Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Babylon
College of Education for Pure Sciences
Department of Mathematics

Almost-Injective and Almost-Projective Semimodules

A Dissertation
Submitted to the Council of the College of Education for Pure Sciences /University of Babylon in Partial Fulfillment of the Requirements for the Degree of Doctor of philosophy in Education / Mathematics

By

Khitam Sahib Hamzah Abed

Supervised by

Prof. Dr. Asaad Mohammed Ali Alhossaini

2023 A.D. **1444 A.H.**

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

يَا أَيُّهَا

الَّذِينَ آمَنُوا إِذَا قِيلَ لَكُمْ تَفَسَّحُوا فِي الْمَجَالِسِ فَافْسَحُوا يَفْسَحِ اللَّهُ لَكُمْ ۖ

وَإِذَا قِيلَ انشُزُوا فَانشُزُوا يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا

الْعِلْمَ دَرَجَاتٍ ۚ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ

صَدَقَ اللَّهُ الْعَلِيُّ الْعَظِيمُ

[سورة المجادلة ، اية(١١)]

Acknowledgements

First of all praise and thanks go to Allah Almighty, I am most grateful to my supervisor Professor Asaad Mohammed Ali Alhossaini for his recommendation, guidance, useful suggestions, and encouragement throughout the period I actually have worked underneath his direction. I also wish to express my thanks to the staff of the Department of Mathematics. I additionally want to thank the College of Education for Pure Sciences for giving me the opportunity to pursue my graduate work.

Finally, I might wish to thank to my family and my friends for encourage and supporting me throughout the product of this work.

Declaration Associated with this thesis

i. K. S. Aljebory, A. M. Alhossaini, „Almost Injective Semimodules„ Iraqi J. Sci. Accepted for publication. 2023, 64(9).

ii. K. S. Aljebory, A. M. Alhossaini , „Almost Projective Semimodules„ Baghdad Sci. J., 2023.

iii. K. S. Aljebory, Alhossaini AM. „Almost self-injective Semimodules„ Int. J. of Health Sci.2022, 6(S5).5565-5573.

iv. K. S. Aljebory, A. M. Alhossaini „Almost self-projective Semimodules„ J. of Discrete Math. Sci. and Cryptography. Accepted for publication, October, 2023.

v. K. S. Aljebory, A. M. Alhossaini „ Some Concepts Related to Almost Injective(Projective)Semimodules, Journal of University of Babylon for Pure and Applied Sciences, 31(1), 2023.

Contents

List of Symbols and abbreviations

Symbol	The meaning
R	Semiring
M	Semimodule
Hom (M, N)	The set of all homomorphisms from M into N
$\mathcal{S} = End_R(M)$	The semiring of all homomorphisms from M into itself.
$\cong$	Isomorphic
$\oplus$	Direct sum
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}_n$	The set of integers modulo n
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\leq$	Subsemimodule
$\leq_e$	Essential subsemimodule
E(M)	The injective hull of a semimodule M
ker (ψ)	The kernel of a homomorphism ψ
$\psi(M)$	The image of a homomorphism ψ
Im(ψ)	Extending image of ψ
$\longmapsto$	Image of element
$\Longrightarrow$	The first direction also Implication
$\Longleftarrow$	The second direction
Soc(M)	The socle of semimodule M
$\mathcal{Z}(M)$	The singular subsemimodule of M

Rad(M)	The Jacobson radical of M
ann(M)	Annihilator of M
CS-semimodule	Extending semimodule
■	The end of proof

Abstract

In this work, injective (resp. projective) semimodule has been generalized to almost-injective (resp. almost projective) and almost self-injective (resp. almost projective) semimodule. The aim of this work is to study some generalizations of injective and projective semimodules also the basic properties of these concepts for semimodule and some concepts related to them. Let M and N be two semimodules, a semimodule M is called almost N-injective semimodule if, for each subsemimodule A of N and each homomorphism $\xi: A \to M$, either there is a homomorphism $\zeta: N \to M$ such that $\zeta i = \xi$, or there is a homomorphism $\gamma: M \to Y$ such that $\gamma\xi = \pi$, where Y is a nonzero direct summand of N, and π is the projection map. A semimodule M is almost injective semimodule if it is almost injective relative to all semimodules. An R-semimodule M is called almost self-injective, if it is almost M-injective semimodule. Some related concepts have been studied and investigated as well. The relationship of these concepts to some concepts have also been clarified as, uniform semimodule, π-injective semimodule, W (ʆ), Z(ʆ) and Rad(ʆ).

Also this work introduces and investigates the dual notion to almost injective semimodule which is almost projective semimodule that is expanded for projective semimodules taking into account the differences between modules and semimodules, it is mainly derived from their definitions. A semimodule M is said to be almost N-projective, if for each epimorphism $\alpha: N \longrightarrow X$ and every homomorphism $\delta: M \longrightarrow X$, either there is $\psi: M \longrightarrow N$ such that $\alpha\psi = \delta$, or there is $\gamma: Y \longrightarrow M$ where Y is a nonzero direct summand of N such that $\delta\gamma = \alpha|_Y$. An R-semimodule M is called almost self-projective, if it is almost M-projective semimodule. Some results on this notion have been obtained. Also, almost self-injective

semiring has been studied. The concepts, generalized *N*- injective, generalized *N*- projective, essentially *N*-injective semimodule, QI-semiring (every quasi-injective semimodule is injective) and AQI-semiring (every almost self-injective semimodule is injective) have been presented as concepts related to almost injective(resp. projective) semimodules

Introduction

One of the most important areas of study of mathematics is algebra and one of its important branches is ring theory. One of the topics discussed at the present time that is module which have a relationship with to rings theory which attention of many researchers at this time, because of its importance in many applications. Since every ring is a semiring, and every module is a semimodule therefore, so can convert many concepts from modules into semimodules with necessary action.

Semimodule over semiring can be defined similarly as module over ring. The book, ″The theory of semirings with applications in mathematics and theoretical computer science″ by J. S. Golan [1], can be considered as a first book on semiring written with algebraic point of view having computer applications. It is very helpful to work on semirings and semimodules and it has several applications. Semirings and semimodules and their applications grow in different branches of mathematics, computer sciences, physics and many other areas of modern science.[1]

In this work, some generalizations of injective and projective semimodules is defined and studied. In 1989, Harada and Tozaki defined " almost M-projective module" which generalized from the concept "M-projective" also they introduced some relations between this concept and a weaker condition[2]. In the same year, Baba introduced the concept "almost N-injective module" [3], the class of modules N, for which M is almost N- injective, is not closed under direct sums. Baba gave necessary and sufficient condition under which a uniform, finite length module U is almost V-injective, where V is a finite direct sum of uniform, finite length modules.

Later, Singh (2016) has determined some conditions under which U is almost V- injective([4], Theorem 1.12) which generalize Baba's result. Also Alahmad discussed some characterizations of almost-injective modules[5]. In 1990 Harada discussed almost M-projective and almost M- injective for any finitely generated module[6].

As regards semimodule, in 1998 Huda Althani gave an analog definition of projective semimodules, which reduces to that in module theory. Also she studied the structure of injective semimodules[7]. The concepts of injective and projective semimodules have been extended to some generalizations, almost-injective, almost projective, almost self-injective and almost self-projective semimodules. Also some characterizations of these notions will be discussed and some concepts related to them. Also the conditions which are needed to get properties and characterizations similar or related to the case in modules will be discussed.

Throughout this work, R denotes a semiring with identity and M a unitary left R-semimodule. This thesis consists of five chapters.

Preliminary results and some definitions that we needed in this thesis are provided in Chapter One. Some different facts between module and semimodule have been clarified as, direct sum of semimodules, injective hull, direct summand of free semimodule need not be projective unlike

module theory, also in module theory closed and complement submodule are equivalent. But in class of semimodules the two concepts are not

equivalent. There are important properties that semimodule possesses make up for having a subtraction in module as the essential difference them like, cancellative, subtractive and semisubtractive.

The Second Chapter is divided into three sections, one for introduction, in the second section almost-injective semimodule is defined. Some characterization of it and other concepts with related to this concept are discussed. In the third section the concept almost self-injective semimodule has been dealt as a generalization of quasi-injective semimodule and some of the results have been generalized. From the results obtained in this chapter, Proposition 2.2.18 by adding properties (having injective hull and semisubtractive), Lemmas 2.2.20 and 2.2.21 with (subtractive, cancellative and semisubtractive), Lemma 2.2.22(2) with semisubtractive and cancellative, Proposition 2.2.27 and Corollary 2.2.28 with semisubtractive and cancellative. Remark 2.3.11 and Proposition 2.3.12 by adding injective hull, Corollary 2.3.13 shows adding a condition to indecomposable to become uniform semimodule. In Proposition 2.3.15 and Corollary 2.13.16 have been added conditions semisubtractive and cancellative for semimodule. Propositions 2.3.18, 2.3.21 and 2.3.22 show some properties of the endomorphism semiring, also Proposition 2.3.23 has been added cancellative and semisubtractive for semimodule.

The Third Chapter consists of three sections, one for introduction, section two contains the definition of the concept almost-projective semimodules and some of its characterizations have been discussed, as well as some results have been generalized from projective semimodules. Section three includes the concept almost self-projective semimodule. In fact this notion is a generalization of quasi-projective semimodule as well as investigating some properties of this concept. Some conditions have been added for semimodules to get desired results similar to results found in module. From the results gotten in this chapter, Proposition 3.2.10 explain an important characteristic which is finite direct sum of almost N-projective semimodule. It is known that, every projective semimodule is almost-projective but the convers is not true. In proposition 3.2.21, we put the conditions for two semimodules M and N to get N-projective from almost N-projective. In Proposition 3.3.16, if M is almost self-projective uniserial which is indecomposable, then $\mathcal{S}$ is a left uniserial semiring. Propositions 3.3.19 and 3.3.20 explain relationship between the concepts almost self-projective semimodule and mini-injective $\mathcal{S}$-semimodule

The Fourth Chapter comprise of three sections, section one included the introduction. In section two almost-injective semiring defined as follows: a semiring R is said to be almost self-injective semiring, if ${}_RR$ is almost self-injective semimodule. In addition some notions related to almost-injective semiring have presented.

In section three, the concepts " generalized N- injective, generalized N-projective, essentially N-injective, " have been defined and some of relationships with each other are discussed. Some concepts related to almost-injective (resp. almost projective) semimodules have been discussed. The semiring having the properties that every quasi- injective semimodule is injective (QI-semiring) and every almost self-injective semimodule is injective (AQI-semiring) have been defined. Other relationship between these concepts are discussed. From the results are obtained in this chapter, Proposition 4.2.3 conditions have been added into semiring R which are yoked and cancellative, Proposition 4.2.4 with conditions, injective hull, semisubtractive and cancellative, Propositions 4.2.10 and 4.2.13 with conditions yoked and cancellative. The Fifth Chapter includes the conclusions and future works.

These concepts have also been defined for class of semimodules depending on literatures of modules with adding more conditions suitable to the case of semimodules.

1.1 Basic Preliminaries

This chapter, some concepts of semirings and semimodules will be represented. Some definitions, remarks and propositions that are needed will be introduced. Some conditions on semimodule that are used to compensate for subtraction in modules are proposed.

Definition 1.1.1[1,p.1]: A **monoid** is a nonempty set S with operation is associative and for all s in S, there is element e in S such that $s\times e= e \times s =s$. That is a monoid is a semigroup with identity element.

Remarks 1.1.2:

(1) The set of natural numbers $\mathbb{N}$ under addition is a commutative monoid with identity element 0. Also $\mathbb{N}\backslash\{0\}$ is a commutative monoid under multiplication.

(2) Every monoid is semigroup, but the converse is not true, for example, the set of positive integers under addition is a semigroup but not monoid.

Definition 1.1.3[6]: A **semiring** is a triple $(R, +, .)$ consisting of a nonempty set R with two operations, addition and multiplication satisfying the following conditions:

(1) $(R,+)$ is a commutative monoid with identity 0.

(2) $(R, .)$ is monoid with identity 1.

(3) Multiplication distributes over addition from either sides.

(4)$0r = 0 = r0$ for all r in R.

If the monoid $(R, .)$ is commutative, then R is called a commutative semiring.

Examples 1.1.4:

(1) Every ring is semiring but the converse is not true, for instance($\mathbb{N}$, +,.) is a commutative semiring which is not a ring.

(2) The tropical semiring ($\mathbb{R} \cup \{+\infty\}$, min, +, ∞, 0) is a commutative semiring with identity element 0.[1, p.16]

Definition 1.1.5 [8]: A nonempty subset I of a semiring R is called left **ideal**, if it satisfies the following conditions:

(1)If $x, y \in I$, then $x+y \in I$.

(2)If $x \in I$ and $r \in R$, then $rx \in I$. A right ideal is defined similar to the left ideal. I is called ideal of R if it is both left and right ideal.

Remarks 1.1.6:

(1) Some of ideals of a semiring ($\mathbb{N}$, +,.) of the form $k\mathbb{N}$, k =0, 1, 2,… and {0, k, k+1, k+2,…}[1,p. 66].

(2)The set of all ideals of R is denoted by $ideal(R)$. Note that ($ideal(R)$, +, .) forms a semiring where $I + J = \{i+j \mid i \in I, j \in J\}$and $I.J=\{\sum_{k=1}^{n} i.j \mid$ for any $i \in I, j \in J\}$, for any $I, J \in$ ideal(R) [1, p. 72].

Definition 1.1.7 [9, p. 8]: A semiring R is called **yoked**, if for all r, t in R, there is an element s in R such that $r + s = t$ or $t + s = r$.

Definition 1.1.8[10]: A left ideal I of a semiring R is said to be **subtractive** if e, $e+h \in I$ imply $h \in I$. A semiring R is called subtractive if each of its ideals is subtractive.

Definition 1.1.9[1, p.3]: A subset E of a semiring R is called **subsemiring**, if it contains 0 and 1 and closed under addition and multiplication.

Example 1.1.10: The set $(\mathbb{Z}^+, +, .)$ of nonegative integers is subsemiring of a semiring $(\mathbb{Q}^+, +, .)$ of set of nonnegative rational numbers.

Definition 1.1.11[1, p.49]: An element r in R is called **additively cancellable** element, if for every s, t in R such that $r+s = r+t$, then s= t. The set of cancellable elements of R is denoted by $K^+(R)$. If $K^+(R)= R$, then R is called cancellative semiring.

Definition 1.1.12[1, p.3]: An element r in R is called **additively idempotent** element, if $r+r = r$. The set of all additively idempotent elements denoted by $I^+(R)$. If $I^+(R)=R$, then R is called an additively idempotent semiring.

Definition 1.1.13[1, p.3]: An element r in R is called a **multiplicatively idempotent** element, if $r \cdot r = r$. The set of multiplicatively idempotent

elements of R is denoted by $I^\times(R)$. If $I^\times(R)=R$, then R is called idempotent semiring.

Definition 1.1.14 [1, p.48]: An element r in R is called **additive inverse**, if there is t in R such that $r+t= 0$. The set of additive inverse of R is denoted by $V(R)$. If $V(R)=\{0\}$, then R is called **zerosumfree.**

Remarks 1.1.15:

(1) If $V(R)= R$, then R is ring.

(2) Every additively idempotent semiring is zerosumfree, in fact, if $r+s = 0$, then $r = r+0 = r+(r+s)=(r+r)+s = r+s=0$, similarly, we have $s=0$ and hence R is zerosumfree.

Definition 1.1.16[1, p.4]: An element r in R is called a left **zero divisor** of R if and only if rs=0 for some nonzero element s of R, and it is right zero divisor if and only if sr=0. It is zero divisor if and only if either a left or a right zero divisor. If R has no zero devisors, then it is called **entire**.

Definition 1.1.17[**9**]: A semiring R is called **semidomain**, if $rs = rt$ implies that $s = t$ for r, s, t in R and r is nonzero element.

Remark 1.1.18:[**10**]: Every semidomain is entire but the convers is not true, if R is an entire yoked cancellative semiring , then it is semi-domain.

Examples 1.1.19:

(1) Every ring is cancellative semiring.

(2) ($ideal(R)$, +, .) forms zerosumfree semiring.

(3) The set ($\mathbb{N}$, +, .) of nonnegative integers is a commutative cancellative zerosumfree entire semiring which is not ring.

(4) The Boolean algebra $\mathbb{B}$={0, 1}is an idempotent semiring.

(5) ($\mathbb{N}$, +, .) with $n+ m= \max\{n, m\}$ and $n. m= \min\{n, m\}$for all n, m in R is an idempotent commutative semiring.

Definition 1.1.20[**11**]:A commutative semiring R is said to be **semifield** if, every nonzero element has a multiplicative inverse. That is ($R\backslash\{0\}$, .) is a group.

Example 1.1.21: The set ($\mathbb{R}^+$, +, .) of positive real numbers is a semi-field. Also ($\mathbb{Q}^+$, +, .) is a semifield.

Definition 1.1.22 [1, p.149]: Let R be a semiring. A left R **-semimodule** M is a commutative monoid (M, +) for which we have a map $R\times M\longrightarrow M$

defined by $(r, m) \longmapsto r\, m$ such that for all r, r' in R and m, n in M, the following conditions are satisfied

(1) $(r\, r')\, m = r\, (r'\, m)$.

(2) $r\, (m + n) = r\, m + r\, n$.

(3) $(r + r')\, m = r\, m + r'\, m$.

(4) $r0_M = 0_M = 0_R\, m$.

The semimodule M is called unitary if the condition $1n = n$, for all n in M.

<u>Definition 1.1.23</u> [1, p.150]: A nonempty subset U of a semimodule M is called a **subsemimodule** of M, if it is closed under addition and scalar multiplication and denoted by $U \leq M$.

<u>Definition 1.1.24</u>[1, p.149]: A subsemimodule U of a semimodule M is called **subtractive** if each m, n in M with $m, m + n$ in U implies n belong to U. A semimodule M is said to be subtractive, if all its subsemimodules are subtractive.

<u>Definition 1.1.25</u>[12]: A semimodule M is said to be **semisubtractive**, if for any $x, y \in M$ there is $z \in M$ such that $x + z = y$ or $y + z = x$.

<u>Examples 1.1.26:</u>

(1) Every semiring is semimodule over itself.

(2) Every commutative monoid is an $\mathbb{N}$- semimodule.

(3) If R is a semiring, then the left ideals of R are exactly the subsemimodules of the left R-semimodule R.

(4) In the semimodule (ℕ, +, .) the subtractive subsemimodules are of the form kℕ [13, p.32]. Also it is cancellative semisubtractive semimodule.

Definition 1.1.27[**14**]**:** Let X and Y be two subsemimodules of R-semimodule M. M is said to be a **direct sum** of X and Y, if each $m \in M$ uniquely written as $m= x+ y$ where $x\in X$, $y\in Y$, denoted by $M=X\oplus Y$, each X and Y is called a direct summand of M.

Remark 1.1.28: If M is a direct sum of subsemimodules X and Y, then $X\cap Y=0$ and $M=X+Y$, but the converse is not true. For example, let $K_4=\{0, 1, a, b\}$, then (K_4 ,+) is monoid with addition defined by; $x+ x= x$, $x+ y=1$ and $x+1=1$ for all $x, y\in K_4$), K_4 is $\mathbb{B}$ -semimodule where $\mathbb{B}$ is the Boolean algebra with 1+1=1. Take a subsemimodule $X=\{0, 1, a\}$, then $X=\{0, 1\}+\{0, a\}$ and $\{0,1\}\cap\{0, a\}=\{0\}$ but 1 can be written by, 1=1+0 and 1= 1+a this means 1 has no uniquely representation. The convers is true if the semimodule M has the conditions; semisubtractive and cancellative[14].

While, for the modules, its known that if, a module B is direct sum of submodules C and D, then $C\oplus D$ if and only if $C+D= B$ and $C\cap D=0$.

Definition 1.1.29[**15**]**:**A set E is said to be **generating set** of a semimodule M if, M is the smallest subsemimodule containing E, written $M=\langle E\rangle$.

Remarks 1.1.30[**15**]**:**

(1)A semimodule $M=\langle E\rangle$ if and only if $\forall n\in M$, $n=\sum_{j=1}^{k} r_j e_j$, $r_j\in R$, $e_j\in E$.

(2)If $E=\{x\}$ and $M=\langle E\rangle$, then M is called a **cyclic** semimodule generated by element x [10].

Definition1.1.31[15]: A set E is called a **free set** if for each $\{e_1,e_2,e_3,\ldots,e_k\}\subseteq E$ (k is a positive integer), the combination $\sum_{j=1}^{k} r_j e_j=0$ implies $r_j=0\ \forall j$.

Definition 1.1.32[15]: A set E is called a **basis** of the semimodule M if it is a free generating set of M.

Definition 1.1.33[15]: A semimodule M is called a **free** if it has a basis.

Remark 1.1.34[13]**:** An R-semimodule M is **free** if and only if it is isomorphic to a direct sum of copies of R.

Example 1.1.35[13]**:**The R-semimodule R is free (by Remark 1.1.34).

Definition 1.1.36[16]**:** A subsemimodule X of M is said to be **essential**, if $X\cap Y \neq 0$ for any nonzero subsemimodule Y of M and denoted by $X \leq_e M$.

Examples1.1.37:

(1) Every semimodule is essential in itself.

(2)$\mathbb{Z}$ as $\mathbb{N}$-semimodule is essential subsemimodule of $\mathbb{Q}$ as $\mathbb{N}$-semimodule.

(3) In $\mathbb{Z}_{12}$ as $\mathbb{N}$ −semimodule, a subsemimodule $2\mathbb{Z}_{12} \leq_e \mathbb{Z}_{12}$.

Definition 1.1.38[17]: A subsemimodule X of M is said to be **small**, if $X+Y \neq M$ for any proper subsemimodule Y of M and denoted by $X \leq_s M$.

Examples 1.1.39:

(1)$\{0\}$ is the only small subsemimodule of the $\mathbb{N}$ -semimodule $\mathbb{Z}$.

(2) In $\mathbb{Z}_4$ as $\mathbb{N}$-semimodule, $2\mathbb{Z}_4 \leq_s \mathbb{Z}_4$.

Definition 1.1.40[18]: A semimodule M is said to be **uniform** if, every nonzero subsemimodule of it is essential.

Examples 1.1.41:

(1) The $\mathbb{N}$- semimodule $\mathbb{Z}$ is uniform.

(2) $\mathbb{Z}_8$ as $\mathbb{N}$ −semimodule is uniform.

Definition 1.1.42[19]: A semimodule M is said to be **hollow**, if every proper subsemimodule of it is small.

Examples 1.1.43:

$\mathbb{Z}_4$ as $\mathbb{N}$-semimodule is hollow.

Definition 1.1.44 [**20**]: A semimodule M is said to be **indecomposable** if the direct summands of its are only $\{0\}$ and M.

It is clear that, every uniform semimodule is indecomposable.

Definition 1.1.45 [21]: Let M be an R -semimodule and m in M. The left **annihilator** of m is defined by $ann(m)=\{ r \in R \mid rm = 0\}$.

It is clear that $ann(m)$ is a left ideal of R. If U is a subsemimodule of M , then $ann(U) =\{ r \in R \mid ru = 0, u\in U \}$. For instance, in a semimodule $\mathbb{Z}_{12}$ over itself, $ann(\bar{2})=\{\bar{0}, \bar{6}\}= ann(\overline{10})$, $ann(\bar{3})=\{ \bar{0}, \bar{4}, \bar{8}\}= ann(\bar{9})$, $ann(\bar{4}) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}= ann(\bar{8})$, $ann(\bar{6})=\{ \bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$, $ann(\bar{1})=ann(\bar{5}) = ann(\bar{7}) =ann(\overline{11})= \bar{0}$.

Definition 1.1.46 [**22**] :A nonzero semimodule M is called **simple**, if M has no nonzero proper subsemimodule.

Remarks 1.1.47:

(1)The $\mathbb{N}$-semimodule $\mathbb{N}/_{p\mathbb{N}}$ where p is prime number is simple semimodule.

(2) Every simple semimodule is indecomposable but the converse is not true, for example $\mathbb{N}$ as $\mathbb{N}$-semimodule is indecomposable, but not simple since have subsemimodules of the form k$\mathbb{N}$ where k in $\mathbb{N}$

Definition 1.1.48 [23]: A semimodule *M* is called **semisimple**, if it is a direct sum of simple subsemimodules in *M*.

Remark 1.1.49[24]: A semimodule *M* is semisimple if every subsemimodule of it is a direct summand in *M*.

Remarks 1.1.50:

(1) The semimodule $\mathbb{Z}_6$over $\mathbb{N}$ is semisimple.

(2) Every simple semimodule is semisimple, but the converse is not true, for example $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule is semisimple but not simple semimodule.

Definition 1.1.51 [25]: Let *M* be a semimodule, the **socle** of *M* is the sum of all simple subsemimodule of *M* and denoted by Soc(*M*). If *M* has no simple subsemimodule, then Soc(*M*)=0.

Remarks 1.1.52: Let *M* be a semimodule. Then:

(1)Soc(*M*) = $\cap$ {*X*| $X \leq_e M$}[25].

(2) If Soc(*M*)= *M*, then *M* is semisimple semimodule. For example, $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule, Soc($\mathbb{Z}_6$)= $\mathbb{Z}_6$, then it is semisimple.

