Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for pure Science
Department of Mathematics

ON DUAL SOFT TOPOLOGICAL SPACES

A Thesis

Submitted to the Council of the College of Education, for Pure Sciences in the University of Babylon

In Partial Fulfillment of the Requirements for the Degree of Master in Education /Mathematics

By

Jannat Yaseen Rasheed Hameed AL-Qaraawi

Supervised By

Prof. Luay Abd- AL-Haine Jabar AL-Swidi (Ph. D.)

2023 A. D. 1444 A. H.

بسم الله الرحمن الرحيم

وعنده مفاتح الغيب لا يعلمها إلا هو ويعلم ما في البر والبحر

صدق الله العلي العظيم

الأنعام آية 59

Contents

Dedication.....

To the one who lead the hearts and minds of humanity to haven safety ,the first teacher of mankind , Peace be upon him

To my dear mother and dear father

To the one who gave me the greatest support, who was my ally in the most of difficult times my dear husband

To the seed of heart and the hope of tomorrow, my beloved children

To my dear brother and dear sister

To the soul of my dear aunt who I wished to be with me at this time

To all my **Teachers** who taught me to come to this stage of learning.

To everyone who Supports me and Stay beside me in my Life.

Jannat Yaseen Rasheed

2023

Acknowledgements……..…….

In the name of Allah, the most merciful the most compassionate all praise be to Allah the lord of the worlds and prayers and peace be upon Mohamed his servant and messenger and to his good and pure household

I would like to express my deep and sincere gratitude to my supervisor Prof. Dr. **Luay Abd Al Hani AL-Swidi** for his continuous support to my research, for his patience, encouragement and invaluable suggestions throughout the entire period of my work.

Also, I would like to thank all the members of Mathematics Department, College of Education for Pure Sciences, **University of Karbala**, who are credited for my arrival at this stage.

Special thanks are extended to the College of Education for Pure Sciences of **Babylon University** for giving me the chance to complete my postgraduate study.

I am extremely and sincerely thankful to **my husband** for his love, care and sacrifice that enabled me to reach this level of learning.

Finally, I want to thank **my collegue Dr. Raghad Hamed** for her valuable assistance, support and encouragement me.

Jannat Yaseen Rasheed

2023

.....List of the symbols .....

Symbol	
X_{DS}	The dual soft universal set
$\emptyset_{DS}$	The dual soft empty set
DP_e	The dual soft point
$\cup$	The stander union of set theory
$\cap$	The stander intersection of set theory
$\cup_{DS}$	The dual soft union
$\cap_{DS}$	The dual soft intersection
$\cup_E$	The extension union
$\cap_E$	The extension intersection
$\in$	Belonging to
$\in_{DS}$	Belonging of dual soft set
$\subseteq$	The stander subset
$\subseteq_{DS}$	The dual soft subset
T_E	The dual soft topological space
$DS-nhd$	The dual soft neighborhood
DS_i	The dual soft interior point
DS_e	The dual soft exterior point
DS_{fr}	The dual soft frontier point
DS_d	The dual soft limit point
DS_{cl}	The dual soft closure point
DS_{adh}	The dual soft adherent point

Abstract.....

The aim of this thesis is to introduce a new spaces called dual soft topological spaces which are defined over two initials universes sets with a fixed set of parameters our study gradually started by defining new generalization of the dual soft set dual soft union, dual soft intersection , extension complement, dual soft point , in order to construct the dual soft topological space in addition we generalized the dual soft open set and dual soft closed set.

So the idea of thesis is crystallized in three paragraphs:

The first paragraph contains new concepts such as dual soft neighborhood, dual soft interior point dual soft exterior point , dual soft frontier point , dual soft closure point , dual soft limit point, dual soft adherent point and dual soft isolated point and it investigates the characteristics of these concepts. and some of important relationships between these notions as well as the definition of dual soft subspace

In second paragraph the definition of dual soft super condensed set, dual soft semi condensed set, dual soft condensed set is generalized and investigated the characteristics of these concepts. The definition of dual soft border set is introduced. Further more classified the dual soft set into three classes.

The third paragraph introduce the concepts of the dual soft dense set ,dual soft nowhere dense set and dual soft somewhere dense set . The most important properties and characteristics of these notions and relationships are investigated.

This study also discuss the definition of dual soft Baire space , dual soft meager set , dual soft non meager set , dual soft co-meager set with most important properties.

Finally the notions such as dual soft resolvable spaces, dual soft irresolvable spaces, dual soft hereditary irresolvable, dual soft hyper connected space, dual soft submaximal space are generalized , with some of properties and results.

Introduction.....

The scientific and life problems that human faces in different times and eras are considered the main pillar of growth process in all mathematical sciences and through them the rest of the sciences grow and flourish . Here the researchers put themselves in front of difficult and important challenges to build and modify some mathematical concepts in a way that some mathematical concepts in a way that is consistent with facilitating on finding appropriate solutions to these problems . One of the most important concepts that gained brilliance and wide distinction is what Zadeh introduced in 1965 [1] which are the fuzzy sets that invaded all science with out exception and played an important role in the process of growth and development that the human world is witnessing .

In 1999 Molodtsov [2] started with soft sets theory ,which were compatible with the path of fuzzy sets ,being define in $X \times 2^E$space while fuzzy sets were define in $X \times [0,1]$ spaces .

He being with soft set theory as mathematical tools to solve many complicated problems in engineering, physics, economics and environment which can not find exact solution by classical methods because the uncertainty of data.

Maji,R.Biswas , and Roy[3] studied a soft set theory in 2003 and introduce numerous new concepts such as the formula for empty set in this theory, the equality of two sets, as well as the soft inter section and the soft union with some significant results and properties .

In 2007 Aktas and N.Cagman [4] provide a study on soft sets and soft groups.

In 2009 Kong etal, [5] comment on a fuzzy soft set theoretic approach to decision making problems .
In 2009 ifran Ali , F.feng , X liu W.Keunmin and M. shabir [6] debated new operations in the soft set theory .

In 2010Majumdar and Samanta [7] present generalized of fuzzy sets . And many recent study on fuzzy theory[8,9,10,11]

In 2011Cagman , Serkan Karatas ,Serdar Enginoghllu [12] studied soft topology after Molodtsov was successful in applying the soft set theory in many directions including the smoothness of functions, game theory, operation research, Reman integration, Perron integration , probability, theory of measurement and so forth . New definition of soft intersection , soft union and new type of set are introduced along with several results and properties.

The most important issues which were discussed using the soft set theory is the soft topological space where the researchers Shabir [13] in 2011 used the concept of soft topological space which is composed of a collection of the soft sets on the set X and accomplished some of the provisions that have been made . Then they defined some of the main idea behind this concept , for example , the soft open set, soft closed set , soft neighborhood , soft separation axioms and others.

Hussain and Ahmed [14,15] have proved many properties for soft open and soft closed sets. Also, they defined the notion of soft exterior and soft interior and addressed some of their characteristics which can be viewed as the

fundamentals of this research in the field of soft topological which the gradually improve bases of soft topological spaces.

In 2016Cagman and Enginoghllu [16] provide a study of a soft set theory and uni-int decision making European journal of operational research. And many recent studies on soft sets and soft topology [17,18, 19, 20, 21,22] .

In 2018 Fatima [23,24] defined a kind of limit points on the ideal soft topological spaces, which she invested in defining density.

Among the studies carried out by the researchers under supervision Dr . Luay AL.Swaidi:

Ahmed Saadi Abd Oon [25]in 2009 studied a new separation axioms called fuzzy y −separation axiom.

Thu-Al fiqar Fakher Nassef Al-aamery [26]in 2016 studied the soft sets theory and dividing the types to four families , to make a comparison between them and identify similarities and differences among them, all of this to get the concepts of the resolvability and irresolvability in soft topological spaces .

Sameer Abbas [27] in 2017 concentrated on the first kind only by discussing and studying the main properties of this kind.

Raghad Hamid Abbas Hassan[28]in 2019 have reexamined the notionof the fuzzy local functions . she also presented different kindsof fuzzy Ψ_i-operators whose definitions are based on fuzzy local functions. And introduced a new kind of fuzzy closure where the relation of the new kind

and different types of fuzzy local function have been investigate Mohammed Majid Najm Abdallah [29] in 2020studied the notion of the fuzzy positional function is introduced based on the fuzzy filter and the kinds of the fuzzy positional function are based on the fuzzy filter.

In 2022 [30] AL.Swaidi expanded these sets into sets that he called dual soft sets, which depend on the parameters E of the elements of the universal sets U_1 and U_2, which are two functions $F : E \to U_1$, $G : E \to U_2$. In this case, the number of elements of this dual soft space is grater than number of elements of soft spaces that is the spaces have been expanded, which allows us to work easily and conveniently with these spaces

This study introduces and construct a dual soft topological spaces which are defined over an initial universes sets U_1 and U_2 with affixed set of parameter E With an explanation of the most important characteristics of the dual soft topological spaces and the most important theorems as well as examples

This work consists three chapters . Chapter one divided into two sections:
Section one includes the basic definitions of dual soft set theory such as dual soft set, dual soft union, dual soft intersection, extension union and extension intersection, dual soft complement and extension complement and dual soft singular point with their types as well as the most important examples and properties

Section two contains the definition of dual soft topological spaces with examples, the characteristic of dual soft open and dual soft closed set, we construct dual soft neighborhood system and we present new concepts such as the dual soft interior point , dual soft exterior point , dual soft boundary

point , dual soft closure ,dual soft limit point ,dual soft adherent point ,dual soft isolated point with basic theorems and examples.
Finally the dual soft subspaces of dual soft topological space are defined and inherent concepts as well as the characterization of dual soft open and dual soft closed sets in dual soft subspaces are investigated.

Chapter two introduce new concepts dual soft super condensed set, dual soft semi condensed set and dual soft condensed set as well as dual soft regular open set and dual soft regular closed set with examples and opposite examples together with the most important properties and characteristics that link the dual soft interior point with the dual soft closure point and these characteristics are consider the pillar on which the chapter depends.
Moreover introduce the concepts of dual soft border set which is different from dual soft boundary of a set which we present in chapter one and discuss about the relations between these new concepts. Furthermore the dual soft sets of the dual soft topological space are classified into three classes as well as its examples and properties .
Chapter three contains three sections , section one introduces the concepts of dual soft dense set, dual soft nowhere dense set and dual soft somewhere dense set with the most important examples and properties as a main concepts for the chapter from which we set out clarify the concepts and proposition of the second and third sections and we links these concepts with concept of dual soft super condensed set ,dual soft semi condensed set and dual soft condensed set.

Section two provides the definition of dual soft Baire space with the more important properties as well as linking with dual soft meager set ,dual soft

non meager set and dual soft co-meager set moreover , presented some important properties.

Section three contains several important concepts in general topology as dual soft resolvable space ,dual soft irresolvable space,dual soft irresolvable subspace , dual soft hyper connected space and dual soft submaximal space with some of properties and results .

The appendix provide four illustrative examples.

Finally, the conclusions and suggestions for future work in this field. We would also like to point out that the ranking of the references will be according to the reference's appearance in the text. We would also like to point out that since the symbols used in this work in order to unify them because they differ from one source to another so we had to standardize them.

I this chapter, we will discuss a new concept of the soft set theory that is the dual soft set, which is depend over the fixed set of parameters E of the elements of the universal set U_1 and U_2 and give some basic and important definitions and examples. Based on that, we will define the dual soft topological spaces. The notions of dual soft open sets dual soft closed sets, dual soft neighborhood, dual soft interior, dual soft exterior, dual soft boundary, dual soft closure and dual soft drive set, dual soft subspace are introduced and their basic properties and examples are investigated .

1.1. Dual Soft Set Theory

In this section we will provide some basic notions of the dual soft set theory, such as dual soft set, dual soft intersection, dual soft union , dual soft complement with examples and the most important properties.

Definition 1.1.1[30]

let U_1 andU_2 are initial universes sets and E be the set of all possible parameters under consideration with respect to U_1 and U_2 .Whereas parameters are description features on properties of members of the initials universes sets. The triple (A, F, G) is called dual soft over to U_1 and U_2

where F, G are functions, $A \subseteq E$, $F : A \rightarrow P(U_1)$, $G : A \rightarrow P(U_2)$. i.e. $A_{FG} = \{(e, F(e), G, (e) \forall e \in A , (e, \emptyset, \emptyset) \forall e \in E/A)\}$ .The collection of all dual soft sets denoted by $DS(U_1, U_2)_E$.

Examples 1.1.2

1.Let U_1={h_1 Arabman, h_2 Americanman,h_3 Chinese man ,
h_4Indian man , h_5Negro man} are five men

U_2={c_1Arab woman, c_2American woman , c_3Chinese woman ,
c_4Indian woman , c_5Negro woman} be five women

E ={$e_1$hair loos , $e_2$multiple sclerosis , $e_3$osteoporosis, $e_4$stroke , $e_5$depression and anxiety, $e_6$acne } Six diseases affecting men and women

The(A, F, G)=
$\{(e_1, F(e_1), G(e_1)), (e_2, \emptyset, \emptyset), (e_3, F(e_3), G(e_3)), (e_4, F(e_4), \emptyset), (e_5, F(e_5), G(e_5)), (e_6, \emptyset, G(e_6)), \}$

$F: A \to P(U_1)$ represent age $G: A \to P(U_2)$ represent weight

$F(e_1)$= $\{h_1, h_3\}$ $G(e_1)$ ={ c_2, c_3}

$F(e_3)$= $\{h_2, h_4, h_5\}$ $G(e_3)$ ={c_3, c_4}

$F(e_4)$= $\{h_2, h_5\}$ $G(e_4) = \emptyset$

$F(e_5)$= $\{h_1, h_5\}$ $G(e_5)$ ={c_1, c_4, c_5}

$F(e_6)$= $\emptyset$ $G(e_6)$ ={ c_4}

It can be explained in the following table

E \ U_1 / U_2	Arab man / Arab woman	American Man / American woman	Chinese man / Chinese Woman	Indian man / Indian Woman	Negro man / Negro Woman
e_1 Hair loos	h_1	c_2	h_3 c_3		
e_2 Multiple sclerosis					
e_3 Osteoporosis		h_2	c_3	h_4 c_4	h_5
e_4 Stroke		h_2			h_5
e_5 Depression and Anxiety	h_1 c_1			c_4	h_5 c_5
e_6 Acne				c_4	

2.Let U_1 ={h_1Chevrolet, h_2Dodge, h_3Ford, h_4Chrysler }are American cars

U_2={c_1Kia , c_2Daewoo, c_3Sangyong }are Korean cars

E={$e_1$colours , $e_2$seating capacity}

The<A, F, G>= {(e_1,$F(e_1), G(e_1)$),(, (e_2,$F(e_2), G(e_2)$), **}**

F: $A \rightarrow P(U_1)$ represent price $G : A \rightarrow P(U_2)$ represent fuel economy standard

$F(e_1)$= {h_1**,** h_3, h_4**}** $G(e_1)$ ={ c_1**,** c_3**}**

$F(e_2)$= {h_2**,** h_3**}** $G(e_2)$ ={ c_2**}**

It can be explained in the following table

U_1 / E / U_2	Chevrolet / Kia	Dodge / Daewoo	Ford / Ssang yong	Chrysler
e_1 Colours	h_1 / c_1		h_3 / c_3	h_4
e_2 Fuel economy Standard		h_2 / c_2	h_3	

Definition 1.1.3[30]

The extension union of two dual soft sets (A, F_1, G_1) , (B, F_2, G_2) over a common universes U_1 and U_2 and A , B are subset of parameter E of a member of U_1 and U_2 is defined as the dual soft set (C, H_1, H_2) where

$C = A \cup B$ and

$$\big(c, H_1(c), H_2(c)\big) = \begin{cases} \big(c, F_1(c), G_1(c)\big) & if\ c \in A/B \\ \big(c, F_2(c), G_2(c)\big) & if\ c \in B/A \\ \big(c, F_1(c) \cup F_2(c)\big), \big(G_1(c) \cup G_2(c)\big) & if c \in A \cap B \\ (c, \emptyset, \emptyset) & if\ c \in E/C \end{cases}$$

This denoted by $(A, F_1, G_1) \cup_E (B, F_2, G_2) = \{\big(c, H_1(c), H_2(c)\big)\ c \in E\}$

Example 1.1.4

Let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1, c_2, c_3\}$ $E = \{e_1, e_2, e_3\}$

$(A, F_1, G_1) = \{(e_1 F_1(e_1), G_1(e_1)), \big(e_2, \{h_2, h_3\}, G_1(e_2)\big), ((e_3, \emptyset, \emptyset)\}$

$= \{(e_1, \{h_1, h_2\}, \{c_2, c_3\}), (e_2, \{h_2, h_3\}, \{c_3\})\}$

$(B, F_2, G_2) = \{(e_1, F_2(e_1), G_2(e_1)), (e_2, \emptyset, \emptyset), ((e_3, \emptyset, \emptyset)\}$

$= \{(e_1, \{h_1, h_2\}), \{c_2\})\}$

$F_1(e_1) = \{h_1, h_2\}$ $G_1(e_1) = \{c_2, c_3\}$

$F_1(e_2) = \{h_2, h_3\}$ $G_1(e_2) = \{c_3\}$

$F_2(e_1) = \{h_1, , h_2\}$ $G_2(e_1) = \{c_2\}$

(C, H_1, H_2) $= \{(e_1, \{h_1, h_2\}, \{c_2, c_3\}), (e_2, \{h_2, h_3\}, \{c_3\}), ((e_3, \emptyset, \emptyset)\}$

Definition 1.5[30]

The extension intersection of two dual soft sets (A, F_1, G_1) , (B, F_2, G_2) over a common universes U_1 and U_2 and A , B are subset of parameter E of a member of U_1 and U_2 is defined as the dual soft set (C, H_1, H_2) where

$C = A \cup B$ and

$$\big(C, H_1(c), H_2(c)\big) = \begin{cases} \big(c, F_1(c), G_1(c)\big) & if\ c \in\ A/B \\ \big(c, F_2(c), G_2(c)\big) & if\ c \in\ B/A \\ \big(c, F_1(c) \cap F_2(c)\big), \big(G_1(c) \cap G_2(c)\big) & if c \in A \cap B \\ (c, \emptyset, \emptyset) & if\ c \in\ E\ /C \end{cases}$$

This denoted by $(A, F_1, G_1) \cap_E (B, F_2, G_2) = \{\big(c, H_1(c), H_2(c)\big)\ c \in E\}$

By example 1.1.4

$(C, H_1, H_2) = \{(e_1, \{h_1, h_2\}, \{c_2\}), (e_2, \{h_2, h_3\}, \{c_3\}), ((e_3, \emptyset, \emptyset)\}$.

Definition 1.1.6[30]

The dual soft union of two dual soft sets (A, F_1, G_1) and (B, F_2, G_2) over the common universes sets $U1$ and $U2$ with A, B are subsets of parameters E over a member for U_1 and U_2 is defined as the dual soft set (C, H_1, H_2)

where $C = A \cup B$ and

$$\big(c, H_1(c), H_2(c)\big) = \begin{cases} \big(c, F_1(c) \cup F_2(c)\big), \big(G_1(c) \cup G_1(c)\big) & if c \in A \cup B \\ (c, \emptyset, \emptyset) & if c \in\ E\ /C \end{cases}$$

denoted by $(A, F_1, G_1) \cup_{DS} (B, F_2, G_2) = \{\big(c, H_1(c), H_2(c)\big)\ \forall c \in E\}$

Definition 1.1.7[30]

The dual soft intersection of two dual soft sets (A, F_1, G_1) and (B, F_2, G_2) over a common universes sets U_1 and U_2 with A, B are subsets of parameters E of members for U_1 and U_2 is defined as dual soft set (C, H_1, H_2)

where $C = A \cup B$ and

$$\big(c, H_1(c), H_2(c)\big) = \begin{cases} \big(c, F_1(c) \cap F_2(c)\big), \big(G_1(c) \cap G_1(c)\big) & if c \in A \cup B \\ (c, \emptyset, \emptyset) & if\ c \in\ E\ /C \end{cases}$$

denoted by $(A, F_1, G_1) \cap_{DS} (B, F_2, G_2) = \{(c, H_1(c), H_2(c)) \; \forall c \in E\}$

Definition 1.1.8[30]

Let U_1 and U_2 are initial universes sets and E be the set of all possible parameters under consideration with respect to U_1 and U_2

1- The dual soft empty set $\emptyset_{Ds} = \{(e, \emptyset, \emptyset) \; ; \forall \; e \in E\}$.

2- The dual soft absolute set $X_{Ds} = \{(e, U_1, U_2); \; \forall \; e \in E\}$

We will change the symbol of second type soft set as follows A_{FG}.

Definition 1.1.9[30]

The extension complement of A_{FG} is denoted by ${A_{FG}}^{e} = A_{F^e G^e}$

Where $F^e : E \to P(U_1)$, $G^e : E \to P(U_2)$ and $F^e(e) = U_1 - F(e)$

$G^e(e) = U_2 - G(e)$

In other word ${A_{FG}}^{e} = \{(e, U_1 - F(e), U_2 - G(e)) \; \forall e \in E\}$

If we use example 1.1.4 we can find

${F^e}_1(e_1) = \{h_3\}$ ${G^e}_1(e_1) = \{c_1\}$

${F^e}_1(e_2) = \{h_1\}$ ${G^e}_1(e_2) = \{c_1, c_2\}$

${F^e}_2(e_1) = \{h_3\}$ ${G^e}_2(e_1) = \{c_1, c_3\}$

${A_{FG}}^{e} = \{(e_1, \{h_3\}\{c_1\}, (e_2), \{h_1\}, \{c_1, c_2\}(e_3, \emptyset, \emptyset))$

Definition 1.1.10[30]

The singular dual soft point for any subset A for parameter E of member for the universes sets U_1 and U_2 The singular $(e, F(e), G, (e))$ of dual soft set (A, F, G) is called singular dual soft point denoted by $SD_e \; at \; e \in E$; $SD_e = (e, F(e), G, (e))$.

Definition 1.1.11[30]

The type *I* of dual soft point

The type I of dual soft point over the universes sets $X$and Y for $x$and y

1- The dual soft point of the form DP_e is defined by

$$DP_e\ (\acute{e}) = \begin{cases} (e, F(e), G(e)) \text{ for } \acute{e} = e \\ (e, \emptyset, \emptyset) \text{ if } \acute{e} \neq e \end{cases}$$

with the condition that the subset $F(e)$and $G(e)$ of $X$and Y respectively which not empty

By example 1.1.4

DP_{e1} ={$(e_1,\{h_1, h_2\}, \{c_2, c_3\}), (e_2, \emptyset, \emptyset), ((e_3, \emptyset, \emptyset)$ **}**

2- The dual soft point of the form $DP(e, x)$ is defined by

$$DP(e,x)_{\acute{e}} = \begin{cases} (\acute{e}, \{x\}, \emptyset) \text{ if } \acute{e} = e \\ (e, \emptyset, \emptyset) \text{ if } \acute{e} \neq e \end{cases} \text{ with } x \in X$$

By example 1.1.4

$DP(e_1, h_2)$={$(e_1,\{h_2\}, \emptyset), (e_2, \emptyset, \emptyset), ((e_3, \emptyset, \emptyset)$ **}**

3- The dual soft point of the form $DP(e, y)$ is defined by

$$DP(\acute{e}) = \begin{cases} (\acute{e}, \emptyset, \{y\}) \text{ if } \acute{e} = e \\ (e, \emptyset, \emptyset) \text{ if } \acute{e} \neq e \end{cases} \text{ with } y \in Y$$

By example 1.1.4

$DP(e_1, c_2)$={$(e_1,\emptyset,\{c_2\}), (e_2, \emptyset, \emptyset), ((e_3, \emptyset, \emptyset)$ **}**

4- The dual soft point of the form $DP(e, x, y)$is defined by

$DP_{\acute{e}}(e, x, y)$=$(\acute{e},\{x\},\{y\})$ if $\acute{e} = e$ $(e, \emptyset, \emptyset)$ if $\acute{e} \neq e$

By the same example

$DP(e_1, h_2, c_2)$={$(e_1,\emptyset,\{ h_2\},\{ c_2\}), (e_2,\emptyset,\emptyset),((e_3,\emptyset,\emptyset)$ **}**

The type *II* of the dual soft point over universes sets *X* and *Y* with parameter *E* of the members for *X* and *Y*

1- The dual soft point of the form DP_x is defined by

DP_x={$(e,\{x\},\emptyset)\forall\ e \in\ E$ where $x\ \in X$

By example 1.1.4

DP_{h_2} ={$(e_1,\{h_2\},\emptyset),(e_2,\{h_2\},\emptyset),\ (e_3,\{h_2\},\emptyset)$}

2- The dual soft point of the form DP_y is defined by

DP_y={$(e,\emptyset,\{ y\})\forall\ e \in\ E$ where $y\ \in Y$

By example 1.1.4

DP_{c_2} ={$(e_1,\emptyset,\{c_2\}),(e_2,\emptyset,\{c_2\}),\ (e_3,\emptyset,\{c_2\})$}

3- The dual soft point of the form $DP_{x,y}$ is defined by

$DP_{x,y}$={$(e,\{ x\},\{ y\})\forall\ e \in\ E$ where $x\ \in X$ and $y\ \in Y$

By example 1.1.4

DP_{h_2,c_2} ={$(e_1,\{h_2\},\{c_2\}),(e_2,\{h_2\},\{c_2\}),\ (e_3,\{h_2\},\{c_2\})$}

Note 1.1.12[30]

The each kind of the dual soft points is dual soft sets. but the singular dual soft point is not dual soft set. so when we deal with singular soft point and their relationship with the dual soft set on the basis of classical belong to with difference.

But when we deal with different types of the dual soft points and their relationship dual soft set.

Definition 1.1.13[30]

Let A_{F1G1} , $B_{F2G2} \in_{DS} DS\,(U_1, U_2)_E$. B_{F2G2} is called dual soft subset of A_{F1G1} if $F_2(e) \subseteq F_1(e)$, $G_2(e) \subseteq G_1(e) \; \forall \; e \in E$

By example 1.1.4

$F_2(e_1) \subseteq F_1(e_1)$, $F_2(e_2) \subseteq F_1(e_2)$

$G_2(e_1) \subseteq G_1(e_1)$, $G_2(e_2) \subseteq G_1(e_2)$ Then $B_{F2G2} \subseteq_{DS} A_{F1G1}$

Remark 1.1.14[30]

1- Each dual soft set if it union (classical) of its singular soft point .
2- Each dual soft set if its extension union of different types of its dual soft points .
3- Each dual soft set if its dual soft union of different types of its dual soft point .

Proposition 1.1.15[30]

For any dual soft set A_{FG} of $DS(X, Y)_E$ the following properties are hold :

1-$A_{FG} \cap_{ES} A_{FG} = A_{FG}$.

2-$A_{FG} \cap_{DS} A_{FG} = A_{FG}$

3-$A_{FG} \cup_{ES} A_{FG} = A_{FG}$.

4-$A_{FG} \cup_{DS} A_{FG} = A_{FG}$.

5-$A_{FG} \cap_{ES} \emptyset_{DS} = \emptyset_{Ds}$.

6-$A_{FG} \cap_{DS} \emptyset_{DS} = \emptyset_{Ds}$.

7-$A_{FG} \cup_{ES} \emptyset_{DS} = A_{FG}$.

8-$A_{FG} \cup_{DS} \emptyset_{DS} = A_{FG}$.

9-$A_{FG} \cup_{ES} X_{DS} = X_{DS}$.

10-$A_{FG} \cup_{DS} X_{Ds} = X_{DS}$.

11-$A_{FG} \cap_{ES} A_{FG}{}^{e} = \emptyset_{DS}$.

12-$A_{FG} \cap_{DS} A_{FG}{}^{e} = \emptyset_{DS}$.

13-$A_{FG} \cup_{ES} A_{FG}{}^{e} = X_{DS}$.

14-$A_{FG} \cup_{DS} A_{FG}{}^{e} = X_{DS}$.

Remark 1.1.16[30]

1- The abelian and associative law under the extension union and extension intersection will be satisfy .
2- The abelian and associative law under the dual union and dual intersection will be satisfy .

1.2 Dual Soft Topological Spaces

Topology is one of the substantial branches of mathematics. In this section, we will present the basic definition of dual soft topology as well as dual soft open and closed set, in addition construction of the dual soft neighborhood, dual soft interior, dual soft exterior, dual soft frontier, dual soft closure, dual soft adherent point, dual soft singular, dual soft subspace with examples.

Definition 1.2.12 Dual Soft Topological Spaces

The Quadruple (U_1, U_2, E, T_E) is called dual soft topological space if T_E satisfies, first $\emptyset_{Ds}$, X_{Ds} $\in_{DS}$ X_{Ds} second T_E is closed under finite dual soft intersection and finally closed under dual soft union of any sub collection of T_E .

Simply we denoted the dual soft topological space (X_{Ds}, T_E) and any dual soft set A_{FG} $\in_{DS}$ T_E is called dal soft open set and the extension complement of the dual soft open set is called dual soft closed set.

Example 1.2.2

Let U_1 $=\{h_1, h_2, h_3\}$ U_2 $=\{c_1,, c_2, c_3\}$ $E=\{e_1, e_2\}$

$A_{1F1G\,1} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \{h_1\}, \{c_1\})\}$

$A_{2F2G2} = \{(e_1, \{h_3\}, \{c_1\}), (e_2, \{h_2\}, \{c_3\})\}$

$B_{F3G\,3} = \{(e_1, \{h_2, h_3\}, \{c_1, c_2\})\}, (e_2, \{h_1, h_2\}, \{c_1, c_3\})\}$

T_E $=\{\emptyset_{Ds}, X_{Ds}, A_{1F1G\,1}, A_{2F2G\,2}, B_{F3G\,3}\}$.

Proposition 1.2.3

Let $\{T_{E\lambda}, \lambda \in \Lambda\}$ where Λ is an arbitrary set, be a collection of dual soft topology on X_{Ds} . then $\cap_{DS}$ $\{T_{E\lambda}, \lambda \in \Lambda\}$ is also dual soft topology on X_{Ds}.

Proof

Let $\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$ is a collection of dual soft topology on X_{Ds} . we have to show that $\cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$ is also dual soft topology on X_{Ds}.

If $\Lambda = \emptyset$ then $\cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$= $P(X_{Ds})$ all the power dual soft set of X_{Ds}

So First X_{Ds} , $\emptyset_{Ds} \in T_E$.

Second since the dual soft intersection of two dual soft subsets of X_{Ds} is again a dual soft subset of X_{Ds}.

Finally the dual soft union of any collection of dual soft subsets of X_{Ds} is again dual soft subset of X_{Ds} so in this case the dual soft intersection of dual soft topology is dual soft topology.

Now let $\Lambda \neq \emptyset$ since $T_{E\lambda}$ is a dual soft topology $\forall\ \lambda \in \Lambda$ it follows that

First X_{Ds} ,$\emptyset_{Ds} \in_{DS} T_{E\lambda}\ , \lambda \in \Lambda$

but $\emptyset_{Ds} \in_{DS} T_{E\lambda}\ \forall\ \lambda \in \Lambda$then $\emptyset_{Ds} \in_{DS} \cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$

and $X_{Ds} \in_{DS} T_{E\lambda}\ , \lambda \in \Lambda$then$X_{Ds} \in_{DS} \cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$

Second let $A_{1F1G\,1}$, $A_{2F2G2}\ \in_{DS} \cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$

then $A_{1F1G\,1}$, $A_{2F2G2}\ \in_{DS} T_{E\lambda}\ \forall\ \lambda \in \Lambda$ since $T_{E\lambda}$ is a dual soft topology of X_{Ds}

$\forall\ \lambda \in \Lambda$ it follows that$A_{1F1G\,1}\ \cap_{DS}\ A_{2F2G2}\ \in_{DS} T_{E\lambda}\ \forall\ \lambda \in \Lambda$

Hence $A_{1F1G\,1}\ \cap_{DS}\ A_{2F2G2}\ \in_{DS} \cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$

Finally let $A_{\alpha F\alpha G\alpha}\ \in_{DS} \cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$for $\alpha \in \Lambda$ where Λ is an arbitrary set

Then $A_{\alpha F\alpha G\alpha}\ \in_{DS}\ T_{E\lambda}\ \forall\ \lambda \in \Lambda$ and $\forall\ \alpha \in \Lambda$. since $T_{E\lambda}$ is a dual soft topology of X_{Ds} it follows that $\cup_{DS}\ \{A_{\alpha F\alpha G\alpha} : \alpha \in \Lambda\} \in_{DS} T_{E\lambda}\ \forall\ \lambda \in \Lambda$.

Hence$\cup_{DS}\ \{A_{\alpha F\alpha G\alpha} : \alpha \in \Lambda\} \in_{DS} \cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$

Thus $\cap_{DS}\{T_{E\lambda}\ , \lambda \in \Lambda\ \}$ is dual soft topology on X_{Ds}.

The next example shows that the dual soft union of two dual soft topology is not necessary a dual soft topology

Example 1.2.4

Let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1,, c_2, c_3\}$ $E = \{e_1, e_2\}$

$A_{1F1G\,1} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \{h_1\}, \{c_1\})\}$

$A_{2F2G2} = \{(e_1, \{h_3\}, \{c_1\}), (e_2, \{h_2\}, \{c_3\})\}$

$T_{E1} = \{\emptyset_{Ds}, X_{Ds}, A_{1F1G\,1}\}$.

$T_{E2} = \{\emptyset_{Ds}, X_{Ds}, A_{2F2G\,2}\}$.

Now $T_{E1} \cup_{DS} T_{E2} = \{\emptyset_{Ds}, X_{Ds}, A_{1F1G\,1}, A_{2F2G\,2}\}$ which is not dual soft topology on X_{Ds} because $A_{1F1G\,1} \cup_{DS} A_{2F2G\,2} \notin_{DS} T_{E1} \cup_{DS} T_{E2}$.

Proposition 1.2.5

If $\{A_{\lambda F\lambda G\lambda} : , \lambda \in \Lambda\}$ is any collection of dual soft closed subset of X_{Ds} then $\cap_{DS} \{A_{\lambda F\lambda G\lambda} : \lambda \in \Lambda\}$ is dual soft closed set

Proof

$A_{\lambda F\lambda G\lambda}$ is dual soft closed $\forall\, \lambda \in \Lambda$

Then $X_{Ds} - A_{\lambda F\lambda G\lambda}$ is dual soft open $\forall\, \lambda \in \Lambda$

so $\cup_{DS} \{X_{Ds} - A_{\lambda F\lambda G\lambda} : \lambda \in \Lambda\}$ is dual soft open by definition 1.2.1

thence $X_{Ds} - [\cap_{DS} \{A_{\lambda F\lambda G\lambda} : \lambda \in \Lambda\}$ is dual soft open by De- Morgan law

Thus $\cap_{DS} \{A_{\lambda F\lambda G\lambda} : \lambda \in \Lambda\}$ is dual soft closed .

Proposition 1.2.6

If A_{1F1G1} , A_{2F2G2} be two dual soft closed subsets of X_{Ds} . Then $A_{1F1G1} \cup_{DS} A_{2F2G2}$ is dual soft closed set

proof

Let A_{1F1G1} , A_{2F2G2} be tow dual soft closed subsets of X_{Ds}

Then $X_{Ds} - A_{1F1G1}$, $X_{Ds} - A_{2F2G2}$ are dual soft open subsets of X_{Ds}

So $X_{Ds} - A_{1F1G1} \cap_{DS} X_{Ds} - A_{2F2G2}$ is dual soft open subsets of X_{Ds} by definition 1.2.1

Thence $X_{Ds} - (A_{1F1G1} \cup_{DS} A_{2F2G2})$ is dual soft open by De- Morgan law

Therefore $A_{1F1G1} \cup_{DS} A_{2F2G2}$ is dual soft closed

Note that if A_{1F1G1} , A_{2F2G2} ,…, A_{nFnGn} be a finite number of dual soft closed subset of X_{Ds} then their dual soft union will be also dual soft closed subset of X_{Ds}. This follows at once by repeated application of the above proposition.

Now we will construction the concept of dual soft neighborhood and introduce some important topological concepts with their examples and properties.

Definition 1.2.7 Dual Soft Neighborhood

let (X_{Ds}, T_E) be dual soft topological space the dual soft set A_{FG} is the dual soft neighborhood simply $Ds - nhd$ of the dual soft point DP_e if ∃ a dual soft open set B_{F1G1} s.t $DP_e \in_{DS} B_{F1G1} \subseteq_{DS} A_{FG}$ .The collection of all $DS - nhd$ of the dual soft point DP_e is defined by $N(DP_e)$.

Example 1.2.8

let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1, c_2, c_3\}$ $E = \{e_1, e_2\}$

Let $A_{F1G1} = \{(e_1, \{h_1, h_2\}, \{c_1, c_2\}), (e_2, \{h_2\}, \{c_2\})\}$

$B_{F2G2} = \{(e_1, \{h_1\}, \{c_1\}, (e_2, \emptyset, \emptyset)\}$

$T_E = \{\emptyset_{Ds}, X_{Ds}, A_{1F1G1}, B_{2F2G2}\}$

Assume $C_{F3G3} = \{(e_1, \{h_1, h_2\}, \{c_1, c_2\}), (e_2, \{h_2\}, \{c_2, c_3\})\}$

$DP_{e1} = \{(e_1, \{h_1, h_2\}, \{c_1 \ , c_2\}) , (e_2, \ \emptyset , \emptyset)\}$

C_{F3G3} is $DS - nhd$ of DP_{e1} since$DP_{e1} \in_{DS} A_{F1G1} T_E$ and$A_{F1G1} C_{F3G3}$.

Proposition 1.2.9

A dual soft subset of dual soft topological space is dual soft open if and only if it is $DS - nhd$ of each its dual soft points.

Proof

Let $A_{FG} \subseteq_{DS} X_{Ds}$ be a dual soft open then every $DP_e \in_{DS} A_{FG}$,

$DP_e \in_{DS} A_{FG} \subseteq_{DS} A_{FG}$ and therefore $A_{FG}$is $DS - nhd$ of each of each its dual soft points.

