodules. Since $(K + L) \cap M_1 = [(K \cap M_1 + K \cap M_2) + (L \cap M_1 + L \cap M_2)] \cap M_1 =$

$[(K \cap M_1 + L \cap M_1) + (K \cap M_2 + L \cap M_2)] \cap M_1$, thus $(K + L) \cap M_1 = K \cap M_1 + L \cap M_1 + 0$. Similarly $(K + L) \cap M_2 = K \cap M_2 + L \cap M_2$. Now,

$$K + L = K \cap M_1 + L \cap M_1 + K \cap M_2 + L \cap M_2$$

and $(N \cap M_1) \cap (K + L) = (N \cap M_1) \cap [M_1 \cap (K + L)] = (N \cap M_1) \cap [K \cap M_1 + L \cap M_1] = N \cap K \cap M_1 + N \cap L \cap M_1$....(1) , similarly $(N \cap M_2) \cap (K + L) = N \cap K \cap M_2 + N \cap L \cap M_2$...(2). Since $N \cap (K + L)$ is fully invariant subsemimodule, then

$N \cap (K + L) = N \cap (K + L) \cap M_1 + N \cap (K + L) \cap M_2 = (N \cap M_1) \cap (K + L) + (N \cap M_2) \cap (K + L)$ . By (1) and (2) then $N \cap (K + L) = N \cap K \cap M_1 + N \cap L \cap M_1 + N \cap K \cap M_2 + N \cap L \cap M_2 = (N \cap K) + (N \cap L)$. ■

## 3.2 Weakly distributive semimodules

Let $\mathcal{D}$ be an $R-$semimodule, and $\mathcal{A} \in L(\mathcal{D})$. We shall call $\mathcal{A}$ a weakly distributive subsemimodule of $\mathcal{D}$ if $\mathcal{A} = \mathcal{A} \cap \mathcal{H} + \mathcal{A} \cap \mathcal{K}$ or $\mathcal{A} + (\mathcal{H} \cap \mathcal{K}) = (\mathcal{A} + \mathcal{H}) \cap (\mathcal{A} + \mathcal{K})$ for all $\mathcal{H}, \mathcal{K} \in L(\mathcal{D})$ such that $\mathcal{H} + \mathcal{K} = \mathcal{D}$. While $\mathcal{D}$ is a weakly distributive if each $\mathcal{A} \in L(\mathcal{D})$ is a weakly distributive subsemimodule.

**Remark 3.2.1**. Let $(R, +, .)$be the semiring with identity, then the R$-$semimodule R is weakly distributive.

**Proof.** If $\mathcal{K} \in L(\mathrm{R})$, and $R = \mathcal{A} + \mathcal{B}$, for some $\mathcal{A}, \mathcal{B} \in L(R)$, we need prove $\mathcal{K} \subseteq \mathcal{K} \cap \mathcal{A} + \mathcal{K} \cap \mathcal{B}$. Since $1 \in R$ then $1 = s + p$, for some $s \in \mathcal{A}, p \in \mathcal{B}$ we have each $k \in \mathcal{K}$ implies that $k.1 = k(s + p) = ks + kp$, where $ks \in \mathcal{K} \cap \mathcal{A}$, and $kp \in \mathcal{K} \cap \mathcal{B}$, hence $k \in \mathcal{K} \cap \mathcal{A} + \mathcal{K} \cap \mathcal{B}$. ■

**Proposition 3.2.2**. Let $\mathcal{D}$ be an $R-$ semimodule, $\mathcal{D}$ is distributive if and only if each subsemimodule of $\mathcal{D}$ is weakly distributive $R-$semimodule.

**Proof.** Let $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{C} \in L(\mathcal{D})$, such that $\mathcal{Q} + \mathcal{C} = \mathcal{D}$, if $\mathcal{D}$ is a distributive, then

$\mathcal{F} \cap (\mathcal{Q} + \mathcal{C}) = \mathcal{F} \cap \mathcal{Q} + \mathcal{F} \cap \mathcal{C}$, hence $\mathcal{F} \cap \mathcal{D} = \mathcal{F} = \mathcal{F} \cap \mathcal{Q} + \mathcal{F} \cap \mathcal{C}$, hence $\mathcal{F}$ is weakly distributive.

In the other direction, let $\mathcal{G}, \mathcal{T}, \mathcal{K} \in L(\mathcal{D})$, then

$$\mathcal{G} \cap (\mathcal{T} + \mathcal{K}) = \mathcal{G} \cap (\mathcal{T} + \mathcal{K}) \cap (\mathcal{T} + \mathcal{K})$$

$$= [\mathcal{G} \cap (\mathcal{T} + \mathcal{K})] \cap \mathcal{T} + [\mathcal{G} \cap (\mathcal{T} + K)] \cap \mathcal{K}$$

$$= [(\mathcal{G} \cap \mathcal{T}) \cap (\mathcal{T} + K)] + [(\mathcal{G} \cap K) \cap (\mathcal{T} + K)] = (\mathcal{G} \cap \mathcal{T}) + (\mathcal{G} \cap K)$$

, since $\mathcal{G} \cap (\mathcal{T} + K) \in L(\mathcal{T} + K)$, and $\mathcal{T} + \mathcal{K}$ is weakly distributive . ■

The following Lemma mentioned in [36], will be proved under different conditions.