Definition 1.1.53 [21]: The **singular** subsemimodule of *M* is defined by $\mathcal{Z}(M)$ = { $m \in M$| $\mathrm{ann}_R(m)$ is an essential ideal in *R*}. If $\mathcal{Z}(M)$ =*M*, then *M* is called singular semimodule and if $\mathcal{Z}(M)$ = 0, *M* is called nonsingular.

Example 1.1.54: $\mathcal{Z}(\mathbb{Q}_{\mathbb{Z}})$=0, then $\mathbb{Q}_{\mathbb{Z}}$ is nonsingular semimodule, while $\mathcal{Z}(\mathbb{Z}_n)$= $\mathbb{Z}_n$, then $\mathbb{Z}_n$ as $\mathbb{Z}$-semimodule is singular.

<u>Definition 1.1.55</u> [26]: A subsemimodule X of semimodule M is said to be **complement** of a subsemimodule Y if $X \cap Y = 0$ and X is a maximal with this property.

<u>Definition 1.1.56</u> **[27]:** A subsemimodule X of semimodule M is said to be **closed** if, $X \leq_e Y \leq M$ implies $X = Y$.

In module theory closed and complement submodule are equivalent. But in class of semimodules the two concepts do not have to be equivalent, the following remark explain the relationship between these concepts.

<u>Remarks 1.1.57:</u> The following relationship between the concepts, direct summand, closed and complement subsemimodules, hold.

(1) Every complement subsemimodule of M is closed[26].

(2) If M is a cancellative ,semisubtractive semimodule, then every closed subsemimodule of M is a complement[28, p 14].

(3) In general, not every closed subsemimodule of M is direct summand. For example, let $M=\mathbb{Z}_9 \oplus \mathbb{Z}_3$ as $\mathbb{N}$-semimodule with subsemimodules; $A_1 = 0 \oplus \mathbb{Z}_3 = \langle(\bar{0}, \overline{2})\rangle$, $A_2 = \mathbb{Z}_9 \oplus 0 = \langle(\bar{2}, \bar{0})\rangle = \langle(\bar{4}, \bar{0})\rangle = \langle(\bar{5}, \bar{0})\rangle = \langle(\bar{7}, \bar{0})\rangle = \langle(\bar{8}, \bar{0})\rangle$, A_3 $\langle(\bar{3}, 0)\rangle = \langle(\bar{6}, \bar{0})\rangle$, $A_4 = \langle(\bar{1}, \bar{1})\rangle$, $A_5 = \langle(\bar{0}, \bar{1}), (\bar{3}, \bar{0})\rangle = A_1 \oplus A_3 = \langle(\bar{0}, \bar{1}), (\bar{6}, \bar{0})\rangle$, $A_6 = \langle(\bar{3}, \bar{1})\rangle$, $A_7 = \langle(\bar{3}, \bar{2})\rangle = \langle(3, 2)\rangle$ we note that A_1 and A_2 are direct summand of M, A_4 is complement of A_1 hence closed, but not a direct summand. Also A_4 is complement of A_6 and A_7 but not direct summand of it. If M is (k-c) that is every closed subsemimodule is subtractive, then every direct summand of M is closed [28, p.14].

<u>Definition 1.1.58</u> **[25]:** A semimodule M is said to be **extending** semimodule (CS-semimodule) if every subsemimodule of M is essential in a direct summand of M.

Proposition 1.1.59 [25]: A semimodule M is extending (CS)-semimodule if and only if, every closed subsemimodule is a direct summand of M.

Remark 1.1.60:

The semimodule $\mathbb{Z}\oplus\mathbb{Z}_2$ as $\mathbb{N}$ −semimodule is extending semimodule.

Definition 1.1.61 [15]: Let M be a semimodule. An element m in M is said to be **torsion element** if $rm=0$ for some $0 \neq r$ in R. We say M is **torsion** semimodule if each element is torsion.

Definition 1.1.62 [27]: A semimodule M is said to be **torsion-free**, whenever $r \in R$ and $m \in M$ with $rm=0$ implies that $r = 0$ or $m = 0$.

Example 1.1.63: The semimodule $\mathbb{Q}_{\mathbb{N}}$ is torsion free.

Definition 1.1.64 [15]: Let U be a subsemimodule of a semimodule M, a semimodule M/U is called a **quotient** semimodule of M by U and defined by $M/U=\{[m] | m\in M\}$.

Definition 1.1.65 [12]: **The radical** of a semimodule M defined by Rad(M)= $\bigcap\{ U$: U is maximal subsemimodule of $M\}$.

Remark 1.1.66 [12]: Rad(M)=$\cap\{ U$: U is maximal subsemimodule in $M\}$= $\sum\{S$: S is small subsemimodule in $M\}$. For instance, take a semimodule M= $\mathbb{Z}_8$ over itself Rad(M)=$\{ \bar{0}, \bar{2}, \bar{4}, \bar{6}\}= < \bar{2} >$.

Definition 1.1.67 [13, p.9]: Let E be subset of R-semimodule M. The set of all elements of the form $V = \sum_{e\in E} r_e e (r_e \in R)$ such that all but a finite number of terms in the sum are zero, then V is called the subsemimodule of M generated by E, denoted by $V =< E >$. If there is finite subset of R-semimodule M such that $V =< E >$, then M is called finitely generated.

Proposition 1.1.68 [12]: Let M be a semimodule, if M is finitely generated, then Rad(M) is **superfluous** (small) subsemimodule of M.

Definition 1.1.69 [29]: A semiring R is called a left **local** semiring if it has a unique maximal left ideal. For example, a semiring $\mathbb{Z}_4$ is local and a semiring $(\mathbb{N}, +, .)$ with maximal ideals is not local semiring.

Theorem 1.1.70[29]: Let R be a semiring, then the following statements are equivalent:

(a) R is local.

(b)Rad(R) is a maximal ideal of R.

(c)The set of all non-invertible elements is an ideal of R.

(d) Rad(R) = $\{r \in R |\ r$ is non-invertible $\}$.

Definition 1.1.71[30]: Let R be a semiring. An element $r \in R$ is called **nilpotent** if there is positive integer n such that $r^n = 0$. An ideal $\mathcal{J}$ of R is called nil if each of its elements is nilpotent .

Definition 1.1.72[18]: A semimodule M is said to be **uniserial** if for any two subsemimodules of M, one of them contained in the other.

Examples 1.1.73:

(1)Every simple semimodule is uniserail.

(2)$\mathbb{Z}_8$ as $\mathbb{N}$-semimodule is uniserail.

Definition 1.1.74[31]: Let M and N be R-semimodules. A **homomorphism** from M to N is a map $\varphi: M \longrightarrow N$ such that:

1- $\varphi(m+n) = \varphi(m) + \varphi(n)$ and

2- $\varphi(rm) = r\varphi(m)$ $\forall\, m, n \in M$ and $r \in$R.

The set of all R-homomorphisms from M to N, denoted by Hom(M, N) has R-semimodule structure for the operations:

(1)$(\alpha + \varphi)(m) = \alpha(m) + \varphi(m)$, $m \in M$ and $\alpha, \varphi \in$ Hom(M, N).

(2)$(r\,\alpha)(m) = r\alpha(m)$, $r \in R$ [15].

An R-homomorphism from M into itself is called an R-endomorphism of M, denoted by End(M) for the set of all R-endomorphisms of M. The triple (End(M), +, °) is a semiring with identity element [32]. It is clear that (End(M), +) is an ℕ- semimodule.

Remark 1.1.75 [33]: Let α:$M\longrightarrow N$ be an R-homomorphism, then:

1.ker(α)= {$m\in M$| $\alpha(m)$= 0} = $\alpha^{-1}\{0\}$ which is subtractive subsemimodule of M.

2. $\alpha(M)$ ={ $\alpha(m)$ | $m\in M$}

3.Im(α)={ $n\in N$ |n + $\alpha(m)$= $\alpha(m')$ for some m, $m'\in M$}which is subtractive subsemimodule of N.

Definition 1.1.76 [**32**]**:** A subsemimodule X of M is called **fully invariant** if for each endomorphism $\emptyset$:$M\rightarrow M$, then $\emptyset(X)\subseteq X$.

Example 1.1.77: Every subtractive subsemimodule of ℕ as ℕ-semimodule is fully invariant. Since the only subtractive subsemimodules of ℕ are of the form k ℕ, $k\in$ ℕ let $\emptyset$:ℕ→ ℕ, then $\emptyset(k\mathbb{N})= k\emptyset(\mathbb{N})\subseteq k\mathbb{N}$.

Definition 1.1.78 [**34**]**:** The sequence of R-semimodules $\ldots\longrightarrow X_i\overset{\alpha_i}{\rightarrow} X_{i+1} \overset{\alpha_{i+1}}{\longrightarrow} X_{i+2} \longrightarrow\ldots$ is said to be:

1.Exact, if Im α_i =ker α_{i+1}, $\forall i\in I$

2.Proper exact, if $\alpha_i(X_i)$= ker $\alpha_{i+1}, \forall i\in I$.

Remarks1.1.79: Let α:$M\longrightarrow N$ be an R-homomorphism,

1. It is well known in module theory ker α={0} if and only if α is monic, this is not hold in general for semimodule, take α:$K_4\longrightarrow X$ defined by $\alpha(0)$ = 0, $\alpha(a)$=a and $\alpha(b)$= $\alpha(1)$ = 1 where K_4={0, 1, a, b} is 𝔹-semimodule

(the scalar multiplication of $\mathbb{B}$ on K_4 is defined by 0.s=0 and 1.s=s, ∀s∈ K_4)and its subsemimodule X ={0, 1, a}, then ker α={0} but α is not monic[34]. In [13], if M is semisubtractive and N is cancellative we have ker α={0} if and only if α is monic.

2. For module always, $\alpha(M)$ =Im(α) but for semimodule not always true, if we take the inclusion map of the function in (1) i: $X \longrightarrow K_4$ since $i(X) = X$ and Im(i) = K_4. $\alpha(M)$ =Im(α) if and only if $\alpha(M)$ is subtractive subsemimodule of N[8].

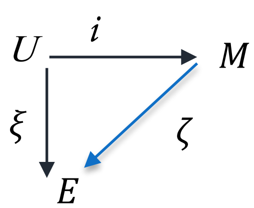
Definition1.1.80[8]: If M and N are R- semimodules, then an R-homomorphism α: :$M \longrightarrow N$ is:

1. Surjective (epic, or onto) if for each element n in N, there is element in M such that $\alpha(m) = n,$ that is Im(α) $= N$.

2. Injective (monic, or one to one) if for all m, m' in M, if $\alpha(m) = \alpha(m')$implies $m = m'$.

3. Isomorphism if is both surjective and injective.

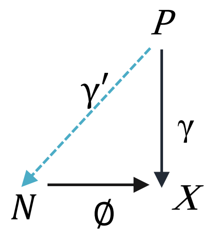
Definition 1.1.81[35]: Let M be a semimodule, an R-semimodule E is called M-**injective** (E is injective relative to M) if for every subsemimodule U of M and any R- homomorphism ξ:$U \longrightarrow E$ can be extended to an R- homomorphism ζ: $M \longrightarrow E$.



A left R-semimodule E is called injective if it is injective relative to every left R-semimodule**.** A semimodule E is called **quasi-injective** if it is E-injective[14].

<u>Remark 1.1.82</u>[13, p.17]: Every injective semimodule is quasi-injective but the converse is not true. For example, $\mathbb{Z}/2\,\mathbb{Z}$ as $\mathbb{N}$-semimodule is quasi-injective but it is not injective.

<u>Definition 1.1.83</u>[36]: A left R-semimodule P is said to be N**-projective** if for every epimorphism $\phi: N \rightarrow X$ and for every homomorphism γ:$P\rightarrow X$ there is a homomorphism γ': $P\rightarrow N$ such that the following diagram commutes:



A semimodule P is **projective** if it is projective relative to every left R-semimodule, and it is **quasi-projective** if, P is P-projective semimodule [37]. It is clear that every simple semimodule is quasi-projective

<u>Proposition 1.1.84</u>[7]: Every free R-semimodule is projective. In particular every semiring with identity is projective over itself.

<u>Remark 1.1.85:</u>The convers of Proposition 1.1.84 is not true, for instance, if R= $\mathbb{Z}/_{10\mathbb{Z}}$. Obviously, R is free over itself and $\mathbb{Z}/_{10\mathbb{Z}}$= $10\mathbb{Z}/_{2\mathbb{Z}} \oplus 10\mathbb{Z}/_{5\mathbb{Z}}$, $10\mathbb{Z}/_{2\mathbb{Z}}$ as R-semimodule is projective (since every direct summand of projective semimodule is projective) but is not free.

<u>Remark 1.1.86:</u> It is well known in the class of module, the following facts are verified:

(1)Any Projective module is a direct summand of free module, this is not true for semimodule see, Example 2.3 in [36].

(2)Any direct summand of free module is projective, but this is not true for semimodules[36].

In [38], Deore set condition to achieve that property as follows:

Lemma 1.1.87[38]: Any direct summand of semisubtractive free R-semimodule is projective.

Definition 1.1.88[14]: A semimodule M is said to be **principally self-generator** if for every element $m \in M$, there exists an epimorphism $\alpha: M \rightarrow Rm$.

Examples 1.1.89:

(i) Every cyclic semimodule is principally self-generator.

(ii) The semiring R is principally self-generator R-semimodule.

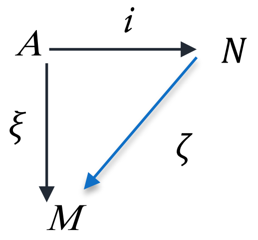
2.1 Introduction

In this chapter, the concept of injective semimodule has been extended to generalizations, almost-injective semimodule and almost self-injective semimodule. Some characterizations of these notions and some concepts related to them will be discussed. Also, the conditions which want to get properties and attributes similar or related to the case in modules will be discussed.

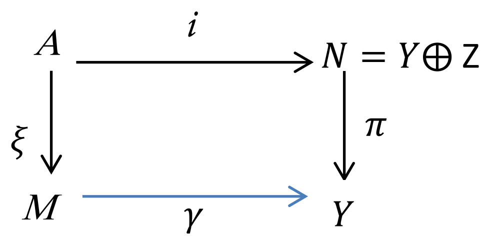
2. 2 Almost Injective Semimodules

In this section, the concept of almost N-injective semimodule will be presented as a generalization of injective semimodule as well as investigating some properties of this notion.

Definition 2.2.1: Let M and N be two left R-semimodules. A semimodule M is called almost N-injective semimodule if, for each subsemimodule A of N and each R-homomorphism $\xi : A \rightarrow M$, either there exists an R-homomorphism ζ such that the diagram(i) commutes



(i)



(ii)

Or there exists a homomorphism $\gamma : M \rightarrow Y$ such that the diagram (ii) commutes, where $0 \neq Y \leq_{\oplus} N$, and π is the projection map.

An R-semimodule M is almost injective semimodule if it is almost injective relative to every R-semimodule.

Examples 2.2.2:

(1) Every injective semimodule is almost injective, but the converse is not true.

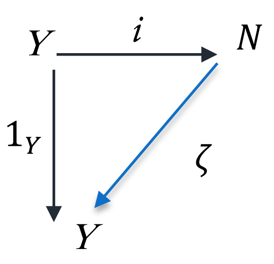
(2) If M is simple, then every semimodule is injective relative to M and hence it is almost M- injective.

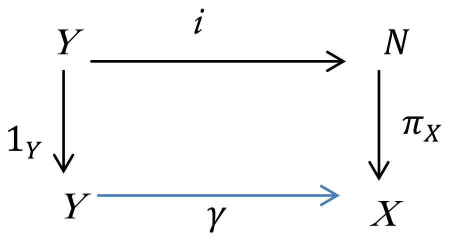
(3) If N is semisimple, then every R-semimodule M is N-injective and then almost N-injective semimodule.

A monomorphism f is called a split monomorphism if it has a left inverse [20].

Proposition 2.2.3: Let Y be a subsemimodule of N such that Y is almost N-injective semimodule, then either Y is a direct summand of N, or there is homomorphism $\gamma: Y \to X$ where X is nonzero direct summand of N such that $\gamma 1_Y = \pi_X i$.

Proof. Assume that Y is an almost N-injective semimodule, from the following diagrams; we have either there is $\zeta: N \to Y$ such that $\zeta i = 1_Y$ implies that i is split monomorphism and hence Y is a direct summand of N, or there is $\gamma: Y \to X$ where X is nonzero direct summand of N such that $\gamma 1_Y = \pi_X i$.





■

Remarks 2.2.4:

In Proposition (2.2.3),

(1)If N is indecomposable semimodule, then the first diagram is not satisfied and hence Y is embedded in N.

Proof. Assume that Y is almost N-injective semimodule, we have either there is $\zeta: N \to Y$ such that $\zeta i= 1_Y$ implies that i is split monomorphism and hence Y is direct summand of N and this contradiction, then the second diagram is satisfied, this means there is $\gamma: Y \to N$ such that $\gamma 1_Y = 1_N i = i$ which implies γ is one to one. ■

(2)If the only proper subsemimodule of N is Y, then we get Y is a direct summand of N.

Recall that, a maximal subsemimodule U of M is a subsemimodule which is not contained properly in any other proper subsemimodule of M [13].

Proposition 2.2.5: A semimodule M is almost N -injective if and only if for each R-homomorphism ξ: $U\to M$ has no extension from N to M where U is subsemimodule of N, there exists decomposition $N=Y\oplus Z$ with $Y \neq 0$ and R-homomorphism ω: $M \to Y$ such that $\omega\, \xi(u) = \pi(u)$ for any u in U , where π: $N \to Y$ is a projection with kernel Z.

Proof. The Definition 2.2.1 implies to the condition is clear. Conversely, let δ: $K\to M$ be an R- homomorphism where K is subsemimodule of N, if δ can be extended to N, it is done, otherwise let U be maximal subsemimodule of N containing K such that ξ: $U\to M$ is extension of δ, by assumption there exists decomposition $N=Y\oplus Z$ with $Y\neq 0$ and R-homomorphism ω: $M \to Y$ such that $\omega\, \xi(u) = \pi(u)$ for any u in U. Therefore M is almost N -injective semimodule. ■

Remark 2.2.6: Let M be almost N-injective semimodule, if N is indecomposable semimodule, then either M is N-injective, or there is subsemimodule of N is embedded in M.

Proof. Assume that M is not N -injective semimodule, then there exists a subsemimodule A of N and homomorphism $\xi : A \rightarrow M$ cannot be extended to N. Hence there exists an R-homomorphism $\gamma: M \rightarrow N$ (since N is indecomposable, then it has no proper direct summand) such that $\gamma\xi = i$. Assume that $\xi(a) = \xi(a')$, where $a, a' \in A \Longrightarrow \gamma\xi(a) = \gamma\xi(a') \Longrightarrow i(a) = i(a') \Longrightarrow a = a'$, then ξ is one to one. ■

Recall that, a semimodule M is said to be maximal essential extension of subsemimodule L if N is a proper extension of M, then N is not essential extension of L. An R-semimodule N is said to be injective hull of semimodule M, if N is injective and it is essential extension of M, denoted by $E(M)$ [19].

Note.[1]. It is well-known that every module over a ring has an injective hull, but this does not hold in general, for semimodules over a semiring.

Lemma 2.2.7: Let M be a torsion-free semimodule and N a semimodule having injective hulls $E(M)$ and $E(N)$ respectively, let θ: $E(N) \rightarrow E(M)$ be a homomorphism and $\theta(N) \nsubseteq M$. If $\mathfrak{l} = \{n \in N | \theta(n) \in M\}$ and $\vartheta = \theta|_{\mathfrak{l}}$, then ϑ has no extension $\alpha: Z \rightarrow M$ with $\mathfrak{l} < Z \leq N$.

Proof. Firstly, $\mathfrak{l} = N \cap \theta^{-1}(M)$. Suppose there is $\alpha: Z \rightarrow M$ with $\mathfrak{l} < Z \leq N$ extending for ϑ. There is z in Z such that $\theta(z) \notin M$, then $\alpha(z) \neq \theta(z)$. Now, since $N \leq_e E(N)$ and $\theta^{-1}(M) \leq_e E(N)$ implies $N \cap \theta^{-1}(M) \leq_e E(N)$, so there is $\mathfrak{r}$ in $\ddot{R}$ such that $0 \neq \mathfrak{r}z \in \mathfrak{l}$ implies $\theta(\mathfrak{r}z) \in M$ and then $\mathfrak{r}\alpha(z) =$

$\theta(ɽz)$ this means $ɽ\alpha(z) = ɽ\theta(z)$ since M is torsion-free, then $\alpha(z) = \theta(z)$ this a contradiction. ■

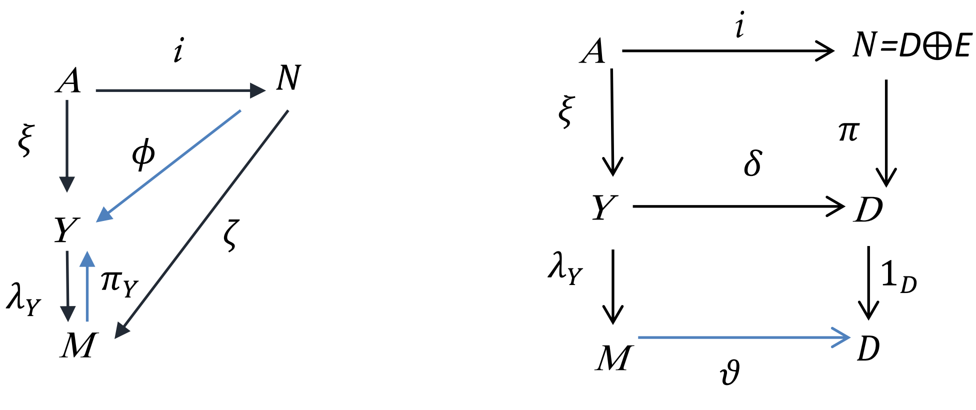
Proposition 2.2.8: Let M be a torsion-free semimodule and N be any semimodule having injective hulls E(M) and E(N) respectively, then M is almost N-injective if and only if for any homomorphism θ: E(N)→ E(M), either $\theta(N) \subseteq M$, or for ɭ$=\{n \in N \mid \theta(n) \in M\}$, there is decomposition Ɲ$=Y \oplus Z$ with $Y \neq 0$ and homomorphism $\omega: M \to Y$ such that $\omega\,\xi = \pi i$ where $\xi = \theta|_{ɭ}$ and $\boldsymbol{i}$:ɭ→N is the inclusion map, $\pi: N \to Y$.

Proof. From Proposition 2.2.5 and Lemma 2.2.7 the result obtained. ■

The following propositions give some properties of almost N-injective semimodules.

Proposition 2.2.9: If M is an almost N-injective semimodule and Y is any direct summand of M, then Y is almost N-injective.

Proof. Let Y be a direct summand of M and consider the following diagrams:

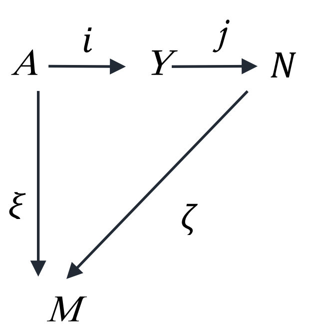


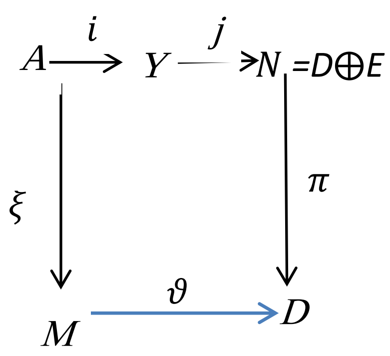
where $A \leq N$ and $\lambda_Y : Y \to M$ be the injection map, since M is almost N-injective, either there exists $\zeta : N \to M$ such that $\zeta i = \lambda_Y\,\xi$. Define $\emptyset: N \to Y$ such that $\emptyset = \pi_Y\,\zeta$, then $\emptyset i = \pi_Y\,\zeta i = \pi_Y\,\lambda_Y\,\xi = \xi$. Or, there exists $\vartheta: M \to$ D where D is nonzero direct summand of N such that $\vartheta\,\lambda_Y\,\xi = \pi$.

Define $\delta: Y \to D$ such that $\delta = \vartheta\lambda_Y$, we have $\delta\xi = \vartheta\lambda_Y \xi = \pi$. Then Y is almost N -injective. ■

Proposition 2.2.10: If M is an almost N -injective semimodule and Y a fully invariant direct summand of N, then M is almost Y-injective .