Conversely, let A_{FG} be $DS - nhd$ of each of each its dual soft points

if $A_{FG} = \emptyset_{Ds}$ then it is dual soft open

if $A_{FG} \neq \emptyset_{Ds}$ then to each $DP_e \in_{DS} A_{FG}$ there exist a dual soft open set $A_{FG\,DP_e}$

such that $DP_e \in_{DS} A_{FG\,DP_e} \subseteq_{DS} A_{FG}$ it follows that

$A_{FG} = \cup_{DS} \{A_{FG\,DP_e} : DP_e \in_{DS} A_{FG}\}$.

Hence A_{FG} is dual soft open set being the dual soft union of dual soft open set.

Properties 1.2.10

Let (X_{Ds}, T_E) be dual soft topological space then the following facts are verified:

1- Every dual soft point has a least one $DS - nhd$.
2- Every $DS - nhd\ A_{FG}$ of DP_econtain DP_e.
3- If N_{FG} is $DS - nhd$ of $DP_e$and $N_{FG} \subseteq M_{F1G1}$ then $\exists\ H_{F2G2} \in_{DS} T_E \ni DP_e \in H_{F2G2} \subseteq_{DS} N_{FG} \subseteq_{DS} M_{F1G1}$.

4- If $N_{FG} \in_{DS} N(DP_e)$, $M_{F1G1} \in_{DS} N(DP_e)$ then $N_{FG} \cap_{DS} M_{F1G1} \in_{DS} N(DP_e)$.

5- If N_{FG} is $DS-nhd$ of DP_e then their exist $DS-nhd$ M_{F1G1} of DP_e which is sub set of N_{FG} s.t M_{F1G1} is $DS-nhd$ ie $N_{FG} \in_{DS} N(DP_e)$ then $\exists$ a dual soft open set $V_{F1G1} \ni DP_e \in_{DS} V_{F1G1} \subseteq_{DS} N_{FG}$ *But* V_{F1G1} is $DS-nhd$ of each its dual soft points

The dual soft topology T_E defined on $E \times P(U_1) \times P(U_2)$ is a composite of two soft topologies The first defined on p (U_1) and The Second defined on $E \times P(U_2)$

i.e. There is one to one corresponding between dual Soft topology defined on $E \times p(U_1) \times p(U_2)$ and The soft topologies defined on $E \times p(U_1)$ and $E \times p(U_2)$ respectively.

Proof

1- Since X_{DS} is $DS-nhd$ so it is $DS-nhd$ of every $DP_e \in_{DS} D(U_1,U_2)_E$ hence their exist at least $DS-nhd$ $D(U_1,U_2)_E$ for every $DP_e \in_{DS} D(U_1,U_2)_E$ hence $N(DP_e) \neq \emptyset_{DS} \forall DP_e \in_{DS} D(U_1,U_2)_E$

2- If $N_{FG} \in_{DS} N(DP_e)$ then N_{FG} is $DS-nhd$ of DP_e so by definition of $DS-nhd$ $DP_e \in_{DS} N_{FG}$

3- If $N_{FG} \in_{DS} N(DP_e)$ then there is dual soft open set $A_{F1G1} \ni DP_e \in_{DS} A_{F1G1} \subseteq_{DS} N_{FG}$

4- Let $N_{FG} \in_{DS} N(DP_e)$ and $M_{F1G1} \in_{DS} N(DP_e)$ then by definition $DS-nhd$ their exist A_{F2G2} and B_{F3G3}(dual soft open sets) $\ni DP_e \in_{DS} A_{F2G2}$ and $DP_e \in_{DS} B_{F3G3}$ so $DP_e \in_{DS} A_{F2G2} \cap_{DS} B_{F3G3} \subseteq_{DS} N_{FG} \cap_{DS} M_{F1G1}$ .....(1)

Since $A_{F2G2} \cap_{DS} B_{F3G3}$is dual soft open set it follows from (1) that $N_{FG} \cap_{DS} M_{F1G1}$ is $DS - nhd$ of DP_e

Thence $N_{FG} \cap_{DS} M_{F1G1} \in_{DS} N(DP_e)$

5- If $N_{FG} \in_{DS} N(DP_e)$ then there exist is a dual soft open set $M_{F1G1} \ni DP_e \in_{DS} M_{F1G1} \subseteq_{DS} N_{FG}$ since M_{F1G1} is dual soft open set it is $DS - nhd$ of each its dual soft points therefor $M_{F1G1} \in_{DS} N(DP_e) \forall DP_{\acute{e}} \in_{DS} M_{F1G1}$

Now we present the definitions of dual soft interior, exterior and boundary points with most important properties and examples.

Definition 1.2.11 Dual Soft Interior Point

let (X_{Ds}, T_E) be dual soft topological space we call the dual soft point DP_e is a dual soft interior point of a dual soft set A_{FG} if ∃ a dual soft open set B_{F1G1} s.t $DP_e \in_{DS} B_{F1G1} \subseteq_{DS} A_{FG}$

We shall denoted to the set of all dual soft interior point of A_{FG} by $DS_i(A_{FG})$

$DS_i(A_{FG}) = \cup_{e \in E} \{DP_e \in X_{Ds}; \exists U_{F1G1} \in_{DS} T_E \ni DP_e \in_{DS} U_{F1G1} \subseteq_{DS} A_{FG}\}$

Example 1.2.12

Let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1,, c_2, c_3\}$ $E = \{e_1, e_2\}$

$A_{FG}\{(e_1, \{h_2\}, \{c_2\}), (e_2, \{h_1, h_3\}, \{c_1, c_2\})\}$

Now let $A_{1F1G1} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \{h_1\}, \{c_1\})\}$

$A_{2F2G2} = \{(e_1, \{h_3\}, \{c_1\}), (e_2, \{h_2\}, \{c_3\})\}$

$B_{F3G3} = \{(e_1, \{h_2, h_3\}, \{c_1, c_2\})\}, (e_2, \{h_1, h_2\}, \{c_1, c_3\})\}$

$T_E = \{\emptyset_{DS}, X_{DS}, A_{1F1G1}, A_{2F2G2}, B_{F3G3}\}$

$A_{1F1G\,1} \subseteq_{DS} A_{FG}$ then all dual soft interior point (by example 4.1) are

DP_{12e1}= $\{(e_1, \emptyset, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

DP_{26e1}= $\{\{(e_1 \{h_2\}, \emptyset)\}, (e_2, \emptyset, \emptyset)\}$

DP_{27e1}= $\{(e_1 \{h_2\}, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

DP_{75e1}= $\{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{c_1\})\}$

DP_{81e1}= $\{(e_1, \emptyset, \emptyset)\}, (e_2, \{h_1\}, \emptyset)\}$

DP_{82e2}= $\{(e_1, \emptyset, \emptyset)\}, (e_2, \{h_1\}, \{c_1\})\}$

So $DS_i(A_{FG}) = DP_{12e1} \cup_{DS} DP_{26e1} \cup_{DS} DP_{27e1} \cup_{DS} DP_{75e1} \cup_{DS} DP_{81e1} \cup_{DS} DP_{82e2} = A_{1F1G1}$

Now we present equivalent theorem for the definition of dual soft interior point

Proposition 1.2.13

Let (X_{Ds}, T_E) be dual soft topological space and let $A_{FG} \subseteq_{DS} X_{Ds}$ then

$DS_i(A_{FG})$=$\cup_{DS}$ $\{A_{1F1G\,1} \subseteq_{DS} A_{FG} : A_{1F1G\,1}$ is a dual soft open set $\}$

Proof

$DP_e \in_{DS} DS_i(A_{FG})$

If and only if A_{FG} is $Ds - nhd$ of DP_e

If and only if there exist dual soft open set $A_{1F1G\,1}$

$\ni DP_e \in_{DS} A_{1F1G\,1} \subseteq_{DS} A_{FG}$ If and only if

$DP_e \in_{DS} \cup_{DS} \{A_{1F1G\,1} \subseteq_{DS} A_{FG} : A_{1f1G\,1}$ is a dual soft open set $\}$

Hence $DS_i(A_{FG})$=$\{A_{1F1G\,1} : A_{1F1G\,1} \subseteq_{DS} A_{FG}\}$

Proposition 1.2.14

Let (X_{DS}, T_E) be a dual soft topological space, and let$A_{FG} \subseteq_{DS} X_{DS}$ then

1- $DS_i(A_{FG})$ is dual soft open set .

2- $DS_i(A_{FG})$ is the largest dual soft open set contained inA_{FG}.

3- $DS_i(A_{FG})$ is dual soft open set if and only if $DS_i(A_{FG})$=A_{FG}.

Proof

1 and 2- By proposition 1.2.13

$DS_i(A_{FG}) = \cup_{DS} \{A_{1F1G\,1} \in_{DS} T_E \; ; A_{1F1G\,1} \subseteq_{DS} A_{FG}\}$

And the dual soft union of dual soft open sets is dual soft open set .

Thus $DS_i(A_{FG})$ and the dual soft union of all dual soft open sets contained inA_{FG} is the largest dual soft open set.

3- let $DS_i(A_{FG})$=A_{FG} by (1) $DS_i(A_{FG})$ is dual soft open set then A_{FG} is dual soft open set.

Conversely let A_{FG} is dual soft open set then surly A_{FG} is identical with largest dual soft open set of A_{FG}

But by (2) $DS_i(A_{FG})$ is the largest dual soft open set of A_{FG}

Hence $DS_i(A_{FG})$=A_{FG} .

The following theory gives the advantages or characteristics of the dual soft interior point

Proposition 1.2.15

Let(X_{Ds} , T_E) be dual soft topological space and let A_{FG} , $B_{F1G1} \subseteq_{DS} X_{Ds}$ then :

1- $DS_i\,(X_{Ds}) = X_{Ds}$, $DS_i\,(\emptyset_{Ds}) = \emptyset_{Ds}$

2- $DS_i\;(A_{FG}) \subseteq_{DS} A_{FG}$

3- If $A_{FG} \subseteq_{DS} B_{F1G1}$ then $DS_i\,(A_{FG}) \subseteq_{DS} DS_i(B_{F1G1})$

4- $DS_i\,(A_{FG}) \cup_{DS} DS_i(B_{F1G1}) \subseteq_{DS} DS_i\;(A_{FG} \cup_{DS} B_{F1G1})$

5- $DS_i\;(A_{FG} \cap_{DS} B_{F1G1}) = DS_i\,(A_{FG}) \cap_{DS} DS_i(B_{F1G1})$

6- $DS_i\ [DS_i\ (A_{FG})\ ] = DS_i\ (A_{FG})$

Proof

1-Since X_{Ds} and $\emptyset_{Ds}$ are dual soft open sets then by proposition 1.2.14 part (3) $DS_i\ (X_{Ds}) = X_{Ds}$ and $DS_i\ (\emptyset_{Ds}) = \emptyset_{Ds}$.

2-$DP_e \in_{DS} DS_i(A_{FG})$ then DP_e is dual soft interior point of A_{FG}

So A_{FG} is $DS-nhd$ of DP_e therefor $DP_e \in_{DS} (A_{FG})$

Thus$DS_i\ (A_{FG}) \subseteq_{DS} A_{FG}$.

3-Let $DP_e \in_{DS} DS_i(A_{FG})$ then DP_e is dual soft interior point of A_{FG} and so A_{FG} is $DS-nhd$ of DP_e

Since $A_{FG} \subseteq_{DS} B_{F1G1}$ then by proposition 1.2.14 part (2) B_{F1G1} is $Ds-nhd$ of DP_e .So $DP_e \in_{DS} DS_i(B_{F1G1})$

Thus $DS_i\ (A_{FG}) \subseteq_{DS} DS_i(B_{F1G1})$.

4-By part (3) $A_{FG} \subseteq_{DS} (A_{FG} \cup_{DS} B_{F1G1})$.

Then $DS_i\ (A_{FG}) \subseteq_{DS} DS_i(A_{FG} \cup_{DS} B_{F1G1})$.

Also $B_{F1G1} \subseteq_{DS} (A_{FG} \cup_{DS} B_{F1G1})$.

So $DS_i(B_{F1G1}) \subseteq_{DS} DS_i(A_{FG} \cup_{DS} B_{F1G1})$.

Thus $DS_i\ (A_{FG}) \cup_{DS} DS_i(B_{F1G1}) \subseteq_{DS} DS_i(A_{FG} \cup_{DS} B_{F1G1})$.

5- Since $A_{FG} \cap_{DS} B_{F1G1} \subseteq_{DS} A_{FG}$, $A_{FG} \cap_{DS} B_{F1G1} \subseteq_{DS} B_{F1G1}$

Then $DS_i\ (A_{FG} \cap_{DS} B_{F1G1}) \subseteq DS_i(A_{FG})$

and $DS_i\ (A_{FG} \cap_{DS} B_{F1G1}) \subseteq DS_i\ (B_{F1G1})$ ..... (1)

again let $DP_e \in DS_i(A_{FG})$and $DP_e \in DS_i(B_{F1G1})$

hence DP_e is dual soft interior point of each A_{FG} and B_{F1G1}it

follows that A_{FG} and B_{F1G1} are $DS-nhd$ of DP_e so that the dual soft intersection of that A_{FG} and B_{F1G1} is also $Ds-nhd$

of DP_e thus $DP_e \in_{DS} DS_i\ (A_{FG} \cap_{DS} B_{F1G1})$

so $DS_i\ (A_{FG}) \cap_{DS} DS_i(B_{F1G1}) \subseteq_{DS} DS_i$

$(A_{FG} \cap_{DS} B_{F1G1})$…….. (2)

From (1) and (2) we get

$DS_i\ (A_{FG} \cap_{DS} B_{F1G1}) = DS_i\ (A_{FG}) \cap_{DS} DS_i(B_{F1G1})$.

6-Now by theorem 1.2.14 part (1) $DS_i(A_{FG})$ is dual soft open set

Hence by part (3) of the same proposition $DS_i\ [DS_i(A_{FG})] = DS_i\ (A_{FG})$.

The next example shows that

$DS_i\ (A_{FG} \cup_{DS} B_{F1G1}) \not\subseteq_{DS} DS_i\ (A_{FG}) \cup_{DS} DS_i(B_{F1G1})$

Example 1.2.16

Let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1, c_2, c_3\}$ $E=\{e_1, e_2\}$

$A_{1F1G\,1} = \{(e_1, \{h_1\}, \{c_1\}), (e_2, \{h_2\}, \{c_2\})\}$

$A_{2F2G2} = \{(e_1, \{h_1, h_2\}, \{c_1, c_3\}), (e_2, \{h_1, h_2\}, U_2)\}$

$T_E = \{\emptyset_{Ds}, X_{Ds}, A_{1F1G\,1}, A_{2F2G\,2}\}$

Assume $B_{F3G\,3} = \{(e_1, \{h_1\}, U_2)\}, (e_2, U_1, \{c_1, c_2\})\}$

$C_{F4G4} = \{(e_1, U_1, \{c_1, c_3\}), (e_2, \{h_1, h_2\}, \{c_2, c_3\})\}$

$B_{F3G\,3} \cup_{DS} C_{F4G4} = X_{Ds}$

$DS_i(B_{F3G\,3} \cup_{DS} C_{F4G4}) = DS_i(X_{Ds}) = X_{Ds}$

$DS_i(B_{F3G\,3}) \cup_{DS} DS_i(C_{F4G4}) = (A_{1F1G\,1}) \cup_{DS} (A_{1F1G\,1}) =$
$A_{1F1G\,1} \subseteq_{DS} X_{Ds}$.

We see that

$DS_i(B_{F3G\,3}) \cup_{DS} DS_i(C_{F4G4}) \not\supseteq_{DS} DS_i(B_{F3G\,3} \cup_{DS} C_{F4G4})$.

Definition 1.2.17 Dual Soft Exterior Points

Let(X_{Ds}, T_E) be dual soft topological space and let $A_{FG} \subseteq_{DS} X_{Ds}$we call the dual soft point DP_e is dual soft exterior point of dual soft set A_{FG} if DP_e is dual soft interior point of $X_{Ds} - A_{FG}$ we shall denoted by $DS_e(A_{FG})$ $= \cup_{e \in E} \{DP_e \in_{DS} X_{Ds} ; \exists U_{F1G1} \in T_E \ni DP_e \in_{DS} U_{F1G1} \subseteq_{DS} X_{Ds} - A_{FG}\}$

Example 1.2.18

Let $U_1 = \{h_1, h_2\}$ $U_2 = \{c_1, c_2\}$ $E = \{e_1, e_2, e_3\}$

$A_{FG} = \{(e_1, \{h_2\}, U_2), (e_2, \{h_1\}, \{c_1\}), (e_3, \{h_2\}, \{c_2\})\}$

$A_{1F1G1} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \emptyset), (e_3, \emptyset, \{c_1\})\}$

$A_{2F2G2} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \{h_1\}, \emptyset), (e_3, \emptyset, \emptyset)\}$

$B_{F3G3} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, U_1, \emptyset), (e_3, \emptyset, \{c_1\})\}$

$X_{Ds} - A_{FG} = \{(e_1, \{h_1\}, \emptyset), (e_2, \{h_2\}, \{c_2\}), (e_3, \{h_1\}, \{c_1\})\}$

$T_E = \{\emptyset_{Ds}, X_{Ds}, A_{1F1G\,1}, A_{2F2G2}, B_{F3G3}\}$

$A_{1F1G1} \subseteq X_{Ds} - A_{FG}$ then by example (4.3) the exterior dual soft point of A_{FG} $DP_{29e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{37e3} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \{c_1\})\}$

$DS_e(A_{FG}) = DP_{29e2} \cup_{DS} DP_{37e3} = A_{1F1G1}$.

The following theory gives the advantages or characteristics of the dual soft exterior point

proposition 1.2.19

Let(X_{Ds} , T_E) be dual soft topological space and let A_{FG} , $B_{F1G1} \subseteq_{DS} X_{Ds}$ then

1- $DS_e(X_{Ds}) = \emptyset_{Ds}$, $DS_e(\emptyset_{Ds}) = X_{Ds}$

2- $DS_e(A_{FG}) \subseteq_{DS} X_{Ds} - A_{FG}$

3- $DS_e(A_{FG}) = DS_e[X_{Ds} - DS_e(A_{FG})]$

4- If $A_{FG} \subseteq_{DS} B_{F1G1}$ then $DS_e(B_{F1G1}) \subseteq_{DS} DS_e(A_{FG})$

5- $DS_i(A_{FG}) \subseteq_{DS} DS_e[DS_e(A_{FG})]$

6- $DS_e(A_{FG} \cup_{DS} B_{F1G1}) = DS_e(A_{FG}) \cap_{DS} DS_e(B_{F1G1})$

7- $DS_e(A_{FG}) \cap_{DS} DS_i(A_{FG}) = \emptyset_{Ds}$

Proof

1-$DS_e(X_{Ds}) = DS_i(X_{Ds} - X_{Ds}) = DS_i(\emptyset_{Ds}) = \emptyset_{Ds}$

$DS_e(\emptyset_{Ds}) = DS_i(X_{Ds} - \emptyset_{Ds}) = X_{Ds}$

2-$DS_e(A_{FG}) = DS_i(X_{Ds} - A_{FG}) \subseteq X_{Ds} - A_{FG}$

3-$DS_e[X_{Ds} - DS_e(A_{FG})] = DS_e[X_{Ds} - DS_i(X_{Ds} - A_{FG})]$

$= DS_i[X_{Ds} - (X_{Ds} - DS_i(X_{Ds} - A_{FG}))]$

$= DS_i[DS_i(X_{Ds} - A_{FG})] = DS_i(X_{Ds} - A_{FG})$

$= DS_e(A_{FG})$

4-$A_{FG} \subseteq_{DS} B_{F1G1}$ then $X_{Ds} - B_{F1G1} \subseteq X_{Ds} - A_{FG}$

So $DS_i(X_{Ds} - B_{F1G1}) \subseteq_{DS} DS_i(X_{Ds} - A_{FG})$

Thus $DS_e(B_{F1G1}) \subseteq_{DS} DS_e(A_{FG})$

5- By part 2we have $DS_e(A_{FG}) \subseteq_{DS} X_{Ds} - A_{FG}$ then by part (4)

$DS_i(A_{FG}) = DS_e(X_{Ds} - A_{FG}) \subseteq_{DS} DS_e[DS_e(A_{FG})]$

Thus $DS_i\ (A_{FG}) \subseteq_{DS} DS_e[DS_e\ (A_{FG}\ )])$

6-$DS_e\ (A_{FG}\ \cup_{\mathrm{DS}}\ \mathrm{B_{F1G1}}) = DS_i\ [X_{Ds} - (A_{FG}\ \cup_{\mathrm{DS}}\ B_{F1G1}\ )]$

$= DS_i[(X_{Ds} - A_{FG}\ ) \cup_{\mathrm{DS}}\ (X_{Ds} - B_{F1G1}\ )]$

$= DS_e\ (A_{FG}) \cap_{DS}\ DS_e(B_{F1G1})$

7-Since $DS_e\ (A_{FG}) = DS_i(\ X_{Ds} - A_{FG}\ )$ and $DS_i\ (A_{FG}) \subseteq_{DS} A_{FG}$

Also $DS_e\ (A_{FG}) = DS_i(\ X_{Ds} - A_{FG}\ ) \subseteq_{DS} X_{Ds} - A_{FG}$

But $A_{FG}\ \cap_{DS} X_{Ds} - A_{FG}\ = \emptyset_{Ds}$

It follows that $DS_e\ (A_{FG}) \cap_{DS}\ DS_i(\ A_{FG}) = \emptyset_{Ds}$.

Proposition 1.2.20

Let$(X_{Ds}\ , T_E\ )$ be dual soft topological space and let $A_{FG} \subseteq_{DS} X_{Ds}$ then $DS_e\ (A_{FG}) = \cup_{e\,\in\,E}\ \{A_{1F1G1} \in T_E\ \ni\ A_{1F1G1}\ \subseteq_{DS} X_{Ds} - A_{FG}\ \}$

Proof

By definition 1.2.17 $DS_e\ (A_{FG}) = DS_i(\ X_{Ds} - A_{FG}\ )$

But by definition 1.2.11 $DS_i(\ X_{Ds} - A_{FG}\ ) = \cup_{e\,\in\,E}\ \{A_{1F1G1} \in T_E\ \ni\ A_{1F1G1}\ \subseteq_{DS} X_{Ds} - A_{FG}\ \}$

Hence $DS_e\ (A_{FG}) = \cup_{e\,\in\,E}\ \{A_{1F1G1} \in_{DS}\ T_E\ \ni\ A_{1F1G1}\ \subseteq_{DS} X_{Ds} - A_{FG}\ \}$

Definition1.2.21 Dual Soft Boundary (Frontier) Point

Let$(X_{Ds}\ , T_E\ )$ be dual soft topological space and let $A_{FG} \subseteq_{DS}\ X_{Ds}$ the dual soft point DP_e is said to be dual soft frontier point if DP_e is neither dual soft interior point nor dual soft exterior point we shall denoted by $DS_{fr}(A_{FG})$

i.e.$\forall\ DP_e \in_{DS} X_{Ds}\ \ni\ DP_e \notin_{DS} DS_i(A_{FG})$ and $DP_e \notin_{DS} DS_e\ (A_{FG})$ then $DP_e \in_{DS} DS_{fr}(A_{FG})$in other word $DS_{fr}\ (A_{FG}) = \cup_{e\,\in\,E}\{DP_e\ \in_{DS} X_{Ds}\ ;\ DP_e \notin_{DS} DS_i\ (A_{FG})$ and $DP_e \notin_{DS} DS_e\ (A_{FG})\}$

if we use example1.2.12

$DS_{fr}(A_{FG})$ = all dual soft extension union ofA_{FG} except the dual soft interior point of $A_{FG} = DP_{12e1},\ DP_{26e1},\ DP_{27e1},\ DP_{75e2}\ DP_{81e2},\ DP_{82e2}$

And the dual soft exterior point of $A_{FG} = DP_{11e1},\ DP_{34e1},\ DP_{35e1},\ DP_{76e2}$ $DP_{89e2},\ DP_{92e2}$

Proposition 1.2.22

Let(X_{DS}, T_E) be dual soft topological space and $A_{FG} \subseteq_{DS} X_{Ds}$ then $DP_e \in_{DS} D(U_1, U_2)_E$ be a dual soft frontier point of A_{FG} if and only if every $Ds - nhd$ of DP_e intersect both $A_{FG}$and $X_{Ds} - A_{FG}$. Also dual soft frontier set is dual soft closed set of any dual soft set A_{FG}

Proof

$DP_e \in_{DS} DS_{fr}(A_{FG})$

If and only if $\{ DP_e \notin_{DS} DS_i(A_{FG})$ and $DP_e \notin_{DS} DS_e(A_{FG})\}$

If and only if $\forall\ DS - nhd\ N_{F1G1}$ of $DP_e \ni N_{F1G1} \cap_{DS} A_{FG} \neq \emptyset_{Ds}$ and

$N_{F1G1} \cap_{DS} X_{Ds} - A_{FG} \neq \emptyset_{Ds}$. by definition 1.2.21

Corollary1.2.23

$DS_{fr}(A_{FG}) = DS_{fr}(X_{Ds} - A_{FG})$

Proof

We have $DP_e \in_{DS} DS_{fr}(A_{FG})$

If and only if every $DS - nhd$ of DP_e intersect both $A_{FG}$and $X_{Ds} - A_{FG}$

If and only if every $DS - nhd$ of DP_e intersect both $X_{Ds} - (X_{Ds} - A_{FG})$ and $X_{Ds} - A_{FG}$ (by $X_{Ds} - (X_{Ds} - A_{FG}) = A_{FG}$

If and only if $DP_e \in_{DS} DS_{fr}(X_{Ds} - A_{FG})$.

Corollary1.2.24

$DS_{fr}(A_{FG}) \cap_{DS} DS_i(A_{FG}) = \emptyset_{Ds}$ and $DS_{fr}(A_{FG}) \cap_{DS} DS_e(A_{FG}) = \emptyset_{Ds}$

Proof

Let $DP_e \in_{DS} DS_{fr}(A_{FG})$

If and only if $DP_e \notin_{DS} DS_i(A_{FG})$ and $DP_e \notin_{DS} DS_e(A_{FG})$

If and only if $DP_e \notin_{DS} DS_i(A_{FG}) \cup_{DS} DS_e(A_{FG})$

If and only if $DP_e \in_{DS} X_{Ds} - [DS_i(A_{FG}) \cup_{DS} DS_e(A_{FG})]$

Then $DS_{fr}(A_{FG}) = X_{Ds} - [DS_i(A_{FG}) \cup_{DS} DS_e(A_{FG})]$

So $DS_{fr}(A_{FG}) \cap_{DS} [DS_i(A_{FG}) \cup_{DS} DS_e(A_{FG})] = \emptyset_{Ds}$

If and only if $[DS_{fr}(A_{FG}) \cap_{DS} DS_i(A_{FG})] \cup_{DS} [DS_{fr}(A_{FG}) \cap_{DS} DS_e(A_{FG})] = \emptyset_{Ds}$ If and only if$DS_{fr}(A_{FG}) \cap_{DS} DS_i(A_{FG}) = \emptyset_{Ds}$

If and only if $DS_{fr}(A_{FG}) \cap_{DS} DS_e(A_{FG}) = \emptyset_{Ds}$.

Proposition 1.2.25

The dual soft set X_{Ds} is decomposition into three pairwise disjoint dual soft interior set, dual soft exterior set and dual soft frontier set

Proof

directly by proposition 1.2.22and corollary 1. 2.24 and 1.2.19 part (7)

Now we will construct the definitions of limit point, closure set, adherent point and singular point within dual soft topological space as well as some examples and properties

Definition1.2.26 Dual Soft Limit Point

Let(X_{Ds}, T_E) be dual soft topological space a dual soft point DP_e is said to be dual soft limit point of A_{FG}if $\forall\ DP_e \in_{DS} B_{F1G1} \in_{DS} T_E$

$$A_{FG} \cap_{DS} B_{F1G1} \cap_{DS} (X_{Ds} - DP_e) \neq \emptyset_{Ds}$$

The dual soft set of all dual soft limit point of A_{FG} is called dual soft derived set we shall denoted by

$DS_d(A_{FG}) = \cup_{e \in E} \{DP_e \in_{DS} X_{Ds}; \forall B_{F1G1} \in_{DS} T_E, DP_e \in_{DS} B_{F1G1} \ni A_{FG} \cap_{DS} B_{F1G1} \cap_{DS} (X_{Ds} - DP_e) \neq \emptyset_{Ds}\}$

Example1.2.27

let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1,, c_2, c_3\}$ $E = \{e_1, e_2\}$

Assume $A_{FG} = \{(e_1, \{h_2\}, \{c_1\}), (e_2, \{h_1, h_3\}, U_2)\}$

Let $A_{1F1G1} = \{(e_1, \{h_1, h_2\}, \{c_1, c_2\}), (e_2, \{h_1\}, \{c_2\})\}$

$A_{2F2g2} = \{(e_1, \{h_2\}, \{c_1\}), (e_2, \{h_2, h_3\}, \{c_2\})\}$

$A_{3F3g3} = \{(e_1, \{h_1, h_2\}, \{c_1, c_2\}), (e_2, U_1, \{c_2\})\}$

$A_{4F4G4} = \{(e_1, \{h_2\}, \{c_1\}), (e_2, \emptyset, \{c_2\})\}$

$T_E = \{X_{Ds}, \emptyset_{Ds}, A_{1F1G1}, A_{2F2G2}, A_{3F3G3}, A_{4F4G4}\}$

Then from example (4.4) $DP_{15e1} = \{(e_1, \{h_2\}, \{c_1\}), (e_2, \emptyset, \emptyset)\}$

$$X_{Ds} - DP_{15e1} = \{(e_1, \{h_1, h_3\}, \{c_2, c_3\}), (e_2, U_1, U_2)\}$$

$DP_{15e1} \in_{DS} A_{1F1G1}$ and for all $A_{iFiGi} \in_{DS} T_E \quad i = 1,2,3,4$

$A_{FG} \cap_{DS} A_{iFiGi} \cap_{DS} (X_{Ds} - DP_{15e1}) \neq \emptyset_{Ds}$

Then DP_{15e1} is a limit point for A_{FG}

And all these dual soft points are dual soft limit points for A_{FG}

$DP_{7e1}, DP_{14e1}, DP_{15e1}, DP_{36e2}, DP_{41e2}, DP_{43e2}, DP_{45e2}, DP_{47e2},$

$DP_{49e2}, DP_{51e2}, DP_{e61e2}, DP_{63e2}$

$DS_d(A_{FG}) = DP_{7e1} \cup_{DS} DP_{14e1} \cup_{DS} DP_{15e1} \cup_{DS} DP_{36e2} \cup_{DS}$

$DP_{41e2} \cup_{DS} DP_{43e2} \cup_{DS} DP_{45e2} \cup_{DS} DP_{47e2} DP_{49e2} \cup_{DS} DP_{51e2}$

$\cup_{DS} DP_{61e2} \cup_{DS} DP_{63e2}$=$\{(e_1, \{h_2\}, \{c_1\}), (e_2, U_1, U_2)\}$

Proposition 1.2.28

Let (X_{DS}, T_E) be dual soft topological space and $A_{FG} \subseteq_{DS} X_{DS}$ then A_{FG} is dual soft closed set if and only if $DS_d(A_{FG}) \subseteq_{DS} A_{FG}$

Proof

Let A_{FG} is dual soft closed set then $X_{DS} - A_{FG}$ is dual soft open set so for each $DP_e \in_{DS} X_{DS} - A_{FG}$ that is $DP_e \notin_{DS} A_{FG}$ and there exists $DS - nhd$ $N_{F1G1}$of D_{pe} s.t $N_{F1G1} \subseteq_{DS} X_{DS} - A_{FG}$

Since $A_{FG} \cap_{DS} X_{DS} - A_{FG} = \emptyset_{Ds}$ then N_{F1G1}contain : no dual soft point of A_{FG} and so no dual soft limit point of A_{FG} Thus no dual soft point of $X_{DS} - A_{FG}$ can be dual soft limit point

Hence $DS_d(A_{FG}) \subseteq_{DS} A_{FG}$

Conversely let $DS_d(A_{FG}) \subseteq_{DS} A_{FG}$

And let $DP_e \in_{DS} X_{DS} - A_{FG}$ then $DP_e \notin_{DS} A_{FG}$

Since $DS_d(A_{FG}) \subseteq_{DS} A_{FG}$ then $DP_e \notin_{DS} DS_d(A_{FG})$

Hence there exist a $DS - nhd$ $N_{F1G1}$of $DP_e \ni N_{F1G1} \cap_{DS} A_{FG} = \emptyset_{Ds}$

So that $N_{F1G1} \subseteq_{DS} X_{DS} - A_{FG}$

Thus $X_{DS} \_ A_{FG}$ contains $DS - nhd$ of each of its dual soft point

So $X_{DS} \_ A_{FG}$ is dual soft open set. Therefor A_{FG} is dual soft closed set.

Proposition 1.2.29

Let (X_{DS}, T_E) be dual soft topological space and let $A_{FG}, B_{F1G1} \subseteq_{DS} X_{DS}$ then

1- $DS_d\ (\emptyset_{DS}\ )=\emptyset_{DS}$

2- If $A_{FG}\ \subseteq_{DS}\ B_{F1G1} then\ DS_d\ (A_{FG})\ \subseteq\ DS_d(B_{F1G1})$

3- $DS_d(A_{FG}\cap B_{F1G1})\ \subseteq_{DS}\ DS_d(A_{FG})\cap_{DS}\ DS_d\ (B_{F1G1})$

4- $DS_d\ (A_{FG}\cup_{DS} B_{F1G1})=DS_d\ (A_{FG})\cup_{DS} DS_d(B_{F1G1})$

Proof:

1-Since $\emptyset_{Ds}$is closed then $DS_d\ (\emptyset_{Ds})\subseteq_{DS}\ \emptyset_{Ds}$.

but $\emptyset_{Ds}$is dual soft subset of every dual soft set so $\emptyset_{Ds}\subseteq_{DS}\ DS_d\ (\emptyset_{Ds})$

Hence $DS_d\ (\emptyset_{Ds})=\emptyset_{Ds}$

2- Let $DP_e\in_{DS}\ DS_d\ (A_{FG})$ so DP_e is a dual soft limit point of A_{FG}

then every D_S – nhd $N_{F1G1}$of DP_e contains a dual soft point of A_{FG} different from DP_e and since $A_{FG}\subseteq_{DS}\ B_{F1G1}$ so every $DS-nhd$ of DP_e also contained a dual soft point diffident from DP_e

Then DP_e is also dual soft limit point of B_{F1G1}

So $DP_e\in_{DS}\ DS_d\ (B_{F1G1})$ thus $A_{FG}\subseteq_{DS}\ B_{F1G1}$

Then $DS_d(A_{FG})\ \subseteq_{DS}\ DS_d\ (B_{F1G1})$

3-Since $DS_d(A_{FG}\cap_{DS}\ B_{F1G1})\subseteq_{DS}\ DS_d(A_{FG})$

$DS_d\ (A_{FG}\cap_{DS}\ B_{F1G1})\subseteq_{DS} DS_d(B_{F1G1})$

And consequently $DS_d(A_{FG}\ \cap_{DS}\ B_{F1G1})\subseteq_{DS} DS_d(A_{FG})\cap_{DS}\ DS_d\ (B_{F1G1})$

4-Since $A_{FG}\subseteq_{DS} A_{FG}\ \cup_{DS} B_{F1G1}$and $B_{F1G1}\subseteq_{DS} A_{FG}\ \cup_{DS}\ B_{F1G1}$ it follows from(2) $DS_d(A_{FG})\ \subseteq_{DS} DS_d\ (A_{FG}\ \cup_{DS} B_{F1G1})$

$DS_d\ (B_{F1G1})\subseteq_{DS} DS_d\ (A_{FG}\cup_{DS}\ B_{F1G1})$

Hence $DS_d(A_{FG}) \cup_{DS} DS_d\ (B_{F1G1}) \subseteq_{DS} DS_d\ (A_{FG}\ \cup_{DS}\ B_{F1G1})$

Conversely if $DP_e \notin DS_d(A_{FG}) \cup_{DS} DS_d\ (B_{F1G1})$ Then $DP_e \notin_{DS} DS_d\ (A_{FG})$ and $\notin DS_d\ (B_{F1G1})$

That is DP_e nither dual soft limit point of A_{FG} nor a dual soft limit point of B_{F1G1}

Hence There exist $DS-nhd\ N_{2F2G2}$, $M_{F3G3}$of DP_e

$\ni N_{2F2G2} -\ DP_e \cap_{DS}\ A_{FG} = \emptyset_{Ds}$ and $M_{F3G3} \_ DP_e \cap_{DS}\ B_{f1G1} = \emptyset_{Ds}$ ….. (1)

$N_{F1G1} = M_{F3G3} \cap_{DS}\ N_{2F2G2}$ is $DS-nhd$ of DP_e which by (1)

contains no dual soft point of $(A_{FG}\ \cup_{DS}\ B_{F1G1})$ except (possibly) DP_e

It follows that $DP_e \notin_{DS} DS_d\ (A_{FG} \cup_{DS} B_{F1G1})$ as required.

Proposition 1.2.30

Let $A_{FG} \subseteq_{DS}\ X_{DS}$ be a dual soft set then $DS_i(A_{FG})$ equal the set of all these dual soft points of A_{FG} which are not dual soft limit points of $X_{DS} - A_{FG}$

Proof

Let DP_e be a dual soft point of A_{FG} which are not limit point of $X_{DS} - A_{FG}$ then there exist $DS-nhd\ N_{F1G1}$of DP_e which contains no dual soft points of $X_{DS} - A_{FG}$and so $N_{F1G1} \subseteq_{DS} A_{FG}$

that is A_{FG}also $DS-nhd$of DP_eso $DP_e \in DS_i\ (A_{FG})$

Conversely let $DP_e \in_{DS} DS_i\ (A_{FG})$and since$DS_i\ (A_{FG})$ is dual soft open set it is $DS-nhd$of DP_eso DP_e also $DS_i\ (A_{FG})$ contains no dual soft points of $X_{DS} - A_{FG}$ It follows that DP_e is not limit point of $X_{DS} - A_{FG}$.