**Lemma 3.2.3.** Let $\mathcal{D}$ be a $R-$semimodule, and $\mathcal{T}, \mathcal{L} \in L(\mathcal{D})$ with $\mathcal{T}$ is weakly distributive.. Then $\mathcal{T}$ is a supplement of $\mathcal{L}$ if and only if $\mathcal{D} = \mathcal{L} + \mathcal{T}$ and $\mathcal{T} \cap \mathcal{L}$ is small in $\mathcal{T}$.

**Proof.** Assume that $\mathcal{T}$ is a supplement of $\mathcal{L}$ and $\mathcal{X} \in L(\mathcal{T})$, where $\mathcal{T} \cap \mathcal{L} + \mathcal{X} = \mathcal{T}$. Since $\mathcal{D} = \mathcal{L} + \mathcal{T}$ by definition supplement then $\mathcal{D} = \mathcal{L} + (\mathcal{T} \cap \mathcal{L} + \mathcal{X}) = \mathcal{L} + \mathcal{X}$ and $\mathcal{X} = \mathcal{T}$ by the minimality of $\mathcal{T}$.

On the other hand, let $\mathcal{D} = \mathcal{L} + \mathcal{T}$ and $\mathcal{T} \cap \mathcal{L}$ small in $\mathcal{T}$. To prove that $\mathcal{T}$ is a supplement of $\mathcal{L}$, if $C \subseteq \mathcal{T}$ and $\mathcal{D} = \mathcal{L} + C$, then $\mathcal{T} = \mathcal{T} \cap \mathcal{D} = \mathcal{T} \cap (\mathcal{L} + C) = \mathcal{T} \cap \mathcal{L} + \mathcal{T} \cap C = \mathcal{T} \cap \mathcal{L} + C$ and $C = \mathcal{T}$ since $\mathcal{T} \cap \mathcal{L}$ small in $\mathcal{T}$. Hence $\mathcal{T}$ is a supplement of $\mathcal{L}$. ■

**Proposition 3.2.4.** Let $\mathcal{D}$ be an $R-$ semimodule, $\mathcal{Q}, \mathcal{F}$, and $\mathcal{A} \in L(\mathcal{D})$. Then

i) If $\mathcal{D} = \mathcal{Q} + \mathcal{F} = \mathcal{Q} + \mathcal{A}$ and $\mathcal{F}$ is a weakly distributive subsemimodule, then $\mathcal{D} = \mathcal{Q} + (\mathcal{F} \cap \mathcal{A})$.

ii) If $\mathcal{Q}$ is a weakly distributive and supplement of $\mathcal{F}$ and $\mathcal{A}$, then $\mathcal{Q}$ is a supplement of $\mathcal{F} \cap \mathcal{A}$.

iii) If $\mathcal{D} = \mathcal{A} + \mathcal{B}$ and $\mathcal{B}$ is a weakly distributive subsemimodule, then every supplement of $\mathcal{A}$ is contained in $\mathcal{B}$.

**Proof.**

i) Since $\mathcal{D} = \mathcal{Q} + \mathcal{A}$, we have $\mathcal{F} = \mathcal{F} \cap \mathcal{D} = \mathcal{F} \cap (\mathcal{Q} + \mathcal{A}) = (\mathcal{F} \cap \mathcal{Q}) + (\mathcal{F} \cap \mathcal{A})$. but $\mathcal{D} = \mathcal{Q} + \mathcal{F}$ then $\mathcal{D} = \mathcal{Q} + (\mathcal{F} \cap \mathcal{Q}) + (\mathcal{F} \cap \mathcal{A}) = \mathcal{Q} + (\mathcal{F} \cap \mathcal{A})$.

ii)By hypothesis $\mathcal{D} = \mathcal{Q} + \mathcal{F} = \mathcal{Q} + \mathcal{A}$, by Lemma3.2.3 $\mathcal{Q} \cap \mathcal{F}$ and $\mathcal{Q} \cap \mathcal{A}$ are small in $\mathcal{Q}$, since $\mathcal{Q} \cap \mathcal{F} \cap \mathcal{A} \subseteq \mathcal{Q} \cap \mathcal{F}$, then $\mathcal{Q} \cap \mathcal{F} \cap \mathcal{A}$ is small in $\mathcal{Q}$. By Lemma3.2.3 $\mathcal{Q}$ is a supplement of $\mathcal{F} \cap \mathcal{A}$.

iii) Let $\mathcal{D} = \mathcal{A} + \mathcal{B}$, and $\mathcal{O}$ is a supplement of $\mathcal{A}$ in $\mathcal{D}$, then $\mathcal{D} = \mathcal{A} + \mathcal{O}$. Hence $\mathcal{B} = \mathcal{B} \cap \mathcal{A} + \mathcal{B} \cap \mathcal{O}$ and so $\mathcal{D} = \mathcal{A} + \mathcal{B} \cap \mathcal{A} + \mathcal{B} \cap \mathcal{O} = \mathcal{A} + \mathcal{B} \cap \mathcal{O}$ this implies $\mathcal{B} \cap \mathcal{O} = \mathcal{O}$ and $\mathcal{O} \subseteq \mathcal{B}$. ■

**Theorem 3.2.5.** If $\mathcal{D}$ is a distributive supplemented $R-$ semimodule. Then $\mathcal{D}$ is an amply supplement, and every subsemimodule has a unique supplement.

**Proof.** Let $\mathcal{D} = \mathcal{M} + \mathcal{S}$, so $\mathcal{M}$ has a supplement in $\mathcal{D}$, say $\mathcal{F}$ . By Proposition3.2.4 (iii) the subsemimodule $\mathcal{F}$ is contained in $\mathcal{S}$. Then $\mathcal{D}$ is an amply supplemented.