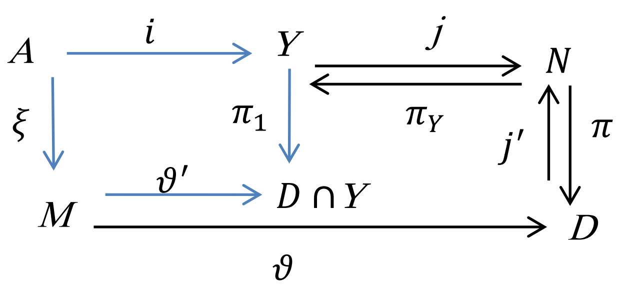
Proof. Suppose Y is a direct summand of N and M is an almost N-injective semimodule. Consider the diagrams:





Since M is almost N –injective, then either $\exists\ \zeta: N \to M$ such that $\zeta ji = \xi$, or $\exists\ \vartheta: M \to D$ where D is a nonzero direct summand of N such that $\vartheta\ \xi$

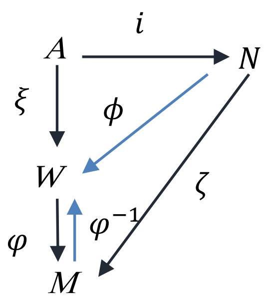
$= \pi ji$. Since $N = D \oplus E$, then $Y = (Y \cap D) \oplus (Y \cap E)$ by fully invariant, we have the following diagram:

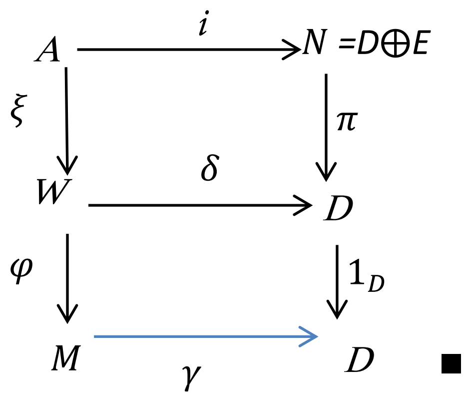


From the above diagram, we get that $\vartheta'\xi = \pi_1\pi_Y\, j'\vartheta\ \xi = \pi_1\pi_Y\, (j'\,\pi) j\ i =$ $\pi_1(\pi_Y j)i = \pi_1 i$. Hence M is an almost Y-injective semimodule. ■

Proposition 2.2.11: Let M be an almost N-injective semimodule, then any semimodule isomorphic to M is almost N-injective.

Proof. Suppose M is an almost N -injective semimodule and φ: $W\rightarrow M$ is an isomorphism where W is any semimodule, assume that $\xi: A\rightarrow W$ is homomorphism, since M is almost N-injective, then either there exists, $\zeta : N \rightarrow M$ such that $\zeta i = \varphi\xi$. Define $\emptyset: N \rightarrow W$ such that $\emptyset = \varphi^{-1}\zeta$, then $\emptyset i = \varphi^{-1}\zeta i = \varphi^{-1}\varphi \xi = \xi$. Or, there exists $\gamma: M \rightarrow D$ where D is nonzero direct summand of N such that $\gamma \varphi\xi = \pi$. Define δ: W $\rightarrow D$ such that $\delta = \gamma\varphi$, we have $\delta\xi = \gamma\varphi\xi = \pi$. Then W is almost N-injective. As in the following diagrams:



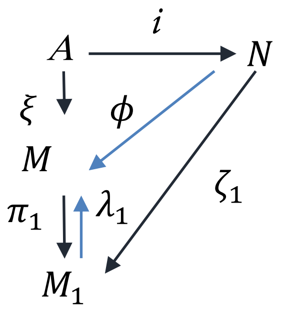


■

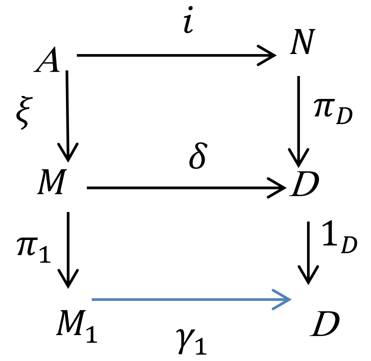
The following proposition explain an important characteristic which is finite direct sum of almost N-injective semimodule. Where the proof was divided into six cases depending on our definition.

Proposition 2.2.12: Let M_1, M_2 and N be semimodules, then $M=M_1\oplus M_2$ is almost N-injective if and only if each M_i (i=1, 2) is almost N-injective semimodule.

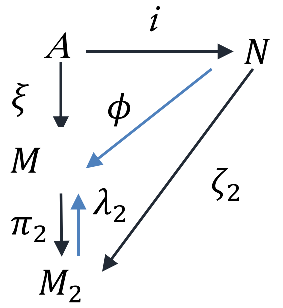
Proof. Consider the below diagrams and let ξ:$A\longrightarrow M$, where A is subsemimodule of N. Suppose each M_i (i=1, 2) is almost N-injective. Then there are six cases: **case1**/ if the diagrams (a) and (b) are satisfying, then the Definition 2.2.1 is achieved **case2**/ the two diagrams (c) and (d) similar to case1. **Case3**/ the two diagrams (a) and (d) are satisfying, there is ζ_1:$N\longrightarrow M_1$ such that $\zeta_1 i= \pi_1\xi$, and there is $\gamma_2 : M_2 \longrightarrow D$ where D is a nonzero direct summand of N such that $\gamma_2\pi_2\xi = 1_D\pi_D i$, define $\delta: M \longrightarrow D$ by $\delta = 1_D\gamma_2\pi_2$, then $\delta\xi = 1_D\gamma_2\pi_2\xi = \pi_D i$.



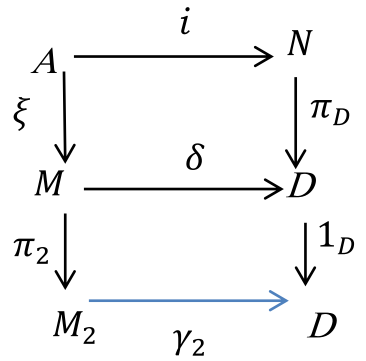
(a)



(b)



(c)



(d)

Case4/suppose the two diagrams (a) and (c) are satisfying, then there is $\zeta_1: N \longrightarrow M_1$ such that $\zeta_1 i = \pi_1\xi$ and $\zeta_2: N \longrightarrow M_2$ such that $\zeta_2 i = \pi_2\xi$, then $(\zeta_1 + \zeta_2)i = \zeta_1 i + \zeta_2 i = \pi_1\xi + \pi_2\xi = (\pi_1 + \pi_2)\xi$.

Case5/ assume the diagrams (b) and (c) are satisfying, similar to case1.

Case6/ suppose that the diagrams (b) and (d) are satisfying, similar to cases 3 and 4. From the previous six cases, M is almost N-injective semimodule. Conversely, suppose that M is almost N-injective, from Proposition 2.2.9, we get M_1 and M_2 are almost N-injective semimodules.■

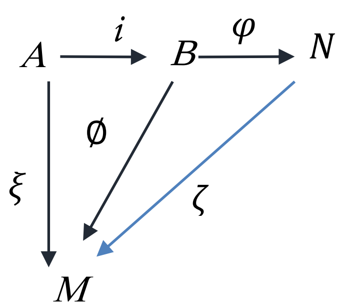
Colloraly 2.2.13: Let $\{M_i\}_{i=1}^{n}$ be a family of semimodules, then M_i is almost N-injective semimodules if and only if $\oplus_i M_i$ is almost N-injective for fixed semimodule N and $1 \leq i \leq n$.

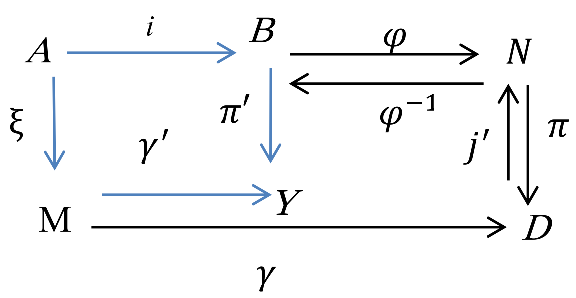
Proof. From Proposition 2.2.12, the result is gotten. ■

Proposition 2.2.14: Let M be an almost N-injective semimodule and B any semimodule which is isomorphic to N, then M is almost B-injective.

Proof. Let M be an almost N–injective semimodule and φ: $B \rightarrow N$ be an isomorphism where B is any semimodule, assume ξ: $A \rightarrow M$ is homomorphism, since M is almost N-injective, then either there exists, $\zeta : N \rightarrow$ M such that $\zeta\varphi i = \xi$. Define $\emptyset: B \rightarrow M$ such that $\emptyset = \zeta\varphi$, then $\emptyset i = \zeta\varphi\, i = \xi$. Or, there exists $\gamma: M \rightarrow D$ where D is nonzero direct summand of N such that $\gamma\, \xi = \pi\varphi i$. Define $\gamma': M \rightarrow Y$ where Y is nonzero direct summand of B, such that $\gamma' = \pi'\varphi^{-1}j'\gamma$, we have

$\gamma'\xi = \pi'\varphi^{-1}(j'\pi)\varphi i = \pi'(\varphi^{-1}\varphi)i = \pi'$. Hence M is almost B-injective semimodule. As the following diagrams:





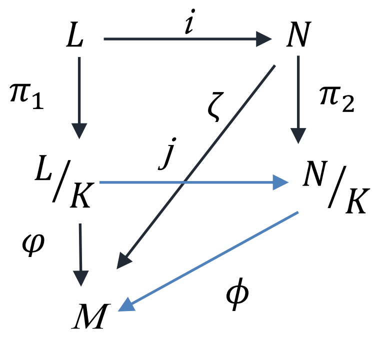
■

Lemma 2.2.15: Let $N = Y \oplus Z$ be semimodule and K a fully invariant subsemimodule of N, then $N/_{K} = (Y+K)/_{K} \oplus (Z+K)/_{K}$.

Proof. It is clear that $N/_K = {(Y+K)}/_K + {(Z+K)}/_K$. Now, to prove the unique representation of the elements of $N/_K$. Since K is fully invariant, then $K = (Y \cap K) + (Z \cap K)$, hence $\forall k \in K$, $k = k_1+k_2$ where $k_1 \in Y \cap K$, $k_2 \in Z \cap K$. Assume that $n+K \in N/_K$, with $n+K = (y+z)+K = (y'+z')+K \ldots (*)$ where $y, y' \in Y+K$ and $z, z' \in Z+K$ it can be assumed that y, $y' \in Y$ and $z, z' \in Z$, then $(y+z)+k = (y'+z')+k'$ for some $k, k' \in K$ by (*) $k = k_1+k_2$ and $k' = k'_1+k'_2$ such that $k_1, k'_1 \in Y \cap K$ and $k_2, k'_2 \in Z \cap K$, $(y+z)+(k_1+k_2) = (y'+z')+(k'_1+k'_2)$ implies $(y+k_1)+(z+k_2) = (y'+k'_1)+(z'+k'_2)$, by unique representation of the elements of $Y \oplus Z$, it follows $y+k_1 = y'+k'_1$, $z+k_2 = z'+k'_2$ where $k_1, k'_1, k_2, k'_2 \in K$, then $y+K = y'+K$ and $z+K = z'+K$, therefore $N/_K = {(Y+K)}/_K \oplus {(Z+K)}/_K$. ■

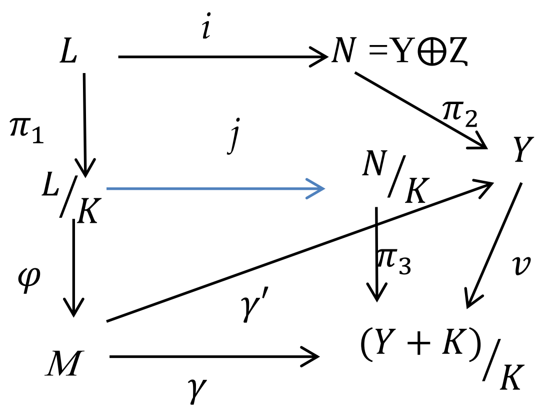
Proposition 2.2.16: If M is an almost N-injective semimodule and K a fully invariant subsemimodule of N, then M is almost $N/_K$ -injective.

Proof. Let $L/_K$ be any subsemimodule of $N/_K$, i.e. $K \leq L \leq N$ and let $\varphi: L/_K \rightarrow M$ be homomorphism . Consider the diagram where i and j are inclusion maps , π_1 and π_2 are natural epimorphisms.



Since M is almost N -injective, either there exists $\zeta: N \to M$ such that $\zeta i = \varphi\pi_1$.Define $\emptyset: {}^{N}/_{K} \to M$ by $\emptyset(n + K) = \zeta(n)$, for each $n + K \in {}^{N}/_{K}$, then

$\emptyset(l + K) = \zeta(l) = \varphi\pi_1(l) = \varphi(l + K)$. Or, there exists $\gamma': M \to Y$, where $0 \neq Y \leq_{\oplus} N$ and $\gamma'\varphi\pi_1 = \pi_2 i$.



Define v:$Y \to {}^{(Y+K)}/_{K}$ by $y \mapsto y+K$ and γ: M $\to {}^{(Y+K)}/_{K}$ by $m \mapsto v\gamma'(m + K)$, then $\gamma = v\,\gamma'$, where ${}^{Y+K}/_{K}$ is a direct summand of ${}^{N}/_{K}$ by Lemma 2.2.15. Then $\gamma\varphi(l + K) = \gamma\varphi(\pi_1(l)) = v(\gamma'\varphi(\pi_1(l))$

$=v(\pi_2 i\,(l)) = v\,(y) = y + K = \pi_3 j(l + K)$, $\forall\, l + K \in L/K$ such that $\gamma\varphi = \pi_3 j$.

Hence M is almost N/K -injective. ■

Remark 2.2.17: If M is a uniform semimodule, then the injective hull of M if there exists is indecomposable.

Proof. Suppose that $E(M) = M_1 \oplus M_2$, if $M_1 \neq 0$, then $0 \neq M_1 \cap M \leq_e M \leq_e E(M)$. But $(M_1 \cap M) \cap M_2 = 0 \Longrightarrow M_2 = 0$ and $E(M) = M_1$, therefore $E(M)$ is an indecomposable. ■

Proposition 2.2.18: Let M and N be uniform semimodules having injective hulls $E(M)$ and $E(N)$ respectively with $E(N)$ is semisubtractive, then M is almost N-injective semimodule if and only if for every $\vartheta \in \mathrm{Hom}(E(N), E(M))$, either $\vartheta(N) \subseteq M$ or ϑ is isomorphism and $\vartheta^{-1}(M) \subseteq N$.

Proof. Assume M is an almost N-injective and let $\vartheta \in \mathrm{Hom}(E(N), E(M))$ and $X = \{b \in N \mid \vartheta(b) \in M\} = N \cap \vartheta^{-1}(M)$, let $h = \vartheta|_X : X \to M$. Since M is almost N -injective, then one of the diagrams (i) or (ii) hold. If (i) holds, there exists $\omega: N \to M$ which extends h to N.

Claim: $Y = \{x \in E(M) \mid x + \omega(b) = \vartheta(b)$ for some $b \in N\} = 0$. Let $x \in M \cap Y$, then $x + \omega(b) = \vartheta(b)$, so $\vartheta(b) \in M$. Hence $b \in X$, so $\vartheta(b) = h(b) = \omega(b)$, this implies $x = 0$ and $M \cap Y = 0$. But M is essential in $E(M)$, so $Y = 0$. Therefore, $\vartheta(b) = \omega(b)$ for all $b \in N$, that is $\vartheta(N) \subseteq M$. If (ii) holds , then there exists $\emptyset : M \to N$ such that $\emptyset\, h = 1_X$. Hence ϑ is one to one (since ker

$\vartheta|_X = \ker\vartheta \cap X = 0 \Rightarrow \ker\vartheta \cap N = 0$ but $N \leq_e E(N)$, then $\ker\vartheta = 0$ hence ϑ is one to one). Also ϑ is onto because $\vartheta(N) \cong E(N)$ and $\vartheta(N)$ is injective subsemimodule of $E(M)$, but $E(M)$ is indecomposable from Remark 2.2.17, then $\vartheta(N) = E(M)$ so ϑ is isomorphism.

Clearly $\emptyset|_{\vartheta(X)}=\vartheta^{-1}|_{\vartheta(X)}$…. (*). Claim: $Z = \{ y \in E(N) | \vartheta^{-1}(a)= y + \emptyset(a)$ for some $a \in M\}= 0$. Let $y \in N \cap Z$, then $\vartheta^{-1}(a)= y + \emptyset(a)$, then $\vartheta^{-1}(a) \in N$, apply ϑ to both sides , we have $\vartheta \vartheta^{-1}(a)= \vartheta(y) + \vartheta\emptyset(a)$

from (*) we get $\vartheta(y) = 0$, then $a \in \vartheta(X)$ and $y = 0$, since N is essential in $E(N)$, we have $Z = 0$ and $\vartheta^{-1}(a)= \emptyset(a)$ for all $a \in M$. Hence $\vartheta^{-1}(M) \subseteq N$. The convers is clear. ■

Remarks 2.2.19:

In case M is a uniform, N is an indecomposable semimodule , then:

(1) If M is almost N-injective and there is a homomorphism $\vartheta : E(N) \to E(M)$ such that $\vartheta(N) \nsubseteq M$, then ϑ is an isomorphism and $M \subseteq \vartheta(N)$ is essential in $E(M)$ implies $N \cong \vartheta(N)$ is uniform(since M is uniform).

(2) If N is not uniform and M is almost N-injective semimodule, then M is N-injective semimodule.

(3) If N is uniform, then M is almost N-injective semimodule if and only if for any homomorphism $\vartheta : E(N) \to E(M)$ with $\vartheta(N) \nsubseteq M$, ϑ is an isomorphism and $M \subseteq \vartheta(N)$.

Lemma2.2.20: Let U and V be subtractive subsemimodules of a cancellative semisubtractive semimodule M, and N any semimodule. If $\alpha: U \to N$ and $\beta: V \to N$ are homomorphisms such that $\alpha(x) = \beta(x)$ for

all x in $U \cap V$, then there is a homomorphism $\gamma: U + V \to N$ which extends both α and β.

Proof. Define $\gamma: U + V \to N$ by $\gamma(u + v) = \alpha(u) + \beta(v)$ where $u \in U$, $v \in V$. Assume that $u + v = u' + v'$….(*), $u, u' \in U$ and $v, v' \in V$, by semisubtractive, there exist x, $y \in M$ such that either u+ x = u′ or u = x + u′ and either v + y= v′ or v′ + y= v.

Claim: In all cases, α (u)+ β(v)= α(u′) +β(v′), that is γ is well-defined. First, by assumption $x \in U$, $y \in V$(by subtractive).

There are four cases, it is enough to discuss one case, the other are similar. Let u′ = x + u and v′+ y= v, then u + v = u′ + v′ implies u + x +v′ = u + v′ + y by cancellative we have $x = y$ and hence x , $y \in U \cap V$, then α(x) = β(y), then α(u) + β (v) = α(u) + β(v′)+β(y) and α(u′) + β (v')= α(u) + α(x) + β(v′). By (*) α(u) + β (v) = α(u′) + β(v′). Hence our claim is true, and γ is an extension of both α and β. ■

<u>Lemma 2.2.21:</u> Let $N = Y \oplus Z$ and M be two semimodules such that N is subtractive, cancellative and semisubtractive and ϑ: $L \rightarrow M$ be an R-homomorphism such that $L < N$, has no extension ψ: $X \rightarrow M$ with $L < X \leq N$. Then, $\vartheta_1 = \vartheta|_{Y \cap L}$ has no extension ψ′: $E \rightarrow M$ with $Y \cap L < E \leq Y$.

<u>Proof</u>. Suppose an extension ψ': $E \rightarrow M$ of ϑ_1 exists where $Y \cap L < E \leq Y$. It is clear that $E \cap L = Y \cap L$ and $L < L+E$. Now for $a \in E \cap L$, $\vartheta(a) = \vartheta_1(a) = \psi'(a)$. By Lemma 2.2.20, the mapping μ: $L + E \rightarrow M$, μ(l + e) = $\vartheta(l)$ + ψ′(e) , $l \in L$, e ∈ E is well defined, on the other hand μ is an extension of ϑ to $L+E$ with $L < L+E$, we have a contradiction. ■

<u>Lemma 2.2.22:</u> Let M and N be any semimodules, and ξ: U→ M be R-homomorphism has no extension from N to M, where U is subsemimodule of N, let $N = Y \oplus Z$ with $Y \neq 0$ and R-homomorphism ω: $M \rightarrow Y$ such that ω ξ(u) = π(u) for any u in U, where π: $N \rightarrow Y$ is a projection map with kernel Z. Then :

(1) ξ is monomorphism on $U \cap Y$ and ξ($U \cap Y$) is closed subsemimodule in M .

(2) If M is semisubtractive and cancellative semimodule, then ker(ω) is complement of $\xi(U \cap Y)$ in M.

(3) $\xi(U \cap Z) \subseteq \ker(\omega)$.

(4) If M is CS semimodule, then $\xi(U \cap Y)$ and ker(ω) are summands of M.

Proof. (1) Since $\omega\, \xi(u) = u$ for any $u \in U \cap Y$, which gives $\xi(U \cap Y) \cap \ker(\omega) = 0$ [if $y \in \xi(U \cap Y) \cap \ker(\omega)$, this mean $\omega(y) = 0$ and $\xi(a) = y$ for some $a \in U \cap Y$, $\omega\, \xi(a) = \omega(y) = 0$, but $\omega\, \xi(a) = a$, hence $y = 0$], let K be subsemimodule of M containing $\xi(U \cap Y)$ with $K \cap \ker(\omega) = 0$. Then $\omega|_K$ is monic and $U \cap Y \subseteq \omega(K) \subseteq Y$. Define $\upsilon: \omega(K) \rightarrow K$, $\upsilon\omega(k) = k$ for any $k \in K$. Then υ extends $\xi|_{U \cap Y}$ and υ is an isomorphism implies that $\xi|_{U \cap Y}$ is monomorphism. By Lemma 2.2.21, $\omega(K) = U \cap Y$ which proves that $\xi(U \cap Y) = K$. Therefore K is complement and then closed implies that $\xi(U \cap Y)$ is closed subsemimodule of M.

(2) Let Ų be a complement of $\xi(U \cap Y)$ containing ker(ω), if $V \leq$ Ų and $V \cap \ker(\omega) = 0$, then$(\xi(U \cap Y) + V) \cap \ker(\omega) = 0$ implies that $\xi(U \cap Y) + V = \xi(U \cap Y)$, then $V = 0$ and hence ker(ω) is essential in Ų. Now ,if $u \in$ Ų and $u \notin \ker(\omega)$, there exists $r \in R$ such that $0 \neq ru \in \omega^{-1}(U \cap Y) \cap$ Ų

(since $\omega^{-1}(U \cap Y) \cap$ Ų$\leq_e$ Ų) implies $0 \neq \omega(ru) \in U \cap Y$. Since $\omega\, \xi(U \cap Y) = U \cap Y$, there is $a \in \xi(U \cap Y)$ such that $\omega(ru) = \omega(a)$, since M is semisubtractive , there exists $v \in M$ and two cases :

Case 1/ $ru = a + v \Longrightarrow \omega(ru) = \omega(a) + \omega(v)$ by cancellative $\omega(v) = 0 \Longrightarrow v \in \ker(\omega)$.

Case2/ $a = v + ru$, similar to case 1/ implies that $v \in \ker(\omega) \subseteq$ Ų and $a \in$ Ų(by semisubtractive) but $a \in \xi(U \cap Y)$, then $a \in \xi(U \cap Y) \cap$ Ų $= 0$ this is a contradiction $\Longrightarrow$ Ų $= \ker(\omega)$ and hence ker(ω) is complement of $\xi(U \cap Y)$.

(3)Let $a \in \xi(U \cap Z) \Longrightarrow \xi(u) = a$ for some $u \in U \cap Z$, $\omega(a) = \omega\,\xi(u) = \pi(u) = 0$ (since $u \in Z$) $\Longrightarrow a \in \ker(\omega) \Longrightarrow \xi(U \cap Z) \subseteq \ker(\omega)$.

(4) Since M is CS-semimodule and both $\xi(U \cap Y)$ and $\ker(\omega)$ are complements hence closed subsemimodules of M, then $\xi(U \cap Y)$ and $\ker(\omega)$ are direct summands of M. ■

Definition 2.2.23 [26]: A semimodule M is called π-injective if for any two subsemimodules D and E of M with $D \cap E = 0$, there exist $\vartheta, \delta \in \mathrm{End}(M)$ such that, ϑ and δ are idempotent; $\vartheta + \delta = 1_M$; $D \subseteq \ker(\vartheta)$ and $E \subseteq \ker(\delta)$.

The above definition is equivalent to:

Remark 2.2.24: A semimodule M is said to be π- injective semimodule if for any two subsemimodules $Ų_1$ and $Ų_2$ of M with $Ų_1 \cap Ų_2 = 0$, the projections π_i: $Ų_1 \oplus Ų_2 \to Ų_1$ or $Ų_2$ can be lifted to an endomorphism of M. In fact, if M is π-injective and $Ų_1$, $Ų_2$ are subsemimodules with $Ų_1 \cap Ų_2 = 0$, then π_i: $Ų_1 \oplus Ų_2 \to Ų_i$ (i=1, 2) can be extended to an idempotent endomorphism with sum equal to 1_M.