Definition 1.2.31 Dual Soft Closure

Let (X_{DS}, T_E) be dual soft topological space and let $A_{FG} \subseteq_{DS} X_{DS}$ then the dual soft closure of $A_{FG}$is dual soft closed set defined as $DS_{cl}(A_{FG}) = \cap_{e\in E}$ $\{A_{1F1G1}, A_{1F1G1}$is dual soft closed set , $A_{FG} \subseteq_{DS} A_{1F1G1}\}$

Or $DS_{cl}\ (A_{FG}) = \cup_{e\in E}\ \{ DP_e \in X_{Ds}, \forall\ U_{F1G1} \in T_E \ni U_{F1G1} \cap_{DS} A_{FG} \neq \emptyset_{Ds} \}$

Example 1.2.32

Let $U_1 = \{ h_1, , h_2, h_3\}$ $U_2 = \{ c_1 , c_2, c_3\}$ $E = \{ e_1, , e_2\}$

assume $A_{FG} = \{(e_1, \{ h_1\}, \{ c_2, c_3\}), (e_2, \emptyset , \{ c_1\})\}$

$A_{1F1G1} = \{(e_1 , \{h_2 \}, \{c_2\}), (e_2 , \{h_3\}, \{ c_1\})\}$

$A_{2F2G2} = \{(e_1, \{ h_1\}, \{ c_3 \}),\ (e_2, \{ h_1\}, \{ c_3 \})\}$

$A_{3F3G3} = \{(e_1 , \{h_1, h_2\}, \{ c_2 , c_3\}), (e_2 , \{h_1, h_3\}, \{ c_1, c_3\})\}$

$T_E = \{ X_{DS}, \emptyset_{Ds} , A_{1F1G1,} A_{2F2G2}, A_{3F3G3}\}$

$X_{DS} - A_{1F1G1} = \{(e_1 , \{h_1, h_3\}, \{ c_1 , c_3\}), (e_2 , \{h_1, h_2\}, \{ c_2, c_3\})\}$

$X_{DS} - A_{2F2G2} = \{(e_1 , \{h_2, h_3\}, \{ c_1 , c_2\}), (e_2 , \{h_2, h_3\}, \{ c_1, c_2\})\}$

$X_{DS} - A_{3F3G3} = \{(e_1 , \{h_3\}, \{c_1\}), (e_2 , \{h_2\},)\}$

Now assume $B_{F5G5} = \{(e_1 , \{h_1 \}, \emptyset), (e_2 , \{h_1\}, \{ c_3\})\}$

$C_{F6G6} = \{(e_1 , \{h_2\}, \{c_2\}), (e_2 , \{h_3\}, \{ c_2\})\}$

$DS_{cl}(A_{FG}) = X_{DS}$

$DS_{cl}(B_{F5G5}) = X_{DS} - A_{1F1G1}$

$DS_{cl}(C_{F6G6}) = X_{DS} - A_{2F2G2}$

Definition 1.2.33 Dual Soft Adherent Point

Let (X_{DS}, T_E) be dual soft topological space a dual soft point DP_e is called dual soft adherent point $of\, A_{FG}$ iff $\forall\ H_{F1G1} \in_{DS} T_E \ni DP_e \in_{DS} H_{F1G1}$ and $H_{F1G1} \cap_{DS} A_{FG} \neq \emptyset_{Ds}$

The set of all dual soft adherent point of A_{FG} DS_{adh} (A_{FG})= $\cup_{DS}\{ DP_e; \forall\ H_{F1G1} \in_{DS} T_E; DP_e \in_{DS} H_{F1G1}$ and $H_{F1G1} \cap_{DS} A_{FG} \neq \emptyset_{Ds}\}$

Example 1.2.34

let U_1= { h_1,, h_2} U_2= { C_1 , C_2} E ={ e_1,, e_2}

assume A_{FG}= {(e_1, U_1 , C_2 }), (e_2,{ h_2},{ c_1})}

$A_{1F1G1} = \{(e_1,\{ h_1\},\{c_2\}), (e_2,\{ h_2\}\ \{ c_1\})\}$

$A_{2F2G2} = \{(e_1,\emptyset,\emptyset), (e_2,\{ h_1\},\emptyset)\}$

$A_{3F3G3} = \{(e_1 ,\{h_1\},\{c_2\}),(e_2 , U_1,\{ c_1\})\}$

$T_E = \{ X_{DS}, \emptyset_{Ds}, A_{1F1G1,} A_{2F2G2}, A_{3F3G3}\}$

from example (4.2) The dual soft adherent point of A_{Fu}

DP_{8e1} , DP_{10e1}, DP_{12e1} , $DP_{18e1}, DP_{19e1}, DP_{20e2}, DP_{22e2}$, $DP_{25e2}, DP_{26e2}, DP_{29e2}, DP_{30e2}$.

proposition1.2.35

Let $A_{FG} \subseteq_{DS} X_{DS}$ be a dual soft set of the dual soft topological space (X_{DS}, T_E) then

1- $DS_{cl}(A_{FG})$is the smallest dual soft closed set containingA_{FG}.

2- $A_{FG}$is dual soft closed set if and only if $DS_{cl}(A_{FG})$=A_{FG}.

Proof

1- Directly by definition 1.2.31

2- If $A_{FG}$is dual soft closed set then$A_{FG}$is itself smallest dual soft

closed set containsA_{FG} hence $DS_{cl}(A_{FG})$=A_{FG}

Conversely let $DS_{cl}(A_{FG})$=A_{FG} then by part (1) $DS_{cl}(A_{FG})$ is dual soft closed set. Thus $A_{FG}$is dual soft closed set.

proposition 1.2.36

$DS_{cl}(A_{FG})$=$A_{FG} \cup_{DS} DS_d(A_{FG})$.

Proof

let $DP_e \in_{DS} X_{DS} - A_{FG} \cap_{DS} X_{DS} - DS_d(A_{FG})$

Then $DP_e \notin_{DS} A_{FG}$and $DP_e \notin_{DS} DS_d(A_{FG})$

So $\exists H_{F1G1} \in_{DS} T_E \ni DP_e \in_{DS} H_{F1G1}$and $H_{F1G1} \cap_{DS} A_{FG} = \emptyset_{Ds}$

Thus $H_{F1G1} \subseteq_{DS} X_{DS} - A_{FG}$

Now $\forall$ $DP_e \notin_{DS} X_{DS} - DS_d(A_{FG})$

Then $DP_e \in_{DS} DS_d(A_{FG})$

So $\forall K_{F2G2} \in_{DS} T_E \ni DP_{\grave{e}} \in K_{F2G2}$, $K_{F2G2} \cap_{DS} A_{FG} \neq \emptyset_{Ds}$

But $H_{F1G1} \cap_{DS} A_{FG} = \emptyset_{Ds}$then $DP_{\grave{e}} \notin H_{F1G1}$

So $H_{F1G1} \subseteq_{DS} X_{DS} - DS_d(A_{FG})$

we have $H_{F1G1} \subseteq_{DS} DS_d(A_{FG}) \cap_{DS} X_{DS} - A_{FG}$

Thus $X_{DS} - DS_d(A_{FG}) \cap_{DS} X_{DS} - A_{FG}$ contains D_S – nhd of each of each of its dual soft points and consequently

$X_{DS} - DS_d(A_{FG}) \cap_{DS} X_{DS} - A_{FG}$ is dual soft open set

we now show that $DS_{cl}(A_{FG}) = A_{FG} \cup_{DS} DS_d(A_{FG})$.

Since $A_{FG} \cup_{DS} DS_d(A_{FG})$ is dual soft closed set contains $A_{FG}$and $DS_{cl}(A_{FG})$is the smallest dual soft closed set containsA_{FG}we have $DS_{cl}(A_{FG}) \subseteq_{DS} A_{FG} \cup_{DS} DS_d(A_{FG})$

Again since $DS_{cl}(A_{FG})$ is dual soft closed set contains all dual soft limit points ofA_{FG} so that$DS_d(A_{FG}) \subseteq_{DS} DS_{cl}(A_{FG})$

Also$A_{FG} \subseteq_{DS} DS_{cl}(A_{FG})$ so $A_{FG} \cup_{DS} DS_d(A_{FG}) \subseteq_{DS} DS_{cl}(A_{FG})$

Thus we acquire $DS_{cl}(A_{FG})$=$A_{FG} \cup_{DS} DS_d(A_{FG})$.

Corollary1.2.37

$DS_{cl}(A_{FG}) = DS_{adh}(A_{FG})$.

Proof

$DP_e \in DS_{adh}(A_{FG})$

If and only if every D_S – nhd of DP_e intersect A_{FG}

If and only if $DP_e \in_{DS} A_{FG}$or every D_S – nhd of DP_e contains a dual soft point

of A_{FG} rather than DP_e

If and only if $DP_e \in_{DS} A_{FG}$or $DP_e \in_{DS} DS_d(A_{FG})$

If and only if $DP_e \in_{DS} A_{FG} \cup_{DS} DS_d(A_{FG})$

If and only if $DP_e \in_{DS} DS_{cl}(A_{FG})$.

proposition1.2.38

Let(X_{Ds}, T_E) be dual soft topological space and $A_{FG}, B_{F1G1} \subseteq_{DS} X_{Ds}$ then

1- $DS_{cl}(\emptyset_{Ds}) = \emptyset_{Ds}$, $DS_{cl}(X_{Ds}) = X_{Ds}$.

2- $A_{FG} \subseteq_{DS} DS_{cl}(A_{FG})$

3- If $A_{FG} \subseteq_{DS} B_{F1G1}$ then $DS_{cl}(A_{FG}) \subseteq_{DS} DS_{cl}(B_{F1G1})$.

4- $DS_{cl}(A_{FG} \cup_{DS} B_{F1G1}) = DS_{cl}(A_{FG}) \cup_{DS} DS_{cl}(B_{F1G1})$.

5- $DS_{cl}(A_{FG} \cap_{DS} B_{F1G1}) \subseteq_{DS} DS_{cl}(A_{FG}) \cap_{DS} DS_{cl}(B_{F1G1})$.

6- $DS_{cl}[DS_{cl}(A_{FG})] = DS_{cl}(A_{FG})$.

Proof

1-Since $\emptyset_{Ds}$and X_{Ds} are dual soft closed set so we have $DS_{cl}\ (\emptyset_{Ds}) = \emptyset_{Ds}$, $DS_{cl}\ \ (X_{Ds}) = X_{Ds}$.

2-By proposition 1.2.35 part(1) $DS_{cl}(A_{FG})$is the smallest dual soft closed set containing A_{FG} and $A_{FG}\ \subseteq DS_{cl}\ (A_{FG})$

3-By part (2) $B_{F1G1} \subseteq_{DS}\ DS_{cl}(B_{F1G1})$and since $A_{FG} \subseteq_{DS} B_{F1G1}$we have $A_{FG}\ \subseteq_{DS}\ \ DS_{cl}(B_{F1G1})$

But by proposition 1.2.29 part (2) $DS_d(A_{FG}) \subseteq DS_d(B_{F1G1})$

Then $DS_{cl}(A_{FG})$ = $A_{FG} \cup_{DS} DS_d(A_{FG}) \subseteq DS_d(B_{F1G1})\ \cup_{DS} B_{F1G1}$

=$DS_d(B_{F1G1})$then $DS_{cl}\ (A_{FG}) \subseteq\ DS_{cl}(B_{F1G1})$

4-Since $A_{FG}\ \subseteq_{DS} A_{FG}\ \cup_{DS}\ \ B_{F1G1}$ and $B_{F1G1} \subseteq_{DS} A_{FG}\ \cup_{DS}\ \ B_{F1G1}$

$DS_{cl}\ (A_{FG}) \subseteq_{DS} DS_{cl}\ (A_{FG} \cup_{DS}\ \ B_{F1G1})$

$DS_{cl}(B_{F1G1}) \subseteq_{DS} DS_{cl}\ (A_{FG}\ \ \cup_{DS}\ \ \ B_{F1G1})$. By part (3)

$DS_{cl}\ (A_{FG})\ \cup_{DS}\ DS_{cl}(B_{F1G1}) \subseteq_{DS} DS_{cl}(A_{FG} \cup_{DS}\ B_{F1G1})$ …….(1)

Since $DS_{cl}\ (A_{FG})$, $DS_{cl}(B_{F1G1})$ are dual soft closed set then

$DS_{cl}\ (A_{FG})\ \cup_{DS}\ DS_{cl}(B_{F1G1})$also dual soft closed set

Furthermore $A_{FG}\ \subseteq_{DS}\ \ DS_{cl}\ (A_{FG})$, $B_{F1G1} \subseteq_{DS}\ \ DS_{cl}(B_{F1G1})$

Then $A_{FG}\ \cup_{DS}\ B_{F1G1} \subseteq_{DS}\ \ DS_{cl}\ (A_{FG}) \cup_{DS}\ DS_{cl}(B_{F1G1})$

Thus $DS_{cl}\ \ (A_{FG})\ \cup_{DS} DS_{cl}(B_{F1G1})$ is dual soft set contains $A_{FG}\ \cup_{DS}\ B_{F1G1}$and since $DS_{cl}\ (A_{FG}\ \ \cup_{DS}\ \ B_{F1G1})$ is smallest dual soft set containing $A_{FG}\ \cup_{DS}\ B_{F1G1}$ we have

$DS_{cl}\ (A_{FG}\ \cup_{DS}\ B_{F1G1}) \subseteq_{DS}\ DS_{cl}\ (A_{FG}) \cup_{DS}\ DS_{cl}(B_{F1G1})$ .....(2)

From (1) and (2) we obtain

$DS_{cl}\ (A_{FG}\ \cup_{DS}\ B_{F1G1}) =\ DS_{cl}\ (A_{FG}) \cup_{DS}\ DS_{cl}(B_{F1G1})$

5- $A_{FG}\ \cap_{DS}\ B_{F1G1} \subseteq_{DS} A_{FG}$ then $DS_{cl}\ (A_{FG}\ \cap_{DS}\ B_{F1G1}) \subseteq_{DS} DS_{cl}\ (A_{FG})$

But ($A_{FG}\ \cap_{DS}\ B_{F1G1}) \subseteq_{DS} B_{F1G1}$

Then $DS_{cl}\ (A_{FG}\ \cap_{DS}\ B_{F1G1}) \subseteq_{DS} DS_{cl}(B_{F1G1})$

Thus $DS_{cl}\ (A_{FG}\ \cap_{DS}\ B_{F1G1}) \subseteq_{DS}\ DS_{cl}\ (A_{FG}) \cap_{DS}\ DS_{cl}(B_{F1G1})$

If we look to example1.2.32 we find

$DS_{cl}\ (C_{F6G6}\ \cap_{DS}\ B_{F5G5}) = \emptyset_{Ds}$

$DS_{cl}\ (C_{F6G6}) \cap_{DS}\ DS_{cl}(B_{F5G5})$={$(e_1, \{h_3\}, \{c_1\}), (e_2, , \{h_2\}, \{c_2\})$}

Then $DS_{cl}\ (C_{F6G6}\ \cap_{DS}\ B_{F5G5}) \not\supseteq_{DS} DS_{cl}\ (C_{F6G6}) \cap_{DS}\ DS_{cl}(B_{F5G5})$

6- Since $DS_{cl}\ (A_{FG})$ is dual soft closed set then we have by proposition1.2.35part (2) $DS_{cl}\ (DS_{cl}\ (A_{FG}))\ = DS_{cl}\ (A_{FG})$.

Proposition1.2.39

Let $A_{FG} \subseteq_{DS}\ X_{DS}$ be a dual soft set of the dual soft topological space (X_{DS}, T_E) then the following statements are equivalent

1- A_{FG} is dual soft closed set

2- $DS_{cl}\ (A_{FG}) =\ A_{FG}$

3- A_{FG} contains all dual soft limit points

Proof

(1→2) A_{FG} is dual soft closed set then by proposition 1.2.35 part 2 $DS_{cl}\ (A_{FG}) =\ A_{FG}$

(2→3)$DS_{cl}\ (A_{FG}) = \ A_{FG}$ then by proposition 1.2.36 A_{FG}= $A_{FG} \cup_{DS} DS_d(A_{FG})$

So $DS_d(A_{FG}) \subseteq_{DS}\ A_{FG}$ hence A_{FG} contains all dual soft limit points

(3 →1) A_{FG} contains all its dual soft limit points

Then $DS_d(A_{FG}) \subseteq_{DS}\ A_{FG}$

Thus A_{FG}= $A_{FG} \cup_{DS} DS_d(A_{FG})$ then by proposition 1.2.35 part 2 A_{FG} is dual soft closed set.

Definition 1.2.40 Dual Soft Isolated Point

Let (X_{DS}, T_E) be dual soft topological space a dual soft point DP_e is called dual soft isolated point of A_{FG}if DP_e is not dual soft limit point of A_{FG}

i.e.∃ D_S – nhd $N_{1F1G1,}$of $DP_e \ni N_{1F1G1}$ contains no dual soft limit point of A_{FG} other than DP_e

Definition 1.2.41 Dual Soft Perfect Set

A dual soft closed set which has no dual soft isolated point is said to be dual soft perfect .

Remark 1.2.42

It follows that a dual soft adherent point is either dual soft limit point of A_{FG} or dual soft isolated point of A_{FG}

Then there are the following exclusive possibilities:

1- Every $DS-nhd$of DP_e contains a dual soft point of A_{FG} rather than DP_e in this case DP_e is dual soft limit point of A_{FG}

2- $DP_e \in_{DS} A_{FG}$ and there is some $DS-nhd$of DP_e which contains no dual soft point of A_{FG} except DP_e in this case DP_e is dual soft isolated point.

Example 1.2.43

let U_1= { $h_1,,h_2,h_3$} U_2= { C_1 ,C_2,C_3} E ={ $e_1,,e_2$}

assume A_{FG}= {(e_1,{ h_1},{ c_1 }), (e_2,{ h_2},{ c_2})}

$A_{1F1G1} = \{(e_1,,\{h_1,h_2\},\{ c_1 ,c_2\}),(e_2 ,\{h_1,h_2\},\{ c_1,c_3\})\}$

$A_{2F2G2} = \{(e_1,\{ h_1\},\{ c_1 \}),\ (e_2,\emptyset,\emptyset)\}$

$T_E = \{ X_{DS}, \emptyset_{Ds} ,\ A_{1F1G1,} A_{2F2G2}\}$

From example (1.1) the DP_{19e1} ={$(e_1 ,\{h_1\},\{c_1\}),(e_2 ,\emptyset,\emptyset)$} is dual soft isolated point since $A_{FG} \cap_{DS} A_{4F4G4} \cap_{DS} (X_{DS} - DP_{19e1})= \emptyset_{Ds}$.

Proposition 1.2.44

Let (X_{DS}, T_E) be dual soft topological space and let $A_{FG} \subseteq_{DS} X_{DS}$ then

$A_{FG}= DS_d(A_{FG})$ if and only if A_{FG} is dual soft perfect

Proof

Suppose that $A_{FG}= DS_d(A_{FG})$ and let DP_e be any arbitrary dual soft point of $X_{DS}$and $DP_e \in_{DS} X_{DS} - A_{FG}$

Then $DP_e \notin_{DS} A_{FG}$ and so $DP_e \notin_{DS} DS_d (A_{FG})$ as $A_{FG}=DS_d(A_{FG})$

So DP_e is not dual soft limit point of A_{FG}

Then $\exists DS-nhd\ N_{F1G1}$ of DP_e disjoint from A_{FG} $\ni N_{F1G1} \subseteq_{DS} X_{DS} - A_{FG}$.Thus $X_{DS} - A_{FG}$ is D_S – nhd of each its dual soft points

Since DP_e is arbitrary dual soft point of $X_{DS} - A_{FG}$

Then $X_{DS} - A_{FG}$ is T_E dual soft open set

Therefor A_{FG} is T_E dual soft closed set

Also let $DP_{\grave{e}} \in_{DS} A_{FG}$ then$DP_{\grave{e}} \in_{DS} DS_d(A_{FG})$

Then$DP_{\grave{e}}$ is dual soft limit point of A_{FG} so no dual soft point of $A_{FG}$is dual soft isolated point of A_{FG}since$DP_{\grave{e}}$ is arbitrary dual soft point of A_{FG}

then A_{FG} is dual soft closed set having no dual soft isolated point

Hence A_{FG} is perfect

Conversely suppose A_{FG} is dual soft perfect set then by definition 1.2.41 is dual soft closed set has no dual soft isolated points

Then $DS_{adh}(A_{FG})$=$DS_d(A_{FG})$ by remark 1.2.42

So $DS_{cl}(A_{FG}) = DS_{adh}(A_{FG})$ by corollary 1.2.37

But A_{FG} is dual soft closed set then $A_{FG} = DS_{cl}(A_{FG}) = DS_d(A_{FG})$.

Proposition 1.2.45

Let $A_{FG} \subseteq_{DS} X_{DS}$ be a dual soft set of the dual soft topological space(X_{DS}, T_E) a dual point $DP_e \in_{DS} X_{DS}$ is dual soft exterior point of A_{FG} if and only if DP_e is not dual soft adherent point of A_{FG} that is $DP_e \in_{DS} X_{DS} - DS_{cl}(A_{FG})$.

Proof

Let DP_e be a dual soft exterior point of A_{FG} then DP_e is dual soft interior point of $X_{DS} - A_{FG}$ so that $X_{DS} - A_{FG}$ is $DS - nhd$of DP_e contains no dual soft point of A_{FG} it follows that DP_e is not dual soft adherent point of A_{FG} that is $DP_e \in_{DS} X_{DS} - DS_{cl}(A_{FG})$.

conversely suppose that DP_e is not dual soft adherent point of A_{FG} then their existD_S – nhd N_{F1G1} of DP_e which contains no dual soft point A_{FG} This implies that $DP_e \in_{DS} N_{F1G1} \subseteq_{DS} X_{DS} - A_{FG}$

It follows that $X_{DS} - A_{FG}$ is$DS - nhd$ of DP_e and consequently DP_e is dual soft interior point of $X_{DS} - A_{FG}$ that is $DP_e$is dual soft exterior point of A_{FG}

Corollary 1.2.46

It follows from above theorem $DS_e(A_{FG}) = X_{DS} - DS_{cl}(A_{FG})$ from this we can conclude this

$DS_i(A_{FG}) = X_{DS} - (DS_{cl}(X_{DS} - A_{FG}))$.

Proposition 1.2.47

Let(X_{Ds}, T_E) be dual soft topological space and let $A_{FG} \subseteq_{DS} X_{Ds}$ then

1- $DS_i(A_{FG}) = X_{DS} - [DS_{cl}(X_{DS} - A_{FG})]$.
2- $DS_{cl}(X_{DS} - A_{FG}) = X_{DS} - [DS_i(A_{FG})]$.
3- $DS_{cl}(A_{FG}) = X_{DS} - [DS_i(X_{DS} - A_{FG})]$.
4- $DS_i(X_{DS} - A_{FG}) = X_{DS} - [DS_{cl}(A_{FG})]$.

Proof

1-Directly by corollary 1.2.46

2-Taking dual soft complement to part (1)we have

$X_{DS} - DS_i(A_{FG}) = X_{DS} - [X_{DS} - (DS_{cl}(X_{DS} - A_{FG}))]$

$X_{DS} - DS_i(A_{FG}) = DS_{cl}(X_{DS} - A_{FG})$. taking dual soft complement again

$X_{DS} - [X_{DS} - DS_i(A_{FG})) = X_{DS} - (DS_{cl}(X_{DS} - A_{FG})]$

So we obtain $DS_i(A_{FG}) = X_{DS} - (DS_{cl}(X_{DS} - A_{FG}))$.

3-By part (2)we have $DS_{cl}(X_{DS} - A_{FG}) = X_{DS} - DS_i(A_{FG})$

replaying A_{FG} by $X_{DS} - A_{FG}$ we get

$DS_{cl}[X_{DS} - (X_{DS} - A_{FG})) = X_{DS} - [DS_i(X_{DS} - A_{FG})]$

Thus $DS_{cl}(A_{FG}) = X_{DS} - [DS_i(X_{DS} - A_{FG})]$.

4-By part (1) $DS_i(X_{DS} - A_{FG}) = X_{DS} - [DS_{cl}(X_{DS} - (X_{DS} - A_{FG}))]$

$= X_{DS} - [DS_{cl}(A_{FG})]$

Proposition 1.2.48

Let $A_{FG} \subseteq_{DS} X_{DS}$ be a dual soft set of the dual soft topological space(X_{DS}, T_E) then $DS_{cl}(A_{FG}) = DS_i(A_{FG}) \cup_{Ds} DS_{fr}(A_{FG})$.

Proof

By definition 1.2.31 we have $DS_{cl}(A_{FG}) = \cap_{e\in E} \{A_{1F1G1}, A_{1F1G1}$is dual soft closed set , $A_{FG} \subseteq_{DS} A_{1F1G1}\}$

Then by De-Morgan law $X_{DS} - DS_{cl}(A_{FG}) = \cup_{e\in E} \{X_{DS} - A_{1F1G1}, X_{DS} - A_{1F1G1}$is dual soft open set , $X_{DS} - A_{1F1G1} \subseteq_{DS} X_{DS} - A_{FG}\} = DS_e(A_{FG})$ by proposition 1.2.20

Taking dual soft complement we get $X_{DS} - (X_{DS} - DS_{cl}(A_{FG}))$

$= X_{DS} - DS_e(A_{FG}) = DS_i(A_{FG}) \cup_{Ds} DS_{fr}(A_{FG})$Hence $DS_{cl}(A_{FG}) = DS_i(A_{FG}) \cup_{Ds} DS_{fr}(A_{FG})$

Corollary 1.2.49

$DS_{cl}(A_{FG}) = A_{FG} \cup_{Ds} DS_{fr}(A_{FG})$.

Proof

Since $A_{FG} \subseteq_{DS} DS_{cl}(A_{FG})$

And by proposition1.2.48 $DS_{fr}(A_{FG}) \subseteq_{DS} DS_{cl}(A_{FG})$

So we have$A_{FG} \cup_{Ds} DS_{fr}(A_{FG}) \subseteq_{DS} DS_{cl}(A_{FG})$.....(1)

Also $DS_{fr}(A_{FG}) = X_{DS} - [DS_i(A_{FG}) \cup_{Ds} DS_e(A_{FG})]$

$= X_{DS} - DS_i(A_{FG}) \cap_{Ds} X_{DS} - DS_e(A_{FG})$Again since$DS_i(A_{FG}) \subseteq_{DS} A_{FG}$

and $DS_{cl}(A_{FG}) = DS_i(A_{FG}) \cup_{Ds} DS_{fr}(A_{FG})$

$DS_{cl}(A_{FG}) \subseteq_{DS} A_{FG} \cup_{Ds} DS_{fr}(A_{FG})$ ..... (2)

from (1)and (2) we require $DS_{cl}(A_{FG}) = A_{FG} \cup_{Ds} DS_{fr}(A_{FG})$.

Proposition 1.2.50

Every dual soft closed set of dual soft topological space is the disjoint dual soft union of its dual soft interior and dual soft frontier in the sense that is contains these dual soft sets they are disjoint and it is their dual soft union

Proof

Let A_{FG}be a dual soft closed set so $A_{FG} = DS_{cl}(A_{FG})$ by proposition1.2.35

and $A_{FG} = DS_i(A_{FG}) \cup_{Ds} DS_{fr}(A_{FG})$ by proposition1.2.48

Hence $DS_i(A_{FG}) \cap_{Ds} DS_{fr}(A_{FG}) = \emptyset_{Ds}$.

Proposition1.2.51

Let(X_{Ds}, T_E) be dual soft topological space and let $A_{FG}, B_{F1G1} \subseteq_{DS} X_{Ds}$ then

1- $DS_{fr}(A_{FG})$=$DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})$
=$DS_{cl}(A_{FG}) - DS_i(A_{FG})$.

2- $DS_i(A_{FG})$=$A_{FG} - DS_{fr}(A_{FG})$.

3- $X_{Ds} - DS_{fr}(A_{FG}) = DS_i(A_{FG}) \cup_{Ds} DS_i(X_{Ds} - A_{FG})$.

4- $DS_{fr}[DS_i(A_{FG})] \subseteq_{DS} DS_{fr}(A_{FG})$.

5- $DS_{cl}[DS_{fr}(A_{FG})] \subseteq_{DS} DS_{fr}(A_{FG})$.

6- $DS_{fr}(A_{FG} \cup_{Ds} B_{F1G1}) \subseteq_{DS} DS_{fr}(A_{FG}) \cup_{Ds} DS_{fr}(B_{F1G1})$.

7- $DS_{fr}(A_{FG} \cap_{Ds} B_{F1G1}) \subseteq_{DS} DS_{fr}(A_{FG}) \cup_{Ds} DS_{fr}(B_{F1G1})$.

Proof

1-We have$DS_{fr}(A_{FG}) = X_{Ds} - [DS_i(A_{FG}) \cup_{Ds} DS_e(A_{FG})]$

= $X_{DS} - DS_i(A_{FG}) \cap_{Ds} X_{DS} - DS_e(A_{FG})$ De-Morgans law

= [$X_{DS} - X_{DS} - DS_{cl}(X_{Ds} - A_{FG})] \cap_{Ds} [X_{DS} - X_{DS} - DS_{cl}(A_{FG})]$

$= DS_{cl}(X_{Ds} - A_{FG}) \cap_{Ds} DS_{cl}(A_{FG})$

Now $DS_{cl}(X_{Ds} - A_{FG}) \cap_{Ds} DS_{cl}(A_{FG}) = DS_{cl}(A_{FG}) \cap_{Ds} X_{DS} - DS_i(A_{FG})$

$= DS_{cl}(A_{FG}) - DS_i(A_{FG})$ by proposition1.2.47 part(2)

Hence $DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) \cap_{Ds} X_{Ds} - A_{FG}$

$= DS_{cl}(A_{FG}) - DS_i(A_{FG})$.

2- $A_{FG} - DS_{fr}(A_{FG}) = A_{FG} - (DS_{cl}(A_{FG}) - DS_i(A_{FG}))$ by part (1)

$= DS_i(A_{FG})$ since $DS_i(A_{FG}) \subseteq_{DS} A_{FG}$.

3- We have this relation

$X_{DS} - DS_{fr}(A_{FG}) = X_{DS} - (DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}))$

$= X_{DS} - DS_{cl}(A_{FG}) \cup_{Ds} X_{DS} - (DS_{cl}(X_{Ds} - A_{FG}))$ De-Morgan law

Then by proposition 1.2.47 part (2)and(4)

$X_{DS} - [(X_{Ds} - A_{FG})] = DS_i(A_{FG})$

so $DS_i(X_{Ds} - A_{FG}) = X_{Ds} - DS_{cl}(A_{FG})$

Hence $X_{DS} - DS_{fr}(A_{FG}) = DS_i(X_{Ds} - A_{FG}) \cup_{Ds} DS_i(A_{FG})$

$= DS_i(A_{FG}) \cup_{Ds} DS_i(X_{Ds} - A_{FG})$

4-$DS_{fr}(DS_i(A_{FG})) = DS_{cl}[(A_{FG})) \cap_{Ds} DS_{cl}(X_{Ds} - DS_i(A_{FG})]$ by proposition 1.2.part (2)

$= DS_{cl}[DS_i(A_{FG})] \cap_{Ds} DS_{cl}[DS_{cl}(X_{Ds} - A_{FG})]$

$= DS_{cl}[DS_i(A_{FG})] \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})$

$\subseteq DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}) = DS_{fr}(A_{FG})$Thus $DS_{fr}(DS_i(A_{FG})) \subseteq_{DS} DS_{fr}(A_{FG})$.

5- $DS_{fr}[DS_{cl}(A_{FG})]= DS_{cl}[DS_{cl}(A_{FG})] \cap_{Ds} DS_{cl}[X_{Ds} - DS_{cl}(X_{Ds} - A_{FG})]$

Now $A_{FG} \subseteq_{DS} DS_{cl}(A_{FG})$then$X_{Ds} - DS_{cl}(A_{FG}) \subseteq_{DS} X_{Ds} - A_{FG}$

Then $DS_{cl}[\ X_{Ds} - DS_{cl}(A_{FG})] \subseteq_{DS} DS_{cl}(X_{Ds} - A_{FG})$

Hence$DS_{fr}(DS_{cl}(A_{FG})) \subseteq_{DS} DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}) = DS_{fr}(A_{FG})$.

6- $DS_{fr}(A_{FG} \cup_{Ds} B_{F1G1}) = [DS_{cl}\ (A_{FG} \cup_{Ds} B_{F1G1})] \cap_{Ds}$

$[DS_{cl}\ (X_{Ds} -(A_{FG} \cup_{Ds} B_{F1G1}))] =[DS_{cl}(A_{FG}) \cup_{Ds} DS_{cl}(B_{F1G1})]$

$\cap_{Ds}[DS_{cl}\ (X_{Ds} - A_{FG}) \cup_{Ds} DS_{cl}\ (X_{Ds} - B_{F1G1})]$ by part (1)

$\subseteq_{DS} [DS_{cl}(A_{FG}) \cup_{Ds} DS_{cl}(B_{F1G1})] \cap_{Ds}[DS_{cl}(X_{Ds} - A_{FG}) \cup_{Ds}$

$DS_{cl}(X_{Ds} - B_{F1G1})]$ by De-Morgan law

$= [(DS_{cl}(A_{FG}) \cap_{Ds} (DS_{cl}(X_{Ds} - A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1}))]$
$\cup_{Ds} [(DS_{cl}(B_{F1G1}) \cap_{Ds} (DS_{cl}(X_{Ds} - B_{F1G1}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}))]$

[by distributive law]

$= [(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1})]$
$\cup_{Ds} [(DS_{cl}(B_{F1G1}) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1})) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})]$

$= [DS_{fr}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1})] \cup_{Ds} [DS_{fr}(B_{F1G1}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})]$ by part (1)

$\subseteq_{DS} DS_{fr}(A_{FG}) \cup_{Ds} DS_{fr}(B_{F1G1})$

by $DS_{fr}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1}) \subseteq_{ds} DS_{fr}(A_{FG})$ etc.

7- $DS_{fr}(A_{FG} \cap_{Ds} B_{F1G1}) =[(DS_{cl}(A_{FG} \cap_{Ds} B_{F1G1}) \cap_{Ds} [\ DS_{cl}(X_{Ds} - (A_{FG} \cap_{Ds} B_{F1G1})]$ by part (1)

$\subseteq_{DS}$ $[(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(B_{F1G1})] \cap_{Ds} DS_{cl}[(X_{Ds} - A_{FG}) \cap_{Ds} (X_{Ds} - B_{F1G1})]$

$= [(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(B_{F1G1})] \cap_{Ds} [DS_{cl}(X_{Ds} - A_{FG})) \cup_{Ds} DS_{cl}(X_{Ds} - B_{F1G1})]$

$=[(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(B_{F1G1})) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})] \cup_{Ds} [(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(B_{F1G1})) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1})]$by distributive law

$= [(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})) \cap_{Ds} DS_{cl}(B_{F1G1})] \cup_{Ds} [(DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(B_{F1G1}) \cap_{Ds} DS_{cl}(X_{Ds} - B_{F1G1})]$

$=[DS_{fr}(A_{FG}) \cap_{Ds} DS_{cl}(B_{F1G1})] \cup_{Ds} [DS_{cl}(A_{FG}) \cap_{Ds} DS_{fr}(B_{F1G1})]$

$\subseteq_{DS}$ $[DS_{fr}(A_{FG}) \cup_{Ds} DS_{fr}(B_{F1G1})]$

Hence $DS_{fr}(A_{FG} \cap_{Ds} B_{F1G1}) \subseteq_{DS} [DS_{fr}(A_{FG}) \cup_{Ds} DS_{fr}(B_{F1G1})]$.

proposition1.2.52

Let(X_{Ds}, T_E) be dual soft topological space and let $A_{FG} \subseteq X_{Ds}$ then

1- If A_{FG} is dual soft open set then $DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) - A_{FG}$.
2- $DS_{fr}(A_{FG}) = \emptyset_{Ds}$ if and only if A_{FG} is dual soft open set as well as dual soft closed set .
3- A_{FG} is dual soft open set if and if $A_{FG} \cap_{Ds} DS_{fr}(A_{FG}) = \emptyset_{Ds}$ that is if $DS_{fr}(A_{FG}) \subseteq_{DS} X_{Ds} - A_{FG}$.
4- A_{FG} is dual soft closed set if and if $DS_{fr}(A_{FG}) \subseteq_{DS} A_{FG}$.

Proof

1-By proposition 1.2.51 part (1) we have $DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) - DS_i(A_{FG})$ and sinceA_{FG} is dual soft open set then $DS_i(A_{FG}) = A_{FG}$

Hence$DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) - A_{FG}$

2-Let $DS_{fr}(A_{FG}) = \emptyset_{Ds}$

so $DS_{fr}(A_{FG})= \emptyset_{Ds}=DS_{cl}(A_{FG})- DS_i(A_{FG})$ by proposition 1.2.51 part (1)

Then $DS_{cl}(A_{FG}) \subseteq_{DS} DS_i(A_{FG}) then\ A_{FG} \cup_{Ds} DS_d(A_{FG}) \subseteq_{DS} A_{FG}$

Thus $DS_d(A_{FG}) \subseteq_{DS} A_{FG}\ so\ A_{FG}$is dual soft closed set.

Again $DS_{fr}(A_{FG})= \emptyset_{Ds}$then $DS_{cl}(A_{FG})- DS_i(A_{FG})= \emptyset_{Ds}$

So $DS_{cl}(A_{FG}) \subseteq_{DS} DS_i(A_{FG})$thence $A_{FG} \cup_{Ds} DS_d(A_{FG}) \subseteq_{DS} DS_i(A_{FG})$

Thus $A_{FG} \subseteq_{DS} DS_i(A_{FG})$ but $DS_i(A_{FG}) \subseteq_{DS} A_{FG}$

Hence $DS_i(A_{FG}) =A_{FG}$then A_{FG} is dual soft open set

Thus we have shown that $DS_{fr}(A_{FG})= \emptyset_{Ds}$ then A_{FG} is dual soft open set as well as dual soft closed set .

Conversely let A_{FG} is dual soft open set as well as dual soft closed set to show that $DS_{fr}(A_{FG})= \emptyset_{Ds}$

Now $DS_{fr}(A_{FG})= DS_{cl}(A_{FG})- DS_i(A_{FG})$ since A_{FG} is dual soft closed set then $DS_{cl}(A_{FG}) = A_{FG}$

Again since A_{FG} is dual soft open set then $DS_i(A_{FG}) =A_{FG}$.