Assume that $\mathcal{X}$ is also a supplement of $\mathcal{M}$ in $\mathcal{D}$, then $\mathcal{D} = \mathcal{M} + \mathcal{F} \cap \mathcal{X}$ by proposition3.2.4 (i), by the minimality $\mathcal{X} = \mathcal{F}$. ■

**Proposition 3.2.6**. If $\mathcal{D} = \mathcal{T}_1 \oplus \mathcal{T}_2$ is a semimodule, and $\mathcal{T}_3 \in L(\mathcal{D})$ is subtractive and weakly distributive, then $\mathcal{D}/\mathcal{T}_3 = (\mathcal{T}_1 + \mathcal{T}_3)/\mathcal{T}_3 \oplus (\mathcal{T}_2 + \mathcal{T}_3)/\mathcal{T}_3$.

**Proof.** Assume that $\mathcal{D} = \mathcal{T}_1 \oplus \mathcal{T}_2$, then it is clear that $\mathcal{D}/\mathcal{T}_3 = (\mathcal{T}_1 + \mathcal{T}_3)/\mathcal{T}_3 + (\mathcal{T}_2 + \mathcal{T}_3)/\mathcal{T}_3$. Now, we need prove the unique representation of the elements of $\mathcal{D}/\mathcal{T}_3$. Since $\mathcal{T}_3$ is weakly distributive, then $\mathcal{T}_3 = (\mathcal{T}_1 \cap \mathcal{T}_3) + (\mathcal{T}_2 \cap \mathcal{T}_3)$, hence $\forall k \in \mathcal{T}_3, k = k_1 + k_2$ where $k_1 \in \mathcal{T}_1 \cap \mathcal{T}_3, k_2 \in \mathcal{T}_2 \cap \mathcal{T}_3$. Assume that $u + \mathcal{T}_3 \in \mathcal{D}/\mathcal{T}_3$, assume $u + \mathcal{T}_3 = (h + t) + \mathcal{T}_3 = (h' + t') + \mathcal{T}_3$ ... $(*)$ Where $h, h' \in \mathcal{T}_1 + \mathcal{T}_3$ and $t, t' \in \mathcal{T}_2 + \mathcal{T}_3$ it can be assumed that $h, h' \in \mathcal{T}_1$ and $t, t' \in \mathcal{T}_2$, then $(h +$

$t) + k = (h' + t') + k'$ for some $k, k' \in \mathcal{T}_3$ by $(*)$ $k = k_1 + k_2$ and $k' = k_1' + k_2'$ such that $k_1, k_1' \in \mathcal{T}_1 \cap \mathcal{T}_3$ and $k_2, k_2' \in \mathcal{T}_2 \cap \mathcal{T}_3$, then $h + k_1 = h' + k_1', t + k_2 = t' + k_2'$ by unique representation of the elements of $\mathcal{T}_1 \oplus \mathcal{T}_2$, it follows $h + k_1 = h' + k_1', t + k_2 = t' + k_2'$ where $k_1, k_1', k_2, k_2' \in \mathcal{T}_3$, then $h + \mathcal{T}_3 = h' + \mathcal{T}_3$ and $t + \mathcal{T}_3 = t' + \mathcal{T}_3$, therefore $\mathcal{D}/_{\mathcal{T}_3} = (\mathcal{T}_1 + \mathcal{T}_3)/_{\mathcal{T}_3} \oplus (\mathcal{T}_2 + \mathcal{T}_3)/_{\mathcal{T}_3}$. ■

**Proposition 3.2.7** Let $U$ and $A$ be $R$-semimodules, μ a $k$-regular homomorphism from $U$ to $A$. If $\mu(U)$ is a weakly distributive subsemimodule of $A$, $W$ and $N$ subsemimodules of $A$ with $W + N = A$ and $\mu^{-1}(W) + \mu^{-1}(N)$ is subtractive in $U$, then $\mu^{-1}(W) + \mu^{-1}(N) = U$.

**Proof.** It is clear that $\mu^{-1}(W) + \mu^{-1}(N) \subseteq U$. Let $x \in U$, then $\mu(x) \in (W + N) \cap \mu(U)$. Since $\mu(U)$ is a weakly distributive subsemimodule, $\mu(x) \in [W \cap \mu(U) + N \cap \mu(U)]$, so Lemma 2.1.4 implies $\mu(x) \in [\mu(\mu^{-1}(W)) + \mu(\mu^{-1}(N))]$ this mean $\mu(x) = \mu(x_1) + \mu(x_2)$ with $x_1 \in \mu^{-1}(W)$, $x_2 \in \mu^{-1}(N)$ by Remark 2.1.5 $\mu(x) = \mu(x_1 + x_2)$.

By hypothesis, μ is a $k$-regular, hence $x + k_1 = x_1 + x_2 + k_2$ for some $k_1, k_2 \in ker\mu$. Now, $x_1 + x_2 + k_2 \in \mu^{-1}(W) + \mu^{-1}(N)$ (since $ker\mu \leq \mu^{-1}(N)$ ), so $x + k_1 \in \mu^{-1}(W) + \mu^{-1}(N)$. But $k_1 \in ker\mu \subseteq \mu^{-1}(W) + \mu^{-1}(N)$, then $x \in \mu^{-1}(W) + \mu^{-1}(N)$(by subtractive property).

**Corollary 3.2.8** Let $U$ be an $R$ - semimodule, $A$ is a subtractive $R$ – semimodule and $\mu, \omega \in Hom\ (U, A)$. If $(\mu + \omega)(U)$ is a weakly distributive subsemimodule in A and $\mu(U) + \omega(U) = A$, then $\omega^{-1}(\mu(U)) + \mu^{-1}(\omega(U)) = U$.