Proof. If $Ų_1 \cap Ų_2 = 0$, where $Ų_1$, $Ų_2$ are subsemimodules of M and π_i: $Ų_1 \oplus Ų_2 \to Ų_i$ (i=1, 2) can be extended to idempotent endomorphism of M, then M is π-injective. Since π_1: $Ų_1 \oplus Ų_2 \to Ų_1$ can be extended to $\rho: M \to M$, then $\pi_1(x + y) = x = \rho(x)$ also π_2: $Ų_1 \oplus Ų_2 \to Ų_2$ can be extended to $\sigma: M \to M$, then $\pi_2(x + y) = y = \sigma(y)$, where x in $Ų_1$ and y in $Ų_2 \Longrightarrow \rho|_{Ų1} = \pi_1$ and $\sigma|_{Ų2} = \pi_2$, $\rho(x) + \sigma(x) = x$, $\forall x \in Ų_1$ (since $Ų_1 = \ker(\pi_2) \subseteq \ker(\sigma)$), also $\rho(y) + \sigma(y) = y$, $\forall y \in Ų_2$ ($Ų_2 = \ker(\pi_1) \subseteq \ker(\rho)$) $\Longrightarrow \rho + \sigma = 1_M$. Thus M is π-injective.■

Example 2.2.25. $\mathbb{Z}_{15}$ as $\mathbb{N}$-semimodule is π-injective, since it has only two proper subsemimodules $5\mathbb{Z}_{15}$ and $3\mathbb{Z}_{15}$ with $5\mathbb{Z}_{15} \cap 3\mathbb{Z}_{15} = 0$, there is

$\pi_1: \mathbb{Z}_{15} \to 5\mathbb{Z}_{15}$ with kernel $3\mathbb{Z}_{15}$and $\pi_2: \mathbb{Z}_{15} \to 3\mathbb{Z}_{15}$ with kernel $5\mathbb{Z}_{15}$ and $\pi_1 + \pi_2 = 1_{\mathbb{Z}_{15}}$.

Recall that a semimodule M with injective hull is said to be quasi-continuous if, for each idempotent $\beta \in$ End(E(M)) implies $\beta(M) \subseteq M$.[26]

The concept, quasi continuous semimodule is different from π-injective semimodule. If conditions, cancellation, semisubtractive and subtractive are added for semimodule, then the two concepts will be equivalent[28]. For instance $\mathbb{N}$ as $\mathbb{N}$-semimodule is π-injective which is not quasi-continuous (since it has no injective hull) while $\mathbb{Z}$ as $\mathbb{Z}$ -semimodule π-injective and quasi-continuous. On the other hand, these concepts are equivalent in module.

Lemma 2.2.26[26]: Any π-injective semimodule is extending.

Proposition 2.2.27: Let M be almost N-injective semimodule which is semisubtractive, cancellative, π-injective semimodule, then for any R-homomorphism $\xi: U \to M$ has no extension from N to M, where U is subsemimodule of N,we get:

(1) There is a decompositions $N = Y \oplus Z$, $M = V \oplus W$ with $Y \neq 0$.

(2) ξ is a monomorphism on $U \cap Y$ and $\xi(U \cap Y) = V$.

(3) $\xi(U \cap Z) \subseteq W$.

(4) $U = (U \cap Y) \oplus (U \cap Z)$.

Proof. Assume that M is almost N- injective semimodule. Since M is π-injective semimodule, then it is CS-semimodule by Lemma 2.2.26.

(1) By Lemma 2.2.22 (4), there is a decomposition $N = Y \oplus Z$ with $Y \neq 0$ and R-homomorphism $\omega: M \to Y$ such that ξ is monomorphism on $U \cap Y$, set $V = \xi(U \cap Y)$ and $W = \ker(\omega)$ are direct summands of M, and $\omega\, \xi(u) =$

$\pi(u)$ for any u in U. As V and W are complements of each other, then $M = V \oplus W$.

(2) From Lemma 2.2.22.

(3) From Lemma 2.2.22 replace $\ker(\omega)$ by W.

(4) Let $u \in U$. Then $u = u_1+u_2$ where $u_1 \in Y$, $u_2 \in Z$. Then $u_1 = \omega\,\xi(u) \in \omega(M) = \omega\,\xi(U \cap Y) = U \cap Y$, in the same way $u_2 \in U \cap Z$. Hence $U = (U \cap Y) \oplus (U \cap Z)$. ■

Corollary 2.2.28: Let M be a uniform semimodule which is semisubtractive, cancellative and N any semimodule, then M is almost N-injective if and only if any R- homomorphism $\xi: U \to M$ with no extension from N to M, where U is subsemimodule of N, then the following hold:

(1) There is a decomposition $N = Y \oplus Z$ such that $\xi(U \cap Y) = M$, $Z = \ker \xi$ and $U = (U \cap Y) \oplus Z$.

(2) There is a decomposition $N = Y \oplus Z$ such that ξ is a monomorphism on $U \cap Y$, $\xi(U \cap Y) = M$ and $U = (U \cap Y) \oplus Z$.

Proof. Since M is a uniform semimodule, then it is π-injective.

(1) Suppose M is almost N-injective semimodule. By Proposition 2.2.27 $N = Y \oplus Z$ with $Y \neq 0$, ξ is monic on $U \cap Y$, $\xi(U \cap Y) = M$ and $\xi(U \cap Z) = 0$, so $\xi|_{U \cap Z} = 0$, it can be extended from Z to M, then by Lemma 2.2.21

$U \cap Z = Z$, $U = (U \cap Y) \oplus Z$.

(2) we get an R-homomorphism $\gamma: Z \to U \cap Y$ such that $\forall z \in Z$, $\gamma(z) = y$, whenever $\xi(z) = \xi(y)$. Then $\mathcal{X} = \{ z \in Z, z = \gamma(z)\} \subseteq \ker \xi$ from (1)

$N = Y \oplus \mathcal{X}$. After then use (1) to get the result. Conversely, suppose the condition is given and from Proposition 2.2.27, we get the result. ■

In [39] the concept total quotient semiring which is R-semimodule(quotient semifield) is studied and discussed.

Corollary 2.2.29: Let D be a commutative semidomain and Q a quotient semifield, then D is almost Q_D -injective semimodule.

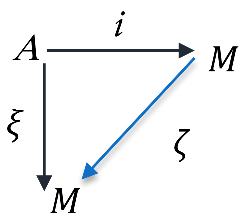
Proof. Let ξ: $U \to$ D has no extension from Q to D, where U is maximal subsemimodule of Q_D, then Q ≠ D , since Q_D is injective, there exists μ: $Q_D \to Q_D$ extension of ξ. Let $Y = \mu^{-1}(D)$, then Y = q D for some q ∈ Q such that μ(q) = 1. It is clear that $U \subseteq Y$. μ (Y) = D, By maximality of ξ, then U = Y and from Corollary 2.2.28 (1), we have D is almost Q_D–injective semimodule.■

2.3 Almost self- injective semimodules

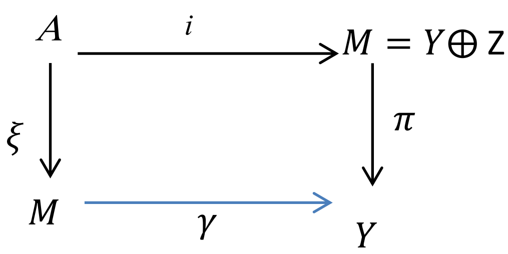
In this section, new generalization of injective semimodule has been presented. An R-semimodule M is called almost self-injective, if M is almost M-injective semimodule. Some properties of this notion have been presented. Also some related notions of this concept have been studied as endomorphism of indecomposable almost self-injective semimodules, and radical of semiring R.

It is mentioned before that the symbol, ʃ denotes End(M), the semiring of endomorphisms of M.

<u>Definition 2.3.1</u>: A semimodule M is said to be almost self-injective, if M is almost M-injective, this means either diagram (i) or diagram (ii) commutes.



(i)



(ii)

<u>Remarks 2.3.2:</u> (1) If M is an indecomposable, almost self-injective semimodule and the diagram(i) of Definition 2.3.1 is not satisfied, then Y=M and $\gamma\xi = 1_M i = i$, this means ξ is a monomorphism.

(2) The following implications hold for any semimodule:

Injective⟶ Almost-injective⟶ Almost self-injective

↓ ↑

Quasi-injective ⟶

Definition 2.3.3 :A valuation semidomain is a semidomain D such that for each element t in semifield of fraction F, either t or t^{-1} belongs to D. It is clear that every semifield is valuation semidomain as $(\mathbb{Q}^+, +, .)$ is semi-field.

Examples 2.3.4:

(1)Notice that by Remarks 2.3.2 (2), every injective semimodule is almost self-injective but the converse is not true, for example $\mathbb{Z}_2$ as $\mathbb{N}$-semimodule is almost self-injective semimodule but not injective.

(2) Every self-injective semimodule is almost self-injective but the converse is not true, for example $\mathbb{N}/_{(p)} \oplus \mathbb{N}/_{(p^2)}$ as $\mathbb{N}$ semimodule where p is a prime number is almost self-injective but it is not self- injective semimodule. Also $\mathbb{N}/_{(p)} \oplus \mathbb{N}/_{(p^2)}$ is not uniform.

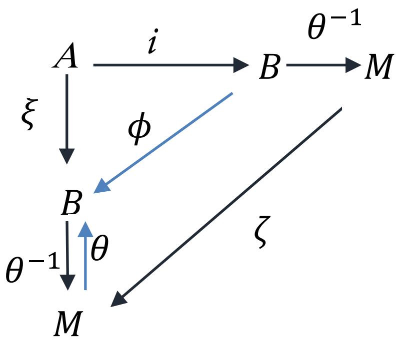
(3) Every valuation semidomain which is not division semiring is almost self-injective but not self -injective semimodule.

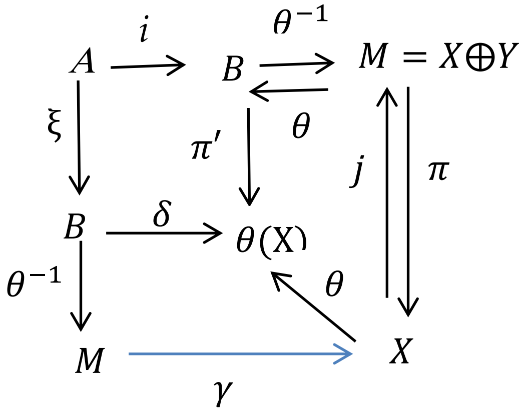
Proposition 2.3.5: Any fully invariant direct summand of almost self-injective semimodule is also almost self-injective.

Proof. Suppose M is an almost self-injective semimodule and Y a fully invariant direct summand of M, this means M is almost M-injective by Proposition 2.2.9, we get Y is almost M-injective and from Proposition 2.2.10, Y is almost Y-injective semimodule, that is Y is almost self-injective .■

Proposition 2.3.6: Let $M \cong B$; M is an almost self-injective semimodule if and only if B is almost self-injective.

Proof. Let $\theta: M \longrightarrow B$ be an isomorphism and M an almost-self-injective semimodule, then either there is $\zeta \in \mathrm{End}(M)$, such that $\zeta\theta^{-1}i = \theta^{-1}\xi$. Define $\emptyset: B \to B$ such that $\emptyset = \theta\,\zeta\theta^{-1}$, then $\emptyset i = \theta\zeta\theta^{-1}i = \theta\theta^{-1}\xi = \xi$, or there exists $\gamma: M \to X$ where X is a nonzero direct summand of M such that $\gamma\,\theta^{-1}\xi = \pi\theta^{-1}i$. Define $\delta: B = \theta(X) \oplus \theta(Y) \longrightarrow \theta(X)$ where $\theta(X)$ is a nonzero direct summand of B, such that $\delta = \theta\gamma\ \theta^{-1}$ we have $\delta\xi = \theta\gamma\theta^{-1}\xi = \theta\pi\theta^{-1}i = \pi' i$. Then B is almost self-injective. As in the following diagrams:





Similarly, if the assumption $\theta: B \longrightarrow M$ and B is an almost self-injective semimodule, then M is almost self-injective again. ■

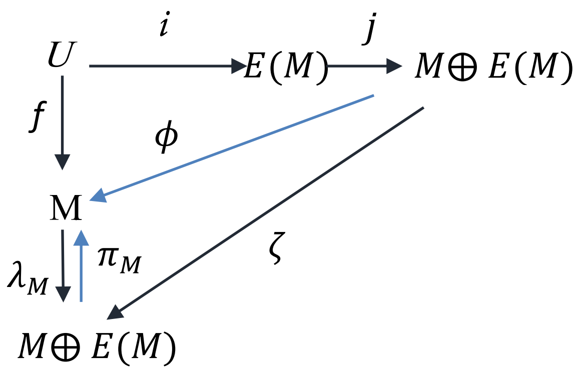
Proposition 2.3.7: Let $M = Y_1 \oplus Y_2$ be an almost self-injective semimodule, where Y_j is fully invariant then Y_i is almost Y_j-injective semimodule for $i, j = \{1, 2\}$.

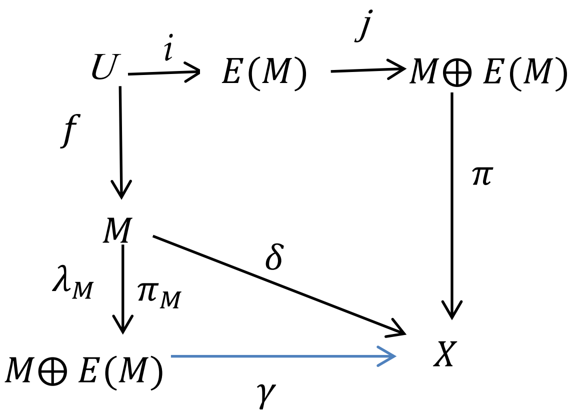
Proof. From hypothesis M is an almost self-injective semimodule, then M is an almost M-injective. Since Y_i is direct summand of M, by Proposition 2.2.10, then Y_i is an almost M-injective semimodule and from Proposition(2.2.10), we have Y_i is an almost Y_j -injective semimodule. ■

Corollary 2.3.8: Let $M=\oplus_{i=1}^{n} Y_i$ be an almost self-injective semimodule, where Y_j is fully invariant, then Y_i is an almost Y_j-injective semimodule for $i, j \in\{1, 2,\ldots, n\}$.

Proposition 2.3.9: If *M* is a simple semimodule with injective hull E(*M*) and *M*⊕ E(*M*) almost self-injective, then *M* is almost *M*⊕ E(*M*)-injective.

Proof. Consider the following diagrams:





Where i and j are the inclusion maps and *U* any subsemimodule of *M*⊕ E(*M*) . Since *M*⊕ E(*M*) is almost self-injective semimodule, then either there is endomorphism ζ on *M*⊕ E(*M*) such that ζji=,$\lambda_M f$ or there is $\gamma: M\oplus E(M) \longrightarrow X$ where *X* is nonzero direct summand of *M*⊕ E(*M*) such that $\gamma\lambda_M f = \pi ji$. From the first diagram can be defined ϕ: $M\oplus$ E$(M) \longrightarrow M$ by $\phi=\pi_M\zeta$, then $\phi ji = \pi_M\zeta ji = \pi_M\lambda_M f_= f$. The second diagram will obtained $\delta: M \longrightarrow X$ such that $\delta = \gamma\pi_M$, then $\delta 1_M = \gamma\pi_M f = \pi ji$. Therefore *M* is almost *M*⊕ E(*M*)-injective semimodule. ■

Remark 2.3.10 [26]: If *M* is a π-injective semimodule having injective hull E(*M*) and E(*M*)= *D*⊕*E*, where *D* and *E* are subsemimodules of E(*M*), then $M = (D \cap M)\oplus(E\cap M)$.

Remark 2.3.11 [26]: If *M* is a π-injective, indecomposable semimodule having injective hull, then *M* is uniform.

Proposition 2.3.12: An indecomposable, almost self-injective semimodule M is π-injective.

Proof. Suppose $Ų_1$, $Ų_2 \neq 0$ are subsemimodules of indecomposable almost self-injective semimodule M with $Ų_1 \cap Ų_2=0$, the projection π: $Ų_1 \oplus Ų_2 \to Ų_i$ either can be extended to endomorphism of M, or there exists an R-homomorphism α of M such that $\alpha\pi = i \Longrightarrow \ker(\pi) = 0$ which is contradiction (since $\ker \pi = Ų_i \neq 0$ where i = 1, 2) hence M is π-injective.■

Corollary 2.3.13: If M is an indecomposable, almost self-injective semimodule with injective hull E(M), then M is uniform.

Proof. From Proposition 2.3.12, we have M is π-injective semimodule and from Remark 2.3.11, we get M is uniform. ■

Proposition2.3.14: If M is an indecomposable, almost self-injective which is nonsingular semimodule, then ʆ is semidomain.

Proof. By Proposition 2.3.12 M is π-injective semimodule and by Lemma 2.2.27 M is an extending semimodule. Now, let $\theta\gamma=0$ where $\theta, \gamma \in$ ʆ, then $\gamma(M) \subseteq \ker(\theta)$. But $\ker(\theta)$ is closed and hence a direct summand of M, then either $\theta=0$ or $\gamma=0$, because M is indecomposable. ■

The following proposition has been proved for module in [40], it will be proved for semimodule after it has been converted to suit the characteristics of semimodules where two conditions were added, cancellative and semisubtractive.

In the following, we mean by symbols ⊂ proper subset and ⊆ subset.

Proposition 2.3.15: Suppose M is an indecomposable, semisubtractive, cancellative, almost self-injective semimodule, then for all $\omega, \varphi \in$ ʆ ,we have:

(1) If $\ker(\omega) \subset \ker(\varphi)$, then $\mathcal{S}\varphi \subset \mathcal{S}\omega$.

(2) If $\ker(\omega) = \ker(\varphi)$, then $\mathcal{S}\varphi \subseteq \mathcal{S}\omega$ or $\mathcal{S}\omega \subseteq \mathcal{S}\varphi$.

Proof. (1) Let $\omega, \varphi \in \mathcal{S}$ such that $\ker(\omega) \subset \ker(\varphi)$. Define ξ: $\omega(M) \longrightarrow \varphi(M)$ by $\xi(\omega(m)) = \varphi(m)$ which is a well define homomorphism, for if $\omega(m) = \omega(n)$ for every m, n $\in M$, by semisubtractive of M, there is ş $\in M$ such that either m = n+ ş or n= m+ ş if m = n+ ş $\Longrightarrow \omega(m) = \omega(n) + \omega(ş)$ by cancellative, $\omega(ş) = 0 \Longrightarrow ş \in \ker(\omega) \subset \ker(\varphi) \Longrightarrow ş \in \ker(\varphi) \Longrightarrow \varphi(m) = \varphi(n)$, the same result hold if n = m+ ş. ξ is not one to one (since $\ker(\omega) \subset \ker(\varphi)$), by hypothesis, ξ can be extended to M. Hence there is $\rho \in \mathcal{S}$ such that $\rho\big(\omega(m)\big) = \xi\big(\omega(m)\big)$ for all m $\in M \Longrightarrow \rho\big(\omega(m)\big) = \varphi(m) \Longrightarrow \mathcal{S}\varphi \subset \mathcal{S}\omega$. As for the second diagram of Definition 2.3.1, since M is indecomposable, there is $\sigma: M \longrightarrow M$ such that $\sigma\xi = 1_M$ this means ξ is one to one and this contradicts that ξ is not one to one).

(2) Suppose $\ker(\omega) = \ker(\varphi)$, then ξ is one to one (in fact, for $\omega(m)$ and $\omega(n)$ in $\omega(M)$ such that $\xi(\omega(m)) = \xi(\omega(n)) \Longrightarrow \varphi(m) = \varphi(n)$,by semisubtractive there exists ş $\in M$ such that either m = n+ ş or n = m+ ş if m = n+ ş $\Longrightarrow \varphi(m) = \varphi(n) + \varphi(ş)$, by cancellative, we have $ş \in \ker(\varphi) = \ker(\omega) \Longrightarrow \omega(n) = \omega(m)$. So either ξ can be extended to an endomorphism $\rho \in \mathcal{S}$, or there exists $\gamma \in \mathcal{S}$ such that $\gamma\xi = 1_{\omega(M)} \Longrightarrow \omega(m) = \gamma(\xi(\omega(m))) = \gamma(\varphi(m)) = \gamma\varphi(m)$ for all m $\in M$. Thus $\mathcal{S}\omega \subseteq \mathcal{S}\varphi$, if $\gamma = \rho$ on $\omega(m)$, then $\mathcal{S}\varphi \subseteq \mathcal{S}\omega$. ■

Corollary 2.3.16: If M is a uniserial, semisubtractive, cancellative, almost self-injective right semimodule, then $\mathcal{S}$ is left uniserial.

Proof. Let X, Y be left ideals of $\mathcal{S}$ such that $Y \nsubseteq X$ and let $\alpha \in X$, $\beta \in Y$ and $\beta \notin X$. since $\ker\alpha$ and $\ker\beta \in M$ and M is uniserail, then either $\ker\alpha \subseteq \ker\beta$ or $\ker\beta \subseteq \ker\alpha$, if $\ker\alpha \subseteq \ker\beta$ from Proposition 2.3.15 we get $\beta\mathcal{S}$

$\subseteq \alpha \mathcal{S}$ implies $\beta \in X$ and this contradiction, the same way if ker$\beta \subseteq$ ker α we get $\alpha \in Y$ implies $X \subseteq Y$. Therefore $\mathcal{S}$ is left uniserial. ■

Lemma 2.3.17: Let M be an indecomposable, semisubtractive, cancellative almost self-injective semimodule, then the left ideal H of $\mathcal{S}$ = End(M) generated by non-isomorphic monomorphisms in $\mathcal{S}$ is two sided ideal.

Proof. Let $\varphi \in \mathcal{S}$ and γ a non-isomorphism with $\ker(\gamma) = 0$, it is enough to show that $\gamma\varphi \in$ H. If ker $(\gamma\varphi) \neq 0$, from Proposition 2.3.15 (ker $(\gamma) \subset$ ker $(\gamma\varphi)$) implies that $\mathcal{S}\gamma\varphi \subset \mathcal{S}\gamma$) then, $\gamma\varphi \in$ H. If ker $(\gamma\varphi) = 0 = \ker(1_M)$, by Proposition 2.3.15, then $\mathcal{S}\gamma\varphi \subseteq \mathcal{S}1_M$ if $\gamma\varphi$ is isomorphism implies that γ is onto, a contradiction. Thus $\gamma\varphi \in$ H. ■

Proposition 2.3.18: The endomorphism semiring of indecomposable, semisubtractive, cancellative almost self-injective semimodule is local.

Proof. Since M is an indecomposable almost self-injective semimodule, then it is uniform by Proposition 2.3.13, let $\tilde{\Omega}$ be the set of all non-isomorphism monomorphism in $\mathcal{S}$. If $\tilde{\Omega} = \emptyset$, then $\psi \in \mathcal{S}$ is an isomorphism if and only if $\ker(\psi) = 0$, let U($\mathcal{S}$) be the monoid of invertible elements of $\mathcal{S}$ and $\alpha + \beta \in$ U($\mathcal{S}$), since M is uniform, either α or β is monomorphism, this mean either α or β is an isomorphism. Therefore $\mathcal{S}$ is local. Now suppose $\tilde{\Omega} \neq \emptyset$, let H= $\sum_{\gamma \in \tilde{\Omega}} \mathcal{S}\gamma$, by Proposition 2.3.15, $\mathcal{S} \setminus$U($\mathcal{S}$)$\subset$H, we must show that $\propto \in$ H is not invertible, $\propto = \sum_{i=1}^{n} f_i \gamma_i$ where $\gamma_i \in \tilde{\Omega}$ and $f_i \in \mathcal{S}$. By Proposition 2.3.15, $\mathcal{S}\gamma_1$, $\mathcal{S}\gamma_2, \ldots, \mathcal{S}\gamma_n$ are linearly ordered by inclusion relation. Hence $\propto = f\gamma_n$ for some $f \in \mathcal{S}$, if $\propto$ is invertible, then γ_n is left invertible, since $\mathcal{S}$ has no nontrivial idempotent, γ_n is invertible and this contradiction with $\gamma_n \in \tilde{\Omega}$. Thus $\mathcal{S} \setminus$U($\mathcal{S}$)=H. Since H is two sided ideal of $\mathcal{S}$ by Lemma 2.3.17, we have $\mathcal{S}$ is local. ■

Lemma 2.3.19 [**13, p.38**]: For any element r of a subtractive semiring R, $r \in \text{Rad}(R)$ if and only if $1+tr$ is invertible for all t in R.

Lemma 2.3.20: If $\int$ =End(M) is local, then M is indecomposable.

Proof. Assume that $M=M_1 \oplus M_2$ and $\pi_i : M \longrightarrow M_i$, $\lambda_i : M_i \longrightarrow M$ (i=1, 2) are the projection and injection maps respectively. Now, $\lambda_i \pi_i$ are noninvertible elements of $\int$. But $\lambda_1 \pi_1 + \lambda_2 \pi_2 = 1_M$ which is invertible and this a contradiction with $\int$ is local. ■

Proposition 2.3.21: Let M be an almost self-injective semimodule, then:

(1) If $\int$ is local, then M is uniform.

(2) If M is uniform, then Z($\int$)⊂ Rad($\int$).

Proof. (1) Suppose $Y \cap X = 0$, where X and Y are subsemimodules of M (since M is almost self-injective semimodule and indecomposable by Lemma 2.3.20 and from Proposition 2.3.12, then M is π-injective and hence there exist $\omega, \varphi \in \int$, such that $X \subseteq \ker\omega$ and $Y \subseteq \ker\varphi$ and $\omega + \varphi = 1_M$. Since $\int$ is local either $\omega \in \text{Rad}(\int)$ or $\varphi \in \text{Rad}(\int)$, if $\omega \in \text{Rad}(\int)$, then φ is invertible implies $\ker\varphi = 0$, since $Y \subseteq \ker\varphi$, hence $Y = 0$.