Thus $DS_{fr}(A_{FG}) = A_{FG} - A_{FG} = \emptyset_{Ds}$

3-By theorem 1.2.51 part (1) $DS_{fr}(A_{FG})= DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})$ let A_{FG}be dual soft open set then $X_{Ds} - A_{FG}$is dual soft closed set

Hence$DS_{cl}(X_{Ds} - A_{FG}) = X_{Ds} - A_{FG}$

Now $A_{FG} \cap_{Ds} DS_{fr}(A_{FG})= A_{FG} \cap_{Ds} (DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}))$

$= A_{FG} \cap_{Ds} (DS_{cl}(A_{FG}) \cap_{Ds} (X_{Ds} - A_{FG})) = \emptyset_{Ds}$

Conversely let$A_{FG} \cap_{Ds} DS_{fr}(A_{FG})= \emptyset_{Ds}$ then by part(1)

$A_{FG} \cap_{Ds} DS_{fr}(A_{FG}) = \emptyset_{Ds}$

So $A_{FG} \cap_{Ds} (DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})) = \emptyset_{Ds}$

Then $(A_{FG} \cap_{Ds} DS_{cl}(A_{FG})) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}) = \emptyset_{Ds}$

Thus $A_{FG} \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG}) = \emptyset_{Ds}$

Then $A_{FG} \subseteq_{DS} X_{Ds} - DS_{cl}(X_{Ds} - A_{FG})$

Hence $A_{FG} \subseteq_{DS} DS_i(A_{FG})$ by $DS_i(A_{FG}) = X_{Ds} - DS_{cl}(X_{Ds} - A_{FG})$

But $DS_i(A_{FG}) \subseteq_{DS} A_{FG}$ hence $DS_i(A_{FG}) = A_{FG}$

It follows that A_{FG} is a dual soft open set

4-let A_{FG} be dual soft closed set then $DS_{cl}(A_{FG}) = A_{FG}$

Hence $DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})$ by part (1)

$= A_{FG} \cap_{Ds} DS_{cl}(X_{Ds} - A_{FG})]$ by part (3)

Therefore $DS_{fr}(A_{FG}) \subseteq_{DS} A_{FG}$

Conversely let $DS_{fr}(A_{FG}) \subseteq_{DS} A_{FG}$

Then $A_{FG} \cup_{Ds} DS_{fr}(A_{FG}) = A_{FG}$ [by corollary 1.2.49]

But $A_{FG} \cup_{Ds} DS_{fr}(A_{FG}) = DS_{cl}(A_{FG})$

It follows that $DS_{cl}(A_{FG}) = A_{FG}$

Hence A_{FG} be dual soft closed set.

Now the dual soft subspaces of dual soft space are defined and inherent concepts as well as the characterization of dual soft open and dual soft closed set in dual soft subspaces are investigated.

Definition 1.2.53 Dual Soft Subspace

Let (X_{Ds}, T_E) be dual soft topological space and let $Y_{FG} \subseteq_{DS} X_{Ds}$ the relative dual soft topology for Y_{FG} is the collection $T_{E_{Y_{FG}}}$ given by:

$T_{E_{Y_{FG}}}$=$\{A_{F1G1} \cap_{DS} Y_{FG}: \forall A_{F1G1} \in T_E\}$

The dual soft topological space $\{Y_{FG}, T_{E_{Y_{FG}}}\}$is called dual soft subspace of (X_{Ds}, T_E).

In order the above definition may be consistence we must show that $T_{E_{Y_{FG}}}$is actually dual soft topology for Y_{FG} . This we will prove in the following proposition

Proposition 1.2.54

Let(X_{Ds}, T_E) be dual soft topological space and let $Y_{FG} \subseteq_{DS} X_{Ds}$ then the collection T_{EY} =$\{A_{F1G1} \cap_{DS} Y_{FG}: A_{F1G1} \in T_E\}$is dual soft topology on Y_{FG}

Proof

First since $\emptyset_{Ds} \in_{DS} T_E$ and $\emptyset_{Ds} \cap_{DS} Y_{FG} = \emptyset_{Ds}$ it follows that $\emptyset_{Ds} \in_{DS} T_{E_{Y_{FG}}}$

again $X_{Ds} \cap_{DS} Y_{FG} = Y_{FG}$ and since $Y_{FG} \subseteq_{DS} X_{Ds}$ and $X_{Ds} \in_{DS} T_E$ we have $Y_{FG} \in_{DS} T_{E_{Y_{FG}}}$

Second let $H_{1F1G1}, H_{2F2G2} \in_{DS} T_{E_{Y_{FG}}}$ then H_{1F1G1}=$A_{1F3G3} \cap_{DS} Y_{FG}$ and

H_{2F2G2}=$A_{2F4G4} \cap_{DS} Y_{FG}$ for some $A_{1F3G3}, A_{2F4G4} \in_{DS} T_E$

Now $H_{1F1G1} \cap_{DS} H_{2F2G2}$=$(A_{1F3G3} \cap_{DS} Y_{FG}) \cap_{DS} (A_{2F4G4} \cap_{DS} Y_{FG})$

=$(A_{1F3G3} \cap_{DS} A_{2F4G4}) \cap_{DS} Y_{FG} \in_{DS} T_{E_{Y_{FG}}}$

Finally let $H_{\lambda F\lambda G\lambda} \in_{DS} T_{E_{Y_{FG}}} \forall \lambda \in \Lambda$ then their exist $A_{\lambda F\lambda G\lambda} \in_{DS} T_E \ni$

$H_{\lambda F\lambda G\lambda}$ =$A_{\lambda F\lambda G\lambda} \cap_{DS} Y_{FG}$ $\forall \lambda$

$\cup_{DS}$ { $H_{\lambda F\lambda G\lambda}$ $\lambda \in \Lambda$ }=$\cup_{DS}\{A_{\lambda F\lambda G\lambda} \cap_{DS} Y_{FG}$; $\lambda \in \Lambda\}$

$=\cup_{DS}\{A_{\lambda F\lambda G\lambda}\,;\lambda\in\Lambda\}\cap_{DS} Y_{FG}\ \in_{DS} T_{E_{Y_{FG}}}$.Thus $T_{E_{Y_{FG}}}$ is dual soft topology on Y_{FG} .

Example 1.2.55

Let $U_1=\{h_1,,h_2,h_3\}$ $U_2=\{C_1,C_2,C_3\}$ $E=\{e_1,,e_2\}$

assume $Y_{FG}=\{(e_1,\{h_1,h_2\},\{c_3,c_2\}),(e_2,\{h_1,h_3\},\{c_1,c_3\})\}$

$A_{1F1G1}=\{(e_1,\{h_1,h_3\},\{c_1,c_3\}\}),(e_2,\{h_2\},\{c_2\})\}$

$A_{2F2G2}=\{(e_1,\{h_2\},\{c_2\}),(e_2,\{h_1,h_3\},\{c_1,c_3\})\}$

$T_E=\{X_{DS},\emptyset_{Ds},A_{1F1G1,}A_{2F2G2}\}$

$X_{Ds}\cap_{DS} Y_{FG}=Y_{FG}$ $\emptyset_{Ds}\cap_{DS} Y_{FG}=\emptyset_{Ds}$

$A_{1F1G1}\cap_{DS} Y_{FG}=\{(e_1,\{h_1\},\{c_3\}),(e_2,\emptyset,\emptyset)\}=C_{F3G3}$

$A_{2F2G2}\cap_{DS} Y_{FG}=\{(e_1,\{h_2\},\{c_2\}),(e_2,\{h_1,h_3\},\{c_1,c_3\})\}=D_{F4G4}$

$T_{E_{Y_{FG}}}=\{\emptyset_{Ds},Y_{FG},C_{F3G3},D_{F4G4}\}$

Proposition 1.2.56

Let $(Y_{FG},T_{E_{Y_{FG}}})$ be a dual soft sub space of (X_{Ds},T_E) then :

1-A subset A_{1F1G1} of Y_{FG} is closed in Y_{FG} if and only if there exist a dual soft closed set A_{2F2G2} in $X_{Ds}\ni A_{1F1G1}=A_{2F2G2}\cap_{DS} Y_{FG}$.

2-For every $A_{1F1G1}\subseteq_{DS} Y_{FG}$ $DS_{cl_{Y_{FG}}}(A_{1F1G1})=DS_{cl_{X_{Ds}}}(A_{1F1G1})\cap_{DS} Y_{FG}$.

3-A subset M_{1F1G1} of Y_{FG} is $a\ DS_{Y_{FG}}-nhd$ of a dual soft point $DP_e\in_{DS} Y_{FG}$ if and only if $M_{1F1G1}=N_{2F2G2}\cap_{DS} Y_{FG}$ for some $DS-nhd\ N_{2F2G2}$ of Y_{FG} .

4-A a dual soft point $DP_e\in_{DS} Y_{FG}$ $DS_{Y_{FG}}$ -limit point of $A_{3F3G3}\subseteq_{DS} Y_{FG}$ if and only if DP_e is DS_d of A_{3F3G3}. Further more

$DS_{d_{Y_{FG}}}=DS_{d_{X_{Ds}}}(A_{3F3G3})\cap_{DS} Y_{FG}$.

5- For every $A_{1F1G1}\subseteq_{DS} Y_{FG}$ $DS_{i_{Y_{FG}}}(A_{1F1G1})\supseteq_{DS} DS_{i_{X_{Ds}}}(A_{1F1G1})$.

6- For every $A_{1F1G1} \subseteq_{DS} Y_{FG}$ $DS_{fr_{Y_{FG}}}(A_{1F1G1}) \subseteq_{DS}$ $DS_{fr\ _{X_{Ds}}}(A_{1F1G1})$.

Proof

1-Let A_{1F1G1} is closed in Y_{FG}

If and only if $Y_{FG} - A_{1F1G1}$is dual soft open in Y_{FG}

If and only if $Y_{FG} - A_{1F1G1} = A_{3F3G3} \cap_{DS} Y_{FG}$for some $A_{3F3G3} \in_{DS} T_E$

If and only if $A_{1F1G1} = Y_{FG} - (A_{3F3G3} \cap_{DS} Y_{FG})$

$=(Y_{FG} - A_{3F3G3}) \cup_{DS} (Y_{FG} - Y_{FG})$

If and only if $A_{1F1G1} = Y_{FG} - A_{3F3G3}$

If and only if $A_{1F1G1} = Y_{FG} \cap_{DS} (X_{Ds} - A_{3F3G3})$

If and only if $A_{1F1G1} = A_{2F2G2} \cap_{DS} Y_{FG}$.

2- By definition 1.2.31

$DS_{cl_{Y_{FG}}}(A_{1F1G1}) = \cap_{e\in E}\{ A_{3F3G3}: A_{3F3G3}$is dual soft closed set in $Y_{FG}$and $A_{1F1G1} \subseteq_{DS} A_{3F3G3}\}$

$=\cap_{e\in E} \{A_{2F2G2} \cap_{DS} Y_{FG}: A_{2F2G2}$ is dual soft closed in $X_{Ds}$and $A_{1F1G1} \subseteq_{DS} A_{2F2G2} \cap_{DS} Y_{FG}\}$

$=\cap_{e\in E}\{A_{2F2G2} \cap_{DS} Y_{FG}: A_{2F2G2}$ is dual soft closed in$X_{Ds}$and $A_{1F1G1} \subseteq_{DS} A_{2F2G2}\}$

$=(\cap_{e\in E} \{A_{2F2G2}: A_{2F2G2}$ is dual soft closed inX_{Ds} and $A_{1F1G1} \subseteq_{DS} A_{2F2G2}\})$ $\cap_{DS} Y_{FG} =$ $DS_{cl\ _{X_{Ds}}}(A_{1F1G1}) \cap_{DS} Y_{FG}$.

3-Let M_{1F1G1}be $DS_{Y_{FG}} - nhd$ of DP_e then there exist a $T_{EY_{FG}}$ –dual soft open set $A_{3F3G3} \ni DP_e \in A_{3F3G3} \subseteq_{DS} M_{1F1G1}$

If and only if $\exists T_E$ –dual soft open set A_{4F4G4}

$\ni\ DP_e \in_{DS} A_{3F3G3} = A_{4F4G4} \cap_{DS} Y_{FG} \subseteq_{DS} M_{1F1G1}$

Let $N_{2F2G2} = M_{1F1G1} \cup_{DS} A_{4F4G4}$.

Then $N_{2F2G2}$is $DS-nhd$of DP_e since A_{4F4G4}isT_E –dual soft open set $DP_e \in_{DS} A_{4F4G4} \subseteq_{DS} N_{2F2G2}$

Further $N_{2F2G2} \cap_{DS} Y_{FG}$ =$(M_{1F1G1} \cup_{DS} A_{4F4G4}) \cap_{DS} Y_{FG}$

=$(M_{1F1G1} \cap_{DS} Y_{FG}) \cup_{DS} (A_{4F4G4} \cap_{DS} Y_{FG})$

=$M_{1F1G1} \cup_{DS} (A_{4F4G4} \cap_{DS} Y_{FG})$= M_{1F1G1} since $M_{1F1G1} \subseteq_{DS} Y_{FG}$

and $A_{4F4G4} \cap_{DS} Y_{FG} \subseteq_{DS} M_{1F1G1}$.

Conversely let $M_{1F1G1} = N_{2F2G2} \cap_{DS} Y_{FG}$ for some $DS-nhd$

$N_{2F2G2}$of DP_e then there exist T_E –dual soft open set $A_{4F4G4} \ni$

$DP_e \in_{DS} A_{4F4G4} \subseteq_{DS} N_{2F2G2}$which implies that

$DP_e \in_{DS} A_{4F4G4} \cap_{DS} Y_{FG} \subseteq N_{2F2G2} \cap_{DS} Y_{FG} = M_{1F1G1}$

Since $A_{4F4G4} \cap_{DS} Y_{FG}$ is $T_{EY_{FG}}$ –dual soft open set , M_{1F1G1} is

$DS_{Y_{FG}} - nhd$ of DP_e

4- DP_e is $DS_{d_{Y_{FG}}}$ of A_{3F3G3}

If and only if $M_{1F1G1} - DP_e \cap_{DS} A_{3F3G3} \neq \emptyset_{DS} \forall\ DS_{Y_{FG}} - nhd M_{1F1G1}$ of DP_e

If and only if $N_{2F2G2} - DP_e \cap_{DS} A_{3F3G3} \neq \emptyset_{DS} \forall\ DS-nhd$ of DP_e

Because $DP_e \in_{DS} Y_{FG}$ so $X_{Ds} - DP_e = Y_{FG} - DP_e$

If and only if DP_e is a dual soft limit point of A_{3F3G3} .

5-$DP_e \in_{DS} DS_{i\ X_{Ds}}(A_{1F1G1})\ so\ DP_e$ is dual soft interior point of A_{1F1G1}

Then $A_{1F1G1}$is $DS-nhd$ of DP_e

So $A_{1F1G1} \cap_{DS} Y_{FG}$is $DS_{Y_{FG}}-nhd$ of DP_e

Thus $DP_e \in_{DS} DS_{iY_{FG}}(A_{1F1G1})$

Hence $DS_{i\ X_{Ds}}(A_{1F1G1}) \subseteq_{DS} DS_{iY}(A_{1F1G1})$.

6-$DP_e \in DS_{fr_{Y_{FG}}}(A_{1F1G1})\ then DP_e$is $T_{E_{Y_{FG}}}$ –dual soft frontier point of A_{1F1G1}

If and only if $\forall\ DS_{Y_{FG}}-nhd$ of DP_e intersect both $A_{1F1G1}$and $Y_{FG}-A_{1F1G1}$ by proposition 1.2.22

If and only if $N_{2F2G2} \cap_{DS} Y_{FG}$ intersect both $A_{1F1G1}$and $Y_{FG}-A_{1F1G1}$

If and only if DP_e is $DS_{fr}(A_{1F1G1})$of T_E

If and only if $DP_e \in_{DS} DS_{fr\ X_{Ds}}(A_{1F1G1})$

Hence $DS_{fr_{Y_{FG}}}(A_{1F1G1}) \subseteq_{DS} DS_{fr\ X_{Ds}}(A_{1F1G1})$.

proposition1.2.57

Let Y_{FG} be a dual soft subspace of topological space (X_{Ds}, T_E) if $A_{F1G1} \subseteq_{DS} Y_{FG}$ is dual soft open (closed) set in X_{Ds}thenA_{1F1G1} is also dual soft open (closed) in Y_{FG}.

Proof

Since $A_{F1G1} \subseteq_{DS} Y_{FG}$, we have $A_{F1G1} = A_{F1G1} \cap_{DS} Y_{FG}$ so that $A_{F1G1}$is the dual soft intersection of $Y_{FG}$with a dual soft open (closed) set in X_{Ds} namely A_{F1G1} . Hence by proposition 1.2.54 and 1.2.56 A_{F1G1} is dual soft open (closed) in X_{Ds} .

proposition1.2.58

Let Y_{FG} be a dual soft subspace of topological space (X_{Ds}, T_E) in order that every dual soft subset $B_{1F1G1}$of Y_{FG} which is dual soft open (closed) in Y_{FG} be dual soft open (closed) in X_{Ds} it is necessary and sufficient that Y_{FG} be dual soft open (closed) in Y_{FG}.

Proof

Necessity: let every dual soft subset $B_{1F1G1}$of Y_{FG} which is dual soft open (closed) in Y_{FG} be dual soft open (closed) in X_{Ds}. Since Y_{FG} is dual soft open (closed) in Y_{FG} it follows that Y_{FG} is dual soft open (closed) in X_{Ds}

Sufficiency: let B_{F1G1} be dual soft subset of Y_{FG} which is dual soft open (closed) in Y_{FG} and let Y_{FG} be dual soft open (closed) in X_{Ds} since B_{F1G1} is dual soft open (closed) in Y_{FG} , there exist a dual soft subset U_{F2G2} open (closed) in X_{Ds} such that $B_{F1G1} = U_{F2G2} \cap_{DS} Y_{FG}$ and since $Y_{FG}$is dual soft open (closed) in X_{Ds} , B_{F1G1} is dual soft open (closed) in X_{Ds} , being the dual soft intersection of two dual soft open (closed) subset of X_{Ds} .

In this chapter we define the following new concepts; dual soft super condensed set, dual soft semi condensed set and dual soft condensed set, using the concepts of dual soft interior point and dual soft closure point only. So it may be advocated that these new concepts are totally basic ones. Further we introduce concepts of dual soft border set which is different from dual soft boundary of a set and discuss about the relations between these new concepts. Furthermore we classify dual soft sets of the dual soft topological space into three classes [31] .

The following proposition characteristic that link the dual soft interior point with dual soft closure point and these characteristics are consider pillar on which the chapter path depends

Proposition 2.1

For any dual soft set $A_{FG} \subseteq_{DS} X_{DS}$ The following relation hold

1- If $U_{F1G1} \in_{DS} T_E$ then $U_{F1G1} \cap_{DS} DS_{cl}(A_{FG}) \subseteq_{DS} DS_{cl}(U_{F1G1} \cap_{DS} A_{FG})$ for any $A_{FG} \in_{DS} X_{DS}$

2- $DS_i[DS_{cl}(DS_i(DS_{cl}(A_{FG})))] = DS_i[DS_{cl}(A_{FG})]$

3- If A_{FG} or $B_{F2G2} \in_{DS} T_E$ then $DS_i[DS_{cl} A_{FG} \cap_{DS} B_{F2G2})] = DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[DS_{cl}(B_{F2G2})]$

4- $DS_{cl}[DS_i(DS_{cl}(DS_i(A_{FG})))] = DS_{cl}[DS_i(A_{FG})]$

5- $DS_i[DS_{cl}(DS_i(A_{FG}))] \cap_{DS} DS_i[DS_{cl}(DS_i(B_{F2G2}))] = DS_i[DS_{cl}(DS_i(A_{FG} \cap_{DS} B_{F2G2}))]$

6- $DS_{cl}[DS_i(DS_{cl}(A_{FG}))] \cup_{DS} DS_{cl}[DS_i(DS_{cl}(B_{F2G2}))] = DS_{cl}(DS_i(DS_{cl}[A_{FG} \cup_{DS} B_{F2G2}]))$

proof

1 - Let DP_e be any dual soft point , $DP_e \in_{DS} U_{F1G1} \cap_{DS} DS_{cl}(A_{FG})$

Then $\forall V_{FiGi} \in_{DS} T_E(DP_e) \ni V_{FiGi} \cap_{DS} A_{FG} \neq \emptyset_{DS}$ …… (1)

If possible $DP_e \notin_{DS} DS_{cl}(U_{F1G1} \cap_{DS} A_{FG})$ so $\exists\ W_{F4G4} \in T_E(DP_e) \ni W_{F4G4} \cap_{DS} (U_{F1G1} \cup_{DS} A_{FG}) = \emptyset_{Ds}$

But $W_{F4G4} \cap_{DS} U_{F1G1} \in_{DS} T_E\ (DP_e)$ which contradict to (1)

Hence $DP_e \in_{DS} DS_{cl}(U_{F1G1} \cap_{DS} A_{FG})$

2 - $DS_i\ [DS_{cl}(A_{FG})] \subseteq_{DS} DS_{cl}\ (A_{FG})$

Then $DS_{cl}[DS_i\ (DS_{cl}\ (A_{FG}))] \subseteq_{DS} DS_{cl}[DS_{cl}\ (A_{FG})] = DS_{cl}\ (A_{FG})$

So $DS_i\ [DS_{cl}\ (DS_i\ (DS_{cl}\ (A_{FG})))] \subseteq_{DS} DS_i[DS_{cl}(A_{FG})]$ ----- (1)

$DS_i\ [DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_{cl}\ (DS_i\ (DS_{cl}\ (A_{FG})))$

Thus $DS_i[DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_i[\ DS_{cl}\ (DS_i\ (\ DS_{cl}\ (A_{FG})))]$ ----- (2)

from (1) and (2) we get

$DS_i\ [DS_{cl}\ (DS_i\ (\ DS_{cl}\ (A_{FG})))] = DS_i\ [DS_{cl}\ (A_{FG})]$

3 - $A_{FG} \cap_{DS} B_{F2G2} \subseteq_{DS} A_{FG}$ and $A_{FG} \cap_{DS} B_{F2G2} \subseteq_{DS} B_{F2G2}$

Then $DS_i[DS_{cl}\ (A_{FG} \cap_{DS} B_{F2G2})] \subseteq_{DS} DS_i\ [DS_{cl}\ (A_{FG})]$

$DS_i\ [DS_{cl}\ (A_{FG} \cap_{DS} B_{F2G2})] \subseteq_{DS} DS_i\ [DS_{cl}\ (B_{F2G2})]$

So $DS_i[DS_{cl}\ (A_{FG} \cap_{DS} B_{F2G2})] \subseteq_{DS} DS_i[DS_{cl}\ (A_{FG})] \cap_{DS} DS_i[DS_{cl}\ (B_{F2G2})]$

If $A_{FG} \in T_E$ then $A_{FG} \cap_{DS} DS_{cl}\ (B_{F2G2}\ ) \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} B_{F2G2})$ by part (1)

Then $DS_i(A_{FG}) \cap_{DS} DS_i[DS_{cl}\ (B_{F2G2})] \subseteq DS_i[DS_{cl}(A_{FG} \cap_{\mathrm{DS}} B_{F2G2})]$

So $A_{FG} \cap_{DS} DS_i[DS_{cl}(B_{F2G2})] \subseteq_{DS} DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F2G2})]$ -------(1)

Either $B_{F2G2} \subseteq_{DS} T_E$

Then$B_{F2G2} \cap_{DS} DS_i[DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_i[DS_{cl}\ (A_{FG} \cap_{DS} B_{F2G2})]$------(2)

Thence $DS_i[DS_{cl}(\ (DS_i(DS_{cl}\ (A_{FG}))\ \cap_{DS}\ B_{F2G2})] \subseteq_{DS} DS_i[DS_{cl}\ (DS_i(DS_{cl}(\ A_{FG}\ \cap_{DS}\ B_{F2G2})))] = DS_i[DS_{cl}\ (A_{FG}\ \cap_{DS}\ B_{F2G2})]$------(3)

But $DS_i[DS_{cl}\ (B_{F2G2})]$ is dual soft open set

Then $DS_{cl}\ (A_{FG})\ \cap_{DS}\ DS_i[DS_{cl}(B_{F2G2})] \subseteq_{DS}\ DS_{cl}[A_{FG}\ \cap_{DS}\ (DS_i\ (DS_{cl}\ (B_{F2G2}))]$

Thus $DS_i[DS_{cl}\ (A_{FG})]\ \cap_{DS} DS_i(DS_{cl}(B_{F2G2})]\ \subseteq_{DS} DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[DS_{cl}(\ B_{F2G2})]$ ----- (4)

From (3) and (4) we get

$DS_i[DS_{cl}\ (A_{FG})] \cap_{DS}\ DS_i[DS_{cl}\ (B_{F2G2})] \subseteq_{DS} DS_i[DS_{cl}(DS_i(DS_{cl}(\ A_{FG}\ \cap_{DS}\ B_{F2G2})))]\ = DS_i[DS_{cl}(\ A_{FG}\ \cap_{\mathrm{DS}}\ B_{F2G2})]$

Therefor$DS_i[DS_{cl}\ (A_{FG})]\ \cap_{DS}\ DS_i[DS_{cl}(B_{F2G2})]\ =\ DS_i[DS_{cl}(A_{FG}\ \cap_{DS}\ B_{F2G2})]$

4 - $DS_i[DS_{cl}\ (DS_i\ (A_{FG}))] \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

Then $DS_{cl}[DS_i(DS_{cl}(DS_i\ (A_{FG})))] \subseteq_{DS} DS_{cl}[DS_{cl}(DS_i(A_{FG}))]$

$=\ DS_{cl}[DS_i(A_{FG})]$

So $DS_{cl}[DS_i\ (DS_{cl}(\ DS_i\ (A_{FG})))] \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$ ……. (1)

Now $DS_i\ (A_{FG})\ \subseteq_{DS} DS_{cl}\ (DS_i\ (A_{FG}))$

So $DS_i\ (A_{FG}) \subseteq_{DS} DS_i[DS_{cl}(DS_i\ (A_{FG})]$

Therefor $DS_{cl}[DS_i\ (A_{FG})] \subseteq_{DS}\ DS_{cl}[DS_i(DS_{cl}(\ DS_i\ (A_{FG})))]$ …… (2)

from (1) and (2) we get

$DS_{cl}[DS_i(DS_{cl}(DS_i\ (A_{FG})))] = DS_{cl}[DS_i\ (A_{FG})]$

5 - since $DS_i\ (A_{FG})$ is dual soft open set then by part (3)

$DS_i[DS_{cl}\ (DS_i(\ A_{FG}))] \cap_{DS}\ DS_i[DS_{cl}\ (DS_i(B_{F2G2}\ ))] = DS_i\ (DS_{cl}[DS_i(\ A_{FG}) \cap_{DS} DS_i(\ B_{F2G2})]) = DS_i(\ DS_{cl}\ (DS_i\ [A_{FG}\ \cap_{DS} B_{F2G2}]))$

6 - $DS_{cl}[DS_i\ (DS_{cl}\ (A_{FG} \cup_{DS} B_{F2G2}))]$

$= DS_{cl}[DS_i\ (X_{DS} -(DS_i\ (X_{DS} - (A_{FG} \cup_{DS} B_{F2G2}))))]$

$= DS_{cl}[DS_i(X_{DS} -(DS_i\ (X_{DS} -\ (A_{FG})) \cap_{DS} (X_{DS} -\ (B_{F2G2}\ ))))]$

$= X_{DS} -[DS_i\ (DS_{cl}\ (DS_i\ (X_{DS} - A_{FG}) \cap_{DS} (\ X_{DS} - B_{F2G2}\ )))]$ ----- (1)

$DS_{cl}\ [DS_i\ (DS_{cl}\ (A_{FG}))] \cup_{DS} DS_{cl}[DS_i\ (DS_{cl}\ (B_{F2G2}\ ))] = DS_{cl}\ (DS_i[X_{DS} - (DS_i\ (X_{DS} - A_{FG}))] \cup_{DS} [X_{DS} - (DS_i\ (X_{DS} - B_{F2G2}\ ))])$

$= X_{DS} -[\ DS_i(DS_{cl}(DS_i(X_{DS} - A_{FG})))] \cup_{DS} (X_{DS} - B_{F2G2})]$

$= X_{DS} -[DS_i\ (DS_{cl}\ (DS_i\ (X_{DS} - A_{FG})))] \cap_{DS} (B_{F2G2})]$ ------(2)

From (1) and (2) we have

$DS_{cl}(DS_i\ (DS_{cl}\ [A_{FG} \cup_{DS} B_{F2G2}])) = X_{DS} - (DS_i(DS_{cl}(DS_i[(X_{DS} - A_{FG}) \cap_{DS} (X_{DS} - B_{F2G2})])) = DS_{cl}(DS_i\ (DS_{cl}\ [A_{FG} \cup_{DS} B_{F2G2}]))$

Now we will introduce a new concept that represents the main focus of our chapter

Definition 2.2

A dual soft set A_{FG} of X_{DS} is said to be

1- Dual soft super condensed set if $DS_i[DS_{cl}\ (A_{FG})] = DS_i\ (A_{FG})$
2- Dual soft semi condensed set if $DS_{cl}[DS_i(A_{FG})] = DS_{cl}(A_{FG})$
3- Dual soft condensed set if $DS_i[DS_{cl}(A_{FG})] = DS_i(A_{FG})$and $DS_{cl}[DS_i\ (A_{FG})] = DS_{cl}\ (A_{FG})$.

Example 2 .3.1

Let $U_1 = \{ h_1 , h_2 , h_3 \}$ $U_2 = \{ c_1 , c_2, c_3\}$ $E = \{ e_1 , e_2 \}$

$A_{F1G1} = \{ (e_1 , \{ h_1 \} , \{ c_1 \}) , (e_2 , \{ h_2 \} , \{ c_2 \}) \}$

$B_{F2G2} = \{(e_1 , \{ h_2 , h_3 \} , \{ c_2 , c_3 \}) , (e_2, \{h_1, h_3 \} ,\{ c_1 , c_3 \}) \}$

$T_E = \{ \emptyset_{DS} , X_{DS}, A_{F1G1} , B_{F2G2}\}$

$DS_{cl}\ (DS_i\ (A_{F1G1}))\ = A_{F1G1}$

$DS_i(DS_{cl}\ (A_{F1G1}))$ = A_{F1G1} therefore A_{F1G1} is dual soft condensed set

Example 2.3.2

Let U_1= { h_1 , h_2 } U_2= { c_1 , c_2 } E= { e_1 , e_2 }

A_{1F1G1} ={(e_1,{ h_1}, U_2) ,(e_2, ∅,{ c_1})}

$X_{DS} - A_{1F1G1}$ ={(e_1,{ h_2},∅),(e_2, U_1, {c_2})

A_{2F2G2} = {(e_1 , { h_2} ,{c_2}) , (e_2 , { h_1}, { c_1})}

$X_{DS} - A_{2F2G2}$ ={(e_1,{ h_1},{c_1}),(e_2,{ h_2}, {c_2})

B_{F2G2} = {(e_1 , U_1 , U_2) , (e_2, {h_1} ,{ c_1})}

$X_{DS} - B_{F2G2}$={(e_1 , ∅, ∅) , (e_2 ,{ h_2}, { c_2})}

C_{F3G3}= {(e_1 , ∅, ,{c_2}) , (e_2 , ∅,{c_1})}

$X_{DS} - C_{F3G3}$={(e_1 , U_1, { c_1}) , (e_2 , U_1, { c_2})}

T_E = { $\emptyset_{DS}$,X_{DS}, A_{1F1G1} , A_{2F2G2}, B_{F2G2}, C_{F3G3} }

Assume A_{FG}= {(e_1, { h_1}, { c_1}), (e_2 , { h_2}, { c_2})} = $X_{DS} - A_{2F2G2}$

Now $DS_i\ (DS_{cl}\ (A_{FG}\ ))$ = $DS_i\ (A_{FG})$ $\emptyset_{DS}$

Then A_{FG} is dual soft super condensed set

Example 2 .3.3

Let U_1= { h_1 , h_2 } U_2= { c_1 , c_2 } E= { e_1 , e_2 }

A_{1F1G1} = { (e_1 ,{ h_1} , { c_1}) , (e_2 , ∅, U_2)}

$X_{DS} - A_{1F1G1}$ = { (e_1 ,{ h_2} , { c_2}) , (e_2, U_1 , ∅)}

A_{2F2G2} = { (e_1 , U_1,{c_2}) , (e_2 , { h_2}, { c_1})}

$X_{DS} - A_{2F2G2}$ = { (e_1 , ∅,{c_1}) , (e_2 , { h_1}, { c_2})}

B_{F2G2} = {(e_1 , U_1 , U_2) , (e_2, {h_2} ,U_2)}

$X_{DS} - B_{F2G2} = \{(e_1\ ,\ \emptyset\ ,\ \emptyset)\ ,(e_2,\{h_1\}\ ,\emptyset)\}$

$C_{F3G3}= \{\ (\ e_1\ ,\{\ h_1\},\emptyset)\ ,\ (e_2\ ,\emptyset,\{\ c_1\})\}$

$X_{DS} - C_{F3G3}= \{\ (\ e_1\ ,\{\ h_2\},U_2)\ ,\ (e_2,U_1,\{\ c_2\})$

$T_E = \{\ \emptyset_{DS}\ ,X_{DS},\ A_{1F1G1}\ ,\ A_{2F2G2},\ B_{F2G2},C_{F3G3}\ \}$

Assume $A_{FG}= \{(\ e_1\ ,\{\ h_1\},U_2)\ ,(e_2\ ,\{\ h_1\},\{\ c_2\})\}$

Now $DS_i\ (DS_{cl}\ (A_{FG}\ ))=\ DS_i\ (X_{DS})= X_{DS}$

$DS_i(A_{FG})= B_{F2G2} \neq X_{DS}$ then $A_{FG}$is not dual soft super condensed set

$DS_{cl}\ (DS_i\ (A_{FG}\ ))=DS_{cl}\ (B_{F2G2})\ =\ X_{DS}\ ,DS_{cl}\ (A_{FG})\ =\ X_{DS}$

Thus A_{FG} is dual soft semi condensed

Example 2.3.4

Let $U_1= \{\ h_1\ ,\ h_2\ \}\ U_2= \{\ c_1\ ,\ c_2\ \}\ E= \{\ e_1\ ,\ e_2\ \}$

$A_{1F1G1}\ = \{\ (\ e_1\ ,\{\ h_1\}\ ,\{\ c_1\}\ )\ ,\ (e_2\ ,\{\ h_2\},\{\ c_2\})\}$

$A_{2F2G2}\ = \{\ (\ e_1,\{\ h_2\},\{\ c_2\}),(e_2\ ,\{\ h_1\},\{\ c_1\})\}$

$T_E = \{\ \emptyset_{DS}\ ,X_{DS}\ ,A_{1F1G1}\ ,A_{2F2G2}\}$

Assume $A_{FG}= \{\ (\ e_1\ ,\{\ h_1\},U_2)\ ,(e_2\ ,\{\ h_2\},\{\ c_2\})\}$

$DS_{cl}\ (DS_i(A_{FG})) = DS_{cl}\ (A_{F1G1}\ )\ = A_{F1G1} \neq DS_{cl}\ (A_{FG})\ =\ X_{DS}$

Then A_{FG} is not dual soft semi condensed set and consequently

$A_{FG}$is not dual soft condensed set since $DS_{cl}\ (DS_i(A_{FG})) \neq DS_{cl}\ (A_{FG})$

Remark 2.4

1- Any dual soft closed set is dual soft super condensed set .

2- Any dual soft open set is dual soft semi condensed set.

Proof

1- Let B_{F2G2} be a dual soft closed set then $DS_{cl}(B_{F2G2})\ =\ B_{F2G2}$by proposition 1.2.35 part (2)

take dual soft interior for both side

$DS_i[DS_{cl}(B_{F2G2})] = DS_i\ (B_{F2G2})$ then B_{F2G2} is dual soft super condensed set.

2- Let A_{FG} be a dual soft open set then $DS_i\ (A_{FG}) = A_{FG}$ by proposition 1.2.13

take dual soft closure for both side

$DS_{cl}\ [DS_i(A_{FG})] = DS_{cl}\ (A_{FG})$ then A_{FG} is dual soft semi condensed set

Remark 2.5

A dual soft set A_{FG} is dual soft condensed set if and only if it is dual soft super condensed set and dual soft semi condensed set.

Proof

Directly by Definition 2.2

The question that arises is what is the relationship between the dual soft condensed sets and theirs complements that is will be answered in the following proposition

proposition 2.6

Let (X_{Ds}, T_E) be a dual soft topological space then the following statements are hold

1- The dual soft complement of a dual soft super condensed set is dual soft semi condensed set.
2- The dual soft complement of a dual soft semi condensed set is dual soft super condensed set.
3- The dual soft complement of dual soft condensed set is also dual soft condensed set.

Proof

1- Let A_{F1G1} be a dual soft super condensed set

Then we have $X_{Ds} - [DS_i(DS_{cl}(A_{F1G1}))] = X_{Ds} - [DS_i(A_{F1G1})]$

Considering the properties of dual soft complement we get

$X_{Ds} - [DS_i(DS_{cl}(A_{F1G1}))] = DS_{cl}[X_{Ds} - (DS_{cl}(A_{F1G1}))] = DS_{cl}[DS_i(X_{Ds} - A_{F1G1})]$

$X_{Ds} - [DS_i(A_{F1G1})] = DS_{cl}(X_{Ds} - A_{F1G1})$

Then $DS_{cl}[DS_i(X_{Ds} - A_{F1G1})] = DS_{cl}(X_{Ds} - A_{F1G1})$

So $X_{Ds} - A_{F1G1}$ is dual soft semi condensed set

2 - Let A_{F1G1} be a dual soft semi condensed set then we have

$X_{Ds} - [DS_{cl}(DS_i(A_{F1G1}))] = X_{Ds} - [DS_{cl}(A_{F1G1})]$

Considering the properties of dual soft complement

$X_{Ds} - [DS_{cl}(DS_i(A_{F1G1}))] = DS_i[X_{Ds} - (DS_i(A_{F1G1}))] = DS_i[DS_{cl}(X_{Ds} - (A_{F1G1}))]$

$X_{Ds} - [DS_{cl}(A_{F1G1})] = DS_i(X_{Ds} - A_{F1G1})$

using these equalities we have the following equality

$DS_i[DS_{cl}(X_{Ds} - A_{F1G1})] = DS_{cl}(X_{Ds} - A_{F1G1})$

This relation implies that $X_{Ds} - (A_{F1G1})$ is dual soft super condensed set

3-Let A_{FG} is dual soft condensed set then $X_{Ds} - (A_{FG})$ is dual soft semi condensed sets by part (1)

And if A_{FG} is dual soft condensed set then $X_{Ds} - A_{FG}$ is dual soft super condensed by part (2)

Connecting the both proposition we have proved that $X_{Ds} - A_{FG}$ is a dual soft condensed set.