**Proof.**

By Proposition 3.2.7 $(\mu + \omega)^{-1}(\mu(U)) + (\mu + \omega)^{-1}(\omega(U)) = U$. Since $A$ is a subtractive then the subsemimodules $\mu(U)$ and $\omega(U)$ are subtractive, by Remark 1.1.30 μ and $\omega$ are $i$-regular, hence by Lemma 2.1.21 $U = (\mu + \omega)^{-1}(\mu(U)) + (\mu + \omega)^{-1}(\omega(U)) = \omega^{-1}(\mu(U)) + \mu^{-1}(\omega(U))$.

**Proposition 3.2.9** Let $U$ and $A$ be $R$-semimodules, μ a $k$-regular homomorphism from $U$ to $A$. If $ker$μ is a weakly distributive subsemimodule in $U$ and $T + S = U$, then

$\mu(T \cap S) = \mu(T) \cap \mu(S)$.

**Proof.** It is clear that $\mu(T \cap S) \subseteq \mu(T) \cap \mu(S)$. Let $y \in \mu(T) \cap \mu(S)$, then $y = \mu(\mathrm{t}) = \mu(\mathrm{s})$, $\mathrm{t} \in T$, $s \in S$. Since , μ is $k$-regular, $t + k_1 = s + k_2$ for some $k_1, k_2 \in ker\mu$. Then $t + k_1 \in (T + ker\mu) \cap (S + ker\mu)$, since $ker$μ is a weakly distributive subsemimodule, then $t + k_1 \in (T \cap S) + ker\mu$ hence $t + k_1 = x + k_3$ where $x \in T \cap S$, $k_3 \in ker\mu$. Then $\mu(\mathrm{t}) = \mu(x)$implies $y = \mu(x) \in \mu(T \cap S)$. Therefore $\mu(T \cap S) = \mu(T) \cap \mu(S)$.

## 3.3 More on supplement property in distributive semimodules

In this section, we aim to define the concepts originally introduced in the context of modules to semimodules, with a focus on generalizing their associated properties related to distributive laws. The definitions we will be exploring and expanding upon include supplement extending [44], principally ⊕-supplement [45], principally supplemented [46], and H-supplemented [47].

**Definition 3.3.1.** An R– semimodule $\mathcal{D}$ is called supplemented extending if every closed subsemimodule is supplement.

**Lemma 3.3.2.** Suppose that $M = M_1 \oplus M_2$ where , $M_1, M_2$ are R– semimodules and $M$ distributive . If $I \in L(M)$ is closed in $M$, then $I \cap M_i$ closed in $M_i$ ($i = 1,2$).

**Proof.** Assume that $I \cap M_i \leq^e {M_i}^* \leq M_i$ where $i = 1,2$ . Then

$I \cap M_1 \oplus I \cap M_2 \leq^e {M_1}^* \oplus {M_2}^* \leq M_1 \oplus M_2 = M$ , thus $I \leq^e {M_1}^* \oplus {M_2}^*$ since $I$ closed in $M$, then $I = {M_1}^* \oplus {M_2}^*$. Hence $I \cap M_1 \oplus I \cap M_2 = {M_1}^* \oplus {M_2}^*$ implies that $I \cap M_1 = {M_1}^*$ and $I \cap M_2 = {M_2}^*$ , so $I \cap M_i$ closed in $M_i (i = 1,2)$. ■

**Proposition 3.3.3.** Let $M = M_1 \oplus M_2$ where $M, M_1, M_2$ are R− semimodules and $M$ distributive , then $M$ is supplemented extending, if $M_1, M_2$ are supplemented extending.

**Proof.** Suppose that $M_1, M_2$ are supplement extending R− semimodules, and $L$ is a closed subsemimodule in $M$. By Lemma 3.3.2 $L \cap M_i$ is closed subsemimodule in $M_i (i = 1,2)$ but $M_i$ is supplement extending , then $L \cap M_i$ is supplement in $M_i$, then by Lemma 3.2.3 there exist a subsemimodule $K_i$ of $M_i$ such that $K_i + (L \cap M_i) = M_i$ and $K_i \cap L \cap M_i = K_i \cap L \ll L \cap M_i \ll L$. Now, $(K_1 + K_2) \cap L = (K_1 \cap L) + (K_2 \cap L) \ll L$ and

$M = [K_1 + (L \cap M_1)] + [K_2 + (L \cap M_2)] \subseteq (K_1 + K_2) + L$ , by Lemma 3.2.3 $L$ is supplement. ■

Following Mohammad and Yassin in [48] we use a concept T-direct summand in the module. We use the definition in semimodule. Let $A, B,$ and $C$ be subsemimodules of R-semimodule $M$. $A$ is called the T-direct sum of $B,$ and $C$ (denoted by $A = B \oplus_T C$) if $A = B + C$ and $B \cap C \in L(T)$. In this case, each of $B,$ and $C$ is called a T-direct summand of $A$.

**Proposition 3.3.4.** Let $A, B$ and $C$ be subsemimodules of cancellative, semisubtractive, and distributive semimodule $M$, such that $M = B \oplus_A C$. Then $M/_A = {(B + A)}/_A \oplus {(C + A)}/_A$.