(2) Let $\varphi \in Z(\int)$ and $0 \neq \omega \in \int$, ann(φ) is essential in $\int$, there exists $f \in \int$ such that $0 \neq f\omega \in \text{ann}(\varphi) \Longrightarrow \varphi(f\omega) = 0$. If ker ($\varphi$)= 0, then φ is monic, then if $\varphi(f\omega) = 0 \Longrightarrow f\omega=0$ and this contradiction. Then ker (φ) $\neq 0$, since M is uniform , then ker (φ) essential in M . Now, since $\ker(\varphi) \cap \ker(1+\omega\varphi) = 0$, in fact, if $x \in \ker(\varphi) \cap \ker(1+\omega\varphi) \Longrightarrow \varphi(x)=0$ and $(1+\omega\varphi)(x) = 0 \Longrightarrow 1(x) = 0$, we have $\ker(1+\omega\varphi) = 0 = \ker(1_M)$ by

Proposition 2.3.18, then there is $h \in \int$ such that $1_M = h(1+\omega\varphi) \Longrightarrow \varphi \in \text{Rad}(\int)$ by Lemma 2.3.19. ■

Proposition 2.3.22: Suppose M is an indecomposable almost self-injective semimodules and $\omega, \varphi \in \mathcal{S}$,

(1) If $\omega(M)$ embeds in $\varphi(M)$, then $\mathcal{S}\omega$ is a homomorphic image of $\mathcal{S}\varphi$.

(2) If $\omega(M) \cong \varphi(M)$, then $\mathcal{S}\omega \cong \mathcal{S}\varphi$.

Proof. (1) Let $\xi: \omega(M) \longrightarrow \varphi(M)$ be a monomorphism, and $i_1: \omega(M) \longrightarrow M$, $i_2: \varphi(M) \longrightarrow M$ be the inclusion maps, since M is almost self-injective, either there exists $\rho \in \mathcal{S}$ such that $\rho\, i_1 = i_2\, \xi$. Define $\mu: \mathcal{S}\varphi \longrightarrow \mathcal{S}\omega$ by $\mu(f\varphi) = f\rho\,\omega$, for all $f \in \mathcal{S}$. μ is well defined (if, $f_1\varphi = f_2\varphi, \forall\, m \in M \Longrightarrow f_1\rho\,\omega(m) = f_1\xi\omega(m) = f_1\varphi(m) = f_2\varphi(m) = f_2\xi\omega(m) = f_2\rho\,\omega(m)$, for all $m \in M$, then $f_1\rho\,\omega(m) = f_2\rho\,\omega(m)$ and hence μ is well defined. Now to show that $\ker(\omega) = \ker(\rho\,\omega)$, let $x \in \ker(\rho\,\omega) \Longrightarrow \rho\omega(x) = 0$, $\xi(\omega(x)) = 0 \Longrightarrow \omega(x) = 0$ because ξ is monic. It is clear that $\ker(\omega) \subset \ker(\rho\,\omega)$, hence $\ker(\omega) = \ker(\rho\,\omega)$ by Proposition 2.3.15, $\mathcal{S}\omega \subset \mathcal{S}\rho$, so $\omega = f\rho\,\omega$ for some $f \in \mathcal{S}$, then $\omega = f\rho\,\omega = \mu(f\varphi)$ implies $\mathcal{S}\omega \subset \mathcal{S}f\rho\,\omega = \mu(\mathcal{S}\varphi)$ this means μ is $\mathcal{S}$-epimorphism, or there is $\beta \in \mathcal{S}$ such that $\beta i_2\xi = 1_{\omega(M)}$ implies $\omega(m) = \beta i_2\xi(\omega(m)) = \beta\, i_2(\varphi(m)) = \beta\varphi(m)$, for every m in $M \Longrightarrow \mathcal{S}\omega \subseteq \mathcal{S}\varphi$, then μ is $\mathcal{S}$-epimorphism.

(2) Let $\xi: \omega(M) \longrightarrow \varphi(M)$ be an R-isomorphism and $i_1: \omega(M) \longrightarrow M$, $i_2: \varphi(M) \longrightarrow M$ be the inclusion maps, since M is almost self-injective, either there exists $\rho \in \mathcal{S}$ such that $\rho\, i_1 = i_2\,\xi$, or $\rho\, i_2\,\xi = 1_{\omega(M)}$. From (1) μ is $\mathcal{S}$-epimorphism, to show μ is a monomorphism, let $f(\varphi) \in \ker(\mu)$, thus $\mu(f(\varphi)) = 0$ so $f\rho\,\omega = 0$ since $\rho\,\omega(M) = \varphi(M)$, $f\rho\,\omega(M) = f\varphi(M)$ hence $f\varphi(M) = 0$ implies $f\varphi = 0$. ■

In [12] the set $W(\mathcal{S}) = \{\varphi \in \mathcal{S} / \ker\varphi$ is essential in $M\}$ is a two sided ideal of $\mathcal{S}$.

Proposition 2.3.23: If M is a cancellative, semisubtractive almost self-injective semimodule, then:

(1) W(ʆ) ⊆ Rad(ʆ).

(2) If ʆ is local, then Rad(ʆ) = {ϑ ∈ʆ : ker (ϑ) ≠0}.

Proof. **(1)**Let α ∈ W(ʆ) and let θ ∈ ʆ since ker (α) ∩ ker(1 + θα) = 0 and ker (α) $\leq_e M$, then ker(1 + θα) = 0=ker(1_M) by Proposition 2.3.15, ʆ ⊆ ʆ(1 + θα), since 1∈ ʆ ⟹ 1 = f (1 + θα) for some f in ʆ and hence α ∈ Rad(ʆ).

(2) Since ʆ is local, ʆα ≠ ʆ for any α ∈ Rad(ʆ), let α ∈ Rad(ʆ) to show that ker(α) ≠0. Suppose that ker(α) =0, define ψ: α(M)⟶ M by ψ(α(ƫ)) = ƫ, ψ is well defined, for any ƫ , e in M, assume that α(ƫ) = α(e) by

semisubtractive, there is ȿ in M such that, either ƫ = e+ ȿ or e = ƫ+ ȿ, we get α(ƫ) = α(e) + α(ȿ) by cancellative α(ȿ) =0 ⟹ȿ ∈ker(α) ⟹ȿ =0, then ƫ = e, the same way for e = ƫ+ ȿ. Since M is almost self-injective semimodule, either there is an endomorphism η of M such that ηi = ψ, then ηα = ηiα =

ψα = 1_M so ker(ηα)= ker(1_M) by Proposition 2.3.15, ʆηα ⊆ ʆα and hence ʆα = ʆ this is a contradiction, then Rad(ʆ) ⊆{ϑ ∈ ʆ |ker (ϑ) ≠0}, or

there is σ ∈ ʆ such that σ ψ = $1_{\alpha(M)}$ ⟹ α(M) =σ ψ(α(M)) =σ(M), this means ʆα =ʆ this contradicts the assumption. Conversely, let α ∈{ϑ ∈ ʆ | ker (ϑ) ≠0}, since ʆ is local, then Rad(ʆ) = { α ∈ ʆ: ʆα ≠ ʆ }. Suppose that ʆα = ʆ⟹ βα = 1_M for some β ∈ ʆ , so ker(βα)= ker (1_M) = 0, then ker(α) ⊆ ker(βα), then ker(α) = 0, this a contradiction.■

Proposition 2.3.24: Let M be a cancellative, semisubtractive almost self-injective semimodule with ʆ = End(M). If α(M) is a simple right R-semimodule, then ʆα is a simple left ʆ-semimodule, where α ∈ ʆ.

Proof. Let D be a nonzero subsemimodule of $\int\alpha$ and $0 \neq \beta\alpha \in D$, then $\int\beta\alpha \subset D$. Suppose $\ker(\beta) \cap \alpha(M) \neq 0$. Since $\alpha(M)$ is simple and $\ker(\beta) \cap \alpha(M) \subset \alpha(M) \Longrightarrow \ker(\beta) \cap \alpha(M) = \alpha(M) \Longrightarrow \alpha(M) \subset \ker(\beta)$, so $\beta\alpha(M) = 0 \Longrightarrow \beta\alpha = 0$, a contradiction so $\ker(\beta) \cap \alpha(M) = 0 \Longrightarrow \ker(\beta\alpha) = \ker(\alpha)$, hence by Proposition 2.3.15, $\int\beta\alpha \subseteq \int\alpha \Longrightarrow \int\beta\alpha \subset D \subset \int\alpha \Longrightarrow D = \int\alpha$. ■

3.1 Introduction

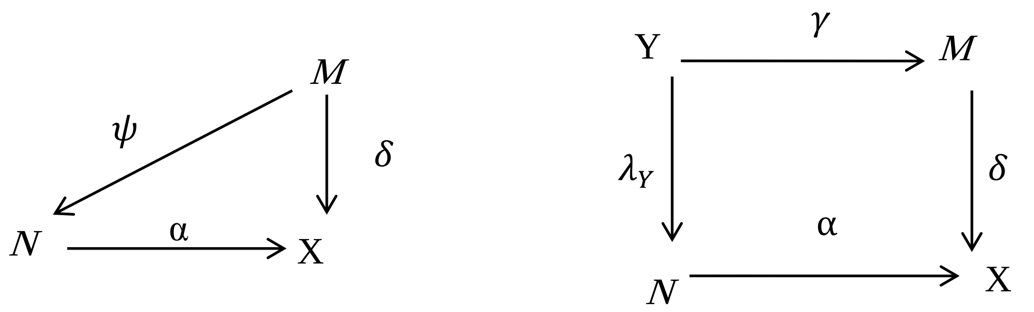
The aim of this chapter is to study the concepts of almost projective and almost self-projective semimodules as generalizations of projective semimodules and quasi-projective semimodules, respectively some of their characterizations have been discussed, as well as some results have been generalized from projective and quasi-projective semimodules.

In this part of thesis, almost projective modules has been expanded for semimodules taking into account the differences between modules and semimodules, which are mainly derived from their definitions.

Some properties of almost-projective semimodule will be investigated in this part.

3.2 Almost projective semimodules

<u>Definition 3.2.1</u>: A semimodule M is said to be almost N-projective, if for each epimorphism $\alpha: N \longrightarrow X$ and every homomorphism $\delta: M \longrightarrow X$, either there is $\psi: M \longrightarrow N$ such that $\alpha\psi = \delta$, or there is $\gamma: Y \longrightarrow M$ where Y is a nonzero direct summand of N such that $\delta\gamma = \alpha\lambda_Y$. As the following diagrams:



(i) (ii)

A semimodule M is called almost-projective if it is almost N-projective for every finitely generated R-semimodule N.

Remarks 3.2.2:

(1) Every projective semimodule is almost-projective.

(2) If M is almost N-projective, N is indecomposable and the diagram(i) of Definition 3.2.1 is not satisfied, then N=Y and there exists $\gamma: N \longrightarrow M$ such that $\delta\gamma = \alpha 1_N$.

(3) If M is almost N-projective and simple semimodule, then any epimorphism α:$N \longrightarrow M$, either splits (take $\delta = 1_M$ implies $\alpha\psi = 1_M$, hence α split epimorphism) or, there is $\gamma: Y \longrightarrow M$ such that $1_M\gamma = \alpha\lambda_Y$ implies $\gamma = \alpha\lambda_Y$ where Y is a nonzero direct summand of N.

Remark 3.2.3: Every semiring with identity is almost-projective semimodule.

Proof. Since every projective semimodule is almost projective and from Proposition 1.1.84. ■

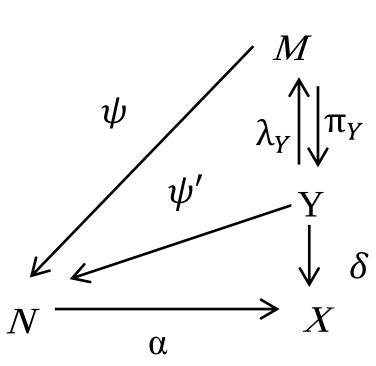
Examples 3.2.4: From Remark 3.2.3, we get:

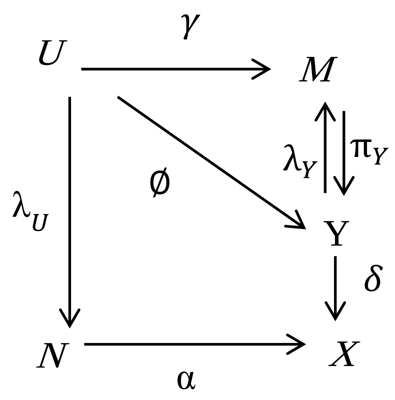
(1)A semiring $\mathbb{N}$ over itself is an almost -projective semimodule.

(2)If $\mathbb{B}$={0,1}where 1+1=1 , then($\mathbb{B}$, +, ·) is a Boolean semiring [1, p.7], with identity 1. $\mathbb{B}$ as $\mathbb{B}$-semimodule is almost-projective.

Proposition 3.2.5: If M is an almost N-projective semimodule and Y a direct summand of M, then Y is almost N-projective also.

Proof. Assume M is an almost N-projective semimodule and Y a direct summand of M. Consider the following diagrams:

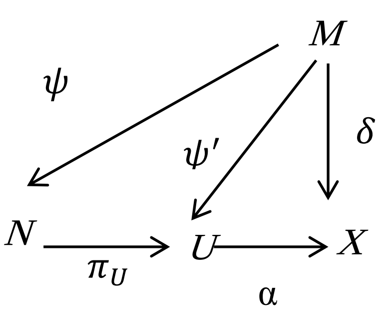


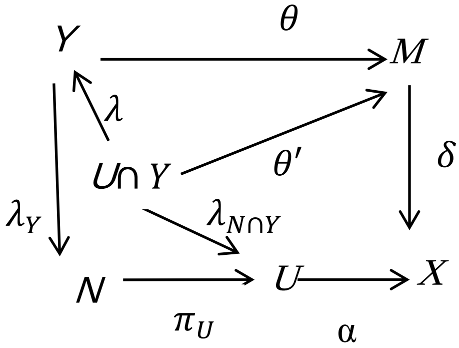


Where π_Y and λ_Y are the projection and injection maps respectively. Since M is almost N- projective semimodule, then either there is ψ:M⟶N such that $\alpha\,\psi = \delta\;\pi_Y$ where α:N⟶X is an epimorphism, or there is γ: U⟶M such that $\delta\;\pi_Y\gamma = \alpha\lambda_U$ where U is nonzero direct summand of N, , in the first diagram take $\psi' = \psi\lambda_Y$, then $\alpha\;\psi' = \alpha\;\psi\lambda_Y = \delta\;\pi_Y\lambda_Y = \delta$. Now from the second diagram, assume $\phi = \pi_Y\gamma$, then $\delta\phi = \delta\pi_Y\gamma = \alpha\lambda_U$.Thus Y is almost N-projective. ■

Proposition 3.2.6: Let M be an almost N-projective semimodule and U a fully invariant direct summand of N, then M is almost U-projective.

Proof. Let M be almost N-projective and U a fully invariant direct summand of N. By the following diagrams:





Since M is an almost N-projective, then either there is ψ:M⟶N such that $\alpha\pi_U\psi = \delta$, or there is θ:Y ⟶M such that $\delta\theta = \alpha\pi_U\lambda_Y$ where Y is a nonzero direct summand of N. Define $\psi' = \pi_U\psi$, we get $\alpha\psi' = \alpha\pi_U\psi = \delta$. In the second diagram, define $\theta': U \cap Y$ ⟶M where $U \cap Y$ is a direct summand of U (by fully invariant) such that $\theta'= \theta\lambda$, then $\delta\theta' = \delta\theta\lambda = \alpha\pi_U\lambda_Y\lambda = \alpha\ \lambda_{N\cap Y}$. Hence M is almost U-projective semimodule. ■

Corollary 3.2.7: If M is an almost N-projective semimodule such that N=$\oplus_{i=1}^{n} N_i$, then M is almost N_i-projective semimodule, where each N_i is fully invariant of N, i=1, 2,…, n.

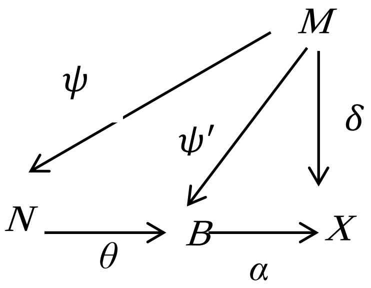
Proof. By Proposition 3.2.6, the result is obtained. ■

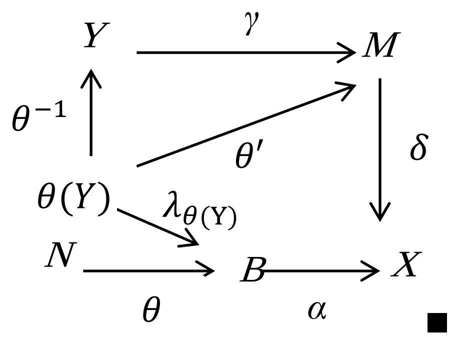
Proposition 3.2.8: Let M and N be R-semimodules and M is almost N-projective, then any semimodule isomorphic to M is also almost N-projective.

Proof. Assume β:M⟶E is an isomorphism and M is almost N-projective, by an argument similar to proof of Proposition 3.2.5 and replacing $\lambda_Y = \beta^{-1}$, $\pi_Y = \beta$ the result is obtained. ■

Proposition 3.2.9: If M is almost N-projective semimodule and $N \cong B$ where B is any semimodule, then M is almost B-projective.

Proof. Suppose θ: N⟶B be an isomorphism, since M is almost N-projective, then either there is ψ:M⟶N such that $\alpha\theta\psi = \delta$ where α is an epimorphism, or there is γ:Y ⟶M such that $\delta\gamma = \alpha\theta|_Y$ where Y is a nonzero direct summand of N. Define $\psi' = \theta\psi$, then $\alpha\psi' = \alpha\theta\psi = \delta$. In the second diagram, define $\theta': \theta(Y)$ ⟶M where $\theta(Y)$ is a direct summand of B such that $\theta'= \gamma\theta^{-1}$, then $\delta\theta' = \delta\gamma\theta^{-1} = \alpha\ \lambda_{\theta(Y)} = \alpha|_{\theta(Y)}$. Hence M is almost B-projective semimodule. As the following diagrams:

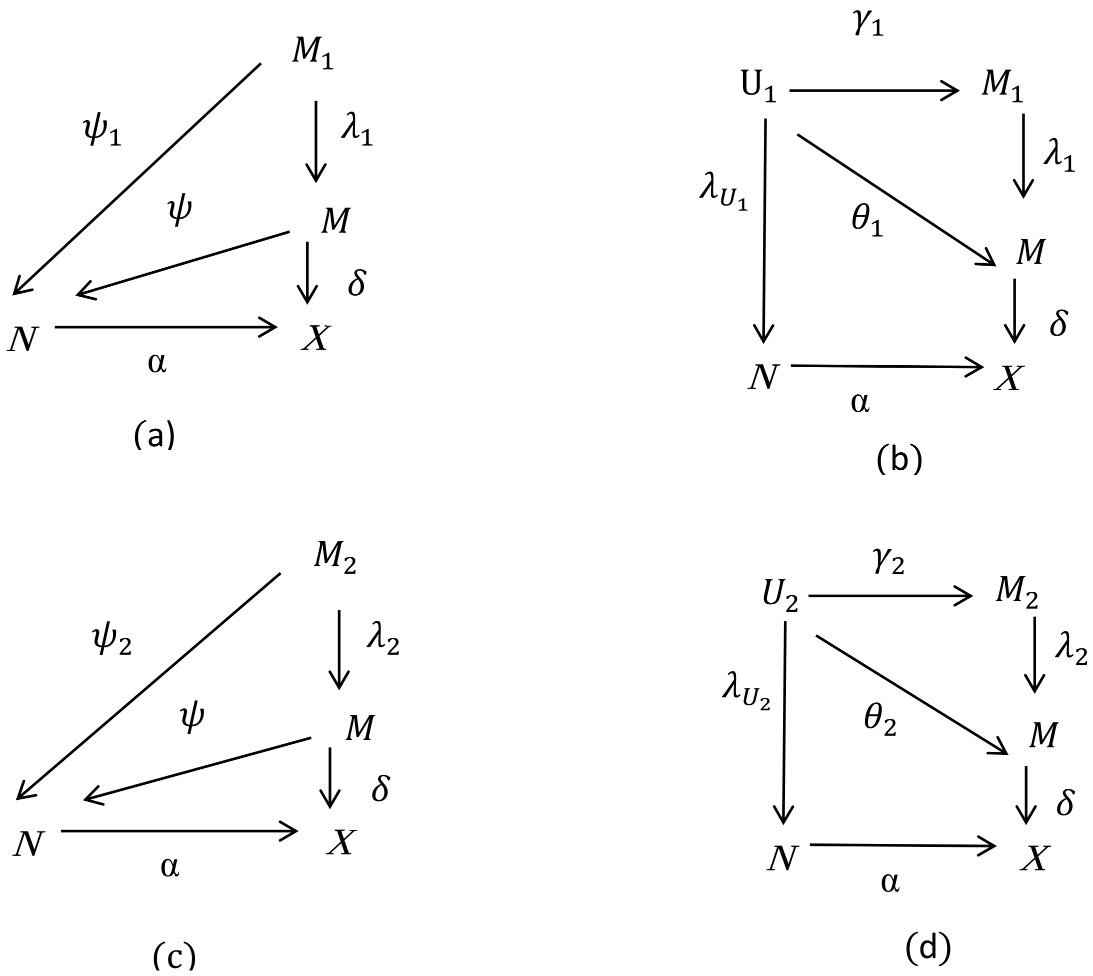




■

Propositin 3.2.10: Let M_1, M_2 and N be R-semimodules, then $M = M_1 \oplus M_2$ is almost N-projective if and only if M_i (i=1, 2) is almost N-projective semimodule.

Proof. Suppose M_i (i=1, 2) are almost N-projective and M=$M_1 \oplus M_2$, consider the following diagrams :



We have six cases: **case1/** if the diagrams(a) and (b) are satisfying, then the Defection 3.2.1 is satisfied. **Case2/** similar to case1, if the two diagrams (c) and (d). **Case3/** suppose the two diagrams (a) and (c) are satisfying, then there is ψ_1: $M_1 \longrightarrow N$ and ψ_2: $M_2 \longrightarrow$ N such that $\alpha\psi_1 = \delta\lambda_1$ and $\alpha\psi_2 = \delta\lambda_2$. Define ψ: $M \longrightarrow N$ by $\psi = \psi_1 + \psi_2$, then $\alpha\,\psi = \alpha(\psi_1 + \psi_2)=$ $\alpha\psi_1 + \alpha\psi_1 = \delta\lambda_1 + \delta\lambda_2 = \delta(\lambda_1 + \lambda_2) = \delta$.

Case4/ suppose the diagram (a) and (d) are satisfying, this means there is ψ_1: $M_1 \longrightarrow N$ such that $\alpha\psi_1 = \delta\lambda_1$ and there is γ_2: $U_2 \longrightarrow M_2$ such that $\delta\lambda_2\gamma_2 = \alpha\lambda_{\mathrm{U2}}$ where U_2 is nonzero direct summand of N, define θ_2: $U_2 \longrightarrow$ M by $\theta_2 = \lambda_2\ \gamma_2$, then $\delta\theta_2 = \delta\lambda_2\gamma_2 = \alpha\lambda_{\mathrm{U2}}$. **Case5/** suppose the two diagrams (b) and (c) are satisfying, similar to case4, by take θ_1: $\mathrm{U}_1 \longrightarrow \mathrm{M}$, then $\delta\theta_1 = \delta\lambda_1\gamma_1 = \alpha\lambda_{\mathrm{U1}}$. **Case6**/ suppose the diagrams (b) and (d) are satisfying, this means there is γ_1:$U_1 \longrightarrow M_1$ and γ_2:$U_2 \longrightarrow M_2$ where U_1 and

U_2 are nonzero direct summands of N such that $\delta\lambda_1\gamma_1 = \alpha\lambda_{\mathrm{U}_1}$ and $\delta\lambda_2\gamma_2 = \alpha\lambda_{\mathrm{U}_2}$. In (b) can be define $\theta_1 = \gamma_1 : U_1 \longrightarrow$ M such that then $\delta\theta_1$ $= \delta\lambda_1\gamma_1 = \alpha\lambda_{\mathrm{U}_1}$, we get the same result in the diagram (d). From the previous six cases, we get, M is almost N-projective. Conversely, suppose that M is almost N-projective. From Proposition 3.2.5, therefore M_1 and M_2 are almost N-projective. ■

<u>Colloraly 3.2.11:</u> Let $\{M_i\}_{i=1}^{n}$be a family of semimodules, then M_i is almost N-projective semimodules if and only if $\oplus_i M_i$ is almost N-projective for fixed semimodule N and $1 \le i \le n$.

<u>Proof.</u> From Proposition 3.2.10, the result is obtained. ■

<u>Proposition 3.2.12:</u> The following statements are equivalent, for a semiring R and semimodule N:

(1) Every almost N-projective R-semimodule is N-projective.

(2) The direct sum of any finite family of almost N-projective R-semimodule is N-projective.

(3) The direct sum of any two almost N-projective R-semimodule is N-projective.

Proof. (1)⟹(2) Suppose that every almost N-projective R-semimodule is N-projective. Since the direct sum of finite family of almost N-projective R-semimodules is almost N-projective by Proposition 3.2.10 and from (1), then this family of semimodules is N-projective.

(2)⟹(3) It is clear.