There are some properties in addition the concept of dual soft condensed, super condensed and semi condensed.

Proposition 2.7

Let (X_{Ds}, T_E) be a dual soft topological space then the following statement are hold

1- A dual soft set A_{F1G1} is dual soft super condensed set if and only if $DS_i[DS_{cl}(A_{F1G1})] \subseteq_{DS} A_{F1G1}$

2- A dual soft set $B_{F2G2}$is dual soft semi condensed set if and only if $DS_{cl}[DS_i(B_{F2G2})] \supset_{DS} B_{F2G2}$

3- A dual soft set A_{F1G1} is dual soft condensed set if and only if

$DS_i[DS_{cl}(A_{F1G1})] \subseteq A_{F1G1} \subseteq_{DS} DS_{cl}[DS_i(A_{F1G1})]$

Proof

1 - Let A_{F1G1} be a dual soft super condensed set then we have $DS_i[DS_{cl}(A_{F1G1})] = DS_i(A_{F1G1}) \subseteq_{DS} A_{F1G1}$

Conversely $DS_i[DS_{cl}(A_{F1G1})] \subseteq_{DS} A_{F1G1}$ taking interior for both side

$DS_i[DS_{cl}(A_{F1G1})] \subseteq_{DS} DS_i(A_{F1G1})$

considering the self- evident relation

$DS_i[DS_{cl}(A_{F1G1})] \supset_{DS} DS_i(A_{F1G1})$

So $DS_i[DS_{cl}(A_{F1G1})] = DS_i(A_{F1G1})$

Therefor A_{F1G1} is dual soft super condensed set.

2- let B_{F2G2} is dual soft semi condensed set then we have

$DS_{cl}[DS_i(B_{F2G2})] = DS_{cl}(B_{F2G2}) \supseteq_{DS} B_{F2G2}$

Conversely let we assume $B_{F2G2} \subseteq_{DS} DS_{cl}(DS_i(B_{F2G2}))$

Taking dual soft closure for both side we have

$DS_{cl}\ (B_{F2G2}\ ) \subseteq_{DS}\ DS_{cl}[DS_i\ (B_{F2G2})]$

Considering the self - evident relation $DS_{cl}\ (DS_i\ (B_{F2G2}\ )) \subseteq_{DS}\ DS_{cl}\ (B_{F2G2})$

We obtain $DS_{cl}\ (B_{F2G2}) =\ DS_{cl}[DS_i\ (B_{F2G2})]$

Then $B_{F2G2}$is dual soft semi condensed set

3 - Directly by definition 2.2

Now we will provide a new definition

Definition 2.8

A dual soft set A_{FG} of X_{Ds} is called

1- Dual soft regular open set if $DS_i[DS_{cl}(A_{FG})] =\ A_{FG}$.

2- Dual soft regular closed set if $DS_{cl}[\ DS_i\ (A_{FG})] = A_{FG}$.

Example 2.9.1

Let U_1= { h_1 , h_2 } U_2= { c_1 , c_2 } E= { e_1 , e_2 }

A_{1F1G1} = {(e_1, { h_1}, { c_1}) , (e_2 , { h_2}, { c_2})}

A_{2F2G2} = { (e_1 , { h_2} ,{c_2}) , (e_2 , { h_1}, { c_1})}

T_E = { $\emptyset_{DS}$, X_{DS}, A_{1F1G1} , A_{2F2G2} }

Now $DS_i[DS_{cl}\ (A_{F1G1})] =\ A_{F1G1}$ then A_{F1G1} is dual soft regular open set

$DS_{cl}[DS_i\ (A_{F1G1})] = A_{F1G1}$ then A_{F1G1} is dual soft regular closed set.

Example 2.9.2

Let U_1= { h_1 , h_2 , h_3} U_2= { c_1 , c_2 } E= { e_1 , e_2 }

A_{1F1G1} = { (e_1 ,{ h_1} ,$\emptyset$) , (e_2, $\emptyset$, {c_1})}

$X_{DS} - A_{1F1G1}$ ={(e_1,{h_2, h_3}, U_2), (e_2, U_1, { c_2})}

A_{2F2G2} = { (e_1 ,{h_2},{c_2}) , (e_2 , { h_1}, { c_2})}

$X_{DS} - A_{2F2G2}$ = { (e_1 ,{h_1, h_3},{c_1}) , (e_2 , { h_2, h_3}, { c_1})}

B_{F3G3} ={(e_1 ,{h_1, h_2}, { c_2}), (e_2, {h_1} ,U_2)}

$X_{DS} - B_{F3G3}$={(e_1 ,{h_3 }, { c_1}), (e_2, {h_2, h_3} , ∅)}

T_E = { $\emptyset_{DS}$, X_{DS}, A_{1F1G1} , A_{2F2G2}, B_{F3G3} }

Assume A_{FG}= {(e_1, { h_1, h_2}, { c_1}), (e_2,{h_2, h_3} ,U_2)}

Now $DS_i[DS_{cl}(A_{FG})] = X_{DS} \neq A_{FG}$ then A_{FG} is not dual soft regular open set.

DS_{cl} $[DS_i$ $(A_{FG})]$ =$X_{DS} - A_{2F2G2} \neq A_{FG}$ then A_{FG} is not dual soft regular closed set.

Remark 2.10

The dual soft whole space X_{DS} and the dual soft empty set $\emptyset_{DS}$ are dual soft regular open and also dual soft regular closed

Proof

$DS_i[DS_{cl}(X_{DS})] = X_{DS}$

DS_i $[DS_{cl}$ $(\emptyset_{DS})]$ =$\emptyset_{DS}$

$DS_{cl}[DS_i$ $(X_{DS}$)] = X_{DS}

DS_{cl} $[DS_i$ $(\emptyset_{DS})]$= $\emptyset_{DS}$

therefor $\emptyset_{DS}$ and X_{DS} are dual soft open and dual soft closed regular sets

Remark 2.11

1- The dual soft complement of dual soft regular open set is dual soft regular closed set.
2- The dual soft complement of dual soft regular closed set is dual soft regular open set.
3- Every dual soft regular open set is dual soft open set.
4- Every dual soft regular closed set is dual soft closed set.

proof

1 - Let A_{FG} is dual soft regular open set then $DS_i[DS_{cl}\ (A_{FG})] = A_{FG}$

take dual soft complement for both side then

$X_{DS} - [DS_i\ (DS_{cl}\ (A_{FG}))] = X_{DS} - A_{FG}$

Then $DS_{cl}[X_{DS} - DS_{cl}\ (A_{FG})] = X_{DS} - A_{FG}$ by properties of dual soft complement

So $DS_{cl}\ [DS_i\ (X_{DS} - A_{FG})] = X_{DS} - A_{FG}$

Hence $X_{DS} - A_{FG}$ is dual soft regular closed set.

2 - Let $B_{FG}$is dual soft regular closed set then $DS_{cl}[DS_i\ (B_{FG})] = B_{FG}$

Take dual soft complement for both side

$X_{DS} - [DS_{cl}\ (DS_i\ (B_{FG}))] = X_{DS} - B_{FG}$

Then $DS_i[X_{DS} - DS_i(B_{FG})] = X_{DS} - B_{FG}$ by properties of dual soft complement

So $DS_i\ [DS_{cl}\ (X_{DS} - B_{FG})] = X_{DS} - B_{FG}$

Therefor $X_{DS} - B_{FG}$ is dual soft regular open set.

3-Let $A_{FG}$is dual soft regular open set then $DS_i[DS_{cl}\ (A_{FG})] = A_{FG}$

If and only if $DS_i[DS_i(DS_{cl}\ (A_{FG}))] = DS_i\ (A_{FG})$

If and only if $A_{FG} = DS_i[DS_{cl}\ (A_{FG})] = DS_i\ (A_{FG})$

Then $A_{FG}$is dual soft regular open set.

4-Let $B_{FG}$is dual soft regular closed set then $DS_{cl}[DS_i\ (B_{FG})] = B_{FG}$

If and only if $DS_{cl}[DS_{cl}(DS_i\ (B_{FG}))] = DS_{cl}(B_{FG})$

If and only if $B_{FG} = DS_{cl}[DS_i\ (B_{FG})] = DS_{cl}(B_{FG})$

Then $B_{FG}$is dual soft regular closed set.

Remark 2 .12

1- The dual soft intersection of two dual soft regular open set is dual soft regular open set.

2- The dual soft union of tow dual soft regular closed set is dual soft regular closed set.

Proof

1 - Let A_{FG} and B_{F1G1} be two dual soft regular open set then :

$DS_i\ (DS_{cl}\ (A_{FG}))$ = A_{FG} , $DS_i\ (DS_{cl}\ (B_{F1G1}))$= B_{F1G1}

$DS_{cl}(A_{FG} \cap_{DS} B_{F1G1}) \subseteq_{DS} DS_{cl}(A_{FG}) \cap_{DS} DS_{cl}\ (B_{F1G1})$by proposition 1.2.38 part(5)

$DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})] \subseteq_{DS} DS_i[DS_{cl}(A_{FG}) \cap_{DS} DS_{cl}\ (B_{F1G1})$

$=DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[DS_{cl}(B_{F1G1})]= A_{FG} \cap_{DS} B_{F1G1}$-----(1)

Also $A_{FG} \cap_{DS} B_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})$

Then $DS_i(A_{FG}) \cap_{DS} DS_i\ (B_{F1G1}) \subseteq DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})]$

But $DS_i(A_{FG}) \cap_{DS} DS_i\ (B_{F1G1})$= $DS_i[\ DS_i(DS_{cl}(A_{FG}))] \cap_{DS} DS_i[\ DS_i(DS_{cl}\ (B_{F1G1}))]= DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[DS_{cl}(B_{F1G1})] = A_{FG} \cap_{DS} B_{F1G1}$

Which imply that $A_{FG} \cap_{DS} B_{F1G1} \subseteq_{DS} DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})]$-----(2)

From (1) and (2) we have the result.

2 - Assume that A_{FG} and B_{F1G1}be dual soft regular closed set then

$DS_{cl}[DS_i\ (A_{FG})] = A_{FG}$

$DS_{cl}[DS_i(DS_{cl}\ (B_{F1G1}))] = B_{F1G1}$

Now by proposition 2.1 part (6)

$DS_{cl}\ (DS_i\ (DS_{cl}\ (A_{FG} \cup_{DS} B_{F1G1}))) = DS_{cl}\ (DS_i\ (DS_{cl}\ (A_{FG})))\cup_{DS} DS_{cl}(DS_i(DS_{cl}\ (B_{F1G1})))$

since A_{FG} , B_{F1G1} is dual soft regular closed set then by remark 2.11 part (3),(4)

$DS_{cl}\ (A_{FG}) = A_{FG}$, $DS_{cl}\ (B_{F1G1}) = B_{F1G1}$

and the union of two closed set is closed set then $(A_{FG} \cup_{DS} B_{F1G1})$ is closed set .Then $DS_{cl}\ (A_{FG} \cup_{DS} B_{F1G1}) = A_{FG} \cup_{DS} B_{F1G1}$

Hence $DS_{cl}[DS_i(DS_{cl}\ (A_{FG} \cup_{DS} B_{F1G1}))] = DS_{cl}[DS_i\ (A_{FG} \cup_{DS} B_{F1G1})]$=$DS_{cl}[DS_i\ (A_{FG})] \cup_{DS} DS_{cl}[DS_i\ (B_{F1G1})] = A_{FG} \cup_{DS} B_{F1G1}$

Therefore the union of two dual soft regular closed set is dual soft regular closed set .

From remark 2.10 and remark 2.12 we set that the collection of all dual soft regular open set building a base for dual soft topology of X_{Ds}

Remark 2.13

for any dual soft set A_{FG} of X_{Ds}

1- $DS_i\ [DS_{cl}\ (A_{FG})]$ is dual soft regular open set
2- $DS_{cl}\ [DS_i\ (A_{FG})]$ is dual soft regular closed set

proof

1 and 2 directly by proposition 2.1 part (3) and (4)

If we want to clarify some of the relationships between the dual soft regular open or closed sets and the dual soft condensed , super condensed , semi condensed sets we can use the following proposition

Proposition 2.14

For any dual soft set $A_{FG} \subseteq_{DS} X_{Ds}$The following relation hold

1- A dual soft set is dual soft regular open set if and only if the dual soft set is dual soft super condensed and dual soft open set.

2- A dual soft set is dual soft regular closed if and only if the dual soft set is dual soft semi condensed and dual soft closed set.

3- A dual soft set is dual soft regular open set if and only if the dual soft set is dual soft condensed and dual soft open set.

4- A dual soft set is dual soft regular closed set if and only if the dual soft set is dual soft condensed and dual soft closed set .

5- A dual soft set A_{FG} is dual soft condensed set if and only if there is a dual soft regular open set U_{F1G1} such that $U_{F1G1} \subseteq_{DS} A_{FG} \subseteq_{DS} DS_{cl}(U_{F1G1})$.

6- A dual soft set B_{F2G2} is dual soft condensed set if and only if there is a dual soft regular closed set V_{F3G3} such that $DS_i(V_{F3G3}) \subseteq_{DS} B_{F2G2} \subseteq_{DS} V_{F3G3}$

proof

1 - Let A_{FG} be a dual soft regular open set then

$DS_i[DS_{cl}(A_{FG})] = A_{FG}$ ---------- (1)

So $DS_i(A_{FG}) = DS_i[DS_i(DS_{cl}(A_{FG}))] = DS_i[DS_{cl}(A_{FG})] = A_{FG}$

this implies that A_{FG} is dual soft open set.

From equation (1) we get

$DS_i[DS_{cl}(A_{FG})] = DS_i(A_{FG})$

Thus A_{FG} is dual soft super condensed set .

Conversely let A_{FG} be dual soft super condensed and open set then we have

$DS_i[DS_{cl}(A_{FG})] = DS_i(A_{FG}) = A_{FG}$. Thence A_{FG} is dual soft regular open set.

2- Let A_{FG} be a dual soft regular closed set

Then $DS_{cl}[DS_i(A_{FG})] = A_{FG}$ ---------- (2)

Using this relation we get $DS_{cl}(A_{FG}) = DS_{cl}[DS_{cl}(DS_i(A_{FG}))] = DS_{cl}[DS_i(A_{FG})] = A_{FG}$

This implies that A_{FG} is dual soft closed set

From equation (2) we get $DS_{cl}[DS_i(A_{FG})] = DS_{cl}(A_{FG})$

Then A_{FG} is dual soft semi condensed set

Conversely let A_{FG} be a dual soft semi condensed and closed set then we have

$DS_{cl}[DS_i(A_{FG})] = DS_{cl}(A_{FG}) = A_{FG}$

Thus A_{FG} is dual soft regular closed set

(3) and (4) directly by Remark 2.4

5 - Let A_{FG} is dual soft condensed set if we take $U_{F1G1} = DS_i[DS_{cl}(A_{FG})]$ is dual soft open set by remark 2.13 then $U_{F1G1} = DS_i[DS_{cl}(A_{FG})] = DS_i(A_{FG}) \subseteq_{DS} A_{FG}$

since A_{FG} is dual soft condensed set then $U_{F1G1} \subseteq_{DS} A_{FG}$

And $DS_{cl}(U_{F1G1}) = DS_{cl}(DS_i(DS_{cl}(A_{FG}))) \supseteq_{DS} DS_i[DS_{cl}(A_{FG})] DS_{cl}(A_{FG}) \supseteq_{DS} A_{FG}$

So U_{F1G1} satisfies $U_{F1G1} \subseteq_{DS} A_{FG} \subseteq_{DS} DS_{cl}(U_{F1G1})$

If we assume that there is a dual soft regular open set U_{F1G1} which is satisfies $U_{F1G1} \subseteq_{DS} A_{FG} \subseteq_{DS} DS_{cl}(U_{F1G1})$ then

$DS_i[DS_{cl}(A_{FG})] \subseteq DS_i[DS_{cl}(DS_{cl}(U_{F1G1}))] = DS_i[DS_{cl}(U_{F1G1})] = U_{F1G1}$

$DS_i[DS_{cl}(A_{FG})] \supseteq_{DS} DS_i[DS_{cl} U_{F1G1})] = U_{F1G1}$

So $U_{F1G1} = DS_i[DS_{cl}(A_{FG})]$. On other hand

$DS_{cl}[DS_i(A_{FG})] \subseteq_{DS} DS_{cl}[DS_i(DS_{cl}(U_{F1G1}))] = DS_{cl}(U_{F1G1})$

$DS_{cl}[DS_i(A_{FG})] \supseteq_{DS} DS_{cl}[DS_i(U_{F1G1})] = DS_{cl}(U_{F1G1})$

So $DS_{cl}(U_{F1G1}) = DS_{cl}[DS_i(A_{FG})]$

Then we have $DS_i[DS_{cl}(A_{FG})] \subseteq_{DS} A_{FG} \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

So $DS_i[DS_i(DS_{cl}(A_{FG}))] = DS_i[DS_{cl}(A_{FG})] \subseteq_{DS} DS_i(A_{FG})$

But $DS_i(A_{FG}) \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

Thus $DS_i(A_{FG}) = DS_{cl}[DS_i(A_{FG})]$

Also $A_{FG} \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$ so $DS_{cl}(A_{FG}) \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

But $DS_{cl}[DS_i(A_{FG})] \subseteq_{DS} DS_{cl}(A_{FG})$

Therefore A_{FG} is dual soft condensed set.

6 - Assume B_{F2G2} is dual soft condensed set and let we take

$V_{F3G3} = DS_{cl}[DS_i(B_{F2G2})]$ then V_{F3G3} is dual soft regular closed set by remark 2.13

Because $V_{F3G3} = DS_{cl}[DS_i(B_{F2G2})] = DS_{cl}(B_{F2G2}) \supseteq_{DS} B_{F2G2}$

And $DS_i(V_{F3G3}) = DS_i[DS_{cl}(DS_i(B_{F2G2}))] \subseteq_{DS} DS_i[DS_{cl}(B_{F2G2})] = DS_i(B_{F2G2}) \subseteq_{DS} B_{F2G2}$

So V_{F3G3} satisfies $DS_i(V_{F3G3}) \subseteq_{DS} B_{F2G2} \subseteq_{DS} V_{F3G3}$

If we assume that there is a dual soft regular closed set V_{F3G3} which satisfies $DS_i(V_{F3G3}) \subseteq_{DS} B_{F2G2} \subseteq_{DS} V_{F3G3}$

Then $V_{F3G3} = DS_{cl}[DS_i(V_{F3G3})] = DS_{cl}[DS_i(DS_i(V_{F3G3}))] \subseteq_{DS} DS_{cl}[DS_i(B_{F2G2})]$

$V_{F3G3} = DS_{cl}[DS_i(V_{F3G3})] \supseteq_{DS} DS_{cl}[DS_i(B_{F2G2})]$

So $V_{F3G3} = DS_{cl}[DS_i(B_{F2G2})]$

On other hand

$DS_i[DS_{cl}(B_{F2G2})] \supseteq DS_i[DS_{cl}(DS_i(V_{F3G3}))] = DS_i(V_{F3G3}) = DS_i[DS_{cl}(B_{F2G2})] \subseteq_{DS}$

$DS_i\ [DS_{cl}\ (V_{F3G3})] = DS_i\ (V_{F3G3}\ )$

So $DS_i\ (V_{F3G3}) = DS_i[DS_{cl}\ (B_{F2G2})]$

Then we have $DS_{cl}[DS_i\ (B_{F2G2})] \supseteq_{DS} B_{F2G2} \supseteq_{DS} DS_i[DS_{cl}\ (B_{F2G2})]$

So B_{F2G2} is dual soft condensed

$DS_{cl}[DS_i\ (B_{F2G2})] = DS_{cl}\ (B_{F2G2}\ )$

$DS_i[DS_{cl}\ (B_{F2G2})] = DS_i\ (B_{F2G2}\ )$.

The question that comes to our mind is how to link by dual soft condensed sets with the dual soft border which we will try to explain through the following proposition

Proposition 2.15

A dual soft set A_{FG} is dual soft condensed set if and only if the dual soft boundary of A_{FG} coincides with

$DS_{cl}\ [DS_i\ (A_{FG})] - DS_i\ [DS_{cl}\ (A_{FG})] = DS_{cl}[DS_i\ (A_{FG})] \cap_{DS} DS_{cl}\ [DS_i\ (X_{Ds} -\ (A_{FG}))]$

Proof

We assume that A_{FG} is a dual soft condensed set

by proposition 2.6 part(3) $X_{Ds} - A_{FG}$ is dual soft condensed set

So we have $DS_{cl}(DS_i\ (A_{FG})) = DS_{cl}(A_{FG})$

$DS_{cl}[DS_i\ (X_{Ds} - A_{FG})] = DS_{cl}(X_{Ds} - A_{FG})$

Therefor $DS_{cl}[DS_i(A_{FG})] \cap_{DS} DS_{cl}[DS_i(X_{Ds} - A_{FG})]$

$= DS_{cl}\ (A_{FG}) \cap_{DS} DS_{cl}\ (X_{Ds} - A_{FG}) = DS_{fr}(A_{FG}\ )$

Now $DS_{fr}\ (A_{FG}\ ) = DS_{cl}[DS_i(A_{FG}\ )] \cap_{DS} DS_{cl}[\ DS_i\ (\ X_{Ds} - A_{FG})]$

Then by proposition 1.2.48 $DS_{cl}(A_{FG}) = DS_i(A_{FG}) \cup_{DS} DS_{fr}\ (A_{FG}\ )$

$= DS_i\ (A_{FG}) \cup_{DS} \{\ DS_{cl}[\ DS_i\ (A_{FG})] \cap_{DS} DS_{cl}[DS_i\ (X_{Ds} - A_{FG})]\}$

$\subseteq_{DS} DS_i\ (A_{FG}) \cup_{DS} DS_{cl}[DS_i\ (A_{FG})] = DS_{cl}[DS_i\ (A_{FG})]$

Considering the evident relation $DS_{cl}(A_{FG}) \supseteq_{DS} DS_{cl}[\ DS_i\ (A_{FG})]$

$DS_{cl}\ (A_{FG}) = DS_{cl}\ [DS_i\ (A_{FG})]$ ---------- (1)

The same relation holds for $X_{Ds} - A_{FG}$because the dual soft boundary of $X_{Ds} - A_{FG}$coincides with that of A_{FG} so we have

$DS_{cl}\ (X_{Ds} - A_{FG}) = DS_{cl}[DS_i\ (X_{Ds} - A_{FG})]$

$DS_{cl}\ (X_{Ds} - A_{FG}\ ) = X_{Ds} - [DS_i\ (A_{FG})]$

$DS_{cl}\ [DS_i\ (X_{Ds} - A_{FG})] = DS_{cl}[X_{Ds} - DS_{cl}\ (A_{FG})] = X_{Ds} - [DS_i\ (DS_{cl}\ (A_{FG}))]$

Thus $X_{Ds} - [DS_i\ (A_{FG})] = X_{Ds} - [DS_{cl}\ (DS_i\ (A_{FG}))]$

Taking dual soft complement for both side we obtain

$DS_i\ (A_{FG}) = DS_i\ (DS_{cl}\ (A_{FG}))$

This equation and equation (1) imply that the dual soft set A_{FG} is dual soft condensed set.

Now we will give a mathematical concept that has a direct impact on the concept that we presented in definition 2.8 and the basic role is the dual soft boundary points

Definition 2.16

The dual soft interior of the dual soft boundary of the dual soft set is called dual soft border of the dual soft set.

Example 2.17

Let $U_1 = \{\ h_1\ ,\ h_2\ \}\ U_2 = \{\ c_1\ ,\ c_2\ \}\quad E = \{\ e_1\ ,\ e_2\ \}$

$A_{1F1G1} = \{(\ e_1\ ,\{\ h_1\}\ ,\emptyset\ )\ ,\ (e_2\ ,\ \{h_2\},\{c_2\})\}$

$A_{2F2G2} = \{(\ e_1\ \ ,\{h_2\},\emptyset)\ ,\ (e_2\ ,\ \emptyset,\ \{\ c_1\})\}$

B_{F3G3} ={(e$_1$, U_1, ∅)(e_2, {h_2} ,U_2) }

T_E = { $\emptyset_{DS}$,X_{DS}, A_{1F1G1} , A_{2F2G2}, B_{F3G3} }

Assume A_{FG}= {(e$_1$, { h_1}, U_2), (e_2, {h_1},∅)}

$X_{DS} - A_{1F1G1}$ ={(e$_1$, { h_2}, U_2), (e_2, {h_1},{ c_1})}

$X_{DS} - A_{2F2G2}$ ={(e$_1$, { h_1}, U_2), (e_2, U_1,{ c_2})}

$X_{DS} - B_{F3G3}$ ={(e$_1$, ∅, U_2), (e_2, {h_1},∅)}

$X_{DS} - A_{FG}$ ={(e$_1$, { h_2}, ∅), (e_2, {h_2},{ U_2})}

$DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) \cap_{DS} DS_{cl}(X_{DS} - A_{FG})$

$=(X_{DS} - A_{2F2G2}) \cap_{DS} (X_{DS} - A_{1F1G1}) = X_{DS} - X_{DS} = \emptyset_{DS}$

$DS_i[DS_{fr}(\emptyset_{DS})] = DS_i(\emptyset_{DS}) = \emptyset_{DS}$.

Remark 2.18

The dual soft border of any dual soft set A_{FG} is denoted by DS_b $(A_{FG})$is expressed by DS_i $(DS_{fr}$ $(A_{FG}$)).

Proposition 2.19

The dual soft border of any dual soft set A_{FG} is dual soft regular open and expressed by

DS_b (A_{FG}) = $DS_i(DS_{cl}(A_{FG}))$− $DS_{cl}(DS_i$ $(A_{FG}))$ = $DS_i(DS_{cl}(A_{FG}))\cap_{DS}$ $DS_i(DS_{cl}$ $(X_{Ds}$ −$(A_{FG})))$

Proof

Considering to the fact that the dual soft boundary of $A_{FG}$is dual soft closed set then we have

DS_b (A_{FG}) = $DS_i[DS_{fr}$ $(A_{FG})]$ = $DS_i[DS_{cl}(DS_{fr}$ $(A_{FG}))]$

By hypotheses $DS_i[DS_{cl}$ $(A_{FG})]$ is dual soft regular open set DS_b (A_{FG})

We conclude that DS_b (A_{FG}) is dual soft regular open set.

Now we will prove the second part of the theorem

$DS_i[DS_{cl}\ (A_{FG})]- DS_{cl}[DS_i(A_{FG})]= DS_i[DS_{cl}\ (A_{FG})] \cap_{\mathrm{DS}} DS_i[DS_{cl}(X_{Ds} - A_{FG})]$

$= DS_i[DS_{cl}\ (A_{FG}) \cap_{\mathrm{DS}} DS_{cl}(X_{Ds} - A_{FG})]$ by proposition1.2.15 part(5)

$= DS_i(DS_{fr}\ (A_{FG}))$ by definition 1.2.21

$= DS_b\ (A_{FG})$ by definition 2.16

proposition 2.20

For any dual soft set $A_{FG} \subseteq X_{Ds}$ then

1- A dual soft set is dual soft super condensed set if and only if

i- The dual soft interior of the dual soft set is dual soft regular open set and

ii- The dual soft border of the dual soft set is dual soft empty

2- A dual soft set is dual soft semi condensed if and only if

i- The dual soft closure of the dual soft set is dual soft regular closed and

ii- The dual soft border of the dual soft set is dual soft empty

3- A dual soft set is dual soft condensed if and only if

i- The dual soft interior of the dual soft set is dual soft regular open

ii- The dual soft closure of the dual soft set is dual soft regular closed set

iii- The dual soft border of the dual soft set is dual soft empty

proof

1 - Let A_{FG} is dual soft super condensed set then $DS_i[DS_{cl}(A_{FG})]=DS_i(A_{FG})$ -----(1) So we get

$DS_i[(DS_{cl}(DS_i\ (DS_{cl}\ (A_{FG}))))]\ = DS_i[(DS_{cl}DS_i(A_{FG}))]$

So the left side of the last equality become

$DS_i[DS_{cl}(DS_i\ (DS_{cl}\ (A_{FG})))]$ = $DS_i[(DS_{cl}A_{FG})]$ by proposition 2.1 part 3

= $DS_i\ (A_{FG})$ by equation (1)

Subsequently $DS_i\ (A_{FG})$ = $DS_i\ [DS_{cl}\ (DS_i\ (A_{FG}))]$

This implies that $DS_i\ (A_{FG})$ is dual soft regular open set using equation (1)

we get $DS_i\ [DS_{cl}(A_{FG})]$ =$DS_i\ (A_{FG}) \subseteq\ DS_{cl}[DS_i\ (A_{FG})]$

$DS_b\ (A_{FG})$ = $DS_i\ [DS_{cl}\ (A_{FG})] - DS_{cl}\ [DS_i\ (A_{FG})]$ = $\emptyset_{DS}$

Conversely suppose that A_{FG} satisfies that

$DS_i[DS_{cl}\ (DS_i\ (A_{FG})]$ = $DS_i\ (A_{FG})$ --------- (2)

And $DS_b\ (A_{FG}) = \emptyset_{DS}$

$DS_i[DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_{cl}[DS_i\ (A_{FG})]$

Taking dual soft interior for both side of the second relation we get

$DS_i\ [DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_i(DS_{cl}(DS_i\ (A_{FG})))$

Combining this relation with (2)we get$DS_i\ (A_{FG}) \supseteq_{DS} DS_i\ [DS_{cl}\ (A_{FG})]$

Considering the self - evident relation $DS_i\ (A_{FG}\ ) \subseteq_{DS}\ DS_i[DS_{cl}(A_{FG})]$

we get $DS_i\ (A_{FG}) = DS_i[DS_{cl}\ (A_{FG})]$

This relation implies that $A_{FG}$is dual soft super condensed set

2 - Let A_{FG} is dual soft semi condensed set

Then $DS_{cl}[DS_i\ (A_{FG})] = DS_{cl}\ (A_{FG})$ ------- (1)

So we get $DS_{cl}[DS_i(DS_{cl}(DS_i(A_{FG})))] = DS_{cl}[DS_i(DS_{cl}\ (A_{FG}))]$

The left side of above equality become

$DS_{cl}[DS_i(DS_{cl}(DS_i(A_{FG})))]$=$DS_{cl}[DS_i(A_{FG})]$by proposition 2.1 part (4)

= $DS_{cl}\ (A_{FG}\ )$ by (1)

Then we get $DS_{cl}\ (A_{FG}\ ) = DS_{cl}[DS_i\ (DS_{cl}\ (A_{FG}))]$

This implies that $DS_{cl}\ (A_{FG}\ )$ is dual soft regular closed set

Using (1) we get $DS_{cl}[DS_i\ (A_{FG})] = DS_{cl}(A_{FG}) \supseteq DS_i[DS_{cl}(A_{FG})]$

Then we get

$DS_b\ (A_{FG}) = DS_i[DS_{cl}\ (A_{FG})] - DS_{cl}\ [DS_i\ (A_{FG})] = \emptyset_{DS}$

Next we proved the sufficiency suppose that A_{FG} satisfies The following conditions $DS_{cl}(DS_i\ (DS_{cl}\ (A_{FG}))) = DS_{cl}(A_{FG})$ ------- (2)

$DS_{cl}[DS_i\ (A_{FG})] \supset_{DS} DS_i[DS_{cl}(A_{FG})]$

And $DS_b\ (A_{FG}) = \emptyset_{DS}$

Taking dual soft closure for the second relation we get

$DS_{cl}[DS_i(DS_{cl}(A_{FG}))] \subseteq_{DS} DS_{cl}[DS_i\ (A_{FG})]$

Combining This relation and relation 2 we get $DS_{cl}\ (A_{FG}) \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

Considering The self - evident relation $DS_{cl}\ (A_{FG}) \supseteq_{DS} DS_{cl}[DS_i\ (A_{FG})]$

we get $DS_{cl}\ (A_{FG}) = DS_{cl}[DS_i\ (A_{FG})]$

Hence A_{FG} is dual soft semi condensed set

3- Let A_{FG} is dual soft condensed set Then $DS_i[DS_{cl}\ (A_{FG})] = DS_i\ (A_{FG})$ ------- (1)

$DS_{cl}\ (DS_i\ (A_{FG})) = DS_{cl}\ (A_{FG})$ --------- (2)

$DS_b\ (A_{FG}) = DS_i\ [DS_{cl}\ (A_{FG})] \cap_{DS} DS_i\ [DS_{cl}\ (X_{DS-}\ A_{FG})]$

And since the dual soft complement of the dual soft condensed set is dual soft condensed set Then $DS_b\ (A_{FG}) = DS_i\ (A_{FG}) \cap_{DS} DS_i(\ X_{DS-}A_{FG})$

$= DS_i\ (A_{FG}) \cap_{DS} DS_e\ (A_{FG}) = \emptyset_{DS}$ ----------(3)

from part (1) and (2) and equation (3) we proved The efficient condition

to prove the sufficient condition let (1) and (2) and (3) conditions are satisfies.

So by definition 2 .2 A_{FG} is dual soft condensed set.

Proposition 2.21

$DS_b[DS_b\ (A_{FG}\ )]\ =\ \emptyset_{DS}$

Proof

By using proposition 2.19 $DS_b\ (A_{FG}\ )$ is dual soft regular open and by using proposition 2.14 part 1 $DS_b\ (A_{FG}\ )$ is dual soft super condensed set

$DS_i\ [DS_{cl}\ (DS_b\ (A_{FG}))] = DS_i[DS_b\ (A_{FG})] \subset_{DS} DS_{cl}(DS_i\ (A_{FG}\ ))$ ------(1)

Again using proposition 2.19 we set

$DS_b[DS_b\ (A_{FG})] = DS_i[DS_{cl}\ (DS_b\ (A_{FG}))]\ /\ DS_{cl}[DS_i(DS_b\ (A_{FG}))]$

$=DS_i[DS_{cl}(DS_b\ \ (A_{FG}))]\ \cap_{DS} X_{DS} -\ \ DS_{cl}[DS_i(DS_b\ \ (A_{FG}))] \subseteq_{DS}$ $DS_{cl}[DS_i(DS_b\ (A_{FG}))] \cap_{DS} X_{DS} - DS_{cl}[DS_i\ (DS_b\ (A_{FG}))]\ = \emptyset_{DS}$

Therefor $DS_b[DS_b(A_{FG})] = \emptyset_{DS}$.

Through what has been studied we can classify the dual soft sets by this definition to the following classification

Definition 2.22

Any dual soft set of X_{DS} can be classified into one of the following three types

1 -$Su(X_{DS})\ \{A_{FG} \subseteq_{DS} X_{Ds}: DS_i[DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_{cl}[DS_i\ (A_{FG})]\}$

2 - $Se(X_{DS})\{B_{FG} \subseteq_{DS} X_{Ds} : DS_i[DS_{cl}\ (B_{FG})] \supsetneq_{DS} DS_{cl}[DS_i(B_{FG})]\}$

3 - $Co(X_{DS})\ \{C_{FG} \subseteq_{DS} X_{Ds}\ : DS_i[\ DS_{cl}\ (C_{FG}\ )]\ , DS_{cl}[DS_i(\ C_{FG}\ )]$ are non-comparable}

Example 2. 23.1

Let U_1= { h_1, h_2, h_3} , U_2= { c_1, c_2, c_3 } , E= { e_1, e_2, e_3}

A_{1F1G1}= { (e_1 , { h_1 } , { c_1}) , (e_2, { h_3} ,{ c_3})}

A_{2F2G2}= { (e_1 , ∅ , { c_3}) , (e_2 , { h_2 }, ∅)}

A_{3F3G3}= { (e_1 , { h_1 } , { c_1 , c_3 }) , (e_2 , {h_2 , h_3} , { c_3}) }

T_E= {$\emptyset_{DS}$, X_{Ds}, A_{1F1G1} , A_{2F2G2} , A_{3F3G3} }

$X_{Ds} - A_{1F1G1}$ = {(e_1 ,{h_2 , h_3} ,{c_2 , c_3}) , (e_2 ,{ h_1 , h_2} ,{c_1 , c_2})}

$X_{Ds} -$ A_{2F2G2} = { (e_1 , U_1 , { c_1 , c_2 }) , (e_2 ,{ h_1 , h_3 }, U_2) }

$X_{Ds} -$ A_{3F3G3} = { (e_1 , { h_2 , h_3 } , { c_2 }) , (e_2 , { h_1 },{ c_1 , c_2})}

Assume A_{FG}= {(e_1, { h_1 , h_2} ,{ c_1 , c_2}) , (e_2, { h_1 , h_3} ,{ c_2 , c_3})}

Now $DS_{cl}[DS_i\ (A_{FG})]$ =$DS_{cl}\ (A_{1F1G1})$ = $X_{Ds} -$ A_{2F2G2}

$DS_i[DS_{cl}(A_{FG})]$ =$DS_i\ (X_{Ds} - A_{2F2G2})$ = $A_{1F1G1}\ \subseteq_{DS} X_{Ds}\ - A_{2F2G2}$

We see $DS_i(DS_{cl}\ (A_{FG})) \subseteq_{DS} DS_{cl}\ (DS_i\ (A_{FG}))$

Therefor A_{FG}belong to $Su(X_{DS})$.

Example 2.23.2

Let U_1= { h_1, h_2} , U_2= { c_1, c_2 } , E= { e_1, e_2}

A_{1F1G1}= { (e_1 , { h_1} , { c_1}) , (e_2, {h_2} ,{c_2})}

A_{2F2G2}= { (e_1 ,{h_2} , { c_2}) , (e_2 , {h_1} , { c_1})}

T_E= {$\emptyset_{DS}$, X_{Ds}, A_{1F1G1} , A_{2F2G2} }

Assume A_{FG}= {(e_1, { h_1} , U_2) , (e_2, { h_2} ,{ c_2})}

Now$DS_i[DS_{cl}\ (A_{FG})]$ =$DS_i\ (X_{Ds})$ =X_{Ds}

$DS_{cl}[DS_i(A_{FG})]$ =$DS_{cl}\ (A_{1F1G1})$ = $A_{1F1G1} \subseteq_{DS} X_{Ds}$

We see $DS_{cl}[DS_i\ (A_{FG})]\ \subseteq_{DS} DS_i[DS_{cl}(A_{FG})]$

Therefor A_{FG}belong to $Se(X_{DS})$.