**Proof.** It's clear $M/_A = {(B \oplus_A C)}/_A = {(B + A)}/_A + {(C + A)}/_A$

and ${(B + A)}/_A \cap {(C + A)}/_A = {[(B + A) \cap (C + A)]}/_A$

$= {[B \cap (C + A) + A \cap (C + A)]}/_A = {[(B \cap C) + (B \cap A) + (A \cap C) + A]}/_A =$

$A/_A = 0$ hence by Lemma 2.2.1 $M/_A = {(B + A)}/_A \oplus {(C + A)}/_A$. ■

**Proposition 3.3.5.** Let $M = M_1 \oplus M_2 = K + N$ be a semimodule and $K \leq M_1$. If $M$ is distributive and $K \cap N$ is small in N, then $K \cap N$ is small in $M_1 \cap$ N.

**Proof.** Let $M_1 \cap N = (K \cap N) + L$, where L is a subsemimodule of $M_1 \cap N$. Since M is distributive, $\mathrm{N} = (M_1 \cap \mathrm{N}) \oplus (M_2 \cap \mathrm{N})$. We have $\mathrm{M} = \mathrm{K} + \mathrm{N} = \mathrm{K} + (M_1 \cap \mathrm{N}) + (M_2 \cap \mathrm{N}) = \mathrm{K} + \mathrm{L} + (M_2 \cap \mathrm{N})$ and $N = (K \cap N) + L + (M_2 \cap N)$. Since $K \cap N$ is small in $N$, we have $N = L \oplus (M_2 \cap N)$. Then $N = (N \cap M_1) \oplus (N \cap M_2)$ and $L \leq M_1 \cap N$ imply $L = M_1 \cap N$. Hence $K \cap N$ is small in $M_1 \cap N$. ■

**Definition 3.3.6.** Let $M$ be an R-semimodule, $m \in M,$ and $L$ a direct summand of $M$, the subsemimodule $L$ is called principally ⊕-supplement of $Rm$ in $M$ if $Rm$ and $L$ satisfy $M = Rm + L$ and $Rm \cap L$ is small in $L$, and the semimodule $M$ is called principally ⊕-supplemented if every cyclic subsemimodule of $M$ has a principally ⊕-supplement in $M$.

**Proposition 3.3.7.** Let $M$ be a semisubtractive, subtractive, cancellative, and distributive principally ⊕-supplemented semimodule. Then every homomorphic image of $M$ is principally ⊕-supplemented.

**Proof.** Let $L$ be a subsemimodule of $M$ and $(Rm + L)/L$ a cyclic subsemimodule of $M/L$. Then there exists a direct summand $A$ of $M$ such that $M = A \oplus B = Rm + A$ and $Rm \cap A$ is small in $A$. We prove $(A + L)/L$ is a principally ⊕-supplement of $(Rm + L)/L$. Now $M/L = (Rm + L)/L + (A + L)/L$ and, since $M$ is distributive, $(Rm + L) \cap (A + L) = L + (Rm \cap A)$. So $(Rm + L)/L \cap (A + L)/L$ is small in $(A + L)/L$. Again by distributivity and $A \cap B = 0$, we have $(A + L) \cap (B + L) = L$. Hence $(A + L)/L$ is a direct summand of $M/L$. ■

Recall that a semimodule $M$ is said to be regular if every cyclic subsemimodule is a direct summand of $M$ (refer Definition3.3 in [49]). In the context of modules, this concept is known as principally semisimple [50].

**Lemma 3.3.8.** Let $M$ be a semisubtractive, subtractive, cancellative semimodule with $Rad(M) = 0$. Then $M$ is principally ⊕-supplemented if and only if $M$ is regular.

**Proof.** Let $M$ be a principally $\bigoplus$-supplemented semimodule and $m \in M$. Then there exists a direct summand $A$ of $M$ such that $M = A + Rm$ and $A \cap Rm$ is small in $A$. Since $A \cap Rm$ is also small in $M$ and $Rad(M) = 0$, then $A \cap Rm = 0$ so, $Rm$ is a direct summand of $M$. Therefore $M$ is regular. The rest is clear. ■

**Proposition 3.3.9.** Let $M$ be a semisubtractive, subtractive, cancellative, and distributive principally $\bigoplus$-supplemented semimodule. Then $M/Rad(M)$ is regular.

**Proof.** Let Ḿ denote the semimodule $M/Rad(M)$. By Proposition 3.3.7, Ḿ is principally $\bigoplus$-supplemented. Since $Rad(\acute{M}) = 0$, Ḿ is regular from Lemma 3.3.8. ■

**Definition 3.3.10.** The semimodule $M$ is called principally supplemented if every cyclic subsemimodule of $M$ has a supplement in $M$.

**Proposition 3.3.11.** Let $M$ be a principally supplemented distributive semimodule. Then every direct summand of M is a principally supplemented semimodule.

**Proof.** Let $M = M_1 \bigoplus M_2$ and $m \in M_1$. There exists a subsemimodule $A$ of $M$ such that $M = Rm + A$ and $(Rm) \cap A$ is small in $A$. Then $M_1 = (Rm) + (M_1 \cap A)$. By Proposition 3.3.5, $(Rm) \cap A$ is small in $M_1 \cap A$. ■

**Proposition 3.3.12.** Let $M$ be a principally supplemented distributive semimodule. Then $M/Rad(M)$ is a regular semimodule.