(3)⟹(1) Let M be almost N-projective R-semimodule, then $R\oplus M$ is almost N-projective, from(3) $R\oplus M$ is N-projective semimodule. Since every direct summand of N-projective semimodule is N-projective, hence M is N-projective semimodule. ■

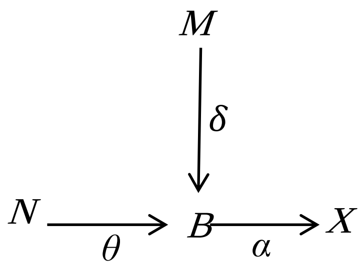
Corollary 3.2.13: The following statements are equivalent, for a semiring R and :

(1)Every almost projective R-semimodule is projective.

(2)The direct sum of finite family of almost projective R-semimodule is projective.

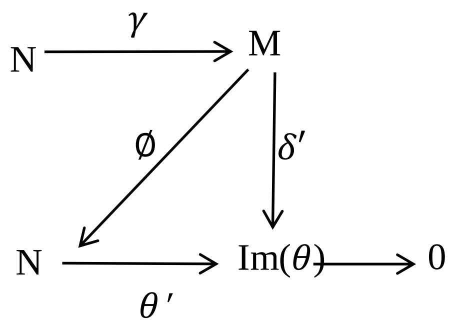
(3)The direct sum of any two almost projective R-semimodule is projective.

Proposition 3.2.14: Let M be an almost N-projective semimodule and N an indecomposable semimodule, any diagram of R-semimodules and R-homomorphisms of the form:



In which the row is exact and $\alpha\delta = 0$, then either there is $\emptyset: M \longrightarrow N$ such that $\theta\emptyset = \delta$, or there is $\gamma: N \longrightarrow M$ such that $\delta\gamma = \theta$.

Proof. Since $\alpha\delta = 0$, then $\mathrm{Im}(\delta) \subseteq \ker(\alpha) = \mathrm{Im}(\theta)$. Let $\theta': N \longrightarrow \mathrm{Im}(\theta)$ where $\theta'(\mathrm{n}) = \theta(\mathrm{n})$, for all $\mathrm{n} \in N$, and $\delta': M \longrightarrow \mathrm{Im}(\theta)$ such that $\delta(\mathrm{m}) = \delta'(\mathrm{m})$, for all $\mathrm{m} \in M$, thus θ' is an epimorphism and we have the following diagram:

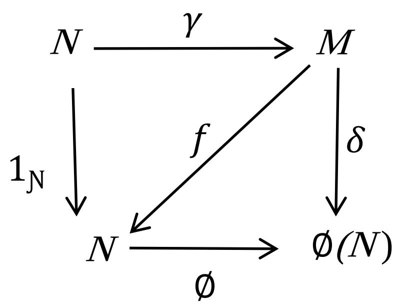


In which the rows is exact, since M is almost N-projective, either there is $\emptyset: M \longrightarrow N$ such that $\theta'\emptyset = \delta'$ implies that $i\theta'\emptyset = i\,\delta'$ where i: $\mathrm{Im}(\theta) \longrightarrow N$ is the inclusion map, then $\theta\emptyset = \delta$, or there is $\gamma: N \longrightarrow M$ such that $\delta'\gamma = \theta'$ implies $i\delta'\gamma = i\theta'$, then $\delta\gamma = \theta$. ■

Proposition 3.2.15: Let M be an almost N-projective semimodule and N an indecomposable semimodule, $\emptyset \in \mathrm{End}(N)$, then for all $\delta \in \mathrm{Hom}(M, \emptyset(N))$, either $\delta \in \emptyset\mathrm{Hom}(M, N)$, or $\delta\ \mathrm{Hom}(N, M) \subseteq \mathrm{Hom}(N, \emptyset(N))$.

Proof. It is clear that $\emptyset\mathrm{Hom}(M, N) \subseteq \mathrm{Hom}(M, \emptyset(N))$, let $\delta \in \mathrm{Hom}(M, \emptyset(N))$ since M is almost N-projective, then either $\emptyset f = \delta$ for some $f \in \mathrm{Hom}(M, N)$

$\Longrightarrow \delta \in \emptyset$Hom($M$, N) ⊆ Hom(M, $\emptyset(N)$), or there is γ: $N \longrightarrow M$ such that $\delta\gamma = \emptyset 1_N$, then $\delta\gamma \in$ Hom(N,$\emptyset(N)$) $\Longrightarrow \delta$Hom (N, M)⊆ Hom(N, $\emptyset(N)$).as the following diagram:



■

Definition 3.2.16 [37]:A short exact sequence $0 \longrightarrow X \overset{\alpha}{\rightarrow} Y \overset{\beta}{\rightarrow} U \longrightarrow 0$ of R-semimodules is said to be split exact sequence if there is map $\emptyset$: $U \longrightarrow Y$ such that $\beta\emptyset = 1_U$.

Example 3.2.17: Consider the sequence of R-semimodules $0 \longrightarrow U \overset{i}{\rightarrow} U \oplus D \overset{\pi}{\rightarrow} D \longrightarrow 0$ where i is the inclusion map and π is the projection map defined by $\pi(u, d) = d$, for all $(u, d) \in U \oplus D$ with ker (π)= U, this sequence is split exact.

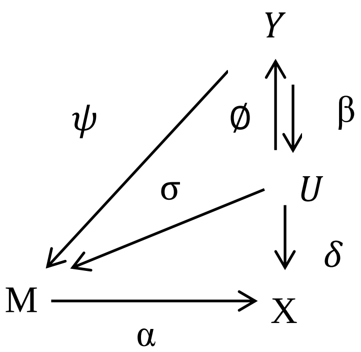
Proposition 3.2.18: Let $0 \longrightarrow X \overset{\alpha}{\rightarrow} Y \overset{\beta}{\rightarrow} U \longrightarrow 0$ be a short exact sequence of semimodules such that U is almost Y-projective, then either this sequence

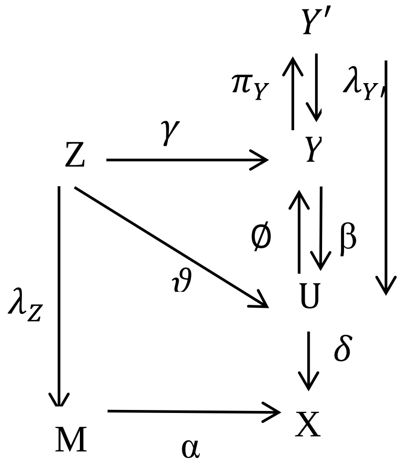
is split, or there is γ:Z$\longrightarrow U$ such that $\gamma = \beta\lambda_Z$, where Z is a nonzero direct summand of Y.

Proof. Since U is almost Y-projective, then there is $\emptyset$: $U \longrightarrow Y$ such that $\beta\,\emptyset = 1_U$ where β: $Y \longrightarrow U$ and hence this sequence splits, or there is γ:Z$\longrightarrow U$ where Z is nonzero direct summand of Y such that $\gamma = \beta\lambda_Z$. ■

Proposition 3.2.19: Let M be R-semimodule and $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\beta} U \longrightarrow 0$ be short exact sequence of R-semimodules with Y is almost M-projective, then U is almost M-projective.

Proof. Consider the following diagrams:





Where $\alpha: M \longrightarrow X$ is an R-epimorphism and $\delta: U \longrightarrow X$ is any R-homomorphism, since Y is almost M-projective, then either there is ψ: $Y \longrightarrow M$ such that $\alpha\psi = \delta\ \beta$, or there is γ:Z$\longrightarrow$ Y where Z is nonzero direct summand of M, such that $\delta\beta\gamma = \alpha\lambda_Z$. By Proposition 3.2.18 either this sequence splits that is there exists $\emptyset$: $U \longrightarrow Y$ such that $\beta\ \emptyset = 1_U$, then in the first diagram define σ: $U \longrightarrow M$ by $\sigma = \psi\emptyset$ hence $\alpha\ \sigma = \alpha\psi\emptyset = \delta\beta\emptyset = \delta$, or there is ρ:$Y' \longrightarrow U$ where Y' is a nonzero direct summand of Y such that

$\rho = \beta\lambda_{Y'}$, define ϑ:Z$\longrightarrow U$ by $\vartheta = \beta\lambda_{Y'}\pi_{Y'}\gamma$. Then $\delta\vartheta = \delta(\beta\lambda_{Y'})\pi_{Y'}\gamma = \delta\rho\pi_{Y'}\gamma = \delta\beta\gamma = \alpha\lambda_Z$. Therefore U is almost M-projective. ■

Proposition 3.2.20:Let M be R-semimodule and $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\beta} U \longrightarrow 0$ be short exact sequence of R-semimodules such that Y is almost M-projective semimodule, then $Y \oplus U$ is almost M-projective.

Proof. From Proposition 3.2.19, U is almost M-projective, and by Proposition 3.2.10, then $Y \oplus U$ is almost M-projective. ■

It is known that, every projective semimodule is almost-projective but the convers is not true. In the following proposition, we put the conditions for two semimodules M and N to get N-projective from almost N-projective.

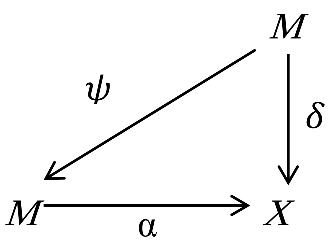
Proposition 3.2.21: Let M and N be hollow semimodules such that M is almost N-projective and there is no epimorphism from N onto M, then M is N-projective.

Proof. Let $\alpha: N \longrightarrow X$ be an epimorphism and $\emptyset: M \longrightarrow X$ be any homomorphism. Assume that the diagram(i) in Definition 3.1.1 does not satisfied, since M is almost N-projective, diagram(ii) must be satisfied. But N is hollow, hence indecomposable, then N is the only nonzero direct summand of N. So, there is $\gamma: N \longrightarrow M$ such that $\emptyset\gamma = \alpha 1_N = \alpha$. by assumption γ cannot be onto, that is $\gamma(N)$ is a proper subsemimodule of M, hence small and $\emptyset(\gamma(N))$ is small in X, but $\emptyset(\gamma(N)) = \alpha(N)$ which is a contradiction. Therefore diagram (ii) of Definition 3.2.1 cannot be satisfied, and only diagram(i) is satisfied that is M is N-projective. ■

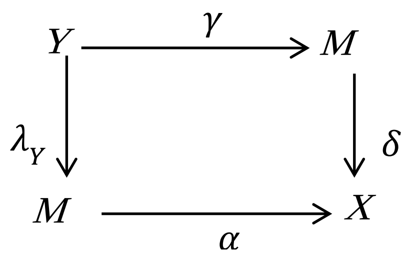
3.3 Almost Self-Projective Semimodules

In this part almost self-projective semimodule has been introduced as generalization of almost projective and some characterization of this notion has been discussed. A semimodule M is said to be almost self-projective if M is almost M-projective semimodule. Some properties of this concept have been discussed.

Definition 3.3.1: A semimodule M is said to be almost self-projective, if for each epimorphism $\alpha: M \longrightarrow X$ where X is any semimodule, and every homomorphism $\delta: M \longrightarrow X$, either there is $\psi: M \longrightarrow M$ such that $\alpha\,\psi = \delta$, or there is $\gamma: Y \longrightarrow M$ where Y is a nonzero direct summand of M such that $\delta\gamma = \alpha\lambda_Y$. As the following diagrams:



(i)



(ii)

Remarks 3.3.2:

(1) Every almost projective semimodule and hence every projective semimodule is almost self- projective, furthermore, every quasi projective semimodule is almost self-projective.

(2) If M is an indecomposable almost self-projective semimodule and if for some X, α and δ, the diagram (i) of Definition 3.3.1 is not satisfied, then $Y = M$ and $\lambda_Y = 1_M$, hence $\delta\gamma = \alpha$ and δ must be onto.

(3)The fact in (2) can be rewritten in this way, if M is an indecomposable almost self-projective semimodule and for some X, $\alpha: M \longrightarrow X$ is an epimorphism, but $\delta: M \longrightarrow X$ is not , then the diagram(i) must be satisfied and there exists endomorphism ψ of M such that $\alpha\, \psi = \delta$.

(4) Every simple semimodule is quasi-projective, hence it is almost-self projective.

Proposition 3.3.3:

(1) If $M \cong N$ and M is an almost self-projective semimodule, then N is almost self-projective .

(2) If Y is fully invariant direct summand of almost self-projective semimodule M, then Y is again almost self-projective.

Proof. Similar to Propositions 3.2.8 and 3.2.6. ■

Proposition 3.2.4: Let M_1, M_2 be fully invariant subsemimodules of M, if M=$M_1 \oplus M_2$ is almost self-projective semimodule, then each M_i is almost self-projective. i=1, 2.

Proof. Assume that $M = M_1 \oplus M_2$is almost self-projective semimodule and M_1, M_2 are fully invariant, this means$M_1 \oplus M_2$is almost $M_1 \oplus M_2$ -projective semimodule, from Proposition 3.2.5 each M_i is almost

$M_1 \oplus M_2$-projectivefor i=1, 2, and from Proposition 3.2.6, we get M_i is almost M_i-projective semimodule for $\boldsymbol{i}$=1,2. ■

Proposition 3.3.5: If $M \oplus M$ is almost self-projective semimodule, then M is almost $M \oplus M$ -projective semimodule.

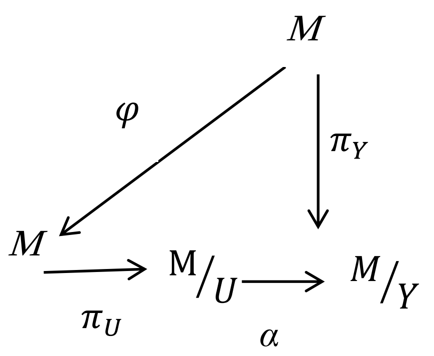
Proof. Since $M \oplus M$ is almost self-projective semimodule, this means $M \oplus M$ is almost $M \oplus M$ -projective semimodule and since M is a direct summand of $M \oplus M$, then by Proposition 3.2.6 M is almost $M \oplus M$ - projective semimodule. ■

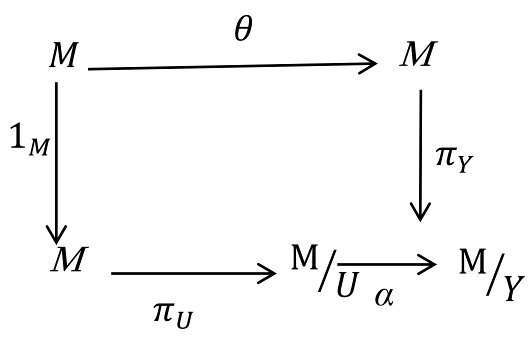
Corollary 3.3.6: For any integer $n \geq 2$, if M^n is an almost self-projective semimodule then M is almost M^n-projective.

Proof. Similar to Proposition 3.3.5. ■

Proposition 3.3.7: Let M be an indecomposable almost self-projective semimodule, U, $Y \leq M$ and U is fully invariant of M. If there is an epimorphism $\alpha: {}^{M}/_{U} \longrightarrow {}^{M}/_{Y}$, then either $U \subseteq Y$, or there is θ ∈ʃ such that $\theta(U) \subseteq Y$.

Proof. Suppose M is almost self-projective semimodule and from the following diagrams:





Since M is almost self-projective, either there is $\varphi: M \longrightarrow M$ such that $\alpha\pi_U\varphi = \pi_Y$ implies $\alpha\pi_U\varphi(U) = 0$(since U is fully invariant, this means $\varphi(U) \subseteq U \Rightarrow \alpha\pi_U\varphi(U) = 0$) and $\pi_Y(U) = \frac{U+Y}{Y}$, then $\frac{U+Y}{Y} = 0$ implies $U+Y=Y$, then $U \subseteq Y$, or there is $\theta: M \longrightarrow M$ such that $\pi_Y\theta = \alpha\pi_U 1_M$, then $\pi_Y(\theta(U)) = \frac{\theta(U)+Y}{Y}$ and $\alpha\pi_U 1_M(U) = \alpha(0) = 0$ implies that $\frac{\theta(U)+Y}{Y} = 0$, then $\theta(U) + Y = Y$ this means $\theta(U) \subseteq Y$. ■

Proposition 3.3.8: If $M= X_1 \oplus X_2$ is almost self-projective semimodule, where X_1 and X_2 are fully invariant subsemimodules of M, then X_i is almost X_j-projective semimodule for $i, j=\{1, 2\}$.

Proof. Suppose that $M= X_1 \oplus X_2$ is an almost self-projective semimodule, then M is almost M-projective semimodule. Since X_i is direct summand of M, then by Proposition 3.2.5 X_i is almost M-projective semimodule and by Proposition 3.2.6 X_i is an almost X_j-projective semimodule. ■

Corollary 3.3.9: If $M= \oplus_{i=1}^{n} X_i$ is almost self-projective semimodule, then X_i is almost X_j-projective semimodule for all distinct $i, j \in \{1, 2, \ldots, n\}$ where X_i is fully invariant in M for each i.

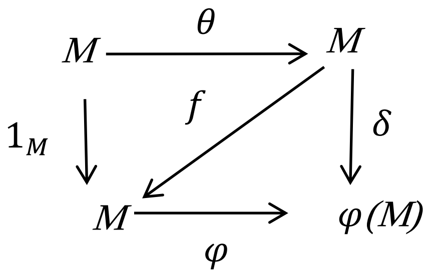
Proof. From Propositions 3.2.5, 3.2.6 and 3.3.8 the result is obtained. ■

Proposition 3.3.10: If $M= X_1 \oplus X_2$ is an almost self-projective semimodule, where X_1 and X_2 are fully invariant subsemimodules of M, then M is almost X_j-projective semimodule for j=1, 2.

Proof. Since M is an almost self-projective semimodule, then M is almost M-projective semimodule by Proposition 3.2.6, M is almost X_j-projective semimodule for j=1, 2. ■

Proposition 3.3.11: Let M be an indecomposable almost self-projective semimodule, $\varphi \in \mathcal{S}$, then there is $\theta \in \mathcal{S}$ such that $\mathrm{Hom}(M, \varphi(M))\theta = \varphi\, \mathcal{S}$. Where $\mathcal{S}$=End(M).

Proof. It is clear that $\varphi\mathcal{S} \subseteq \mathrm{Hom}(M, \varphi(M))\, \theta$, let $\delta \in \mathrm{Hom}(M, \varphi(M))$, since M is almost self-projective, then either $\varphi f = \delta$ for some $f \in \mathcal{S}$, take $\theta = 1_M$, we have $\varphi f = \delta 1_M = \delta$ implies $\delta \in \varphi\, \mathcal{S}$, or there is endomorphism θ such that $\delta\, \theta = \varphi 1_M$, thus $\delta\theta \in \varphi\, \mathcal{S}$. As the following diagram:



■

Proposition 3.3.12: Let M be indecomposable almost self-projective semimodule, for $\alpha, \beta \in$ ʃ if $\beta(M) \subseteq \alpha(M)$, then βʃ $\subseteq \alpha$ ʃ.

Proof. Since $\beta(M) \subseteq \alpha(M)$, Hom($M$, $\beta(M)$) $\subseteq$ Hom(M, $\alpha(M)$) and since M is almost self-projective by Proposition 3.3.11, then βʃ = Hom(M, $\beta(M)$)θ $\subseteq$ Hom(M, $\alpha(M)$) $\theta = \alpha$ ʃ, for some $\theta \in$ ʃ, then βʃ $\subseteq \alpha$ ʃ. ■

Proposition 3.3.13: Let M be almost self-projective semimodule and indecomposable and $\beta, \alpha \in$ ʃ, then, If α (M) is an image of $\beta(M)$, then α ʃ is an image of β ʃ.

Proof. Let $\vartheta: \beta\ (M) \longrightarrow \alpha(M)$ be an epimorphism, since M is almost self – projective semimodule, then either there is g$\in$ ʃ such that αg $= \vartheta\beta$, or there is $\gamma \in$ ʃ such that $\alpha = \vartheta\beta\gamma$. Define μ: βʃ $\longrightarrow \alpha$ ʃ by $\mu(\beta\varphi) = \alpha$g$\varphi$, $\forall\varphi \in$ʃ, μ is well defined since (if $\beta\varphi$=$\beta\varphi$', then αgφ (m)= $\vartheta\beta\varphi$(m)=

$\alpha\varphi$(m)= $\alpha\varphi$'(m)= $\vartheta\beta\varphi$'(m)= αgφ'(m) hence μ is well defined. Since M is almost self-projective semimodule, by Proposition 3.3.11, $\vartheta(\beta(M)) \subseteq \alpha$

(M) implies $\vartheta\beta$ ʃ$\subseteq$ αʃ $\Longrightarrow$ αg ʃ$\subseteq$ αʃ $\Longrightarrow$ $\mu(\beta$ʃ$) \subseteq \alpha$ʃ, hence μ is an epimorphism. ■

Proposition 3.3.14: Let M be an indecomposable almost self-projective semimodule, $\beta \in$ ʃ, if $\beta(M)$ is a simple subsemimodule of M, then βʃ is simple subsemimodule of $\mathcal{S}$. The convers is true if M is a principally self-generator semimodule.

Proof. Suppose that $\beta(M)$ is a simple subsemimodule of M, if $0 \neq \beta\alpha\,\mathcal{S} \neq \beta\,\mathcal{S}$ for some $\alpha \in \mathcal{S}$, then $\beta\alpha(M)$ is a nonzero proper subsemimodule of $\beta(M)$, which is a contradiction with hypotheses, then $\beta\mathcal{S}$ is $\mathcal{S}$-simple semimodule. Conversely, suppose M is a principally self- generator semimodule and $\beta\,\mathcal{S}$ is simple. If $0 \neq \beta(Rm) \neq \beta(M)$, then for some $\alpha \in \mathcal{S}$, $\beta\alpha(M) = \beta(Rm) \neq \beta(M)$ hence by Proposition 3.3.11, $0 \neq \beta\alpha\mathcal{S} \subseteq \beta\mathcal{S}$ which is a contradiction with simplicity of $\beta\mathcal{S}$. ■

Proposition 3.3.15: Let M be an indecomposable self-generator and almost self-projective semimodule. If U is essential in Y with $Y \leq M$, then Hom(M, U) is essential in right $\mathcal{S}$-semimodule Hom(M,Y).

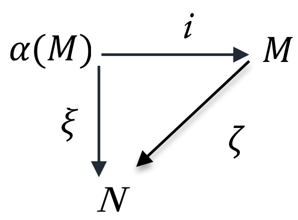
Proof. Let $\alpha \in$ Hom(M,Y) and Hom(M, U)$\cap\, \alpha\,\mathcal{S} = 0$. Assume that $\alpha(m) \in U \cap \alpha(M)$. Since M is self-generator, there exist epimorphisms β: $M \longrightarrow R\alpha(m)$ and γ:$M \longrightarrow Rm$. Then $\beta(M) = R\alpha(m) = \alpha\,\gamma(M)$ from Proposition 3.3.11 we get $\beta\mathcal{S} = \alpha\,\gamma\mathcal{S}$, thus $\beta = \alpha\,\gamma\vartheta$ for some ϑ in $\mathcal{S}$. Whence $\beta \in$ Hom(M,U) $\cap\, \alpha\,\mathcal{S} = 0$ implies $\alpha(m) = 0$, then $U \cap \alpha(M) = 0$, but U is essential in Y, then $\alpha(M) = 0$ and hence $\alpha = 0$. Therefore Hom(M, U) is essential in Hom(M,Y). ■

Proposition 3.3.16: Let M be an indecomposable uniserial semimodule, if M is almost self-projective, then $\mathcal{S}$ is a left uniserial semiring.

Proof. Suppose that X and Y are left ideals of $\mathcal{S}$ such that $Y \not\subseteq X$, let $\alpha \in X$, $\beta \in Y$ and $\beta \notin X$. If $\beta(M) \subseteq \alpha(M)$ by Proposition 3.3.11, then $\beta\mathcal{S} \subseteq \alpha\,\mathcal{S}$ and hence $\beta \in \alpha\mathcal{S} \subseteq X$ this is a contradiction. Since M is uniserial, then $\alpha(M) \subseteq \beta(M)$ and hence $\alpha\,\mathcal{S} \subseteq \beta\,\mathcal{S}$ and this means $X \subseteq Y$. Therefore $\mathcal{S}$ is left uniserial semiring. ■

Definition 3.3.17 [13, p.51]: A subsemimodule U is said to be M-cyclic subsemimodule of M, if it is isomorphic to $M/_Y$ for some subsemimodule Y of M.

Definition 3.3.18: Let M and N be two semimodules. N is said to be M-mini-injective semimodule, if for every homomorphism from simple N-cyclic subsemimodule of M to N can be extended to M, i.e., the following diagram commutes: where $\alpha \in$ End(M).

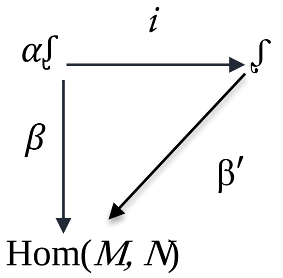


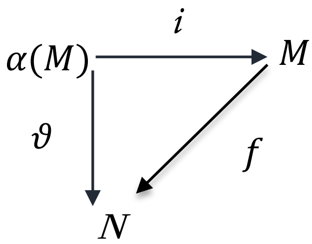
A semimodule N is called a mini-injective semimodule if it mini-injective for any semimodule M and called a quasi mini-injective semimodule, if it N-mini-injective .

Proposition 3.3.19: Let M be an almost self-projective semimodule, if Hom(M, N) is mini-injective as right ʆ -semimodule, then N is M-mini-injective semimodule.