Example 2.23.3

Let U_1= { h_1, h_2} , U_2= { c_1, c_2 } , E= { e_1, e_2}

A_{1F1G1}= { (e_1 , { h_1} , {c_1}) , (e_2, { h_2} , { c_2})}

A_{2F2G2}= { (e_1 , { h_2}, { c_2 }) , (e_2 , { h_1} , {c_1})}

A_{3F3G3}= { (e_1 , ∅,∅) , (e_2 ,∅ , { c_1})}

A_{4F4G4}= { (e_1 , { h_1 } , { c_1}) , (e_2 , {h_2} , U_2) }

T_E= {$\emptyset_{DS}$, X_{Ds}, A_{1F1G1} , A_{2F2G2} , A_{3F3G3} , A_{4F4G4}}

$X_{Ds} - A_{1F1G1}$ = { (e_1 , { h_2}, { c_2 }) , (e_2 , { h_1} , {c_1})}

$X_{Ds} -$ A_{2F2G2} = { (e_1 , { h_1} , {c_1}) , (e_2, { h_2} , { c_2})}

$X_{Ds} -$ A_{3F3G3} = { (e_1 , U_1, U_2) , (e_2 , U_1,{ c_2})}

$X_{Ds} -$ A_{4F4G4} = { (e_1 , { h_2 } , { c_2 }) , (e_2 ,{ h_1},∅)}

Assume A_{FG}= {(e_1, { h_1 } , U_2) , (e_2, U_1,{ c_2})}

Now $DS_{cl}[DS_i\ (A_{FG})]$ =$DS_{cl}\ (A_{1F1G1})$ = A_{1F1G1}

$DS_i[DS_{cl}\ (A_{FG})]$ =$DS_i\ (X_{Ds} - A_{3F3G3})$ = A_{1F1G1}

We see $DS_i[DS_{cl}(A_{FG})] \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$ therefor A_{FG}belong to $Su(X_{DS})$

If we compare the previous definition with what we learned about the concept of dual soft condensed, super condensed and semi condensed sets we will find that

Remark 2.24

1- The dual soft set belong to the $Su(X_{DS})$ if and only if the dual soft border of the dual soft set is dual soft empty
2- A dual soft super condensed set and a dual soft semi condensed set is belong to $Su(X_{DS})$

proof

1-Let (A_{FG}) belong to $Su(X_{DS})$ then$DS_i\ (DS_{cl}\ (A_{FG}))\subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$ ---------- (1)

$DS_b(A_{FG}) = DS_i[DS_{cl}\ (A_{FG})]\cap_{DS} DS_i[DS_{cl}(X_{DS} - A_{FG})] = DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[X_{DS} - DS_i(A_{FG})] = DS_i\ (DS_{cl}\ (A_{FG}))\cap_{DS} X_{DS} - DS_{cl}\ [DS_i\ (A_{FG})] = \emptyset_{DS}$

because $DS_i(DS_{cl}\ (A_{FG}))\ \subseteq_{DS}\ DS_{cl}\ (DS_i\ (A_{FG}))$

Conversely Let $DS_b\ (A_{FG}) = \emptyset_{DS}$

Since $DS_b\ (A_{FG}) = DS_i\ (DS_{cl}\ (A_{FG})) - DS_{cl}(DS_i\ (A_{FG}))$

Then $DS_i\ (DS_{cl}\ (A_{FG}\ )) \subseteq_{DS} DS_{cl}(DS_i\ (A_{FG}))$

Hence A_{FG} belong to the $Su(X_{DS})$

2- If A_{FG} is a dual soft super condensed set

Then $DS_i(DS_{cl}\ (A_{FG})) = DS_i\ (A_{FG})$

So $DS_i\ (A_{FG}\ ) \subseteq_{DS}\ DS_{cl}\ (DS_i\ (A_{FG}\ ))$

Then A_{FG} belong to $Su(X_{DS})$ and if A_{FG} be a dual soft semi condensed set

So $DS_{cl}(DS_i\ (A_{FG})) = DS_{cl}\ (A_{FG}\ )$

Thus $DS_i\ (DS_{cl}\ (A_{FG})\ ) \subseteq_{DS}\ DS_{cl}\ (A_{FG}\ )$

Therefore A_{FG} belong to the $Su(X_{DS})$

Again the question arises : Can we classify the complements of these types of dual soft sets as the original sets and that is what will be do in the following proposition

Proposition 2.25

For any dual soft set A_{FG}belong to $Su(X_{DS})$ or $Se(X_{DS})$ or $Co(X_{DS})$ then $X_{DS} - A_{FG}$belong to $Su(X_{DS})$ or $Se(X_{DS})$or $Co(X_{DS})$ respectively

proof

If we assume that A_{FG} belong to $Su(X_{DS})$

Then $DS_i[DS_{cl}(A_{FG})] \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

Taking dual soft complement for both side we obtain

$X_{Ds} - [DS_i\ (DS_{cl}(A_{FG}))] \supseteq_{DS} X_{Ds} - [DS_{cl}\ (DS_i\ (A_{FG})]$

Using The property of the dual soft complement we get

$DS_i[DS_{cl}(X_{Ds} - A_{FG})] \subseteq_{DS} DS_{cl}[DS_i(X_{Ds} - A_{FG})]$

Hence $X_{Ds} - A_{FG}$ belong to $Su(X_{DS})$

If we assume that B_{F1G1} belong to $Se(X_{DS})$

Then $DS_{cl}[DS_i(B_{F1G1})] \subseteq_{DS} DS_i(DS_{cl}(B_{F1G1})$

Taking dual soft complement for both side we obtain

$X_{Ds} - [DS_i(DS_{cl}(B_{F1G1}))] \subseteq_{DS} X_{Ds} - [DS_{cl}(DS_i(B_{F1G1}))]$

Using the property of the dual soft complement we get

$DS_{cl}[DS_i(X_{Ds} - B_{F1G1})] \subseteq_{DS} DS_i[DS_{cl}\ (X_{Ds} - B_{F1G1})]$

Hence $X_{Ds} - B_{F1G1}$ belong to $Se(X_{DS})$

Next we prove about $Co(X_{DS})$ in this case we must say that $DS_i[DS_{cl}(C_{F2G2})]$ and $DS_{cl}[DS_i(C_{F2G2})]$ are non – comparable

Then $X_{Ds} - [DS_i(DS_{cl}(C_{F2G2}))]$ and $X_{Ds} - [DS_{cl}(DS_i(C_{F2G2}))]$ are also non comparable .

We use reduction absurdum for proving this proposition

Let $DS_i[DS_{cl}(X_{Ds} - C_{F2G2})]$ and $DS_{cl}[DS_i\ (X_{Ds} - C_{F2G2})]$be comparable .

Then $X_{Ds} - [X_{Ds} - (C_{F2G2})] = C_{F2G2}$ must belong to $Su(X_{DS})$ or $Se(X_{DS})$

By proved part of this theorem this is contradiction .

Proposition 2.26

The dual soft set A_{FG} belong to the $Su(X_{DS})$ if

1- $DS_i[\ DS_{cl}\ (A_{FG})] = DS_i[DS_{cl}(DS_i\ (A_{FG}))]$

2- $DS_{cl}[DS_i\ (A_{FG}\ )]\ = DS_{cl}[DS_i\ (\ DS_{cl}\ (A_{FG}))]$

On other hand if one of these equalities holds then A_{FG} belong to $Su(X_{DS})$

proof

1- suppose that A_{FG} belong to the $Su(X_{DS})$ then the following relation holds

$DS_i[DS_{cl}(A_{FG})] \subseteq_{DS} DS_{cl}[DS_i\ (A_{FG})]$

Taking dual soft interior for both side we get $DS_i[DS_{cl}(A_{FG})] \subseteq_{DS} DS_i[DS_{cl}(DS_i(A_{FG}))]$

Considering the self- evident relation

$DS_i[DS_{cl}(A_{FG})] \supseteq_{DS} DS_i[DS_{cl}(DS_i(A_{FG}))]$

we get $DS_i\ (DS_{cl}(\ A_{FG})) = DS_i\ (DS_{cl}(\ DS_i\ (A_{FG})))$

Conversely assume $DS_i(\ DS_{cl}\ (A_{FG})) = DS_i(DS_{cl}(\ DS_i\ \ (A_{FG})))$

Clearly $DS_{cl}[DS_i(A_{FG})] \supseteq_{DS} DS_i[DS_{cl}(DS_i(A_{FG}))] = DS_i[DS_{cl}\ (A_{FG}))$

This implies that A_{FG} belong to $Su(X_{DS})$.

2 - Let A_{FG} belong to the $Su(X_{DS})$

Then $DS_i\ [DS_{cl}\ (A_{FG})] \subseteq_{DS} DS_{cl}[DS_i(A_{FG})]$

Taking dual soft closure for both side we get

$DS_{cl}[DS_i\ (DS_{cl}\ (A_{FG}))]\ \subseteq_{DS} DS_{cl}[DS_i\ (A_{FG})]$

Considering the evident relation

$DS_{cl}[DS_i(A_{FG})] \subseteq_{DS} DS_{cl}[DS_i(DS_{cl}(A_{FG}))]$

We get $DS_{cl}[DS_i(A_{FG})] = DS_{cl}[DS_i(DS_{cl}(A_{FG}))]$

Conversely let $DS_{cl}[DS_i(A_{FG})] = DS_{cl}[DS_i(DS_{cl}(A_{FG}))]$

And clearly $DS_{cl}[DS_i\ (A_{FG})] = DS_{cl}[DS_i\ (DS_{cl}\ (A_{FG}))] \supseteq_{DS} DS_i[DS_{cl}\ (A_{FG})]$.Then A_{FG} belong to $Su(X_{DS})$.

Proposition2. 27

Any dual soft set A_{FG} , B_{F1G1} of $Su(X_{DS})$ satisfy the following relation

1- $DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[DS_{cl}(B_{F1G1})] = DS_i[DS_{cl}\ (A_{FG} \cap_{DS} B_{F1G1})]$

2- $DS_i[DS_{cl}(A_{FG}))] \cup_{DS} DS_i[DS_{cl}\ (B_{F1G1})] = DS_i[DS_{cl}(A_{FG} \cup_{DS} B_{F1G1})]$

Proof

1 - $DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})] \subseteq_{DS} DS_i[DS_{cl}\ (A_{FG})] \cap_{DS} DS_{cl}(B_{F1G1})]$

$= DS_i[DS_{cl}\ (A_{FG}\ )] \cap_{DS} DS_i[DS_{cl}(B_{F1G1})]$

On other hand since A_{FG} , $B_{F1G1} \in Su(X_{DS})$ and by proposition 2.26 we get

$DS_i\ (DS_{cl}\ (A_{FG})) \cap_{DS} DS_i(DS_{cl}\ (B_{F1G1})) = DS_i(DS_{cl}\ (DS_i\ (A_{FG}))) \cap_{DS} (DS_{cl}\ (DS_i\ (B_{F1G1})))$ $Su(X_{DS})$

$= DS_i\ (DS_{cl}\ (DS_i\ (A_{FG} \cap_{DS} B_{F1G1})))$ proposition 2.1 part (5)

$\subseteq_{DS} DS_i\ (DS_{cl}\ (A_{FG} \cap_{DS} B_{F1G1}))$

So $DS_i\ (DS_{cl}\ (A_{FG})) \cap_{DS} DS_i\ (DS_{cl}\ (B_{F1G1}))=DS_i(DS_{cl}\ (A_{FG} \cap_{DS} B_{F1G1}))$

2 -$DS_{cl}[DS_i\ (A_{FG}\ )] \cup_{DS} DS_{cl}\ [DS_i\ (B_{F1G1})] = DS_{cl}[DS_i\ (A_{FG} \cup_{DS} DS_i(B_{F1G1})] \subseteq_{DS} DS_{cl}[DS_i\ (A_{FG} \cup_{DS} B_{F1G1})]$

On other hand

$DS_{cl}[DS_i(A_{FG})] \cup_{DS} DS_{cl}[DS_i(B_{F1G1})]$

$= DS_{cl}[DS_i(DS_{cl}(A_{FG}))] \cup_{DS} DS_{cl}[\ DS_i\ (DS_{cl}\ (B_{F1G1}))]$

$= DS_{cl}\ [DS_i\ (DS_{cl}\ (A_{FG} \cup_{DS} B_{F1G1}))]$ proposition 2.1 part(6)

$\supseteq_{DS} DS_{cl}[DS_i\ (A_{FG} \cup_{DS} B_{F1G1})]$

So $DS_{cl}[DS_i(A_{FG})] \cup_{DS} DS_{cl}\ [DS_i\ (B_{F1G1})] = DS_{cl}\ [DS_i\ (A_{FG} \cup_{DS} B_{F1G1})]$

Finally we can conclude that

Proposition 2.28

The dual soft intersection and dual soft union of any two set of $Su(X_{DS})$belong to $Su(X_{DS})$

Proof

Let A_{FG} , B_{F1G1} belong to $Su(X_{DS})$then

1-$DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})] = DS_i[DS_{cl}(A_{FG})] \cap_{DS} DS_i[DS_{cl}(B_{F1G1})]$ by proposition 2.27 part(1)

$=DS_i[DS_{cl}\ (DS_i\ (A_{FG}))) \cap_{DS} DS_i\ (DS_{cl}\ (DS_i\ (B_{F1G1}))]$ by proposition 2.26part(1) =

$DS_i[DS_{cl}\ (DS_i\ (A_{FG} \cap_{DS} B_{F1G1}))] \subseteq_{DS} DS_i[DS_{cl}(A_{FG} \cap_{DS} B_{F1G1})]$ by proposition 2.1 part (5)

So by definition 2.22 $A_{FG} \cap_{DS} B_{F1G1}$ belong to $Su(X_{DS})$

2-$DS_{cl}[DS_i(A_{FG} \cup_{DS} B_{F1G1})] = DS_{cl}[DS_i\ (A_{FG})] \cup_{DS} DS_{cl}[DS_i\ (B_{F1G1})]$ by proposition 2.27 part (2)

$= DS_{cl}\ [DS_i(DS_{cl}\ (A_{FG}))] \cup_{DS} DS_{cl}[DS_i(DS_{cl}\ (B_{F1G1}))]$ by proposition 2.26 part (2) =

$DS_{cl}\ [DS_i\ (DS_{cl}(A_{FG} \cup_{DS} B_{F1G1}))]$ by proposition 2.1 part (6)

so $A_{FG} \cup_{DS} B_{F1G1}$ belong to $Su(X_{DS})$ by definition 2.22

The concept of density is considered one of the mathematical concepts that is affected directly and indirectly the course of the development of applied sciences. It has played an important role in solving many mathematical and engineering problems. This chapter divided into three sections, the first section contains the concepts of dual soft dense, dual soft nowhere dense and dual soft somewhere dense set, second contains dual soft Baire space, dual soft meager set, third contains dual soft resolvable space, dual soft hyper connected space and dual soft submaximal space as well as numerous new definitions and properties that eventually gave rise to new theorems as applications to it

3.1DenseAnd Nowhere Dense On Dual Soft Topological Space

Here we will give the topological structure of dual soft dense sets, nowhere dense set, somewhere dense set and their important properties, give some necessary examples and discuss some cases and relationships that affect these topological spaces [32,33]**.**

Definition 3.1.1

Let (X_{DS}, T_E) be a dual soft topological space and let $A_{FG} \subseteq_{DS} X_{DS}$ then

1- A_{FG} is said to be dual soft dense inX_{DS} if $DS_{cl}(A_{FG})= X_{DS}$.

It follows that A_{FG} is dual soft everywhere dense if every dual soft point of X_{DS} is a dual soft adherent point of A_{FG} .

2- A_{FG} is said to be dual soft dense in B_{F1G1} if $B_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG})$.

3- A_{FG} is said to be dense in itself if $A_{FG} \subseteq_{DS} DS_d(A_{FG})$

4- A set A_{FG} is perfect if and only if $DS_d(A_{FG}) = A_{FG}$.

Example 3.1.2

Let U_1= { h_1,, h_2, h_3} U_2= { c_1 c_2, c_3} E={e_1,, e_2}

assume A_{FG}={(e_1,{ h_1},{ c_2, c_3}), ($e_2, \emptyset$,{ c_1})}

$A_{1F1G1} = \{(e_1 , \{h_2\}, \{c_2\}), (e_2 , \{h_3\}, \{ c_1\})\}$

$A_{2F2G2} = \{(e_1, \{ h_1\}, \{ c_3 \}),\ (e_2, \{ h_1\}, \{ c_3 \})\}$

$A_{3F3G3} = \ \{(e_1 , \{h_1, h_2\}, \{ c_2 , c_3\}), (e_2 , \{h_1, h_3\}, \{ c_1, c_3\})\}$

$T_E = \{ X_{DS}, \emptyset_{Ds} , A_{1F1G1,} A_{2F2G2}, A_{3F3G3}\}$

$X_{DS} - \ A_{1F1G1}$ $=\{(e_1 , \{h_1, h_3\}, \{ c_1 , c_3\}), (e_2 , \{h_1, h_2\}, \{ c_2, c_3\})\}$

$X_{DS} - \ A_{2F2G2}$ $=\{(e_1 , \{h_2, h_3\}, \{ c_1 , c_2\}), (e_2 , \{h_2, h_3\}, \{ c_2, c_3\})\}$

$X_{DS} - \ A_{3F3G3} = \{(e_1 , \{h_3\}, \{c_1\}), (e_2 , \{h_2\}, \{c_2\})\}$

$DS_{cl}(A_{FG}) = \ X_{DS}$. so$A_{FG}$is dual soft dense set in X_{DS}

The following proposition is equivalent to the definition of dual soft dense set

Proposition 3.1.3:

A subset A_{FG} of topological space is dual soft dense if and only if every non empty dual soft open set B_{F1G1} , $B_{F1G1} \cap_{DS} A_{FG} \neq \emptyset_{DS}$

Proof

Let A_{FG} is dual soft dense in X_{DS} if possible and let $\exists B_{F1G1} \in_{DS} T_E$ if $B_{F1G1} \cap_{DS} A_{FG} = \emptyset_{DS}$

Then $A_{FG} \subseteq_{DS} X_{DS} - B_{F1G1}$

So $DS_{cl}(A_{FG}) \subseteq_{DS} DS_{cl}(X_{DS} - B_{F1G1}\ ) = X_{DS} - B_{F1G1}$ because $X_{DS} - B_{F1G1}$ is dual soft close set

But A_{FG} is dual soft dense in X_{DS}

Then $DS_{cl}(A_{FG}) = X_{DS}$ thus $X_{DS} \subseteq_{DS} \ X_{DS} - B_{F1G1}$

Thence $B_{F1G1} = \emptyset_{DS}$ which is contradiction

Conversely assume $B_{F1G1} \cap_{DS} A_{FG} \neq \emptyset_{DS}$ $\forall \emptyset_{DS} \neq B_{F1G1} \in_{DS} T_E$

Since$DS_{cl}(A_{FG})$ =$\cap_{DS}\{H_{iFiGi}, H_{F2G2}$is dual soft closed ∋ $A_{FG} \subseteq_{DS} H_{F2G2}$ } by definition 1.2.31

In other word the only dual soft closed set contains $A_{FG}$is X_{DS}

So$DS_{cl}(A_{FG})$ = X_{DS}

Hence A_{FG} is dual soft dense in X_{DS} .

Proposition 3.1.4

Let (X_{DS}, T_E) be a dual soft topological space and let $A_{FG} \subseteq_{DS} X_{DS}$ then the following statements are equivalent:

1- A_{FG} is dual soft dense set.
2- only dual soft closed subset of X_{DS} contained A_{FG} is X_{DS} .
3- the only dual soft open set disjoint from A_{FG} is $\emptyset_{DS}$.
4- A_{FG} intersect every non empty dual soft open subset of X_{DS}.

Proof

(1→2) Let A_{FG} is dual soft dense set and let $A_{FG} \subseteq_{DS} S_{F1G1}$ S_{F1G1} is dual soft closed

Then $DS_{cl}(A_{FG}) \subseteq_{DS} DS_{cl}(S_{F1G1})$. So $X_{DS} \subseteq_{DS} DS_{cl}(S_{F1G1})$ = S_{F1G1}

Thus $X_{DS} \subseteq_{DS} S_{F1G1}$.

Hence $X_{DS} = S_{F1G1}$.

(2 → 3) LetU_{F2G2} be any non empty dual soft open set ∋ $A_{FG} \cap U_{F2G2} = \emptyset_{DS}$

Then $A_{FG} \subseteq_{DS} X_{DS} - U_{F2G2}$ which is contradiction.

(3→4) From (3) $A_{FG} \cap_{DS} U_{F2G2} \neq \emptyset_{DS}$ $\forall U_{F2G2}$is non empty dual soft open set if and only if the dual soft empty set is the only dal soft open set disjoint from A_{FG}.

(4→1) By proposition 3.1.3

Proposition 3.1.5

Any dual soft set C_{FG}contain a dual soft dense set D_{F1G1} is dense.

Proof

Since $D_{F1G1} \subseteq C_{FG}\ then\ DS_{cl}(D_{F1G1}) \subseteq_{DS}\ DS_{cl}(C_{FG})$

But $DS_{cl}(D_{F1G1}) = X_{DS}\quad so\ X_{DS} \subseteq_{DS} DS_{cl}(C_{FG})$

Also $DS_{cl}(C_{FG}) \subseteq\ X_{DS}\ \ then\ DS_{cl}(C_{FG}) =\ X_{DS}$

So C_{FG} is dual soft dense set.

Corollary 3.1.6

If A_{FG} is dual soft dense set in X_{DS} and B_{F1G1} is dual soft dense on A_{FG} then B_{F1G1} is dual soft dense set in X_{DS} .

Proof

Directly by proposition 3.1.5

Proposition 3.1.7

Let (X_{DS}, T_E) be a dual soft topological space and let $A_{FG} \subseteq_{DS}\ X_{DS}$ then A_{FG} is dual soft dense set in X_{DS} if and only if $DS_{fr}(A_{FG}) = DS_{cl}(X_{DS} - A_{FG})$.

Proof

Let A_{FG} is dual soft dense set in X_{DS}

Then $DS_{cl}(A_{FG}) = X_{DS}$

But $DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) \cap_{DS}\ DS_{cl}(X_{DS} - A_{FG})$

$=X_{DS} \cap_{DS}\ DS_{cl}(X_{DS} - A_{FG})$

$=DS_{cl}(X_{DS} - A_{FG})$.

Conversely let $DS_{fr}(A_{FG}) = DS_{cl}(X_{DS} - A_{FG})$

But by proposition 1.2.48

$DS_{cl}(A_{FG})$ =$DS_i(A_{FG}) \cup_{DS} DS_{fr}(A_{FG})$

=$[X_{DS} - DS_{cl}(X_{DS} - A_{FG})] \cup_{DS} DS_{cl}(X_{DS} - A_{FG}) = X_{DS}$

Therefore A_{FG} is dual soft dense set in X_{DS}.

Proposition 3.1.8

If $DS_i(A_{FG}) = \emptyset_{DS}$ then $X_{DS} - A_{FG}$ is dual soft dense set.

Proof

Let $DS_i(A_{FG}) = \emptyset_{DS}$ but $DS_i(A_{FG}) = X_{DS} - DS_{cl}(X_{DS} - A_{FG})$ by proposition 1.2.47 part (1)

Then $X_{DS} - DS_{cl}(X_{DS} - A_{FG}) = \emptyset_{DS}$

Take dual soft complement for both side we have

$DS_{cl}(X_{DS} - A_{FG}) = X_{DS}$ thus $X_{DS} - A_{FG}$ is dual soft dense set.

Is there is a relationship between the dual soft dense set and its dual soft exterior points?

A partial answer will be givens in the following proposition

Proposition 3.1.9

If A_{FG} is dual soft dense then $DS_e(A_{FG}) = \emptyset_{DS}$.

proof

let A_{FG} is dual soft dense then $X_{DS} - DS_{cl}(A_{FG}) = X_{DS} - X_{DS}$

Then $X_{DS} - DS_{cl}(A_{FG}) = \emptyset_{DS}$

$X_{DS} - DS_{cl}(A_{FG}) = DS_i(X_{DS} - A_{FG})$ =$DS_e(A_{FG})$=$\emptyset_{DS}$

Proposition 3.1.10

If A_{FG} is dual soft dense set and U_{F1G1} is dual soft open set then $U_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})$

Proof

By proposition 2.1 part (1)

$U_{F1G1} \cap_{DS} DS_{cl}(A_{FG}) \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})$

But $DS_{cl}(A_{FG}) = X_{DS}$ then $U_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})$.

Corollary 3.1.11

If A_{FG} is dual soft dense and U_{F1G1} is dual soft open set then $DS_{cl}(U_{F1G1}) = DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})$.

Proof

Let $U_{F1G1} \in T_E$ and A_{FG} is dual soft dense

Then by proposition 3.1.10

$DS_{cl}(U_{F1G1}) \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} U_{F1G1}) \subseteq_{DS} DS_{cl}(U_{F1G1})$

Thus $DS_{cl}(U_{F1G1}) = DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})$.

Definition3. 1.12 Dual Soft Nowhere Dense Set

A_{FG} is said to be a dual soft nowhere dense in X_{DS} if $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

If we use example 3.1.2 Assume that

$C_{FG} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \{h_3\}, \{c_2\})\}$

$DS_{cl}(C_{FG}) = X_{DS} - A_{2F2G2}$ then $DS_i[DS_{cl}(C_{FG})] = \emptyset_{DS}$

Thus C_{FG} is dual soft nowhere dense set.

Proposition 3.1.13

Let (X_{DS}, T_E) be a dual soft topological space then the following statements are equivalent

1- A_{FG} is dual soft nowhere dense.

2- $DS_{cl}(A_{FG})$contain no dual soft open set.

3- $X_{DS} - DS_{cl}(A_{FG})$ is dual soft dense.

Proof

(1→ 2) Let A_{FG} is dual soft nowhere dense

Then $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

So there is no dual soft open set $U_{F1G1} \ni \emptyset_{DS} \neq U_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG})$.

(2→3) Since $DS_{cl}(A_{FG})$ contain no dual soft open set U_{F1G1}

$\ni \forall \emptyset_{DS} \neq U_{F1G1} \in_{DS} T_E$, $U_{F1G1} \nsubseteq DS_{cl}(A_{FG})$, $U_{F1G1} \cap_{DS} X_{DS} - DS_{cl}(A_{FG}) \neq \emptyset_{DS}$

Therefore $X_{DS} - DS_{cl}(A_{FG})$ is dual soft dense set.

(3→1) Let $X_{DS} - DS_{cl}(A_{FG})$ is dual soft dense set

Then $DS_{cl}[X_{DS} - DS_{cl}(A_{FG})] = X_{DS}$

So $X_{DS} - DS_i[DS_{cl}(A_{FG})] = X_{DS}$

Thus $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$. Hence A_{FG} is dual soft nowhere dense set.

Proposition 3.1.14

If A_{FG} is dual soft nowhere dense then for every dual soft open set $U_{F1G1} \in_{DS} T_E$ the $DS_i(U_{F1G1} - A_{FG})$is not empty.

proof

If possible$DS_i(U_{F1G1} - A_{FG}) = \emptyset_{DS}$

If and only if $U_{F1G1} \cap_{DS} X_{DS} - DS_{cl}(A_{FG}) = \emptyset_{DS}$

If and only if $U_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG})$

Then $U_{F1G1} = DS_i(U_{F1G1}) \subseteq_{DS} DS_i[DS_{cl}(U_{F1G1})] = \emptyset_{DS}$

So $U_{F1G1} = \emptyset_{DS}$ which is contradiction.

Proposition 3.1.15

If a subset A_{FG} of a dual soft topological space is dual soft nowhere dense then for every non-empty dual soft set $U_{F1G1} \in T_E$ $A_{FG} \cap_{DS} U_{F1G1}$ is not dual soft dense in U_{F1G1} .

Proof

Let A_{FG} is nowhere dense then $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

If possible $A_{FG} \cap_{DS} U_{F1G1}$ is dual soft dense in U_{F1G1}

then $U_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})$

So $U_{F1G1} \subseteq_{DS} DS_i[DS_{cl}(A_{FG} \cap_{DS} U_{F1G1})] \subseteq DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

Thus $U_{F1G1} = \emptyset_{DS}$ which is contradiction.

Proposition 3.1.16

Let (X_{DS}, T_E) be a dual soft topological space then:

1- $DS_i(A_{FG})$ is dual soft dense if and only if $X_{DS} - A_{FG}$ is dual soft nowhere dense.
2- $DS_i(X_{DS} - A_{FG})$ is dual soft dense if and only if A_{FG} is nowhere dense.
3- If A_{FG} is dual soft nowhere dense then the complement of the dual soft closure of A_{FG} is dual soft dense.
4- If A_{FG} is dual soft nowhere dense then $DS_{cl}(X_{DS} - A_{FG})$ is dual soft dense.

Proof

1-$DS_i(A_{FG})$ is dual soft dense If and only if$DS_{cl}[DS_i(A_{FG})] = X_{DS}$

If and only if $X_{DS} - DS_{cl}[DS_i(A_{FG})] = \emptyset_{DS}$

If and only if $DS_i[DS_{cl}(X_{DS} - A_{FG})] = \emptyset_{DS}$

If and only if $X_{DS} - A_{FG}$ is dual soft nowhere dense.

2-$DS_i(X_{DS} - A_{FG})$ is dual soft dense

If and only if $DS_{cl}[DS_i(X_{DS} - A_{FG})] = X_{DS}$

If and only if $X_{DS} - [DS_{cl}(DS_i(X_{DS} - A_{FG}))] = \emptyset_{DS}$

If and only if $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

If and only if A is dual soft nowhere dense.

3-Let A_{FG} is dual soft nowhere dense If and only if $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

If and only if $X_{DS} - DS_i[DS_{cl}(A_{FG})] = X_{DS}$

If and only if $DS_{cl}[X_{DS} - DS_{cl}(A_{FG})] = X_{DS}$

Hence $X_{DS} - DS_{cl}(A_{FG})$ is dual soft dense set.

4- A_{FG} is dual soft nowhere dense set then $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

So $DS_{cl}[DS_i(X_{DS} - A_{FG})] = X_{DS}$

But $DS_i(X_{DS} - A_{FG}) \subseteq_{DS} X_{DS} - A_{FG}$

So $X_{DS} = DS_{cl}(X_{DS} - A_{FG})$

Thus $X_{DS} - A_{FG}$ is dual soft dense set.

Proposition 3.1.17

Let(X_{DS}, T_E) be a dual soft topological space and let $U_{F1G1} \in_{DS} T_E$ if $S_{F2G2} \subseteq_{DS} U_{F1G1}$is dual soft nowhere dense in U_{F1G1} then S_{F2G2} is dual soft nowhere dense in X_{DS}

Proof

By proposition 2.1part (1) $DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2})] \cap_{DS} U_{F1G1} \subseteq_{DS}$

$DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2} \cap_{DS} U_{F1G1})] \subseteq_{DS} DS_{i_{U_{F1G1}}}[DS_{cl_{U_{F1G1}}}(S_{F2G2})] = \emptyset_{DS}$ Then $DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2} \cap_{DS} U_{F1G1})] = \emptyset_{DS}$……(1)

Thus $\forall DP_{e1} \in DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2})],\ DP_{e1} \notin_{DS} U_{F1G1}$

So $DP_{e1} \in DS_{cl_{X_{DS}}}(S_{F2G2}) then \forall V_{F3G3} \in T_{(DP_{e1})}$,

$V_{F3G3} \cap_{DS} S_{F2G2} \neq \emptyset_{DS}\ then DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2})] \cap_{DS} S_{F2G2} \neq \emptyset_{DS}$

Then $\exists DP_{e2} \in DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2})]$and $DP_{e2} \in S_{F2G2}$

But $S_{F2G2} \subseteq_{DS} U_{F1G1}$then $DP_{e2} \in_{DS} U_{F1G1}$ which contradiction to (1) Thus $DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(S_{F2G2})] = \emptyset_{DS}$

Therefor S_{F2G2} is dual soft nowhere dense in X_{DS} .

Proposition 3.1.18

Any subset of dual soft nowhere dense is dual soft nowhere dense.

Proof

Let S_{F2G2} is dual soft nowhere dense then $DS_i\big(DS_{cl}(S_{F2G2})\big) = \emptyset_{DS}$ and let $U_{F1G1} \subseteq_{DS} S_{F2G2}$

So $DS[_iDS_{cl}(U_{F1G1})] \subseteq_{DS} DS[_iDS_{cl}(S_{F2G2})] = \emptyset_{DS}$

Then $DS[_iDS_{cl}(U_{F1G1})] = \emptyset_{DS}$

Hence U_{F1G1} is dual soft nowhere dense.

Proposition 3.1.19

The dual soft union of finitely many dual soft nowhere dense is dual soft nowhere dense.

Proof

Let $A_{FG}, B_{F1G1} \subseteq_{DS} X_{DS}$ are dual soft nowhere dense sets

And let $S_{F2G2} = A_{FG} \cup_{DS} B_{F1G1}$ to show S_{F2G2} is dual soft nowhere dense

We will show that by proposition 3.1.13 $X_{DS} - DS_{cl}(S_{F2G2})$is dual soft dense in X_{DS} that is $X_{DS} - DS_{cl}(S_{F2G2})$ intersect every non-empty dual soft open set $U_{F3G3} \subseteq_{DS} X_{DS}$

Now A_{FG}, B_{F1G1} are dual soft nowhere dense

So $U_{F3G3} \cap_{DS} [X_{DS} - DS_{cl}(A_{FG})] \neq \emptyset_{DS}$

$U_{F3G3} \cap_{DS} [X_{DS} - DS_{cl}(B_{F1G1})] \neq \emptyset_{DS}$

Also $X_{DS} - DS_{cl}(A_{FG})$, $X_{DS} - DS_{cl}(B_{F1G1})$ are dual soft open sets

Hence $[U_{F3G3} \cap_{DS} X_{DS} - DS_{cl}(A_{FG})] \cap_{DS} X_{DS} - DS_{cl}(B_{F1G1})] \neq \emptyset_{DS}$

Since $U_{F3G3} \cap_{DS} X_{DS} - DS_{cl}(A_{FG})$is dual soft open and $X_{DS} - DS_{cl}(B_{F1G1})$ is dual soft dense in X_{DS}

Thus $U_{F3G3} \cap_{DS} X_{DS} - [DS_{cl}(A_{FG}) \cup_{DS} DS_{cl}(B_{F1G1})] \neq \emptyset_{DS}$

which implies that $U_{F3G3} \cap_{DS} [X_{DS} - DS_{cl}(S_{F2G2})] \neq \emptyset_{DS}$

Hence $X_{DS} - DS_{cl}(S_{F2G2})$is dual soft dense

Therefor S_{F2G2} is dual soft nowhere dense.

Proposition 3.1.20

The dual soft closure of dual soft nowhere dense set is dual soft nowhere dense set.

proof

Let A_{FG} is dual soft nowhere dense then $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

$= DS_i[DS_{cl}(DS_{cl}(A_{FG}))]$

Hence the dual soft closure of dual soft nowhere dense set is dual soft nowhere dense set.

Proposition 3.1.21

1- If U_{FG} is dual soft open and dense in X_{DS} then $X_{DS} - U_{FG}$is dual soft nowhere dense.

2- If A_{FG} is dual soft closed and nowhere dense set then $X_{DS} - A_{FG}$ is dual soft open and dense.

Proof

1-Let U_{F1G1} is open and dense then$DS_{cl}(U_{FG}) = X_{DS}$

Then $X_{DS} - DS_{cl}(U_{FG}) = \emptyset_{DS}$

So $DS_i(X_{DS} - U_{FG}) = \emptyset_{DS}$ but $X - U_{FG}$ is dual soft closed set

Thus $DS_i[DS_{cl}(X_{DS} - U_{FG})] = \emptyset_{DS}$

Therefore $X_{DS} - U_{FG}$ is dual soft nowhere dense.

2-Let A_{FG} is dual soft closed and nowhere dense then $DS_i(DS_{cl}(A_{FG})) = \emptyset_{DS}$

So $DS_i(A_{FG}) = \emptyset_{DS}$ Thus $X_{DS} - A_{FG}$ is dual soft dense and open.

The following conclusions can be drawn

Proposition 3.1.22

A subset A_{FG} of the topological space $(X_{DS}, T_E)$is dual soft nowhere dense if and only if it satisfies any one and therefore all the following equivalent conditions

1- Its dual soft closure has no dual soft interior points.

2- The dual soft exterior points of A_{FG} is dual soft dense in X_{DS} .

3- $\forall \emptyset_{DS} \neq U_{F1G1} \in_{DS} T_E$then there is non -empty dual soft open set$S_{F2G2} \ni S_{F2G2} \subseteq_{DS} U_{F1G1} \ni S_{F2G2} \cap_{DS} A_{FG} = \emptyset_{DS}$

proof

(1↔ 2) and $DS_{cl}(DS_e(A_{FG}) = X_{DS}$

If and only if $DS_{cl}\big(DS_i(X_{DS} - A_{FG})\big) = X_{DS}$

If and only if $X_{DS} - DS_i(DS_{cl}(A_{FG})) = X_{DS}$

If and only if $DS_i(DS_{cl}(A_{FG})) = \emptyset_{DS}$

If and only if A_{FG} is dual soft nowhere dense set

If and only if $DS_{cl}(A)$has no interior point.