**Proof.** Let $m \in M$. There exists a subsemimodule $M_1$ such that $M = Rm + M_1$ and $(Rm) \cap M_1$ is small in $M_1$. Then $M/Rad(M) = [(Rm + Rad(M))/Rad(M)] + [(M1 + Rad(M))/Rad(M)]$. Now we prove that $(Rm + Rad(M)) \cap (M_1 + Rad(M)) = Rad(M)$. The distributivity of $M$ implies $(Rm + Rad(M)) \cap (M_1 + Rad(M)) = (Rm) \cap M_1 + Rad(M)$. Since $(Rm) \cap M_1$ is small in $M_1$, therefore small in $M$, $(Rm) \cap M_1 \leq Rad(M)$. Hence $M/Rad(M) = [(Rm + Rad(M))/Rad(M)] \bigoplus [(M_1 + Rad(M))/Rad(M)]$ and so every principal submodule of $M/Rad(M)$ is a direct summand. ■

**Definition 3.3.19.** Let $M$ be an $R$-semimodule. $M$ is called H-supplemented if, given any subsemimodule $A$ of $M$, there exists a direct summand $D$ of $M$ such that $M = A + X$ holds if and only if $M = D + X$.

**Lemma 3.3.14.** Let $M$ be a H-supplemented semimodule and $X$ a subsemimodule of $M$. If for every direct summand $D$ of $M$, $(X + D)/X$ is a direct summand of $M/X$, then $M/X$ is H-supplemented.

**Proof.** Let $N/X \leq M/X$. Since $M$ is H-supplemented, there exists a direct summand $D$ of $M$ such that $M = N + Y$ if and only if $M = D + Y$. By hypothesis, $(D + X)/X$ is a direct summand of $M/X$. Then $M/X = N/X + L/X$ if and only if $M/X = (D + X)/X + L/X$ for every $L/X \leq M/X$. ■

**Proposition 3.3.15.** Let $M$ be a semisubtractive, cancellative, and H-supplemented distributive semimodule**.** Then $M/X$ is H-supplemented for every subtractive subsemimodule $X$ of $M$.

**Proof.** Let $D$ be a direct summand of $M$. Then $M = D \oplus D'$ for some subsemimodule $D'$ of $M$. Now $M/X = [(D + X)/X] + [(D' + X)/X]$ and $X = X + (D \cap D') = (X + D) \cap (X + D')$. So $M/X = [(D + X)/X] \oplus [(D' + X)/X]$. By Lemma 3.3.14, $M/X$ is H-supplemented. ■

# Conclusions

In this research, these is focus on the investigation of distributive semimodules and their properties. The concept of distributive property for semimodules has been developed, aiming to extend the understanding of distributive modules to the realm of semimodules. Extensive studies have been conducted to explore the concrete development of this concept and to establish its fundamental properties. Through rigorous analysis and exploration, various characterizations of the distributive property have been presented, accompanied by illustrative examples. Additionally, specific conditions on semirings or semimodules have been identified to obtain interesting and insightful results. The relationship between the distributive semimodule over a local semiring and the hollow semimodule has been explored, providing valuable insights. Connections have been established between the distributive semimodule and distributed homomorphisms over intersection processes or inverse image distributed over addition. A concept of weakly distributive semimodules has been introduced and investigated, revealing that a semimodule is distributive if and only if each of its subsemimodules is weakly distributive. One significant aspect of this research is the integration and application of various concepts and terminologies within the study. Specifically, we explore the practical implications of these concepts, such as the distributive and arithmetical semimodules.

Additionally, are investigate the properties of supplement, amply supplement, supplement extending, principally $\oplus$-supplement, and H-supplemented semimodules. By incorporating and examining these concepts, it is to enhance our understanding of the subject matter and contribute new insights to the field of distributive semimodules. These results contribute to our understanding of distributive semimodules and open up avenues for further research in this area.

# Future works

A work could be on:

- Investigating the applicability of distributive semimodules in specific mathematical frameworks or domains, such as algebraic structures, computational mathematics, or mathematical physics.
- Exploring the potential connections between distributive semimodules and other mathematical concepts, such as vector spaces, tensor products, or category theory, to further enrich our understanding of their properties and applications.
- Extending the study of weakly distributive semimodules by examining their relationships with other related concepts, such as partially distributive structures or weakly distributivity in different algebraic systems.
- Investigating the implications of the distributive property in practical applications, such as optimization problems, coding theory, or data analysis, to uncover potential benefits and develop novel methodologies.
- Analyze the computational aspects of distributive semimodules, including the development of algorithms or computational techniques for efficiently handling and manipulating these structures.
- Exploring the existence and uniqueness of solutions for systems of equations involving distributive semimodules, and investigate their applications in diverse fields, such as mathematical modeling, engineering, or economics.
- Investigating the connections between distributive semimodules and other mathematical frameworks, such as lattice theory, Boolean algebra, or order theory, to understand their interplay and potential applications in those areas.

- Generalizing the concept of distributive semimodules to higher-dimensional structures or noncommutative settings, and explore the implications and properties of such generalizations.

These suggestions aim to inspire further exploration and development of the concept of distributive semimodules and their applications.

## References


[1] J. Saffar Ardabili, S. Motmaen and A. Yousefian Darani, "The Spectrum of Classical Prime Subsemimodules," *Australian Journal of Basic and Applied Sciences,* vol. 7, no. 11, pp. 1824-1830, 2011.

[2] Zaidoon W. Alboshindi and Asaad A.M. Alhossaini, "Fully Prime Semimodule, Fully Essential Semimodule and Semi-Complement Subsemimodules," *Iraqi Journal of Science,* vol. 63, no. 12, pp. 5455-5466, 2002.

[3] L. Fuchs, "Über die Ideale arithmetischer ringe," *Commentarii Mathematici Helvetici,* vol. 23, no. 1, pp. 334-341, . "." 23.11949 .