Proof. Let $\alpha \in$ ʆ and ϑ: α (M) $\longrightarrow N$ be homomorphism where α (M) is simple M-cyclic subsemimodule of M, since M is almost self-projective semimodule from Proposition 3.3.14, α ʆ is simple, then $\vartheta\alpha \in$ Hom (M, N). Let β: αʆ $\longrightarrow$ Hom(M, N) be ʆ -homomorphism defined by $\beta(\alpha\gamma)$= $\vartheta\alpha\gamma$, $\gamma \in$ ʆ . Since Hom(M, N) is mini-injective , then there is β': ʆ$\longrightarrow$ Hom (M, N) such that $\beta' i$= β. Let f = $\beta'(1)$, then $\vartheta\ \alpha = \beta\alpha = \beta'\alpha = \beta'(1)\alpha =$

$f\alpha$. Therefore N is M- mini-injective semimodule. As the following diagrams:



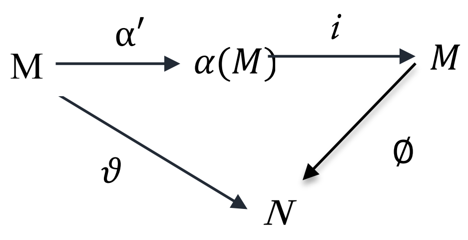


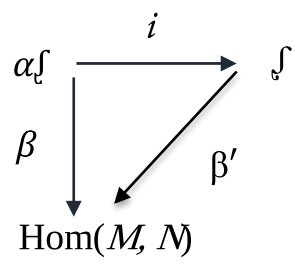
■

<u>Proposition 3.3.20</u>: Let M be an almost self-projective semimodule which is self-generator, if N is M-mini-injective, then Hom (M, N) is mini-injective as right ʃ -semimodule.

<u>Proof.</u> Let α ʃ be a simple subsemimodule of ʃ -semimodule ʃ. Let β: αʃ $\longrightarrow$ Hom(M, N) be ʃ -homomorphism and let $\beta(\alpha)= \vartheta$, $\alpha \in$ ʃ, since M is self-generator, ker $\alpha= \sum_{\gamma\in I}\gamma(M)$ where $I \subseteq$ ʃ. Take any $\gamma \in I$, since $\alpha\gamma$=0, then $\beta(\alpha\gamma)= \beta(\alpha)\gamma=\vartheta\gamma= 0$ and then Imγ $\subseteq$kerϑ. It follows that ker α $\subseteq$kerϑ, consider α as an epimorphism and α':$M\longrightarrow$ $\alpha(M)$, then ker α' $\subseteq$ker α and hence there is homomorphism ψ: $\alpha(M)$ $\longrightarrow M$, such that $\psi\alpha'= \vartheta$, since M is almost self-projective and self-generator semimodule by Proposition 3.3. 14, then $\alpha(M)$ is simple subsemimodule

of M. By assumption ψ can be extended to homomorphism $\emptyset: M \longrightarrow N$ such that $\emptyset i\alpha'=\vartheta$ where i: $\alpha(M)$ $\longrightarrow M$ and then $\emptyset\alpha =\vartheta$. Let β': ʃ $\longrightarrow$ Hom(M, N) defined by $\beta'(\gamma)= \emptyset\, \gamma$, then $\beta'(\alpha)= \emptyset\, \alpha= \vartheta = \beta(\alpha)$. Therefore Hom($M$, N) is mini-injective. As the following diagrams:

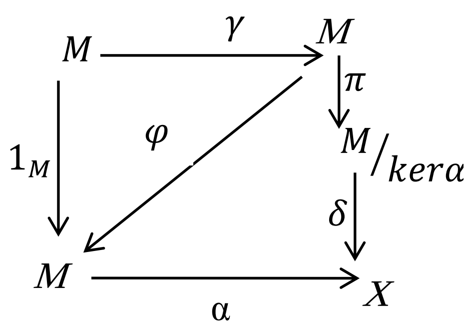




■

<u>Proposition 3.3.21:</u> Let M be an indecomposable almost self-projective semimodule and $\alpha{:}M\longrightarrow X$ be epimorphism, then there is $\varphi \in \int$ such that $\ker\alpha = \ker \alpha\varphi$.

<u>Proof.</u> Since α: $M\longrightarrow X$ is an epimorphism . Consider the following diagram:

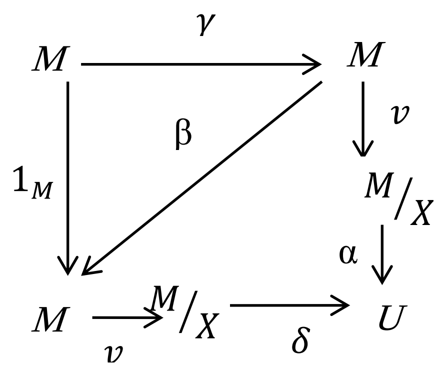


Let δ: $M/_{ker\alpha} \longrightarrow X$ be an isomorphism and $\pi{:}M \longrightarrow M/_{ker\alpha}$ be the natural projection, since M is an indecomposable almost self-projective semimodule, then either there is $\varphi{:}\,M \longrightarrow M$ such that $\alpha\varphi = \delta\,\pi$ or there is $\gamma{:}\,M \longrightarrow M$ such that $\alpha 1_M = \delta\,\pi\gamma$. The first case implies $\ker\alpha\varphi =$

$\ker\delta\,\pi = \pi^{-1}\ker\delta = \pi^{-1}(0) = \ker\alpha$. In the second case, $\ker\alpha = \ker\alpha 1_M = \ker\delta\,\pi\gamma$ can be consider $\varphi = \gamma$, then $\ker\alpha\varphi = \ker\alpha$. ■

<u>Proposition 3.3.22:</u> Let M be indecomposable almost self-projective semimodule and X, Y are invariant subsemimodules of M, then $X\cap Y$ is fully invariant subsemimodule of M.

Proof. Let $\delta: {}^{M}/_{X} \longrightarrow U$ be any epimorphism, $\alpha: {}^{M}/_{X} \longrightarrow U$ be homomorphism and $v: M \longrightarrow {}^{M}/_{X}$ be the natural projection. Now since M is almost self-projective semimodule, either there is $\beta: M \longrightarrow M$ such that $\delta v \beta = \alpha v$ or there is $\gamma: M \longrightarrow M$ such that $\delta v 1_M = \alpha v \gamma$, since $X \cap Y \subseteq \ker \delta v \cap \ker \alpha v$, then $\beta(X \cap Y) \subseteq \beta(X) \subseteq X$ and $\beta(X \cap Y) \subseteq \beta(Y) \subseteq Y$, then $\beta(X \cap Y) \subseteq \beta(X) \cap \beta(Y)$, also $X \cap Y \subseteq \ker \delta v 1_M \cap \ker \alpha v$, then $\gamma(X \cap Y) \subseteq X \cap Y$. This means $X \cap Y$ is a fully invariant subsemimodule of M.



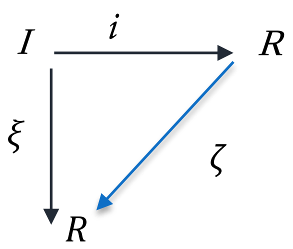
■

4.1 Introduction

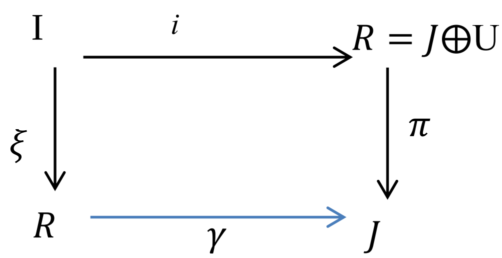
After the concepts "Almost injective, Almost projective, Almost self - injective and Almost self-projective semimodules" have been defined in previous chapters. In this chapter, almost self-injective semirings and some notions related to these concepts will be dealt, as well as their relationship with each other. These concepts also have been defined for class of semimodules depending on what is in the class of modules , with making what needs to be changed based on the difference between the two classes, as it is the basic and fundamental difference between them is the lack of subtraction in class of semimodules and try to replace for some of the qualities that semimodules possesses. In this chapter, the concepts " generalized N- injective, generalized N- projective, essentially N-injective semimodule and almost self-injective semiring have been defined and some of relationships with each other are discussed.

4.2 Almost Self-Injective Semirings

<u>Definition 4.2.1</u>:A semiring R is said to be almost self-injective, if $_RR$ is almost self-injective semimodule. See Definition 2.3.1, this means either diagram (i) or diagram (ii) commutes.



(i)



(ii)

The following Lemma was proved in [15] with conditions yoked, cancellative and subtractive on the semiring *R*. Here a proof will be given without the subtractive condition.

Lemma 4.2.2: A left ideal I of a yoked, cancellative semiring *R* is a direct summand if and only if I= *Re* for some idempotent element e of *R*.

Proof. The proof of the first condition is the same as in [15], which did not need any of the above conditions. Now, assume that I= *Re* and e is idempotent element of *R*, by yoked property, there is r in *R* such that either

e +r =1 or r +1= e. In any case, we get er= re= 0 which leads to $Re \cap Rr = 0$

Now, assume e +r =1 implies $Re+Rr=R$. r +1= e implies rr +r = er $\Longrightarrow$ rr + r =0 so e +r=1 implies (e + r)+ (rr+ r) =1+(rr+r) $\Longrightarrow$ e +rr = 1 , that is $1\in Re+Rr$, hence $Re+Rr=R$. Now, if $r_1e+ r_2r = s_1e+ s_2r \Longrightarrow r_1e\,e + r_2r\,e = s_1e\,e + s_2r\,e \Longrightarrow r_1e = s_1e$, then by cancellative $r_2r = s_2r$, so the representation is unique and $Re\oplus Rr=R$. Therefore *Re* is direct summand of *R*. ■

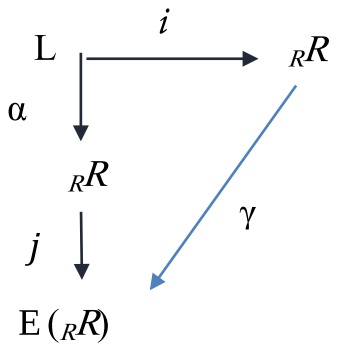
Proposition4.2.3: Let *R* be an almost self-injective semiring, then for each idempotent element *e* of *R*, *R*e is almost self-injective provided that Re is fully invariant.

Proof. Let *R* be almost self-injective this means *R* is almost *R*-injective, then *Re* is direct summand of *R* for some idempotent element e of *R* by Lemma 4.2.2 . Firstly since *R* is almost self-injective this means *R* is almost *R*-injective, then *R* is almost *Re*-injective by Proposition 2.2.11.and hence *Re* is *Re*-injective by Proposition 2.2.10. ■

Proposition 4.2.4: Let *R* be a semiring has no nontrivial idempotent elements with injective hull $E({}_RR)$ which is cancellative, semisubtractive, then R is left almost self-injective semimodule if and only if for all $e \in E({}_RR)$, either $e \in R$ or $\exists r \in R$ such that re = 1.

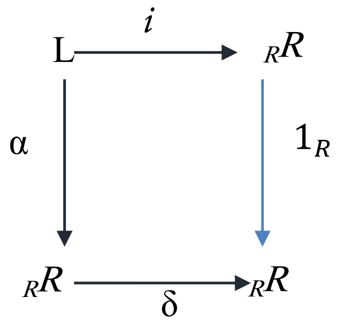
Proof. Suppose that R is a left almost self-injective, since it has no nontrivial idempotent and by Lemma 4.2.2, then ${}_RR$ is indecomposable and hence uniform semimodule (because it is indecomposable almost self-

injective by Proposition 2.3.13, let $e \in E({}_RR)$ and $\mathcal{L}_e$: $R \longrightarrow E({}_RR)$ be the left multiplication homomorphism by e. Then there is endomorphism ψ of $E({}_RR)$ extension of $\mathcal{L}_e$ by Proposition 2.2.19, either $\psi(R) \subseteq R$ or ψ is an isomorphism and $\psi^{-1}(R) \subseteq R$. If $\psi(R) \subseteq R \Longrightarrow e \in R$. If ψ is an isomorphism and $\psi^{-1}(R) \subseteq R \Longrightarrow \psi^{-1}(1) \in R$ and there is $r \in R$ such that $\psi(r) = 1$, so $r\,e = \mathcal{L}_e(r) = \psi(r) = 1$. Conversely, suppose $\forall\, e \in E({}_RR)$, then either $e \in R$ or $\exists r \in R$ such that $re = 1$. Now consider the diagram



By injectivity of $E({}_RR)$, there is $\gamma: {}_RR \to E({}_RR)$ such that $\gamma i = j\alpha$,

since $\gamma(1) \in E({}_RR)$, then by assumption either $\gamma(1) \in {}_RR$ in this case $\gamma({}_RR) \subseteq {}_RR$, that is γ is an extension of α to ${}_RR$ hence the diagram (i) of the Definition 2.3.1 holds. Or , there is $r \in R$ such that $r\,\gamma(1) = 1$. Let $\delta = \gamma|_R R$, then $\forall l$ in L, $\delta\alpha(l)) = \gamma(\alpha(l)) = \alpha(l)\gamma(1) = \alpha(l)r = \gamma(l\,)r = l\,\gamma(r) = l$ that is $\delta\alpha = i = 1_R i$, which means diagram(ii)holds. Therefore ${}_RR$ is almost self -injective. As the following diagrams:



■

Corollary 4.2.5: Let D be a semidomain has injective hull and Q its semiring of quotients, then D is almost self -injective if and only if for every $q \in Q$, either q or $q^{-1} \in D$. This means D is almost self -injective if and only if D is valuation semidomain.

Lemma 4.2.6: Suppose M is an indecomposable almost self-injective semimodule, if $\delta: U \longrightarrow M$, where $U \leq M$ is a homomorphism which cannot be extended to an endomorphism of M, then there is a monic endomorphism θ of M such that θ is left inverse of δ on δ (U), but θ is not invertible.

Proof. From definition there exists an R-homomorphism $\theta: M \longrightarrow M$ such that $\theta\delta(u) = u$, for any u $\in U$. Hence θ is a left inverse of $\delta|_{\delta(U)}$, since M is uniform by Proposition 2.3.13, θ is monic. If θ is invertible, then θ^{-1} would be an extension of δ. ■

Proposition 4.2.7: Let R be a local semiring with Rad(R) nil ideal. If R is almost self-injective, then R is self-injective.

Proof. Suppose R is almost self-injective, we have an R-homomorphism $\theta:K\longrightarrow R$ where $K<_{R}R$ which cannot be extended to an endomorphism of $_{R}R$ from Lemma 4.2.6, there is $\vartheta: {}_{R}R\longrightarrow {}_{R}R$ such that $\vartheta(\theta(x)) = x$, $\forall$ $x\in K$ $\Longrightarrow$ ϑ is not invertible. If it invertible, then $\theta(x) = \vartheta^{-1}(x)$, $\forall$ $x\in K \Longrightarrow \theta$ can be extended to ϑ^{-1} which is contrary with hypotheses, hence $\vartheta(1) = r$ is not invertible $\Longrightarrow r\in \mathrm{Rad}(R)$ and ϑ is monic $\Longrightarrow$ if $\vartheta(t) = t\,\vartheta(1) = t\,r = 0$,

then $t = 0$ but this contradiction with hypotheses, hence R is self-injective. ■

Proposition 4.2.8: Let R be a semiring with no nontrivial idempotent element, then R is almost self-injective semiring if and only if for each left ideal I of R, each homomorphism $\xi:I\rightarrow R$ there is r in R such that either $\xi|_{Rr} = l_r$ or $l_r\xi|_{Rr} = i$. Where l_r is the left multiplication map by element r.

Proof. Assume that R is an almost self-injective semiring with no nontrivial idempotent, then by Lemma 4.2.2, R is indecomposable. In the first diagram of Definition 4.2.1, the result can be obtained by replacing $\zeta = l_r$ and from the second diagram can be obtained by taking $\gamma = l_r$. Conversely, it is clear. ■

Corollary 4.2.9: Let R be almost-injective semiring, then R is π-injective.

Proof. Similar to Proposition 2.3.12, with replacing subsemimodules by ideals of R. ■

Proposition 4.2.10: Let R be a left almost self-injective semiring with no nontrivial idempotent, yoked and cancellative, then for every $a, b \in R$

(1) If $ann(a)=0$ and $ann(b)\neq 0$, then $Rb \subset Ra$.

(2) If $ann(a)=0$ and $ann(b)=0$, then either $Ra \subset Rb$ or $Rb \subset Ra$.

Proof. (1)Define f:$Ra \longrightarrow Rb$ by $f(ra)$=rb. Clearly that f is a well-defined R-homomorphism(if, ra=$r'a$, by yoked there is k in R such that r=r' +k or r+ k=r' $\Longrightarrow ra = r'a + ka$ by cancellative, then ka =0$\Longrightarrow k \in ann(a)$ implies $k = 0 \Longrightarrow rb = r'b$. Now, if ann(b)≠0, then f is not one to one. Since R is a left almost self-injective, f can be extended to R.

Hence there is $s \in R$ such that f=l_s on Ra. so $b = f(a) = sa$ and then $Rb \subset Ra$.

(2) Let $ann(a)$=0 and $ann(b)$=0, then f is one to one. Thus either, f=l_s on Ra implies $Rb \subset Ra$ for some $s \in R$, or there is $s \in R$ such that $l_s f$=1_{Ra} implies a=$l_s f(a)$= $sf(a) = f(sa) = sb \in R$b, thus $Ra \subset Rb$. ■

Lemma 4.2.11 [26]:A semimodule M is $\underset{i\in I}{\oplus} N_i$-injective if and only of M is N_i-injective for every $i \in I$.

A semiring R is said to be von Neumann regular if, for any x in R , there is y in R such that x =xyx [41].

Lemma 4.2.12 [41]: A cancellative, yoked semiring R is von Neumann regular if and only if every principle right ideal of R is direct summand.

Proposition 4.2.13: Let R be left almost self-injective, which is cancellative, yoked and von Neumann regular semiring. If R has a nontrivial idempotent element, then R is left self-injective.

Proof. Let e be a nontrivial idempotent element of R from Lemma 4.2.12, then Re is direct summand of R, from Lemma 4.2.11, it is enough to show that ${}_R R$ is both Re-injective and Ru-injective semimodule where ${}_R R$=$Re \oplus Ru$. Let X be a nonzero subsemimodule of Re and f:$X \longrightarrow R$ be homomorphism. Define g:X⊕Ru $\longrightarrow R$ by g(x+ ru)= f(x), then g is homomorphism which is not one to one. Since R is almost-self-injective, there is h: $R \longrightarrow R$ such that $h|_{X\oplus Ru} = g$. But $h|_X = f$. Hence, R is Re-injective semimodule, similarly R is Ru-injective. Therefore R is R-injective semiring. ■

<u>Proposition 4.2.14:</u> Let R be a left almost self-injective semiring with no nontrivial idempotent element and let T=$\sum Ra$, where $ann(a)$=0 and a is not invertible, then T is two sided ideal of R.

<u>Proof.</u> Assume that T $\neq$ 0, let t $\in R$ and a noninvertible element of R such that $ann(a)$=0 , if $ann(ta) \neq$ 0, then by Proposition 4.2.10, ta ∈T. Now, assume that $ann(ta)$= 0, then ta is not invertible, if possible, there is $x \in R$ such that $xta = tax = 1$. Since R has no nontrivial idempotent, thus $xta = tax$ =1, this is a contradiction with a is not invertible. Hence $ta \in T$. ■

Recall that, U(R) is the monoid of invertible elements of R.

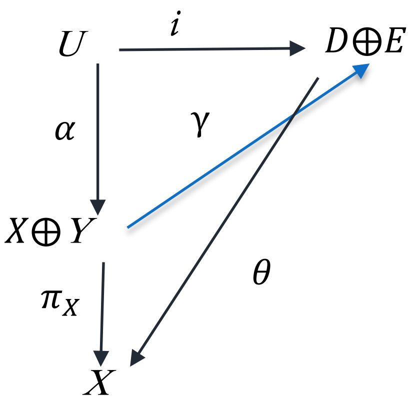
<u>Proposition 4.2.15:</u> Let R be almost self-injective semiring with no nontrivial idempotent, then R is either self-injective or local.

<u>Proof.</u> By Proposition 4.2.13, if R has a nontrivial idempotent, then it is left self-injective. Thus assume that R has no nontrivial idempotent. Since

R is almost self-injective, by Corollary 4.2.9, R is π-injective and from Proposition 2.3.13, hence ${}_RR$ is uniform. Let F={s ∈R| $ann(s) = 0$ and s is not invertible}.If F is empty , then $s \in R$ is invertible if and only if $ann(s) = 0$, since ${}_RR$ is uniform, $\mathcal{Z}({}_RR)$= R \ U(R) is two sided ideal. Hence R is local. If F is not empty, let T =$\sum_{s\in F} sR$ by Proposition 4.2.14, R \ U(R) ⊂ T. Let k ∈T such that k is not invertible, so k= hn for some h∈ F. If k is invertible, then h is left invertible. Since R has no nontrivial idempotents, h is invertible, a contradiction (since h∈F), therefore T=$R \setminus$ U(R) . Since T is two sided ideal of R, hence R is local. ■

4.3 Some concepts related to almost-injective(projective)semimodules

Definition 4.3.1: A semimodule M is said to be generalized N- injective, if for any subsemimodule U of N and any homomorphism α:U→M, there is decompositions N=D⊕E, M=X⊕Y, a homomorphism θ:D→X and monomorphism γ: Y →E such that U⊂ D⊕$\gamma(Y)$ with $\pi_X\alpha = \theta\, \pi_D|_U = \theta\, \pi_D i_U$ and $\gamma\pi_Y\alpha = \pi_E|_U = \pi_E\, i_U$.



A semimodule M is called a generalized self-injective semimodule if, M is generalized M-injective.

Proposition 4.3.2: If M is a generalized N-injective semimodule, then M is almost N-injective.

Proof. Let U be a subsemimodule of N and α:U→M be a homomorphism, then there is decompositions N=D⊕E, M=X⊕Y, a homomorphism θ:D→X and monomorphism γ: Y →E such that U⊂ D⊕$\gamma(Y)$ with $\pi_X\alpha = \theta\, \pi_D|_U$ and $\gamma\pi_Y\alpha = \pi_E|_U = \pi_E\, i_U$ .If α cannot be extended to N, then E≠0, define δ: M→D by $\delta = \gamma\pi_Y$. For every u in U, $\delta\alpha$(u)= $\delta\alpha$(d+e)= $\delta(\theta$(d)+ θ'(e))=

$\gamma\pi_Y(\theta$(d)+ θ'(e))= $\gamma(\theta'$(e))= e = $\pi_E i_U(u)$. Where θ':$E\rightarrow Y$ and π_E: $N\rightarrow E$.

Corollary 4.3.3: If M is generalized self-injective semimodule, then it is almost self-injective-semimodule.

Remarks 4.3.4:

(1)If N is indecomposable and M is generalized N-injective, by Proposition 4.3.2 (2), M is almost N-injective semimodule, and by Remark 2.2.4, either M is N-injective or there is subsemimodule of N embedded in M.

(2)If M, N are indecomposable semimodules and M is generalized N-injective, then there are four cases:

Case1/ M=M⊕0 and N=0⊕N, this means there are θ:0⟶M and γ:0⟶N, hence $\pi_M\alpha$= θ $\pi'_0|_U = \theta$ $\pi'_0 i_U$ and $\gamma\pi_0\alpha$= $\pi_N{}'|_U = \pi_N{}'$ i_U implies $U \subset N\oplus\gamma(0), \alpha = 0$ and $\gamma = 0$ which is impossible.

Case2/ M=M⊕0 and N=N⊕0, this means there are $\theta: N \longrightarrow M$ and γ:0⟶0, then $\pi_M\alpha$= θ $\pi'_N|_U = \theta$ $\pi'_N i_U$ and $\gamma\pi_0\alpha$= $\pi'_0|_U = \pi'_0$ i_U implies that $U \subset N\oplus\gamma(0)$, $\alpha = \theta$ i_U and 0=0.

Case3/ M= 0⊕M and N=0⊕ N, then there are θ:0⟶0 and γ: $M \longrightarrow N$, hence$U \subset N\oplus\gamma(M)$ with $\pi_0\alpha = \theta$ $\pi'_0|_U = \theta$ $\pi'_0 i_U$ and $\gamma\pi_M\alpha =$ $\pi'_N|_U = \pi'_N$ i_U $\Longrightarrow$ 0=0 and $\gamma\alpha$ = i_U implies α is monomorphism.

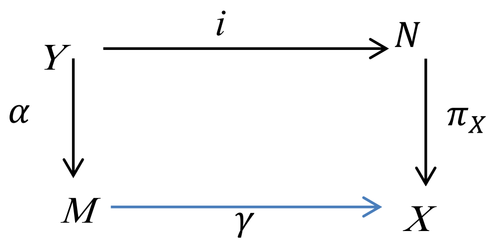
Case4/ M=0⊕ M and N=N⊕0, this means there are θ: $N \longrightarrow 0$ and γ: $M \longrightarrow 0$, hence U⊂N⊕$\gamma(M)$ with $\pi_0\alpha$= θ $\pi'_N|_U = \theta$ $\pi'_N i_U$ and $\gamma\pi_M\alpha$= $\pi'_0|_U = \pi'_0$ i_U implies α = 0 which is impossible.