(3→1) To prove $DS_i(DS_{cl}(A_{FG}) = \emptyset_{DS}$

If possible $DS_i(DS_{cl}(A_{FG})) \neq \emptyset_{DS}$then $\exists\ DP_{e1} \in_{DS} DS_i(DS_{cl}(A_{FG}))$

So $\exists\ U_{F1G1} \in_{DS} T_{DP_{e1}} then\ U_{F1G1} \subseteq DS_{cl}(A_{FG})$ so by assumption

$\exists \emptyset_{DS} \neq S_{F2G2} \subseteq_{DS} U_{F1G1} \subseteq_{DS} DS_{cl}(A_{FG})$

And $A_{FG} \cap_{DS} S_{F2G2} = \emptyset_{DS}\ \ni S_{F2G2} \in_{DS} T_E$

But $\forall\ DP_{e2} \in_{DS} S_{F2G2} \in_{DS} T_E then\ DP_{e2} \in_{DS} U_{F1G1}$
$so\ DP_{e2} \in_{DS}\ DS_{cl}(A_{FG})$

Then $\forall\ W_{F3G3} \in_{DS} T_{DP_{e2}} \ni W_{F3G3} \cap_{DS} A_{FG} \neq \emptyset_{DS}$

And$S_{F2G2} \in_{DS} T_{DP_{e2}}$ so $A_{FG} \cap_{DS} S_{F2G2} \neq \emptyset_{DS}$ which is contradiction

Thus $DS_i(DS_{cl}(A_{FG})) = \emptyset$. therefore $DS_{cl}(A_{FG})$has no interior point.

(1→3) Let $\emptyset_{DS} \neq U_{F1G1} \in T_E$ if possible $\forall \emptyset_{DS} \neq S_{F2G2} \subseteq_{DS} U_{F1G1} \ni A_{FG} \cap_{DS} S_{F2G2} \neq \emptyset_{DS}, S_{F2G2} \in T_E$

Then $S_{F2G2} \cap_{DS} DS_{cl}(A_{FG}) \neq \emptyset_{DS}$

But by (3→1) $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

so $S_{F2G2} \cap_{DS} DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

Now $S_{F2G2} \cap_{DS} DS_{cl}[X_{DS} - DS_{cl}(A_{FG})] \subseteq_{DS} DS_{cl}[S_{F2G2} \cap_{DS} X_{DS} - DS_{cl}(A_{FG})]$ by proposition 3.1.13 $DS_{cl}[X_{DS} - DS_{cl}(A_{FG})] = X_{DS}$

Then $S_{F2G2} \subseteq_{DS} DS_{cl}\left[S_{F2G2} \cap_{DS} \left(X_{DS} - DS_{cl}(A)\right)\right] = \emptyset_{DS}$

Thus $S_{F2G2} = \emptyset_{DS}$ which is contradiction.

Therefore$\exists \emptyset_{DS} \neq S_{F2G2} \in_{DS} T_E, S_{F2G2} \subseteq_{DS} U_{F1G1} \ni A_{FG} \cap_{DS} S_{F2G2} = \emptyset_{DS}$.

But if A_{FG} is dual soft nowhere dense set

If and only if $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$

If and only if $DS_{cl}(A_{FG})$ has no dual soft interior points .

Proposition 3.1.23

If a dual soft closed set A_{FG} is dual soft nowhere dense then it has no

Dual soft interior points.

Proof

Let A_{FG} be a dual soft closed and nowhere dense Then $DS_i(DS_{cl}(A_{FG})) = \emptyset_{DS}$ So $DS_i(A_{FG}) = \emptyset_{DS}$ so$DS_i(A_{FG})$ has no dual soft points.

Corollary 3.1.24

If A_{FG} is dual soft closed then A_{FG} is dual soft nowhere dense if and only if $DS_i(A_{FG}) = \emptyset_{DS}$.

Proof

Let A_{FG} is dual soft closed and nowhere dense then by above

theorem $DS_i(A_{FG}) = \emptyset_{DS}$

Conversely let $DS_i(A_{FG}) = \emptyset_{DS}$. Since $A_{FG}$is dual soft closed

Then $DS_i(DS_{cl}(A_{FG})) = \emptyset_{DS}$. Therefore A_{FG} is dual soft nowhere dense.

Proposition 3.1.25

A dual soft set A_{FG} has empty dual soft interior points in X_{DS} if

for every dual soft point of A_{FG} is a dual soft limit point of $X_{DS} - A_{FG}$.

Proof

Let if $\forall\, DP_{e1} \in A_{FG}$then $DP_{e1} \in DS_d(X_{DS} - A_{FG})$ to prove $DS_i(A_{FG}) = \emptyset_{DS}$

If possible $DS_i(A_{FG}) \neq \emptyset_{DS}$ then $\exists\, DP_{e2} \in_{DS} DS_i(A_{FG})$

So there is dual soft open set$V_{F1G1} \ni DP_{e1} \in_{DS} V_{F1G1} \subseteq_{DS} A_{FG}$

Then $V_{F1G1} \cap_{DS} (X_{DS} - A_{FG}) = \emptyset_{DS}$

So $DP_{e2} \notin DS_d(X_{DS} - A_{FG})$

But$DP_{e2} \in_{DS} A_{FG}$ which is contradiction

Thus $DS_i(A_{FG}) = \emptyset_{DS}$.

Proposition 3.1.26

The dual soft boundary of every dual soft open and

dual soft closed is dual soft nowhere dense.

Proof

If A_{FG} is dual soft open as well as closed

So $DS_{fr}(A_{FG}) = DS_{cl}(A_{FG}) \cap_{DS} DS_{cl}(X_{DS} - A_{FG})$

$= A_{FG} \cap_{DS} X_{DS} - A_{FG} = \emptyset_{DS}$

Then $DS_i[DS_{cl}(DS_{fr}(A_{FG}))] = \emptyset_{DS}$

Hence $DS_{fr}(A_{FG})$ is dual soft nowhere dense.

Proposition 3.1.27

Let A_{FG} be a subset of the dual soft topological space

(X_{DS}, T_E)then the following statements are equivalent:

1- A_{FG} is dual soft closed and $DS_i[DS_{cl}(A_{FG})] = \emptyset_{DS}$.

2- $DS_{fr}(A_{FG}) = A_{FG}$.

3- $DS_{fr}(U_{F1G1}) = A_{FG}$ for some$U_{F1G1} \in_{DS} T_E$.

Proof

(1→2) $A_{FG}$is closed if $DS_i[DS_{cl}(A_{FG})] = DS_i(A_{FG}) = \emptyset_{DS}$

but $DS_{fr}(A_{FG})$=$DS_{cl}(A_{FG}) \cap_{DS} [X_{DS} - DS_i(A_{FG})]$

$= A_{FG} \cap_{DS} X_{DS} - \emptyset_{DS} = A_{FG} \cap_{DS} X_{DS} = A_{FG}$

Hence $DS_{fr}(A_{FG}) = A_{FG}$.

(2→3) $DS_{fr}(A_{FG}) = A_{FG}$then A_{FG} is dual soft closed

So $X_{DS} - A_{FG}$ is dual soft open

But $DS_{fr}(X_{DS} - A_{FG})$=$DS_{fr}(A_{FG}) = A_{FG}$

Thus A_{FG} is equal to the dual soft boundary of some dual soft open set.

(3→1) If $DS_{fr}(U_{F1G1}) = A_{FG}$ for some$U_{F1G1} \in_{DS} T_E$

Then A_{FG} is dual soft closed set

So $X_{DS} - A_{FG}$ is dual soft open set

Put $X_{DS} - A_{FG} = U_{F1G1}$

Then $DS_i[DS_{cl}(A_{FG})]$=$DS_i[DS_{cl}(DS_{fr}(A_{FG}))]$

=$DS_i[DS_{fr}(A_{FG})]$ because $DS_{fr}(A_{FG})$ is dual soft closed set

=$DS_i[DS_{cl}(U_{F1G1})] \cap_{DS} DS_{cl}(X_{DS} - U_{F1G1})]$

=$DS_i[DS_{cl}(U_{F1G1} \cap_{DS} (X_{DS} - U_{F1G1}))]$ because$U_{F1G1} \in_{DS} T_E$

=$DS_i[DS_{cl}(U_{F1G1})] \cap_{DS} DS_i(X_{DS} - U_{F1G1})$

$=DS_i[DS_{cl}(U_{F1G1})] \cap_{DS} X_{DS} - DS_{cl}(U_{F1G1})$

$\subseteq_{DS} DS_{cl}(U_{F1G1}) \cap_{DS} X_{DS} - DS_{cl}(U_{F1G1})=\emptyset_{DS}$

Thus A_{FG} is dual soft nowhere

Proposition3.1.28

$DP_e$is dual soft isolated point of X_{DS}if and only if the dual soft singleton set $\{(e,F(e),G(e))\}$is dual soft open set.

Proof

$DP_e$is dual soft isolated point of X_{DS}

If and only if $\exists U_{FG} \in_{DS} T_{DP_e} \ni U_{FG} \cap_{DS} X_{DS} - \{(e,F(e),G(e))\}=\emptyset_{DS}$

If and only if $U_{FG}=\{(e,F(e),G(e))\}$

If and only if $\{(e,F(e),G(e))\}$ is dual soft open set

Conversely if $\{(e,F(e),G(e))\}$is dual soft open set and $\{(e,F(e),G(e))\} \cap_{DS} \{(e,F(e),G(e))\} = \{(e,F(e),G(e))\}$so $DP_e$is dual soft isolated point of X_{DS}.

Proposition 3.1.29

For every $DP_e \in_{DS} X_{DS}$if $\{(e,F(e),G(e))\}$is dual soft open set then $\{(e,F(e),G(e))\}$is not dual soft nowhere dense set.

Proof

Let$\{(e,F(e),G(e))\}$is dual soft open set if possible$\{(e,F(e),G(e))\}$is dual soft nowhere dense set

Then by proposition 3.1.13 $X_{DS} - DS_{cl}(\{(e,F(e),G(e))\})$is dual soft dense set

Then by proposition 3.1.3 $\forall \emptyset_{DS} \neq U_{FG} \in T_E$

$U_{FG} \cap_{DS} X_{DS} - DS_{cl}(\{(e,F(e),G(e))\}) \neq \emptyset_{DS}$

But $\{(e, F(e), G(e))\}$is dual soft open set

Then $\{(e, F(e), G(e))\} \cap_{DS} X_{DS} DS_{cl}(\{(e, F(e), G(e))\}) \neq \emptyset_{DS}$

So $DP_e \notin DS_{cl}(\{(e, F(e), G(e))\})$which is contradiction.

Therefore$\{(e, F(e), G(e))\}$is not dual soft nowhere dense set.

Corollary 3.1.30

If $DP_e$is dual soft isolated point of X_{DS} then$\{(e, F(e), G(e))\}$ is not dual soft nowhere dense set.

Proof

Directly by proposition 3.1.28and proposition 3.1.29

Definition 3.1.31 Dual soft somewhere dense set

A dual soft subset A_{FG} of topological space is said to be dual soft somewhere dense if $DS_i[DS_{cl}(A_{FG})] \neq \emptyset_{DS}$.

If we use example 3.1.2 Assume that

B_{FG}=$\{(e_1, \{h_1\}, \emptyset), (e_2, \{h_1\}, \{c_3\})\}$

$DS_{cl}(B_{FG}) = X_{DS} - A_{1F1G1}$ then $DS_i(DS_{cl}(C_{FG})) = A_{2F2G2} \neq \emptyset_{DS}$

Then $B_{FG}$is dual soft somewhere dense set.

Proposition 3.1.32

A subset A_{FG} of X_{DS} is dual soft somewhere dense if and only if there is a non -empty dual soft open set$U_{F1G1} \ni A_{FG} \cap_{DS} U_{F1G1}$ is dense in U_{F1G1} .

Proof

Let $U_{F1G1} \in_{DS} T_E \ni DS_{cl_{U_{F1G1}}}(A_{FG} \cap_{DS} U_{F1G1}) = U_{F1G1}$

If and only if $DS_{cl_{X_{DS}}}(A_{FG} \cap_{DS} U_{F1G1}) \cap_{DS} U_{F1G1} = U_{F1G1}$

If and only if $DS_{cl_{X_{DS}}}(A_{FG} \cap_{DS} U_{F1G1}) \supseteq_{DS} U_{F1G1}$

Then $DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(A_{FG} \cap_{DS} U_{F1G1})] \supseteq_{DS} U_{F1G1} \neq \emptyset_{DS}$

So $DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(A_{FG} \cap_{DS} U_{F1G1})] \neq \emptyset_{DS}$

Again let $DS_{i_{X_{DS}}}(DS_{cl_{X_{DS}}}(A_{FG})) \neq \emptyset$

Then $\exists DP_{e1} \in_{DS} DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(A_{FG})] \neq \emptyset_{DS}$

So $\exists\ U_{F1G1} \in_{DS} T_{DP_{e1}} \ni U_{F1G1} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{FG})$

Subsequently $DS_{cl_{X_{DS}}}(A_{FG}) \cap_{DS} U_{F1G1} = U_{F1G1}$

But $U_{F1G1} = DS_{cl_{X_{DS}}}(A_{FG}) \cap_{DS} U_{F1G1} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{FG} \cap_{DS} U_{F1G1})$

Then $DS_{cl_{X_{DS}}}(A_{FG} \cap_{DS} U_{F1G1}) \cap_{DS} U_{F1G1} = U_{F1G1}$

Thus $DS_{cl_{U_{F1G1}}}(A_{FG} \cap_{DS} U_{F1G1}) = U_{F1G1}$

Therefor $A_{FG} \cap_{DS} U_{F1G1}$ is dual soft dense in U_{F1G1} .

Remark 3.1.33

Now if we have linked the concepts dual soft super condensed set, dual soft semi condensed set and dual soft condensed set with the concepts dual soft dense and nowhere dense are deriving the following conclusions:

1- If A_{FG} is dual soft dense set then A_{FG} be dual soft super condensed set if and only if A_{FG}=X_{DS} that if A_{FG} is dual soft dense set then $DS_i\big(DS_{cl}(X_{DS})\big) = DS_i(X_{DS})$

And clearly X_{DS} is dual soft dense set.

If A_{FG} is dual soft dense set then A_{FG} will be dual soft semi condensed set if $A_{FG} = X_{DS}$ or $DS_i(A_{FG})$is also dual soft dense $DS_{cl}\big(DS_i(X_{DS})\big) = DS_{cl}(X_{DS})$, if $DS_i(A_{FG})$is dual soft dense $DS_{cl}\big(DS_i(A_{FG})\big) = X_{DS} = DS_{cl}(A_{FG})$.

From above argument If A_{FG} is dual soft dense set then A_{FG} will be dual soft condensed set if and only if A_{FG}=X_{DS}.

If A_{FG} is dual soft nowhere dense set then A_{FG}will be dual soft super condensed set if$A_{FG} = \emptyset_{DS}$ or $DS_i(A_{FG}) = \emptyset_{DS}$ since$DS_i\big(DS_{cl}(A_{FG})\big)$ =$\emptyset_{DS} = DS_i(A_{FG})$.

And A_{FG} will be dual soft semi condensed set if $A_{FG} = \emptyset_{DS}$

$DS_{cl}\big(DS_i(A_{FG})\big) = DS_{cl}(\emptyset_{DS})$ since the interior of dual soft nowhere dense set is empty. Then ifA_{FG} is dual soft dense set then A_{FG} will be dual soft condensed set if and only if A_{FG}=$\emptyset_{DS}$.

2-If A_{FG} is dual soft dense set then A_{FG} will be dual soft regular open if $A_{FG} = X_{DS}$,since$DS_i\big(DS_{cl}(A_{FG})\big) = X_{DS}$

If A_{FG} is dual soft nowhere dense set then A_{FG} will be dual soft

regular open if $A_{FG} = \emptyset_{DS}$ since$DS_i\big(DS_{cl}(A_{FG})\big) = \emptyset_{DS}$.

3-If A_{FG} is dual soft dense set then A_{FG}belong to $Se(X_{DS})$ by

$DS_{cl}\big(DS_i(A_{FG})\big) \subseteq_{DS} DS_i\big(DS_{cl}(A_{FG})\big) = X_{DS}$.

If A_{FG} is dual soft nowhere dense set then A_{FG} belong to $Su(X_{DS})$ by $DS_i\big(DS_{cl}(A_{FG})\big) = \emptyset_{DS} \subseteq_{DS} DS_{cl}\big(DS_i(A_{FG})\big)$.

3.2 Dual Soft Baire Space

There is a wide variety of dual soft topological space which is dual soft dense subset of dual soft Baire space. Our purpose in this section is to define dual soft Baire space with the most important properties as well as is linking with the concepts dual soft meager set, dual soft non meager set and dual soft co-meager set [34]

Definition 3.2.1

A dual soft topological space is said to be dual soft Baire space if the dual soft countable intersection of each countable family of dual soft open dense subsets is dual soft dense in X_{DS} . On other words:

$\forall A_{iFiGi} \in_{DS} T_E , i \in I\ A_{iFiGi}$ is open and dense then $DS_{cl}(\cap_{DS\ i\in I}\ A_{iFiGi})= X_{DS}$

Proposition 3.2.2

Let (X_{DS}, T_E) be a dual soft Baire space if $\{A_{nFnGn}: n \in N\}$ is any dual soft countable closed sets with empty dual soft interior then $DS_i(\cup_{DS\ n=1}^{\infty}\ A_{nFnGn}) =\emptyset_{DS}$.

Proof

From $DS_i(A_{nFnGn}) =\emptyset_{DS}$, $X_{DS} - A_{nFnGn} \in_{DS} T_E$ for each $n \in N$ it follows that $X_{DS} = X_{DS} - DS_i(A_{nFnGn}) = DS_{cl}(X_{DS} - A_{nFnGn})$ so since X_{DS} is dual soft Baire space .Consequently $X_{DS} = DS_{cl}(\cap_{DS\ n=1}^{\infty}\ X_{DS} - A_{nFnGn})$

Follows $\emptyset_{DS}$=$X_{DS} - DS_{cl}\left(\cap_{DS\ n=1}^{\infty}\ X_{DS} - A_{nFnGn}\right)$

$= X_{DS} - DS_{cl}(X_{DS} -\cup_{DS\ n=1}^{\infty}\ A_{nFnGn}) =DS_i(\cup_{DS\ n=1}^{\infty}\ A_{nFnGn})$

Proposition 3.2.3

Let (X_{DS}, T_E) be dual soft Baire space $\{A_{nFnGn}: n \in Z^+\}$is any countable dual soft closed covering of X_{DS} then at least one of A_{nFnGn} contains a dual soft open set .

Proof

From X_{DS}=$\cup_{DS}{}_{n=1}^{\infty} A_{nFnGn}$ follows $\cap_{DS}{}_{n=1}^{\infty} (X_{DS} - A_{nFnGn})$ =$\emptyset_{DS}$,

So since X_{DS} is a dual soft Baire space not every $X_{DS} - A_{nFnGn}$ can be dual soft dense set .

Consequently $DS_{cl}(X_{DS} - A_{n0Fn0Gn0}) \neq X_{DS}$ for some $n_0 \in N$ that is $DS_i(A_{n0Fn0Gn0})= X_{DS} - DS_{cl}(X_{DS} - A_{n0Fn0Gn0}) \neq \emptyset_{DS}$.

Proposition 3.2.4

Any dual soft open subspace Y_{FG} of dual soft Baire space X_{DS} is itself dual soft Baire space.

Proof

Let { A_{nFnGn}} be any dual soft countable collection of dual soft closed sets of Y_{FG} that have $\forall n \in_{DS} N\ DS_{i_{Y_{FG}}}(A_{nFnGn}) = \emptyset_{DS}$ also

$A_{nFnGn} = DS_{cl_{Y_{FG}}}(A_{nFnGn}) = DS_{cl_{X_{DS}}}(A_{nFnGn}) \cap_{DS} Y_{FG}$.

If possible $DS_{i_{X_{DS}}}(DS_{cl_{X_{DS}}}(A_{nFnGn})) \neq \emptyset_{DS}$ follows $\exists\ \emptyset_{DS} \neq U_{FIGI} \in_{DS} T_E \ni U_{FIGI} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{nFnGn})$

Imply that $U_{FIGI} \cap_{DS} Y_{FG} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{nFnGn}) \cap_{DS} Y_{FG}$=$A_{nFnGn}$

follows that

$DS_{i_{Y_{FG}}}(A_{nFnGn}) \neq \emptyset_{DS}$ contradicted the hypothesis

Now if possible $DS_{i_{Y_{FG}}}(\cup_{DS} A_{nFnGn}) \neq \emptyset_{DS}$ follows $\exists\ \emptyset_{DS} \neq U_{FIGI} \in_{DS} T_{E_{Y_{FG}}}$

$U_{FIGI} \subseteq_{DS} \cup_{DS} (A_{nFnGn}) \subseteq_{DS} \cup_{DS} (DS_{cl_{X_{DS}}}(A_{nFnGn}))$

Since $Y_{FG} \in_{DS} T_E$ so $U_{FIGI} \in_{DS} T_E$ that is $DS_{i_{X_{DS}}}(\cup_{DS_{n\in N}} DS_{cl_{X_{DS}}}(A_{nFnGn})) \neq \emptyset_{DS}$.Which is contradiction that X_{DS} is dual soft Baire space.

Proposition 3.2.5

If every dual soft point of X_{DS}has an open dual soft neighborhood that is dual soft Baire space then X_{DS} is dual soft Baire space.

Proof

if $\{V_{nFnGn}\}$ is a collection of dual soft open dense subset in X_{DS}

To prove $\cap_{DS_{n\in N}} (V_{nFnGn})$ is dual soft dense in X_{DS}

let $DP_e \in X_{DS}$ and let W_{F1G1} be an open dual soft neighborhood of X_{DS} that is dual soft Baire space then $W_{F1G1} \cap_{DS} (\cap_{DS_{n\in N}} V_{nFnGn})$is dual soft dense in W_{F1G1} . because we take $\emptyset_{DS} \neq W_{0F0G0} \in_{DS} T_{E_{W_{F1G1}}}$

Then $\exists\ U_{F2G2} \in_{DS} T_E \ni \emptyset_{DS} \neq W_{0F0G0} = U_{F2G2} \cap_{DS} W_{F1G1}$

So $W_{0F0G0} \cap_{DS} V_{nFnGn} \cap_{DS} W_{F1G1}$

$=U_{F2G2} \cap_{DS} W_{F1G1} \cap_{DS} V_{nFnGn} \cap_{DS} W_{F1G1}$

$=U_{F2G2} \cap_{DS} V_{nFnGn} \cap_{DS} W_{F1G1} \neq \emptyset_{DS}$ because $U_{F2G2}, W_{F1G1} \in_{DS} T_E$ $U_{F2G2} \cap_{DS} W_{F1G1} \neq \emptyset_{DS} \in_{DS} T_E$ and V_{nFnGn} is dual soft dense inX_{DS}

Thus $V_{nFnGn} \cap_{DS} W_{F1G1}$is dual soft dense in$W_{F1G1}$

And $V_{nFnGn} \cap_{DS} W_{F1G1} \in_{DS} T_{E_{W_{F1G1}}} then \cap_{DS_{n\in N}}(V_{nFnGn} \cap_{DS} W_{F1G1})$is dual soft dense in$W_{F1G1}$

Thence$\forall W_{0F0G0} \in_{DS} T_{E_{W_{F1G1}}}$,

$W_{0F0G0} \cap_{DS} (\cap_{DS_{n\in N}}(V_{nFnGn} \cap_{DS} W_{F1G1})) \neq \emptyset_{DS}$

$= (W_{0F0G0} \cap_{DS} W_{F1G1}) \cap_{DS} (\cap_{DS_{n\in N}} V_{nFnGn})$

$= W_{0F0G0} \cap_{DS}(\cap_{DS_{n\in N}} V_{nFnGn}) \neq \emptyset_{DS}$

$\emptyset_{DS} \neq W_{0F0G0} \cap_{DS}(\cap_{DS_{n\in N}} V_{nFnGn}) \subseteq W_{F1G1} \cap_{DS}(\cap_{DS_{n\in N}} V_{nFnGn})$

Then $W_{F1G1} \cap_{DS}(\cap_{DS_{n\in N}} V_{nFnGn}) \neq \emptyset_{DS}$

Thus $\cap_{DS_{n\in N}} V_{nFnGn}$ is dual soft dense in X_{DS}

Hence X_{DS} is dual soft Baire space.

Now we will define and discuss new three types of dual soft set.

Definition 3.2.6

Let (X_{DS}, T_E)be a dual soft topological space then:

1- A subset $A_{FG}$of X_{DS} is called dual soft meager in X_{DS} or the first dual soft category in X_{DS}if it is countable dual soft union of nowhere dense subsets of X_{DS}.

2- A subsetA_{FG} of $X_{DS}$is not dual meager in X_{DS} is called non dual soft meager subset of X_{DS} or the second dual soft category inX_{DS}.

3- A subset $A_{FG}$of $X_{DS}$is called dual soft co-meager in X_{DS} or dual soft residual in X_{DS} if it is the extension complement of first dual soft category .

Proposition3.2.7

Let (X_{DS}, T_E) be a dual soft Baire space then:

1- A dual soft meagre set has an empty dual soft interior points.

2- The dual soft complement of dual soft meagre set is dual soft dense set.

3- A countable dual soft intersection of adual soft open dense sets is not dual soft empty.

Proof

1- Let A_{nFnGn} be a countable family of dual soft nowhere dense subsets and U_{F1G1} dual soft open set $\ni$ $U_{F1G1} \subseteq_{DS} \cup_{DS_{i=1}^{\infty}} A_{nFnGn}$

We will prove that $U_{F1G1} = \emptyset_{DS}$

From $U_{F1G1} \subseteq_{DS} \cup_{DS_{i=1}^{\infty}} A_{nFnGn}$ $U_{F1G1} \subseteq_{DS} \cup_{DS_{i=1}^{\infty}} DS_{cl}(A_{nFnGn})$

We find $\cap_{i=1}^{\infty} [X_{DS} - DS_{cl}(A_{nFnGn})] \subseteq_{DS} (X_{DS} - U_{F1G1})$ and so by proposition 3.1.16 part(3**)** $X_{DS} - DS_{cl}(A_{nFnGn})$ be a dual soft dense and open but $(X_{DS}, T_E)$is a dual soft Baire space that is

$X_{DS} = DS_{cl}\left(\cap_{DS_{i=1}^{\infty}} X_{DS} - DS_{cl}(A_{nFnGn})\right) \subseteq_{DS} DS_{cl}(X_{DS} - U_{F1G1})$

$= X_{DS} - U_{F1G1} then\ X_{DS} = X_{DS} - U_{F1G1}$.

Then $U_{F1G1} = \emptyset_{DS}$.

2− let A_{FG} be a dual soft co-meagre set in X_{DS}

So $X_{DS} - A_{FG}$ be a dual soft meagre set thenby part(1) $DS_i(X_{DS} - A_{FG}) = \emptyset_{DS}$ .Thus A_{FG} is dual soft dense set by proposition 3.1.8

3 − suppose that$\{A_{iFiGi}\}$be a dual soft sequence of dual soft open dense subsets of X_{DS} then $\cap_{DS_{i=1}^{\infty}}(A_{iFiGi})$ is dual soft open

Therefore $\cap_{DS_{i=1}^{\infty}} (A_{iFiGi}) \neq \emptyset_{DS}$.

Proposition 3.2.8

Let(X_{DS}, T_E)be a dual soft topological space then:

1- Any dual soft subset of meager set is meager set.

2- The dual soft union of the dual soft countable many meager sets is dual soft meager set.

Proof

1- Let A_{FG} be a dual soft meager set in T_E and let $B_{F1G1} \subseteq_{DS} A_{FG}$

since A_{FG} is a dual soft meager set then there exist a dual soft countable collection of dual soft $\{U_{nFnGn} : n \in N\}$ nowhere dense set in T_E such that:

$A_{FG} = \cup_{DS_{n\in N}} U_{nFnGn}$

then we have $B_{F1G1} = A_{FG} \cap_{DS} B_{F1G1} = (\cup_{DS_{n\in N}} U_{nFnGn}) \cap_{DS} B_{F1G1}$

$= \cup_{DS_{n\in N}} (U_{nFnGn} \cap_{DS} B_{F1G1})$

$U_{nFnGn} \cap_{DS} B_{F1G1} \subseteq_{DS} U_{nFnGn}$ but by proposition 3.1.18 any dual soft subset of dual soft nowhere dense set is dual soft nowhere dense set then we see thatB_{F1G1} can be written as the dual soft union of the dual soft nowhere dense set in $T_E$so $B_{F1G1}$is a dual soft meager in T_E.

2- Let A_{FG} and B_{F1G1}be a dual soft meager in T_E since A_{FG} is a dual soft meager set then there exist a dual soft countable collection of dual soft $\{U_{nFnGn}: n \in N\}$ nowhere dense set in T_E such that $A_{FG} = \cup_{DS_{n\in N}} U_{nFnGn}$

Since B_{F1G1} is a dual soft meager set then there exist a countable collection of dual soft sets $\{V_{mFmGm} : m \in N\}$ nowhere dense set in T_E such that

$B_{F1G1} = \cup_{DS_{m\in N}} V_{mFmGm}$

Then $A_{FG} \cup_{DS} B_{F1G1} = \left(\cup_{DS_{n\in N}} U_{nFnGn}\right) \cup_{DS} \left(\cup_{DS_{m\in N}} V_{mFmGm}\right)$

The right- hand side is a countable union of dual soft nowhere dense sets in T_E so $A_{FG} \cup_{DS} B_{F1G1}$ is a dual soft meager in T_E.

Proposition 3.2.9

X_{DS} is second dual soft category in itself if and only if the dual soft intersection of every countable family of dual soft open dense set in X_{DS} is not dual soft empty.

Proof

Let $\{U_{iFiGi}\}_{i=1}^{\infty}$ is non empty dual soft dense open subset of X_{DS}

then $\{X_{DS} - U_{iFiGi}\}_{i=1}^{\infty}$ are countable dual soft nowhere dense closed by proposition 3.1.21 so $\cup_{DS} \{X_{DS} - U_{iFiGi}\}_{i=1}^{\infty} \neq X_{DS}$ because X_{DS} is second dual soft category

Hence $X_{DS} - (\cup_{DS} \{X_{DS} - U_{iFiGi}\}_{i=1}^{\infty}) = \cap_{DS}\, {}_{i=1}^{\infty} (U_{iFiGi}) \neq \emptyset_{DS}$.

Conversely if $X_{DS} = \cup_{DS}\ A_{nFnGn}$ each A_{nFnGn} is dual soft nowhere dense Then $X_{DS} - \cup_{DS}\ A_{nFnGn} = \cap_{DS} (X_{DS} - A_{nFnGn}) = \emptyset_{DS}$

Follows $\cap_{DS}\, {}_{i=1}^{\infty} (DS_i (X_{DS} - A_{nFnGn})) = \emptyset_{DS}$

Since A_{nFnGn} is dual soft nowhere dense

So by proposition 3.1.16 part(2) each $DS_i (X_{DS} - A_{nFnGn})$ is dual soft open and dense sets and hence not dual soft empty which is contradiction

Thus$X_{DS} \neq \cup_{DS}\ A_{nFnGn}$ for any sequence of dual soft nowhere dense sets.

Corollary 3.2.10

Every dual soft Baire space is second dual soft category.

Proof

Directly by above proposition.

Proposition 3.2.11

Suppose that $A_{F1G1} \subseteq_{DS} Y_{FG} \subseteq_{DS} X_{DS}$ when Y_{FG} has dual soft open subspace induced from X_{DS} then the following statements hold:

1- If A_{F1G1} is dual soft meagre in Y_{FG} , then A_{F1G1} is dual soft meagre in X_{DS}.
2- If A_{F1G1} is dual soft non meagre in X_{DS} , then A_{F1G1} is dual soft non meagre in Y_{FG} .

Proof

1-suppose that Y_{FG} be a dual soft open subspace of X_{DS} and let $A_{F1G1} \subseteq_{DS} Y_{FG}$ is a dual soft meagre in Y_{FG} then there is a countable of dual soft nowhere dense set in Y_{FG} $A_{nFnGn}\ n \in N \ni A_{F1G1} = \cup_{DS\, n\in N} (A_{nFnGn})$

But $Y_{FG} \subseteq_{DS} X_{DS}$ then by proposition 3.1.17 A_{nFnGn} are dual soft nowhere dense set in X_{DS} for each n

Hence A_{F1G1}=$\cup_{DS_{n\in N}}$ (A_{nFnGn}) can be written as the countable dual soft union of dual soft nowhere dense in X_{DS}

Thus X_{DS} is first dual soft category.

2-let A_{F1G1} is dual soft non meagre in X_{DS} then A_{F1G1} can not be written as the countable dual soft union of dual soft nowhere dense sets in other words A_{F1G1}=$\cup_{DS_{n\in N}}$ (A_{nFnGn}) $n \in N$ at least one of A_{nFnGn} is not dual soft nowhere dense set in X_{DS} say A_{2F2G2}

Then $DS_{i_{X_{DS}}}[DS_{cl_{X_{DS}}}(A_{2F2G2})] \neq \emptyset_{DS}$

Since $Y_{FG} \in_{DS} T_E$ then $DS_{cl\ _{Y_{FG}}}(A_{2F2G2}) =$ $DS_{cl_{X_{DS}}}(A_{2F2G2}) \cap_{DS} Y_{FG} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{2F2G2} \cap_{DS} Y_{FG})=$ $DS_{cl_{X_{DS}}}(A_{2F2G2})$

$\emptyset_{DS} \neq DS_{i_{X_{DS}}}(DS_{cl_{X_{DS}}}(A_{2F2G2}))\subseteq_{DS} DS_{i\ _{Y_{FG}}}(DS_{cl\ _{Y_{FG}}}(A_{2F2G2})$

so $DS_{i\ _{Y_{FG}}}(DS_{cl\ _{Y_{FG}}}(A_{2F2G2}) \neq \emptyset_{DS}$

Then A_{2F2G2} is not dual soft nowhere dense set in Y_{FG}

So A_{F1G1} can not be written as the countable dual soft union of dual soft nowhere dense sets in Y_{FG}

Hence $A_{F1G1}$is dual soft non meagre in Y_{FG} or the second dual soft category.

Proposition 3.2.12

Let (X_{DS}, T_E) be a dual soft Baire space then any dual soft non empty open subset of X_{DS} is not dual soft meagre.

Proof

Let (X_{DS}, T_E) be a dual soft Baire space and let$\emptyset_{DS} \neq A_{FG} \in_{DS} T_E$

If possible that is A_{FG} is dual soft meagre in X_{DS}

Then by proposition 3.2.7 part(1) $DS_i(A_{FG}) = \emptyset_{DS}$

But A_{FG} is dual soft open set then $DS_i(A_{FG})$=$A_{FG} = \emptyset_{DS}$ which is contradiction.

3.3 Dual Soft Resolvability

In this section we generalize several important concepts in general topology as; dual soft resolvable spaces, dual soft irresolvable spaces, dual soft hereditary irresolvable, dual soft hyper connected, dual soft submaximal …etc , with some of properties and results [35,36,37] .

Definition 3.3.1

A dual soft topological space (X_{DS}, T_E) is said to be dual soft resolvable space if there exist two dual soft dense subset A_{FG} and B_{F1G1} such that $A_{FG} \cap_{DS} B_{F1G1} = \emptyset_{DS}$ and $A_{FG} \cup_{DS} B_{F1G1} = X_{DS}$

Otherwise X_{DS} is called dual soft irresolvable space

Definition 3.3.2

Let $Y_{FG} \subseteq X_{DS}$ be a dual soft subspace of X_{DS} if there exist two dual soft dense subset of Y_{FG} $A_{1F1G1}, A_{2F2G2} \ni DS_{cl\ Y_{FG}}(A_{1F1G1}) = Y_{FG}$ $DS_{cl\ Y_{FG}}(A_{2F2G2}) = Y_{FG}$ and $A_{1F1G1} \cup_{DS} A_{2F2G2} = Y_{FG}$, $A_{1F1G1} \cap_{DS} A_{2F2G2} = \emptyset_{DS}$ then Y_{FG} is irresolvable dual soft subspace .

Proposition 3.3.3

Every dual soft open subspace of dual soft resolvable space is dual soft resolvable space.

Proof

Let (X_{DS}, T_E) be dual soft resolvable space then there exist two dual soft dense subset A_{FG} and B_{F1G1} such that $A_{FG} \cap_{DS} B_{F1G1} = \emptyset_{DS}$ and $A_{FG} \cup_{DS} B_{F1G1} = X_{DS}$

Now Let $Y_{FG} \subseteq_{DS} X_{DS}$ be an open dual soft subspace of X_{DS}

so $Y_{FG} = X_{DS} \cap_{DS} Y_{FG} = DS_{cl}(A_{FG}) \cap_{DS} Y_{FG} \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} Y_{FG})$

then $Y_{FG} \subseteq_{DS} DS_{cl}(A_{FG} \cap_{DS} Y_{FG})$ thus $A_{FG} \cap_{DS} Y_{FG}$is dual soft dense set in Y_{FG}

But $(A_{FG} \cap_{DS} Y_{FG}) \cap_{DS} (B_{FG} \cap_{DS} Y_{FG}) = (A_{FG} \cap_{DS} B_{FG}) \cap_{DS} Y_{FG} = \emptyset_{DS}$

$(A_{FG} \cap_{DS} Y_{FG}) \cup_{DS} (B_{FG} \cap_{DS} Y_{FG}) = (A_{FG} \cup_{DS} B_{FG}) \cap_{DS} Y_{FG} = Y_{FG}$

Therefore Y_{FG} is dual soft resolvable subspace.

Proposition 3.3.4

A dual soft topological space **(**X_{DS}, T_E) is dual soft resolvable space if and only if there exist a non empty dual soft dense subset $A_{FG} \ni DS_i(A_{FG}) = \emptyset_{DS}$.

Proof

Let(X_{DS}, T_E) be dual soft resolvable space then then there exist two dual soft dense subset A_{FG} and B_{F1G1} such that $A_{FG} \cap_{DS} B_{F1G1} = \emptyset_{DS}$ and $A_{FG} \cup_{DS} B_{F1G1} = X_{DS}$

Then $X_{DS} - A_{FG} = B_{F1G1}$

But $DS_{cl}(A_{FG}) = DS_{cl}(B_{FG}) = X_{DS}$

Then $DS_{cl}(X_{DS} - A_{FG}) = DS_{cl}(B_{FG}) = X_{DS}$

So $DS_{cl}(X_{DS} - A_{FG}) = X_{DS}$ $then$ $X_{DS} - DS_{cl}(X_{DS} - A_{FG}) = \emptyset_{DS}$

Thus $DS_i(A_{FG}) = \emptyset_{DS}$.