[4] C. U. Jensen, "A remark on arithmetical rings," *Proceedings of the American Mathematical Society,* vol. 15, no. 6, pp. 951-954, 1964 .

[5] Mimouni, A., Kabbour, M., & Mahdou, N, "Trivial ring extensions defined by arithmetical-like properties," *Communications in Algebra,* vol. 41, no. 12, pp. 4534-4548, 2013 .

[6] F. Couchot, "Finitistic weak dimension of commutative arithmetical rings," *Arabian Journal of Mathematics,* vol. 1, pp. 63-67, 2012 .

[7] X. Lu, "Some Properties of Completely Arithmetical Rings," vol. 23, no. 1, p. 83–88, 2016.

[8] W. Stephenson, "Modules whose lattice of submodules is distributive," *Proceedings of the London Mathematical Society,* vol. 3, no. 2, pp. 291-310, 1974.

[9] V. Erdoğdu, "Distributive modules," *Canad. Math. Bull,* vol. 30, no. 2, pp. 248-254, 1987.

[10] A. Tuganbaev, "Distributive extensions of modules," *Journal of Mathematical Sciences,* vol. 149, no. 3, 2008.

[11] N. Zamani, "On torsion free distributive modules," *"," Contributions to Algebra and Geometry,* vol. 50, no. 20, pp. 327-336, 2009.

[12] E. Büyükaşık and Y. M. Demirci, "Weakly distributive modules. Applications to supplement submodules," *Proceedings-Mathematical Sciences,* vol. 5, pp. 525-534, 2010.

[13] A. M. Buhphang and M. B. Rege, "Semi-commutative modules and Armendariz modules," *Arab. J. Math. Sci,* pp. 53-65, 2002.

[14] V. Camillo, "Distributive modules," *J.Algebra,* no. 36, pp. 16-25, 1975.

[15] P. Vamos, "Finitely generated Artinian and distributive modules are cyclic," *Bulletin of the London Mathematical Society,* vol. 10, no. 3, pp. 287-288, 1978.

[16] A. A. Tuganbaev, "Flat modules and distributive rings," *Journal of Mathematical Sciences,* vol. 95, no. 4, 1999.

[17] A. A. Tuganbaev, "Direct sums of distributive modules," *Matematicheskii Sbornik,* vol. 187, no. 12, pp. 137-156, 1996.

[18] A. A. Tuganbaev, "Left and Right Distributive Rings," *Mathematical Notes,* vol. 58, no. 4, 1995.

[19] A. A. Tuganbaev, "Semidistributive modules," *Journal of Mathentatical Scierces,* vol. 94, no. 6, 1999.

[20] J. S. Golan, Semirings and their Applications, Kluwer Academic Publishers, Dordrecht, 1999.

[21] Vishnu Gupta and J.N. Chaudhari, "Some remarks on semirings," vol. 12, pp. 13-18, 2003.

[22] Reza Ebrahimi Atani, and Shahabaddin Ebrahimi Atani, "Ideal Theory in Commutative Semirings," *Buletinul Academiei de Ştiinţe a Moldovei. Matematica,* vol. 57, p. 14–23, 2008.

[23] A. H. Alwan, "Maximal ideal graph of commutative semirings," *International Journal of Nonlinear Analysis and Applications,,* vol. 12, no. 1, pp. 913-926, 2021.

[24] H. Behzadipour and P. Nasehpour, "On 2-absorbing ideals of commutative semirings," *Journal of Algebra and Its Applications,* vol. 19, no. 20, p. 2050034, 2020.

[25] Khitam S. H. Aljebory and Assad M. A. Alhossaini, "Principally Quasi-Injective Semimodules," *Baghdad Science Journal,* vol. 16, no. 4, pp. 928-936, 2019.

[26] Tuyen, Nguyen Xuan, and Ho Xuan Thang, "On superfluous subsemimodules," *Georgian Mathematical Journal,* vol. 10, no. 4, pp. 763-770, 2003.

[27] Abuhlail, Jawad, and Rangga Ganzar Noegraha, "On k-Noetherian and k-Artinian semirings," *arXiv preprint arXiv:1907.06149,* 2019 .

[28] Y. Katsov and T.G. Nam, "Algebra Colloquium," *On Subtractive Semisimple Semirings,* vol. 16, no. 3, p. 415–426, 2009.

[29] Muna M.T. Altaee, Asaad A. M. Alhossaini, "π-Projective Semimodule Over Semiring," *Journal of University of Babylon for Pure and Applied Sciences,* vol. 28, no. 1, pp. 122-137, 2020.

[30] M.T.Altaee and A .M. A. Alhossanini, "Π-Injective Semimodule over Semiring," *Solid State Technolog,* vol. 63, no. 5, pp. 3424-3433, 2020.

[31] E. D. Sow, "On Essential Subsemimodules and Weakly Co-Hopfion," *European Journal of pur and applied Mathematics,* vol. 9, no. 3, pp. 250-265, 2016.

[32] Khitam S. H. Aljebory, and Assad M. A. Alhossaini, "The Jacobson Radical of the Endomorphism Semiring of P.Q.- Principal," *Baghdad Science Journal,* vol. 17, no. 2, pp. 523-529, 2020.

[33] j. R. Tsiba and D. Sow, "On Generators and Projective Semimodules," *International Journal of Algebra,* vol. 4, no. 24, pp. 1153-1167, 2010.

[34] Asaad M. A. Alhossaini, and Zainab A. H. Aljebory, "On p-duo Semimodules," *Modules with fully invariant submodules essential in fully invariant summands,* vol. 26, no. 4, pp. 27-35, 2018.