Conversely let $DS_i(A_{FG}) = \emptyset_{DS}$ and A_{FG} is dual soft dense set

Then $X_{DS} - DS_{cl}(X_{DS} - A_{FG}) = \emptyset_{DS}$

So $DS_{cl}(X_{DS} - A_{FG}) = X_{DS}$

Subsequently $X_{DS} - A_{FG}$ is dual soft dense set

Thence $DS_{cl}(X_{DS} - A_{FG}) = DS_{cl}(A_{FG}) = X_{DS}$

But $(X_{DS} - A_{FG}) \cup_{DS} A_{FG} = X_{DS}$ and$(X_{DS} - A_{FG}) \cap_{DS} A_{FG} = \emptyset_{DS}$

Thus (X_{DS}, T_E) dual soft resolvable space.

Definition 3.3.5

A dual soft topological space (X_{DS}, T_E) is said to be dual soft hereditary irresolvable if every dual soft subspace is dual soft irresolvable .

We say that (X_{DS}, T_E) is said to be dual soft hereditary irresolvable if it does not contain any non -empty dual soft resolvable subspace.

Definition 3.3.6

A dual soft topological space (X_{DS}, T_E) is said to be dual soft hereditary irresolvable if each dual soft open subspace is dual soft irresolvable.

Definition 3.3.7

A dual soft topological space (X_{DS}, T_E) is said to be dual soft hyper connected space if every non empty dual soft open subset of X_{DS} is dual soft dense set

In other words $\forall \emptyset_{DS} \neq A_{FG} \in_{DS} T_E$ then $DS_{cl}(A_{FG}) = X_{DS}$.

Proposition 3.3.8

Let (X_{DS}, T_E) be a dual soft topological space then the following statements are equivalents:

1- (X_{DS}, T_E) is dual soft hyper connected space .

2- No two dual soft non empty open subsets are disjoint.

3- X_{DS} can not be written as the dual soft union of two proper dual soft closed subsets.

4- every non empty dual soft open subset is dual soft dense.

5- The dual soft interior of every proper dual soft closed subset is empty.

Proof

$(1 \rightarrow 2)$ Let (X_{DS}, T_E) be a dual soft hyper connected space and let A_{F1G1}, $B_{F2G2} \in T_E$ then A_{F1G1}, B_{F2G2} are dual soft dense

If possible that $A_{F1G1} \cap_{DS} B_{F2G2} = \emptyset_{DS}$

So $A_{F1G1} \subseteq_{DS} X_{DS} - B_{F2G2}$ and $B_{F2G2} \subseteq_{DS} X_{DS} - A_{F1G1}$

Then $X_{DS} = DS_{cl}(A_{F1G1}) \subseteq_{DS} DS_{cl}(X_{DS} - B_{F2G2}) = X_{DS} - B_{F2G2}$

Thus $B_{F2G2=}\emptyset_{DS}$ which is contradiction

Hence $A_{F1G1} \cap_{DS} B_{F2G2} \neq \emptyset_{DS} \forall A_{F1G1}, B_{F2G2} \in_{DS} T_E$.

$(2 \rightarrow 3)$Let the condition is hold and if possible there exist two proper dual soft closed subsets of X_{DS} $A_{F1G1}, B_{F2G2} \ni A_{F1G1} \cup_{DS} B_{F2G2} = X_{DS}$

Then $(X_{DS} - A_{F1G1}) \cap_{DS} (X_{DS} - B_{F2G2}) = \emptyset_{DS}$

But$(X_{DS} - A_{F1G1}), (X_{DS} - B_{F2G2})$ are dual soft open subsets which is contradiction

Thus X_{DS} can not be written as the dual soft union of two proper dual soft closed subsets.

$(3 \rightarrow 4)$ Let the condition is hold that is $\forall A_{F1G1}, B_{F2G2}$are proper dual soft closed subsets of X_{DS} then$A_{F1G1} \cup_{DS} B_{F2G2} \neq X_{DS}$

Then $(X_{DS} - A_{F1G1}) \cap_{DS} (X_{DS} - B_{F2G2}) \neq \emptyset_{DS}$

But$(X_{DS} - A_{F1G1}), (X_{DS} - B_{F2G2})$ are dual soft open

since $(X_{DS} - A_{F1G1}) \cap_{DS} (X_{DS} - B_{F2G2}) \neq \emptyset_{DS}$

So $(X_{DS} - A_{F1G1}) \not\subseteq_{DS} B_{F2G2}$ and $(X_{DS} - B_{F2G2}) \not\subseteq_{DS} A_{F1G1}$

$\forall (X_{DS} - A_{F1G1}), (X_{DS} - B_{F2G2}) \in_{DS} T_E$

Thus the only dual soft closed subset contains $(X_{DS} - A_{F1G1}), (X_{DS} - B_{F2G2})$ is X_{DS} . Thus $(X_{DS} - A_{F1G1}), (X_{DS} - B_{F2G2})$ are dual soft dense subsets of X_{DS}.

$(4 \rightarrow 5)$ Let $A_{F1G1}$is dual soft closed subset then $X_{DS} - A_{F1G1}$ is dual soft open

So by 4 $X_{DS} - A_{F1G1}$ is dual soft dense subset

But $X_{DS} - A_{F1G1} = DS_i(X_{DS} - A_{F1G1})$

Then $DS_i(X_{DS} - A_{F1G1})$ is dual soft dense subset

Thus $DS_{cl}(DS_i(X_{DS} - A_{F1G1})) = X_{DS}$

If and only if $\emptyset_{DS} = X_{DS} - DS_{cl}(DS_i(X_{DS} - A_{F1G1}))$

$= X_{DS} - X_{DS} - DS_i(DS_{cl}(A_{F1G1}))$

$= DS_i(DS_{cl}(A_{F1G1})) = DS_i(A_{F1G1})$

Hence $DS_i(A_{F1G1}) = \emptyset_{DS}$

$(5 \rightarrow 1)$) Let $A_{F1G1}$is dual soft open subset then $X_{DS} - A_{F1G1}$ is dual soft closed

Then by 5 $DS_i(X_{DS} - A_{F1G1}) = \emptyset_{DS}$

If and only if $X_{DS} - DS_{cl}(A_{F1G1}) = \emptyset_{DS}$

If and only if $DS_{cl}(A_{F1G1}) = X_{DS}$ iff A_{F1G1} is dual soft dense subset.

Proposition 3.3.9

Every open dual soft subspace of dual soft hyper connected space is dual soft hyper connected space.

Proof

Let (X_{DS}, T_E) be a dual soft hyper connected space and $Y_{FG} \in_{DS} T_E$

To prove $(Y_{FG}, T_{E_{Y_{FG}}})$ is dual soft hyper connected space

Let $W_{F1G1} \in_{DS} T_{E_{Y_{FG}}}$ to prove $DS_{cl_{Y_{FG}}}(W_{F1G1}) = Y_{FG}$

Then $\exists\ U_{F2G2} \in_{DS} T_E \ni W_{F1G1} = U_{F2G2} \cap_{DS} Y_{FG}$

But $U_{F2G2} \cap_{DS} Y_{FG} \in_{DS} T_E$ then $U_{F2G2} \cap_{DS} Y_{FG}$ is dual soft dense in X_{DS}

So $DS_{cl_{X_{DS}}}(W_{F1G1}) = DS_{cl_{X_{DS}}}(U_{F2G2} \cap_{DS} Y_{FG}) = X_{DS}$

But $DS_{cl_{Y_{FG}}}(W_{F1G1}) = DS_{cl_{X_{DS}}}(W_{F1G1} \cap_{DS} Y_{FG}) = X_{DS} \cap_{DS} Y_{FG} = Y_{FG}$

Hence W_{F1G1} is dual soft dense in Y_{FG}

Therefore $Y_{FG}$is dual soft hyper connected space .

Definition 3.3.10

A dual soft topological space is said to be dual soft submaximal space if every dual soft dense set is dual soft open set.

On other words $\forall A_{FG} \subseteq_{DS} X_{DS}, DS_{cl}(A_{FG}) = X_{DS}\ then\ A_{FG} \in_{DS} T_E$.

Proposition 3.3.11

Let (X_{DS}, T_E) be a dual soft submaximal space and let $Y_{FG} \subseteq_{DS} X_{DS}$ then $(Y_{FG}, T_{E_{Y_{FG}}})$ is dual soft submaximal space.

Proof

Let $\emptyset_{DS} \neq A_{F1G1} \subseteq_{DS} Y_{FG} \ni A_{F1G1}$ is dual soft dense in Y_{FG}

Follows $Y_{FG} = DS_{cl_{Y_{FG}}}(A_{F1G1}) = DS_{cl_{X_{DS}}}(A_{F1G1} \cap_{DS} Y_{FG})$

So $Y_{FG} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{F1G1})$

But $DS_{cl_{X_{DS}}}[A_{F1G1} \cup_{DS} (X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1})]$

$= DS_{cl_{X_{DS}}}(A_{F1G1}) \cup_{DS} DS_{cl_{X_{DS}}}[X_{DS} - DS_{cl}(A_{F1G1})]$

$= DS_{cl_{X_{DS}}}[DS_{cl_{X_{DS}}}(A_{F1G1})] \cup_{DS} [DS_{cl_{X_{DS}}}(X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1})]$

$= DS_{cl_{X_{DS}}}[DS_{cl_{X_{DS}}}(A_{F1G1}) \cup_{DS} (X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1}))]$

$= DS_{cl_{X_{DS}}}(X_{DS}) = X_{DS}$

Then $A_{F1G1} \cup_{DS} [X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1})]$is dual soft dense set in X_{DS} and by hypothesis $A_{F1G1} \cup_{DS} (X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1}))$is dual soft open in X_{DS}

But $Y_{FG} \cap_{DS} [A_{F1G1} \cup_{DS} (X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1}))]$

$=(Y_{FG} \cap_{DS} A_{F1G1}) \cup_{DS} [Y_{FG} \cap_{DS} (X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1})] = A_{F1G1}$

Because $Y_{FG} \subseteq_{DS} DS_{cl_{X_{DS}}}(A_{F1G1})$

So $[Y_{FG} \cap_{DS} (X_{DS} - DS_{cl_{X_{DS}}}(A_{F1G1})] = \emptyset_{DS}$

And $A_{F1G1} \subseteq_{DS} Y_{FG}$ that is $Y_{FG} \cap_{DS} A_{F1G1} = A_{F1G1}$

Therefore $A_{F1G1} \in T_{E_{Y_{FG}}}$.

Definition 3.3.12

A dual soft set A_{FG} is said to be dual soft locally closed if $DS_{cl}(A_{FG}) - A_{FG}$ is dual soft closed in X_{DS} or A_{FG}

Proposition 3.3.13

If a dual soft topological space $(X_{DS}, T_E)$is dual soft submaximal space then every dual soft dense subset of X_{DS} is dual soft locally closed

Proof

Let (X_{DS}, T_E) be s dual soft submaximal space and let A_{FG}be a dual soft dense subset of X_{DS} then A_{FG} be dual soft open subset of X_{DS}

Since A_{FG} is dual soft dense then $DS_{cl}(A_{FG}) = X_{DS}$

Then $DS_{cl}(A_{FG}) - A_{FG} = X_{DS} - A_{FG}$ is dual soft closed subset of X_{DS}

Thus A_{FG} is dual soft locally closed set.

Example 4.1

Let$U_1 = \{ h_1 , h_2 , h_3 \}$ $U_2 = \{ c_1 , c_2 , c_3 \}$ $E = \{ e_1 , e_2 \}$

The number of all dual soft points:

is $| DS(U_1, U_2)_E | = [n_1(2^{n_2+n_3})] - (n_1 - 1)$ $DS(U_1, U_2)_E$

When $n_1 = |E|$, $n_2 = |U_1|$, $n_3 = |U_2|$

$| DS(U_1, U_2)_E | = 2(2^6) - 1 = 127$

$X_{DS} = \{(e_1, U_1, U_2), (e_2, U_1, U_2)\}$

$DP_{1e1} = \{(e_1, U_1, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{2e1} = \{(e_1, U_1, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{3e1} = \{(e_1, U_1, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{4e1} = \{(e_1, U_1, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{5e1} = \{(e_1, U_1, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{6e1} = \{(e_1, U_1, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{7e1} = \{(e_1, U_1, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{8e1} = \{(e_1, U_1, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{9e1} = \{(e_1, \emptyset, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{10e1} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{11e1} = \{(e_1, \emptyset, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{12e1} = \{(e_1, \emptyset, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{13e1} = \{(e_1, \emptyset, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{14e1} = \{(e_1 \emptyset, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{15e1} = \{(e_1, \emptyset, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{16e1} = \{(e_1, \emptyset, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{17e1} = \{(e_1, \{ h_1 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{18e1} = \{(e_1, \{ h_1 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{19e1} = \{(e_1, \{ h_1 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{20e1} = \{(e_1, \{ h_1 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{20e1} = \{(e_1, \{ h_1 \}, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{21e1} = \{(e_1, \{ h_1 \}, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{22e1} = \{(e_1, \{ h_1 \}, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{24e1} = \{(e_1, \{ h_1 \}, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{25e1} = \{(e_1, \{ h_2 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{26e1} = \{(e_1, \{ h_2 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{27e1} = \{(e_1, \{ h_2 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{28e1} = \{(e_1, \{ h_2 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{29e1} = \{(e_1, \{ h_2 \}, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{30e1} = \{(e_1, \{ h_2 \}, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{31e1} = \{(e_1, \{ h_2 \}, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{32e1} = \{(e_1, \{ h_2 \}, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{33e1} = \{(e_1, \{ h_3 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{34e1} = \{(e_1, \{ h_3 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{35e1} = \{(e_1, \{ h_3 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{36e1} = \{(e_1, \{ h_3 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{37e1} = \{(e_1, \{ h_3 \}, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{38e1} = \{(e_1, \{ h_3 \}, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{39e1} = \{(e_1, \{ h_3 \}, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{40e1} = \{(e_1, \{ h_3 \}, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{41e1} = \{(e_1, \{ h_1 , h_2 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{42e1} = \{(e_1, \{ h_1 , h_2 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{43e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{44e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{45e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{46e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{47e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{48e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{49e1} = \{(e_1, \{ h_1 , h_3 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{50e1} = \{(e_1, \{ h_1 , h_3 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{51e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{52e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{53e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{54e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{55e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{56e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{57e1} = \{(e_1, \{ h_2 , h_3 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{58e1} = \{(e_1, \{ h_2 , h_3 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{59e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{60e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{61e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{62e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_1 , c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{63e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_1 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{64e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_2 , c_3 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{65e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, U_2)\}$

$DP_{66e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \emptyset)\}$

$DP_{67e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_1 \})\}$

$DP_{68e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_2 \})\}$

$DP_{69e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_3 \})\}$

$DP_{70e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_1 , c_2 \})\}$

$DP_{71e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_1 , c_3 \})\}$

$DP_{72e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_2 , c_3 \})\}$

$DP_{73e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, U_2)\}$

$DP_{74e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_1 \})\}$

$DP_{75e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_2 \})\}$

$DP_{76e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_3 \})\}$

$DP_{77e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_1 , c_2 \})\}$

$DP_{78e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_1 , c_3 \})\}$

$DP_{79e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_2 , c_3 \})\}$

$DP_{80e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, U_2)\}$

$DP_{81e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \emptyset)\}$

$DP_{82e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_1 \})\}$

$DP_{83e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_2 \})\}$

$DP_{84e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_3 \})\}$

$DP_{85e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_1 , c_2 \})\}$

$DP_{86e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_1 , c_3 \})\}$

$DP_{87e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_2 , c_3 \})\}$

$DP_{88e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, U_2)\}$

$DP_{89e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \emptyset)\}$

$DP_{90e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_1 \})\}$

$DP_{91e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_2 \})\}$

$DP_{92e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_3 \})\}$

$DP_{93e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_1 , c_2 \})\}$

$DP_{94e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_1 , c_3 \})\}$

$DP_{95e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_2 , c_3 \})\}$

$DP_{96e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, U_2)\}$

$DP_{97e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \emptyset)\}$

$DP_{98e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_1 \})\}$

$DP_{99e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_2 \})\}$

$DP_{100e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_3 \})\}$

$DP_{101e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_1 , c_2 \})\}$

$DP_{102e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_1 , c_3 \})\}$

$DP_{103e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_2 , c_3 \})\}$

$DP_{104e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, U_2)\}$

$DP_{105e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \emptyset)\}$

$DP_{106e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_1\})\}$

$DP_{107e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_2\})\}$

$DP_{108e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_3\})\}$

$DP_{109e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_1, c_2\})\}$

$DP_{110e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_1, c_3\})\}$

$DP_{111e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_2, c_3\})\}$

$DP_{112e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, U_2)\}$

$DP_{113e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \emptyset)\}$

$DP_{114e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_1\})\}$

$DP_{115e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_2\})\}$

$DP_{116e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_3\})\}$

$DP_{117e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_1, c_2\})\}$

$DP_{118e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_1, c_3\})\}$

$DP_{119e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_2, c_3\})\}$

$DP_{120e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, U_2)\}$

$DP_{121e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \emptyset)\}$

$DP_{122e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_1\})\}$

$DP_{123e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_2\})\}$

$DP_{124e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_3\})\}$

$DP_{125e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_1, c_2\})\}$

$DP_{126e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_1, c_3\})\}$

$DP_{127e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_2, c_3\})\}$

Example 4.2

Let$U_1 = \{ h_1 , h_2 \}$ $U_2 = \{ c_1 , c_2 \}$ $E = \{ e_1 , e_2 \}$

$| DS(U_1, U_2)_E |$ =2$(2^4) - 1 = 31$

$X_{DS} = \{(e_1, U_1, U_2), (e_2, U_1, U_2)\}$

$DP_{1e1} = \{(e_1, U_1, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{2e1} = \{(e_1, U_1, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{3e1} = \{(e_1, U_1, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{4e1} = \{(e_1, U_1, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{5e1} = \{(e_1, \emptyset, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{6e1} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{7e1} = \{(e_1, \emptyset, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{8e1} = \{(e_1, \emptyset, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{9e1} = \{(e_1, \{ h_1 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{10e1} = \{(e_1, \{ h_1 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{11e1} = \{(e_1, \{ h_1 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{12e1} = \{(e_1, \{ h_1 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{13e1} = \{(e_1, \{ h_2 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{14e1} = \{(e_1, \{ h_2 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{15e1} = \{(e_1, \{ h_2 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{16e1} = \{(e_1, \{ h_2 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{17e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, U_2)\}$

$DP_{18e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \emptyset)\}$

$DP_{19e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_1 \})\}$

$DP_{20e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_2 \})\}$

$DP_{21e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, U_2)\}$

$DP_{22e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_1 \})\}$

$DP_{23e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_2 \})\}$

$DP_{24e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, U_2)\}$

$DP_{25e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \emptyset)\}$

$DP_{26e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{c_1\})\}$

$DP_{27e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{c_2\})\}$

$DP_{28e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, U_2)\}$

$DP_{29e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \emptyset)\}$

$DP_{30e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{c_1\})\}$

$DP_{31e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{c_2\})\}$

Example 4.3

Let $U_1 = \{h_1, h_2\}$ $U_2 = \{c_1, c_2\}$ $E = \{e_1, e_2, e_3\}$

$|DS(U_1, U_2)_E| = 3(2^4) - 2 = 46$

$X_{DS} = \{(e_1, U_1, U_2), (e_2, U_1, U_2), (e_3, U_1, U_2)\}$

$DP_{1e1} = \{(e_1, U_1, U_2), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{2e1} = \{(e_1, U_1, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{3e1} = \{(e_1, U_1, \{c_1\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{4e1} = \{(e_1, U_1, \{c_2\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{5e1} = \{(e_1, \emptyset, U_2), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{6e1} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{7e1} = \{(e_1, \emptyset, \{c_1\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{8e1} = \{(e_1, \emptyset, \{c_2\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{9e1} = \{(e_1, \{h_1\}, U_2), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{10e1} = \{(e_1, \{h_1\}, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{11e1} = \{(e_1, \{h_1\}, \{c_1\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{12e1} = \{(e_1, \{h_1\}, \{c_2\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{13e1} = \{(e_1, \{h_2\}, U_2), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{14e1} = \{(e_1, \{h_2\}, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{15e1} = \{(e_1, \{h_2\}, \{c_1\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{16e1} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{17e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, U_2), (e_3, \emptyset, \emptyset)\}$

$DP_{18e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{19e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_1 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{20e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_2 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{21e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, U_2), (e_3, \emptyset, \emptyset)\}$

$DP_{22e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_1 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{23e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_2 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{24e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,1}\}, U_2), (e_3, \emptyset, \emptyset)\}$

$DP_{25e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,1}\}, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{26e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,1}\}, \{ c_1 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{27e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,1}\}, \{ c_2 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{28e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,2}\}, U_2), (e_3, \emptyset, \emptyset)\}$

$DP_{29e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,2}\}, \emptyset), (e_3, \emptyset, \emptyset)\}$

$DP_{30e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,2}\}, \{ c_1 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{31e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_{\,2}\}, \{ c_2 \}), (e_3, \emptyset, \emptyset)\}$

$DP_{32e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, U_1, U_2)\}$

$DP_{33e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, U_1, \emptyset)\}$

$DP_{34e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, U_1, \{ c_1 \})\}$

$DP_{35e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, U_1, \{ c_2 \})\}$

$DP_{36e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, U_2)\}$

$DP_{37e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \{ c_1 \})\}$

$DP_{38e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \emptyset, \{ c_2 \})\}$

$DP_{39e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_{\,1}\}, U_2)\}$

$DP_{40e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_{\,1}\}, \emptyset)\}$

$DP_{41e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_{\,1}\}, \{ c_1 \})\}$

$DP_{42e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_{\,1}\}, \{ c_2 \})\}$

$DP_{43e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_{\,1}\}, U_2)\}$

$DP_{44e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_{\,1}\}, \emptyset)\}$

$DP_{45e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_1\}, \{c_1\})\}$

$DP_{46e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset), (e_3, \{h_1\}, \{c_2\})\}$

Example 4.4

Let $U_1 = \{h_1, h_2, h_3\}$ $U_2 = \{c_1, c_2\}$ $E = \{e_1, e_2\}$

$|DS(U_1, U_2)_E| = 2(2^5) - 1 = 63$

$X_{DS} = \{(e_1, U_1, U_2), (e_2, U_1, U_2)\}$

$DP_{1e1} = \{(e_1, U_1, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{2e1} = \{(e_1, U_1, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{3e1} = \{(e_1, U_1, \{c_1\}), (e_2, \emptyset, \emptyset)\}$

$DP_{4e1} = \{(e_1, U_1, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

$DP_{5e1} = \{(e_1, \emptyset, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{6e1} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{7e1} = \{(e_1, \emptyset, \{c_1\}), (e_2, \emptyset, \emptyset)\}$

$DP_{8e1} = \{(e_1, \emptyset, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

$DP_{9e1} = \{(e_1, \{h_1\}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{10e1} = \{(e_1, \{h_1\}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{11e1} = \{(e_1, \{h_1\}, \{c_1\}), (e_2, \emptyset, \emptyset)\}$

$DP_{12e1} = \{(e_1, \{h_1\}, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

$DP_{13e1} = \{(e_1, \{h_2\}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{14e1} = \{(e_1, \{h_2\}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{15e1} = \{(e_1, \{h_2\}, \{c_1\}), (e_2, \emptyset, \emptyset)\}$

$DP_{16e1} = \{(e_1, \{h_2\}, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

$DP_{17e1} = \{(e_1, \{h_3\}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{18e1} = \{(e_1, \{h_3\}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{19e1} = \{(e_1, \{h_3\}, \{c_1\}), (e_2, \emptyset, \emptyset)\}$

$DP_{20e1} = \{(e_1, \{h_3\}, \{c_2\}), (e_2, \emptyset, \emptyset)\}$

$DP_{21e1} = \{(e_1, \{h_1, h_2\}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{22e1} = \{(e_1, \{ h_1 , h_2 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{23e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{24e1} = \{(e_1, \{ h_1 , h_2 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{25e1} = \{(e_1, \{ h_1 , h_3 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{26e1} = \{(e_1, \{ h_1 , h_3 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{27e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{28e1} = \{(e_1, \{ h_1 , h_3 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{29e1} = \{(e_1, \{ h_2 , h_3 \}, U_2), (e_2, \emptyset, \emptyset)\}$

$DP_{30e1} = \{(e_1, \{ h_2 , h_3 \}, \emptyset), (e_2, \emptyset, \emptyset)\}$

$DP_{31e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_1 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{32e1} = \{(e_1, \{ h_2 , h_3 \}, \{ c_2 \}), (e_2, \emptyset, \emptyset)\}$

$DP_{33e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, U_2)\}$

$DP_{34e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \emptyset)\}$

$DP_{35e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_1 \})\}$

$DP_{36e2} = \{(e_1, \emptyset, \emptyset), (e_2, U_1, \{ c_2 \})\}$

$DP_{37e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, U_2)\}$

$DP_{38e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_1 \})\}$

$DP_{39e2} = \{(e_1, \emptyset, \emptyset), (e_2, \emptyset, \{ c_2 \})\}$

$DP_{40e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, U_2)\}$

$DP_{41e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \emptyset)\}$

$DP_{42e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_1 \})\}$

$DP_{43e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1\}, \{ c_2 \})\}$

$DP_{44e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \emptyset)\}$

$DP_{45e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_1 \})\}$

$DP_{46e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2\}, \{ c_2 \})\}$

$DP_{47e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, U_2)\}$

$DP_{48e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \emptyset)\}$

$DP_{49e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{ c_1 \})\}$

$DP_{50e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{c_2\})\}$

$DP_{51e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_3\}, \{c_2, c_3\})\}$

$DP_{52e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, U_2)\}$

$DP_{53e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \emptyset)\}$

$DP_{54e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_1\})\}$

$DP_{55e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_2\}, \{c_2\})\}$

$DP_{56e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, U_2)\}$

$DP_{57e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \emptyset)\}$

$DP_{58e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_1\})\}$

$DP_{59e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_1, h_3\}, \{c_2\})\}$

$DP_{60e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, U_2)\}$

$DP_{61e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \emptyset)\}$

$DP_{62e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_1\})\}$

$DP_{63e2} = \{(e_1, \emptyset, \emptyset), (e_2, \{h_2, h_3\}, \{c_2\})\}$

Conclusion

This study presents new notions of the soft set theory that is dual soft set, which is depends over a fixed set of parameters E of the elements of universes sets U_1 and U_2 and introduces some basic definitions such as gives two kind of definition of union and intersection dual soft union and extension union , dual soft intersection and extension intersection .we have adopted the dual soft union and dual soft intersection in the thesis to complicate the second type as we have seen in the examples of dual soft union and extension union ,dual soft intersection, extension intersection . Through this study we found several types of dual soft point and we use the type I that is $(e, F(e), G(e))$. As we notice that each kind of the dual soft points is dual soft sets. but the singular dual soft point is not dual soft set. so when we deal with singular soft point and their relationship with the dual soft set on the basis of classical belong to with difference. The number of all dual soft points:

$DS(U_1, U_2)_E$ is $|DS(U_1, U_2)_E| = [n_1(2^{n_2+n_3})] - (n_1 - 1)$

When $n_1 = |E|$, $n_2 = |U_1|$, $n_3 = |U_2|$

We know in the general topology the intersection of two topological spaces represent a topological space , unlike the union it doesn't necessary represent a topological space and the same case verified in the dual soft topological space, and that is we have discussed through examples .

As in general topology we found in dual soft topological spaces

$$DS_i\ (A_{FG})\ \cup_{DS}\ DS_i(B_{F1G1})\ \subseteq DS_i\ (A_{FG}\ \cup_{DS}\ B_{F1G1})$$

$$DS_{cl}\ (A_{FG}\ \cap_{DS}\ B_{F1G1})\ \subseteq\ DS_{cl}\ (A_{FG}) \cap_{DS}\ DS_{cl}(B_{F1G1})$$

but the opposite trend did not materialize as illustrated by the examples,

the notions dense set , nowhere dense set, somewhere dense set are achievable in dual soft topological spaces and we introduce the most important properties we also found some properties that link concepts dual soft super condensed set ,semi condensed set and condensed set with dual soft dense and dual soft nowhere dense set.

We also represent the most important ideas about dual soft Baire space and dual soft resolvability spaces but in general terms due the complexity of the ideas.

Future work

The current study represent a starting point for the framework to expand the study of different aspects of dual soft topological spaces such us compactness on dual soft topological space and connectedness on dual soft topological space . we can also link the dual soft topological space with the fuzzy topology This study is just a beginning of new structure and we have studied the main ideas, it will necessary to carry out more theorical research to establish general framework for the practical application

[1] Goguen, J. A. (1973). LA Zadeh. Fuzzy sets. Information and control, vol. 8 (1965), pp. 338–353.-LA Zadeh. Similarity relations and fuzzy orderings. Information sciences, vol. 3 (1971), pp. 177–200. The Journal of Symbolic Logic, 38(4), 656-657.

[2] Molodtsov, D. (1999). Soft set theory—first results. Computers & mathematics with applications, 37(4-5), 19-31.

[3] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. Computers & mathematics with applications, 45(4-5), 555-562.

[4] Kong, Z., Gao, L., & Wang, L. (2009). Comment on "A fuzzy soft set theoretic approach to decision making problems". Journal of computational and applied mathematics, 223(2), 540-542. [5] Aktaş, H., & Çağman, N. (2007). Soft sets and soft groups. Information sciences, 177(13), 2726-2735.

[6]Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. Computers & Mathematics with Applications, 57(9), 1547-1553.

[7] Majumdar, P., & Samanta, S. K. (2010). Generalised fuzzy soft sets. Computers & Mathematics with Applications, 59(4), 1425-1432.

[8] Almohammed, R. (2020). New Concepts of Fuzzy Local Function: Types of fuzzy local function. Baghdad Science Journal, 17(2), 0515-0515.

[9] Abd Al Razzaq, A. S., & Al Swidi, L. A. A. H. (2021, February). An Analytical View for The Combination between The Soft Topological Spaces and Fuzzy Topological Spaces. In Journal of physics: Conference series (Vol. 1804, No. 1, p. 012125). IOP Publishing.

[10] Ali, R. D., Al-Swidi, L. A., & Hadi, M. H. (2022). On fuzzy intense separation axioms in fuzzy ideal topological space. Journal of Interdisciplinary Mathematics, 25(5), 1357-1363.

[11] Chang, C. L. (1968). Fuzzy topological spaces. Journal of mathematical Analysis and Applications, 24(1), 182-190.

[12] Çağman, N., Karataş, S., & Enginoglu, S. (2011). Soft topology. Computers & Mathematics with Applications, 62(1), 351-358.
[13] Ali, M. I., Shabir, M., & Naz, M. (2011). Algebraic structures of soft sets associated with new operations. Computers & Mathematics with Applications, 61(9), 2647-2654.
[14] Hussain, S., & Ahmad, B. (2011). Some properties of soft topological spaces. Computers & Mathematics with Applications, 62(11), 4058-4067.
[15Hussain, S., & Ahmad, B. (2015). Soft separation axioms in soft topological spaces. Hacettepe journal of mathematics and statistics, 44(3), 559-568.

[16] Cagman, N., & Enginoglu, S. (2011). FP-soft set theory and its applications. Ann. Fuzzy Math. Inform, 2(2), 219-226.

[17] Razzaq, A. S. A. A., & Al Swidi, L. A. A. H. (2019). Soft Generalized Vague Sets: An application in Medical Diagnosis. Indian Journal of Public Health Research & Development, 10(2), 901-7.

[18]Al-Swidi, L. A., & Awad, F. S. (2018, October). Analysis on the soft Bench points. In 2018 International Conference on Advanced Science and Engineering (ICOASE) (pp. 330-335). IEEE. [19] Hadi, M. H., & Al-Swidi, L. A. A. (2022). The Neutrosophic Axial Set theory. Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering, 295.

[20] Abd Al Razzaq, A. S., & Al Swidi, L. A. A. H. (2021, February). An Analytical View for The Combination between The Soft Topological Spaces and Fuzzy Topological Spaces. In Journal of physics: Conference series (Vol. 1804, No. 1, p. 012125). IOP Publishing.

[21] Mustafa, H. I., & Sleim, F. M. (2014). Soft generalized closed sets with respect to an ideal in soft topological spaces. Applied Mathematics & Information Sciences, 8(2), 665.

[22Saraf, S. (2013). Survey or review on soft set theory and development. The SIJ Transactions on Computer Science and Engineering and its applications (CSEA), 1(3), 59-66.

[23] Al-Swidi, L. A., & Awad, F. S. (2018, October). Analysis on the soft Bench points. In 2018 International Conference on Advanced Science and Engineering (ICOASE) (pp. 330-335). IEEE.

[24] Al-Swidi, L. A., & Awad, F. S. (2018). On soft turning points. Baghdad science journal, 15(3), 0352-0352.

[25] Abd Oon, A. S., AL-Swidi, A. A.L.: "On Fuzzy y-Separation Axioms in Fuzzy Topological Spaces" . Degree of Master of Science in Math. Babylon University. Dep.pf Math, Coll. Educ. Pure Sci. (2009).

[26]AL-aamery ,T.F., AL-Swidi, A. A.L.:Resolvability In Soft Topological Space . Degree of Master of Science in Math. Babylon University. Dep.pf Math, Coll. Educ. Pure Sci. (2016).

[27]Abbas,S.A., AL-Swidi, A. A.L.: Resolvable In Soft ideal Topological Space . Degree of Master of Science in Math. Babylon University. Dep.pf Math, Coll. Educ. Pure Sci.(2017).

[28]Abbas, R. H. , AL-Swidi, A. A.L.: Combination of fuzzy sets with local functions . Degree of Master of Science in Math. Babylon University. Dep.pf Math, Coll. Educ. Pure Sci. (2019).

[29]Aballah ,M. M., AL-Swidi, A. A.L.: Fuzzy Positional Function . Degree of Master of Science in Math. Babylon University. Dep.pf Math, Coll. Educ. Pure Sci. (2020).

[30] Al-Swidi, L. A., Hadi, M. H., & Ali, R. D. (2022). About the dual soft sets theory. Journal of Discrete Mathematical Sciences and Cryptography, 25(8), 2717-2722.

[31] Isomichi, Y. (1971). New concepts in the theory of topological space—supercondensed set, subcondensed set, and condensed set. Pacific Journal of Mathematics, 38(3), 657-668.

[32] Al-Swidi, L. A., Qahtan, G. A., Hadi, M. H., & Omran, A. A. (2022). On the soft Ds-dense and Dc-dense in soft ideal topological spaces. Journal of Interdisciplinary Mathematics, 25(5), 1341-1346.

[33] Shymapada Modak (2011)Remark on Dense Set.Journal of International Mathematics Forum ,6(44),2153-2158.

[34] King, B. S. (1994). Some sets of first category. Topology and its Applications, 56(2), 105-115..

[35] Rose, D., Sizemore, K., & Thurston, B. (2006). Strongly irresolvable spaces. International journal of mathematics and mathematical sciences, 2006..

[36] Ganster, M., Reilly, I., & Vamanamurthy, M. K. (1987). Dense sets and irresolvable spaces. Ricerche di matematica, 36(2), 163-170.

[37] Ganster, M., Reilly, I., & Vamanamurthy, M. K. (1987). Dense sets and irresolvable spaces. Ricerche di matematica, 36(2), 163-170.

مستخلص الدراسة

الهدف من هذه الدراسة هو إدخال فضائات جديدة تسمى الفضائات الطوبولوجية الناعمة المزدوجة التي يتم تعريفها على مجموعتين شاملتين مع مجموعة ثابتة من المعلمات بدأت دراستنا تدريجيا من خلال تحديد اعمام جديد للمجموعة الناعمة المزدوجة الاتحاد الناعم المزدوج، والتقاطع الناعم المزدوج، والمتممة الموسعة، والنقطة المفردة الناعمة المزدوجة، من أجل بناء الفضاء الطوبولوجي الناعم المزدوج بالإضافة إلى ذلك قمنا بتعميم المجموعة المفتوحة الناعمة المزدوجة والمجموعة المغلقة الناعمة المزدوجة

وتبلورت فكرة الدراسة في ثلاث فقرات :

تحتوي الفقرة الأولى على مفاهيم جديدة مثل الجوار الناعم المزدوج، والنقطة الداخلية الناعمة المزدوجة، والنقطة الخارجية الناعمة المزدوجة، ونقطة الإغلاق الناعمة المزدوجة، ونقطة الحد الناعمة المزدوجة، والنقطة الملتصقة الناعمة المزدوجة، والنقطة المعزولة الناعمة المزدوجة، والتحقيق في خصائص هذه المفاهيم. وبعض العلاقات الهامة بين هذه المفاهيم وكذلك تعريف الفضاء الفرعي الناعم المزدوج

اما في الفقرة الثانية، نقوم بتعميم تعريف المجموعة المكثفة الفائقة الناعمة المزدوجة، والمجموعة شبه المكثفة الناعمة المزدوجة، والمجموعة المكثفة الناعمة المزدوجة، ويحقق في خصائص هذه المفاهيم. كما قدمنا تعريف مجموعة الحدود الناعمة المزدوجة كذلك قدمنا تصنيفا للمجموعة الناعمة المزدوجة إلى ثلاث فئات.

وفي الفقرة الثالثة، قدمنا مفاهيم المجموعة الكثيفة الناعمة المزدوجة، والمجموعة غير الكثيفة المزدوجة الناعمة في أي مكان، والمجموعة الكثيفة المزدوجة الناعمة في مكان ما. تم التحقيق في أهم خصائص وخصائص هذه المفاهيم والعلاقات.

ناقشت هذه الدراسة أيضا تعريف فضائات باير الناعمة المزدوجة وكذلك مجموعة هزيلة ناعمة مزدوجة، ومجموعة مزدوجة ناعمة غير هزيلة، ومتممة المجموعة مزدوجة ناعمة الهزيلة مع اهم الخصائص.

أخيرا، قمنا بتعميم المفاهيم على النحو التالي: فضائات مزدوجة قابلة للحل، و فضائات مزدوجة ناعمة غير قابلة للحل، وراثية مزدوجة ناعمة غير قابلة للحل، ومزدوجة ناعمة متصلة للغاية، ومزدوجة ناعمة دون الحد الأقصى، مع بعض الخصائص والنتائج.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

حول الفضاءات التبولوجيات المرنة المزدوجة

رسالة مقدمة إلى مجلس كلية التربية للعلوم الصرفة – جامعة بابل
كجزء من متطلبات نيل درجة الماجستير في
التربية / الرياضيات

اعداد
جنات ياسين رشيد حميد القرعاوي

بإشراف
أ.د. لؤي عبد الهاني جبارعلي السويدي

2023م ---------- 1444 هـ