[35] Pareigis, Bodo, and Helmut Rohrl, "Remarks on semimodules," *arXiv preprint arXiv:1305.5531,* 2013 .

[36] Muna M. T. Altaee and Asaad M. A. Alhossaini, "Supplemented and π-Projective Semimodules," *iraqi Journal of Science,* pp. 1479-1487, 2020.

[37] Alwan, Ahmed H., and Asaad MA Alhossaini, "On dense subsemimodules and prime semimodules," *Iraqi Journal of Science,* vol. 61, no. 6, pp. 1446-1455, 2020 .

[38] Y. Katsov, T. Nam, and N. Tuyen, "On subtractive semisimple semirings," *In Algebra Colloquium,* vol. 16, pp. 415-426, 2009.

[39] Kasch, Friedrich, and David Alexander Ross Wallace, Modules and rings: a translation of Moduln und Ringe, Academic Press, 1982.

[40] P. M. Cohn, Universal algebra, vol. 6, .: Springer Science & Business Media, 2012.

[41] P. M. Cohn, Free Rings and Their Relations, Academic Press, 1985.

[42] A. A. Tuganbaev, Distributive modules and related topics, vol. 12, : CRC Press, 1999.

[43] R. E. Atani, and Sh. E. Atani, "On subsemimodules of semimodules," *Buletinul Academiei de Ştiinţe a Moldovei. Matematica,* vol. 2, no. 36, pp. 20-30, 2010.

[44] Yaseen, Sahira M., and Mustafa Mamoon Tawfeek, "Supplement extending modules," *Iraqi J. of Science,* vol. 56, no. 3B, pp. 2341-2345, .2015 .

[45] Ungor, Burcu, Sait Halicioglu, and Abdullah Harmanci, "On a class of δ-supplemented modules," *Bull. Malays. Math. Sci. Soc,* vol. 37, no. 3, pp. 703-717, 2014 .

[46] Acar, Ummahan, and Abdullah Harmanci, "Principally supplemented modules," *Albanian J. Math,* vol. 3, no. 4, pp. 79-88, 2010 .

[47] M. T. Ko¸san, "H-Cofinitely Supplemented Modules," *Vietnam Journal of Mathematics,* vol. 35, no. 2, p. 215–222, 2007.

[48] Mohammad, Z. R., and Yassin, S. M, "On small (T-extending) module. In Journal of physics: conference series," *In Journal of physics: conference series,* vol. 1530, no. 1, p. 012059, 2020 .

[49] Asaad Mohammad Ali Husain, and Haider Abbas Abdul AL-Ameer, "Fully Stable Semimodules," *Albahir J,* vol. 5, no. 9, pp. 13-20, 2017.

[50] Ungor, Burcu, Sait Halicioglu, and Abdullah Harmanci, "On a class of δ-supplemented modules," *Bull. Malays. Math. Sci. Soc,* vol. 73, no. 3, pp. 703-717, 2014 .

## الخلاصة

تم دراسة المقاسات التوزيعية وخصائصها على نطاق واسع من قبل العديد من الباحثين. ومع ذلك، يركز هذا العمل على التطوير الفعلي لمفهوم خاصية التوزيع لشبه المقاسات التوزيعية. بالإضافة للتعريف، يتم تقديم توصيفات مختلفة لهذه الخاصية، جنبًا إلى جنب مع أمثلة توضيحية. للحصول على نتائج مثيرة، كانت هناك حاجة لشروط معينة على شبه الحلقات أو شبه المقاسات، مثل **subtractive, semisubtractive, cancellative, k-cyclic, yoked, i-regular, and k-regular**. من خلال استكشاف العلاقة بين شبه المقاسات التوزيعية على شبه حلقة محلية، تم الحصول على رؤى قيمة. وعلاوة على ذلك، تم تأسيس الصلات بين شبه المقاسات التوزيعية وكون التطبيقات توزيعية بالنسبة لعملية التقاطع أو الصورة العكسية التوزيعية مع الجمع. تم إثبات أن أي شبه مقاس جزئي وشبه مقاس القسمة لشبه مقاس توزيعي يحتفظ بخاصية التوزيع. بالإضافة إلى ذلك، تم تقديم مفهوم شبه المقاس التوزيعي الضعيف والتحقيق فيه. تم اكتشاف أن شبه المقاسات تكون توزيعية إذا وفقط إذا كانت كل من وحداتها الفرعية موزعة بشكل ضعيف. علاوة على ذلك، باستخدام المفهوم المكمل، تم استخلاص خصائص مختلفة للنصف الوحدات التوزيعية**.** كذلك تمت الاستفادة من المفهوم التكميلي لإيجاد بعض خصائص شبه المقاس. أخيرًا، تمت دراسة خصائص تقاطع مركبات الجمع الجدائي وجمع مركبات الجمع الجدائي لشبه المقاس التوزيعي بالاستفادة من شروط مناسبة.

**جمهورية العراق**

**وزارة التعليم العالي والبحث العلمي**

**جامعة بابل**

**كلية التربية للعلوم الصرفة**

**قسم الرياضيات**

# حول شبه المقاسات التوزيعية

**رسالة**

**مقدمة الى مجلس كلية التربية للعلوم الصرفة - جامعة بابل كجزء من متطلبات نيل درجة الماجستير في التربية / الرياضيات**

**من قبل**

**علي حسن عبد الخالق كندوح**

**بأشراف**

**أ. د. اسعد محمد علي الحسيني**

**1445هـ**

**2023 م**