(3)From (1), it follows, if M is generalized N-injective semimodule and M, N are indecomposable, then either there is θ: $N \longrightarrow M$ such that $\alpha = \theta$ i_U, or there is $\gamma: M \longrightarrow N$ such that $\gamma\alpha = i_U$ implies α is monomorphism and U is embedded in M.

(4)If there is no monomorphism $\alpha: U \longrightarrow M$ where U is any subsemimodule of N, then Case(3) is not satisfied, so M is N-injective semimodule.

(5) If M is indecomposable and generalized self-injective semimodule, then by Corollary 4.3.3, M is almost self-injective semimodule and then from Remark 2.3.2, if the diagram(i) is not satisfied, then $\gamma\alpha = i_U$ implies that α is monomorphism.

Proposition 4.3.5: Let M be almost N-injective semimodule. Consider the following diagram:



Put K = ker(α). If the second case of Definition 2.3.1 occurs, there is proper direct summand D of N which contains K. In particular, if $K \leq_e N$, then the first case occurs.

Proof. Since the second diagram holds, we get a direct decomposition N=X⊕D and γ:$M \longrightarrow X$ such that $\gamma\alpha = \pi_X i$, then $\pi_X(K) = \pi_X i(K) = \gamma\alpha(K) = 0$, so $K \subseteq \ker(\pi_X) = D$, then D is proper direct summand of M. If $K \leq_e N$, we have $K \cap X=0$ implies $X=0$ which is contradiction. Hence the first case occurs. ■

Definition 4.3.6: A semimodule M is said to be essentially N-injective if, for every subsemimodule X of N, any homomorphism $\alpha:X\longrightarrow M$ with $\ker(\alpha)$is essential in X, can be extended to homomorphism $\beta: N\longrightarrow M$.

Proposition 4.3.7: If M is almost N-injective semimodule, then M is essentially N-injective .

Proof. Let X be a subsemimodule of N and $\alpha:X\longrightarrow M$ be homomorphism with $\ker(\alpha)$is essential in X, then from Proposition 4.3.5, α can be extended to a homomorphism $\beta:N\longrightarrow M$ provided $\ker(\alpha) \leq_e N$ (in fact, if Y is a complement of X in N, define $\vartheta:X\oplus Y \longrightarrow M$ by $\vartheta(x+y) = \alpha(x)$ for every x in X and y in Y. Then $\ker(\vartheta)=\ker(\alpha) + Y \leq_e X\oplus Y \leq_e N$. So $\ker(\vartheta) \leq_e N$ and hence ϑ can be extended to a homomorphism $\beta:N\longrightarrow M$. Clearly, β is an extension of α. ■

From the above propositions, we have:

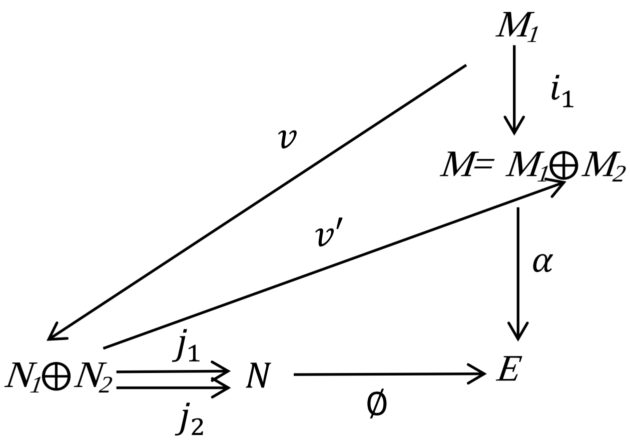
Corollary 4.3.8: If M is a generalized N-injective semimodule, then M is essentially N-injective.

Proof. Since, if M is a generalized N-injective semimodule, then M is almost N-injective by Proposition 4.3.2 and from Proposition 4.3.7, then M is an essentially N-injective semimodule. ■

Dually, generalized projective semimodule can be defined, also the relationship of this concept with almost projective semimodule will be explained.

Definition 4.3.9: A semimodule M is said to be generalized N-projective if, for any homomorphism $\alpha: M\longrightarrow E$ and any epimorphism $\emptyset: N\longrightarrow E$, there are decompositions $M = M_1\oplus M_2$ and $N= N_1\oplus N_2$, a homomorphism $v: M_1\longrightarrow N_1$ and an epimorphism $v': N_2\longrightarrow M_2$ such that $\emptyset j_1 v = \alpha i_1$ and $\alpha i_2 v' = \emptyset j_2$

. Where i_1, i_2 are the inclusion maps from M_1, M_2 into M respectively and j_1, j_2 are the inclusion maps of N_1, N_2 into N.



A semimodule M is called a generalized self-projective semimodule if, M is generalized M-projective.

Remark 4.3.10:

(1)If M is generalized N-projective and M, N are indecomposable semimodules, there are four cases:

Case/a/ $M=M\oplus 0$ and $N=0\oplus N$, this means there are $v:M\longrightarrow 0$ and $v':N\longrightarrow 0$, hence $\emptyset|_0 v = \alpha i_1$ implies $\alpha = 0$ which is impossible and $\alpha|_0 v' = \emptyset j_2$ implies 0 = 0.

Case/b/ $M=M\oplus 0$ and $N=N\oplus 0$, this means there are $v:M\longrightarrow N$ and $v':0\longrightarrow 0$, then $\emptyset j_1 v = \alpha i_1 \Longrightarrow \emptyset v = \alpha$ and $\alpha|_0 v' = \emptyset|_0$ implies that 0=0.

Case/c/ $M= 0\oplus M$ and $N=0\oplus N$, then there are $v:0\longrightarrow 0$ and $v':N\longrightarrow M$, hence $\emptyset|_0 v = \alpha|_0$ implies 0=0 and $\alpha i_2 v' = \emptyset j_2 \Longrightarrow \alpha v' = \emptyset$ implies α is epimorphism.

Case/d/ M=0⊕ M and N=N⊕0, this means there are $v: 0 \longrightarrow N$ and $v': 0 \longrightarrow M$, hence $\emptyset j_1 v = \alpha|_0$ implies $\emptyset = 0$ which is impossible and $\alpha|_0 v' = \emptyset|_0$ implies 0 = 0.

(2)From (1), it follows, if M is a generalized N-projective semimodule and M, N are an indecomposable, then either there is v:M⟶N such that $\emptyset v = \alpha$, or there is v':N⟶M such that $\alpha v' = \emptyset$ implies α is epimorphism.

(3)If there is no epimorphism α: M⟶E where E is any semimodule which is homomorphic image of M, then Case(**c**)in (1) is not satisfied, so M is N-projective semimodule.

Proposition 4.3.11: If M is a generalized N-projective semimodule, then M is almost N-projective.

Proof. Let α: $N \longrightarrow X$ be an epimorphism and δ:M⟶ X be any homomorphism for any semimodule X. From hypothesis, there are decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism ϑ: $M_1 \longrightarrow N_1$ and an epimorphism ϑ': $N_2 \longrightarrow M_2$ such that $\alpha\vartheta = \delta|_{M1}$ and $\delta\vartheta' = \alpha|_{N2}$. Thus M is almost N-projective. ■

Corollary 4.3.12: If M is a generalized self-projective semimodule, then M is almost self-projective semimodule.

Definition 4.3.13 [42]: A semimodule M is called Noetherian (resp. Artinian), if and only if every chain of subsemimodules of M satisfies ACC(DCC). A semiring R is Noetherian (resp. Artinian), if $_RR$ is Noetherian (resp. Artinian) that is every chain of left ideals satisfies ACC(resp. DCC).

Examples4.3.14:

(1)A simple semimodule is Noetherian and Artinian.

(2)The semimodule ℕ over itself is Noetherian but not Artinian.

(3)ℚ as ℤ-semimodule is not Noetherian.

Lemma 4.3.15: Let M be an Artinian semimodule, then M can be written as a finite direct sum of indecomposable subsemimodules.

Proof. Assume M is an Artinian semimodule. Let M cannot be decomposed into direct sum of indecomposable subsemimodules, that is $M = M_1 \oplus M_2$ where M_2 cannot be decomposed into a direct sum of indecomposable subsemimodules. Write $M_2 = M_3 \oplus M_4$, where M_4 cannot de decomposed into a direct sum of indecomposable subsemimodules. Continuing this process, we have an infinite decreasing chain of subsemimodules of M, $M_2 \supseteq M_4 \supseteq \cdots \supseteq M_{\mathrm{i}} \supseteq M_{\mathrm{i}+1} \supseteq \cdots$, this contradiction with assumption, therefore M can be decomposed into a finite direct sum of indecomposable subsemimodules. ■

Proposition 4.3.16: Let M be an Artinian, almost N-projective semimodule, then M can be written as a finite direct sum of indecomposable almost N-projective subsemimodules where N is any semimodule.

Proof. From Lemma 4.3.15, M can be decomposed into a direct sum of indecomposable subsemimodules. Since M is almost N-projective, any direct summand of M is almost N-projective, so M is finite direct sum of indecomposable almost N-projective subsemimodules. ■

Proposition 4.3.17: Let M, N be semimodules and M is Artinian almost N-injective semimodule, then M can be written as a finite direct sum of indecomposable almost N-injective subsemimodules.

QI-ring is defined by many authors as [43], can be generalized to define QI-semiring as follows:

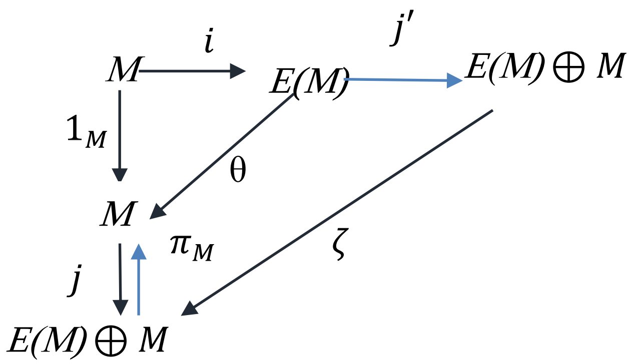
Definition 4.3.18: A semiring R is said to be QI-semiring, if every quasi-injective R-semimodule is injective. It is clear that any semi-simple semiring is QI.

Proposition 4.3.19: If R is QI-semiring, then the direct sum of two quasi-injective R-semimodule is quasi-injective.

Proof. Let A and B be two quasi-injective semimodules, then they are injective from assumption, then $A \oplus B$ is injective semimodule by Lemma 4.2.11, since every injective semimodule is quasi- injective, then a direct sum $A \oplus B$ is quasi-injective . ■

Proposition 4.3.20: If the direct sum of two quasi-injective R-semimodules M_1, M_2 with injective hulls E(M_1), E(M_2) is always quasi-injective, then R is QI-semiring.

Proof. Let M be a quasi-injective semimodule. Consider the following diagram:



Where i is the inclusion map and j, j' are the injection maps from M, E(M) respectively to E(M)⊕M. Since E(M)⊕M is quasi-injective semimodule, there is endomorphism ζ on E(M)⊕M such that $\zeta j' i = j 1_M$.

Take $\theta = \pi_{\mathrm{M}}\zeta\, j'$ hence $\theta i = 1_{\mathrm{M}}$, then M is isomorphic to direct summand of E(M), and hence it is injective semimodule implies that R is QI-semiring. ■

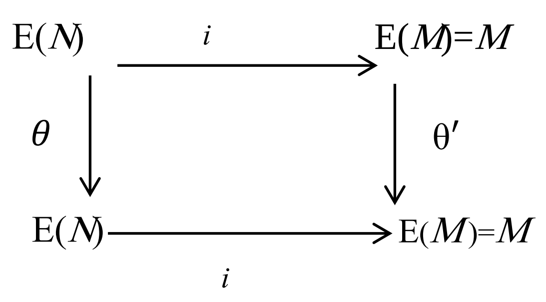
The following result is in modules [44], can be project into semimodules with similar proof.

Lemma 4.3.21: Let M be semimodule with injective hull E(M), then M is quasi-injective if and only if it is fully invariant in the injective hull of M.

Proposition 4.3.22: If R is QI-semiring, then every injective R-semimodule M with injective hull satisfies the property "every fully invariant subsemimodule of M is a direct summand".

Proof. Let M be an injective semimodule and N be fully invariant subsemimodule of M, it suffices to show that N is a quasi-injective, let θ:E(N)→E(N), then θcan be extended to endomorphism θ':E(M)→E(M),

since N is fully invariant in M, then θ(N)= θ'(N)≤N, therefore by Lemma 4.3.21, N is quasi-injective semimodule. As the following diagram:



■

The converse of the above proposition is true, that is:

Proposition 4.3.23: If every injective R-semimodule M with injective hull satisfies the property "every fully invariant subsemimodule of M is a direct summand", then R is QI-semiring.

Proof. Assume that M is a quasi-injective semimodule, from Lemma 4.3.21, M is fully invariant in E(M), so M is direct summand of E(M) by hypothesis. Hence M=E(M). Therefore R is QI-semiring. ■

Definition 4.3.24: A semiring is called AQI-semiring, if every almost self-injective semimodule is injective.

Remark 4.3.25:

Every AQI-semiring is QI-semiring since every quasi-injective semimodule is almost self-injective semimodule.

Proposition 4.3.26: If R is AQI-semiring, then the direct sum of two almost self-injective semimodule is almost self-injective.

Proof. Let A and B be two almost self-injective semimodules, then they are injective from assumption, then $A \oplus B$ is injective semimodule by Proposition 4.2.11, since every injective semimodule is almost-self-injective, then a direct sum $A \oplus B$ is almost-self-injective . ■

Proposition 4.3.27: If the direct sum of two almost self-injective R-semimodules M_1, M_2 with injective hulls E(M_1), E(M_2) is always almost self-injective, then R is AQI-semiring.

Proof. Similar to Proposition 4.3.20. ■

5.1 Conclusions

In this thesis, the concepts, almost-injective (resp. projective) and almost self-injective (resp. projective) modules have been generalized for semimodule over semiring, some basic characteristics which are analogous to ring and module theory have been studied. Since every module is semimodule but the converse is not true, in some result, some conditions were added to semimodules to achieve them.

From the results we obtained in this work, Proposition 2.2.18 by adding properties (injective hull and semisubtractive), Lemmas 2.2.20 and 2.2.21 with (subtractive, cancellative and semisubtractive), Lemma 2.2.22(2) with conditions (semisubtractive and cancellative), Proposition 2.2.27 and Corollary 2.2.28 with (semisubtractive and cancellative). Remark 2.3.11 and Proposition 2.3.12 by adding injective hull, Corollary 2.3.13 shows by adding a condition to indecomposable to become uniform semimodule. In Proposition 2.3.15 and Corollary 2.3.16 have been added conditions semisubtractive and cancellative for semimodule. Propositions 2.3.18, 2.3.21 and 2.3.22 show some properties of the endomorphism semiring, also Proposition 2.3.23: If M is a cancellative, semisubtractive almost self-injective semimodule, then:

(1) $W(\mathcal{S}) \subseteq Rad(\mathcal{S})$.

(2) If $\mathcal{S}$ is local, then $Rad(\mathcal{S}) = \{\vartheta \in \mathcal{S} : \ker(\vartheta) \neq 0\}$.

Also Proposition 3.2.10 explain an important characteristic which is finite direct sum of almost N-projective semimodule. It is known that, every projective semimodule is almost-projective but the convers is not true. In Proposition 3.2.21, we put the conditions for two semimodules M and N to get N-projective from almost N-projective. In Proposition 3.3.16, if M is almost self-projective uniserial which is indecomposable, then $\mathcal{S}$ is a left uniserial semiring. Propositions 3.3.19 and 3.3.20 explain relationship between the concepts almost self-projective semimodule and mini-injective $\mathcal{S}$-semimodule. Proposition 4.2.3 conditions have been added into semiring R which are yoked and cancellative, Proposition 4.2.4 with conditions, injective hull, semisubtractive and cancellative, Propositions 4.2.10 and 4.2.13 with conditions yoked and cancellative. Propositions 4.3.2, 4.3.7 and corollary 4.3.8 explain the relation between the following concepts

generalized N-injective semimodule $\Longrightarrow$ almost N-injective.

almost N-injective semimodule$\Longrightarrow$ essentially N-injective .

generalized N-injective semimodule$\Longrightarrow$ essentially N-injective.

Dually, generalized projective semimodule can be defined, also the relationship of this concept with almost projective semimodule will be explained.

Proposition 4.3.26: If R is AQI-semiring, then the direct sum of two almost self-injective semimodule is almost self-injective. Proposition 4.3.27:If the direct sum of two almost self-injective R-semimodules M_1, M_2 with injective hulls E(M_1), E(M_2) is always almost self-injective, then R is AQI-semiring.

5.2 Future Works

For future works,

(1) It could be interesting to study the relations between the concepts almost injective semimodule and extending semimodule, almost projective and lifting semimodule. And under what conditions these relations may exist.

(2) It is possible to study the notions, almost QF-semirings and almost $QF^{\#}$ -semirings depending on what is in ring theory because every ring is semiring but the converse is not true.

(3) We may work on a subject, hereditary and almost hereditary semimodules. Relation of these concepts and almost-injective (projective)semimodules.

(4) It can to survey the topics, V-semiring, V-semimodule and almost V-semimodule.

During our study of this work, we encountered some issues that we may think about in the future.

<u>Some of these questions</u>.

(1) Whether a “Baer like” criterion holds for almost injective semimodule. In other words, if M is almost injective relative to R, is it true that M is almost injective, and conversely?

(2) If M is an indecomposable almost self-injective semimodule, is it true that the endomorphism semiring of $M/Z(M)$ is semidomain?

(3) If R is right almost self-injective semiring, is R /Rad(R) also right almost self-injective?

References

1. J. S. Golan, “Semirings and Their Applications”, *Kluwer Academic Publishers*, Dordrecht, The Netherlands, 1999
2. M. Harada and A. Tozaki, “Almost M-projectives and Nakayama Rings” J. of Algebra. 122, pp. 447-474,1989.
3. Y. Baba, “Note of Almost M-Injectives,” Osaka J. Math. 26, pp. 687-698, 1989.
4. S. Singh, “Almost Relative Injective Modules” Osaka J. Math. 53, pp. 425-438, 2016.
5. A. Alahmad, S. K. Jain and S. Singh, “Characterizations of Almost Injective Modules” Contemporary Math. 634, pp. 11-17, 2015.
6. M. Harada and T. Mabuchi, “Almost projective Modules” J. of Algebra. 159, pp. 150-157,1993.
7. H. M. Althani , “Projective and Injective Semimodules over Semirings” MSc. [Thesis], University of East London.(1998).
8. S. Qandeel and M. Saleh . “A study on Some Forms of Projectivity and Injectivity”, MSc. [Thesis], Birzeit University, Palestine, 2020.
9. S. Ghalandarzadeh, P. Nasehpour and R. Razavi, “Invertible Ideals and Gaussain Semiring” Arch. Math. (Brno), 53, pp. 179-192, 2017.
10. A. H. Alwan and A. M. Alhossaini, “On Dedekind Semimodules” MSc. [Thesis], University of Babylon: Babylon, Iraq; 2020.
11. G. Tahar, “Ordered Algebraic Structures and Classification of Semifields” Math. AG, 4 pages, 2017.
12. K. S. Aljebory and A. M. Alhossaini, “The Jacobson Radical of the Endomorphism Semirings of P.Q. -Principal Injective Semimodules” Baghdad Sci. J. 17(2), pp. 523-529. 2020.

13. K. S. Aljebory, "Principally Quasi-Injective Semimodules and Principally Pseudo-Injective Semimodules" MSc. [Thesis]. University of Babylon: Babylon, Iraq; 2019.

14. K. S Aljebory and A. M. Alhossaini, "Principally Quasi-Injective Semimodules" Baghdad Sci. J.; 16(4), 928-936, 2019

15. A. Z.Aljebory and A.M. Alhossaini, "Fully dual stable Semimodules" JIKHS.; 1(1), 92-100, 2017

16. E. Diop and D. Sow, "On Essential Subsemimodules and Weakely CO-Hopfian Semimodules" Eur. J. Pure Appl. Math. 9, pp. 250-265, , 2016.

17. N. Tuyen and H. Thang, "On Superfluous Subsemimodules" Geo. Math. Jou., 10, pp. 763-770. 2003.

18. M.T. Altaee and A. M. Alhossaini " π-Projective semimodule over semiring" J. of University of Babylon for Pure and Applied Sciences, 28(1), pp.122-137, 2020.

19.M.T. Altaee and A. M. Alhossaini "Supplemented and π-projective Semimodules" Iraqi J. Sci.; 61(6): 1479-1487, 2020.

20. S. Alhashemi and A. M. Alhossaini, "FI-Extending Semimodule and singularity" Iraqi J. Sci., 63(3), pp. 1277-1284, 2022

21.S. H. Alsaebari and A. M. Alhossaini "On Preradical of Semimodules". Baghdad Sci. J.; 15(4), pp. 472-478, 2018.

22.K. S. Aljebory and A. M. Alhossaini "Principally Pseudo-Injective Semimodule" JUBAS., 27(4): pp. 121-127, 2019.

23.Y. Kastove, T.G. Nam and N. X. Tuyen "on Subtractive Semisimple Semirings" , Algebra Colloquium., 16(3), pp. 415-426, 2009.

24.R. R. Nazari and S. Ghalandazaden, "Content Semimodules", Exta. Math.,; 32(2), pp. 239-254, 2017.

References

25.S. Alhashemi and A. M. Alhossaini, "Extending Semimodules over Semirings," Journal of Physics: Conference Series, 1818, 012074, 2021.

26.M. T. Altaee and A. M. Alhussaini, " π-injective semimodule over semiring" J. Eng. Appl Sci. 63(5), 3424-3433, 2020.

27.R. E. Atani and S. E. Atani, "Spectra of Semimodules", A Republicii Moldova. Mathematica. 3(67), pp. 15-28, 2011.

28.S. Alhashemi and A. M. Alhossaini, "Extending Semimodules and Some of their Generalizations" Ph.D. [Thesis], University of Babylon, Babylon, Iraq, 2022.

29. V. Gupta and J.N. Chaudhari "Right Local Semiring", Asian European J. of Mathematics, 6(1), 5 pages, 2013.

30.G. Nabanita and K. S. Helen, "On Nilpotency Of The Right Singular Ideal Of Semiring" Bol. Soc. Paran. Mat. 37 (2), pp. 123-127, 2019.

31. J. N. Chaudhari and D.R. Bonde "On Exact Sequence of Semimodules Over Semirings" Int. Sch. Res., 1, pp.1-5. http://dx.dio.org/10.1155/2013/156485, 2013.

32. H. Abdulameer and A. M. Husain, "Fully stable semimodules" Al-Bahir Quarterly Adjudicated. Journal for Natural and Engineering Reseach and Studies, 5(9) and (10), pp.13-20, 2017.

33.J. R. Tsiba and D. Sow, "On Generators and Projective Semimodules" Int. J. of Algebra, 4(24): pp. 1153-1167, 2010.

34.N. X. Tuyen and T. G. Nam On Projective Covers of Semimodules in the Category Λ-CSSMod and Their Applications. Southeast Asian Bull. M., 31: pp.363-377, 2007.

35.S.H. Al Saebairi and A.M. Alhossaini, "Nearly Injective Semimodules". JUBAS., 27(1), pp.11-31, 2019.

References

36. H. M. Althani, “Characterizations of Projective and K-Projective Semimodules” IJMMS., 32(7), pp.439-448, 2002.
37. A. shabeir and Weinert. “Charactrizations semiring by semimodule P-injective and projective semimodules”, communication algebra. 26(7), pp. 2199-2209, 1998.
38. R. P. Deore, “A Note on Projective Semimodules”, Int. J.Contemp. Math. Sciences,v. 3(26), pp. 1269-1272, 2008.
39. A. H. Alwan and A. M. Alhossaini, “Dedekined Multiplication Semimodules”, Iraqi J. Sci., 61(6), pp. 1488-1497, 2020.
40. A. Alahmadi and S. K. Jain, “A note on Almost Injective Modules”, Math. J. Okayyama Univ. 51, 101-109, 2009.
41. A. H. Ali and A. M. Alhussaini, “Von Neumann Regular Semiring”, J. of Physics Conference Series, 2020.
42. J. Abuhlail and R. Ganazar, “on k-Notherian and K-Artinian Semirings”, math. R. A., 2019.
43. A. K. Boyle, “Hereditary QI-Rings”, Transactions of the American Math Society, 192,pp.115-120, 1974.
44. S. Sahinkaya and J. Trlifaj, “Generalized Injectivity and Approximation”, Math RT. 6, 2016.

المستخلص

في هذا العمل تم تعميم المقاسات شبه الاغماريه والاسقاطية الى المقاسات شبه الاغمارية غالبا وشبه المقاسات الاسقاطية غالبا . الهدف من هذا العمل هو دراسة الخصائص الأساسية لهذه المفاهيم لشبه المقاسات وبعض المفاهيم المتعلقة بها. يسمى شبه المقاس M اغماري غالبا لشبه المقاس N اذا كان لكل شبه مقاس جزئي من N ولكل تماثل من شبه المقاس الجزئي الى M ، اما يوجد توسعة من N الى M، أو يوجد تماثل من M الى جداء مباشر غير صفري من N حيث ان π هي دالة الاسقاط . شبه المقاس M يسمى اغماري غالبا اذا كان اغماري غالبا لكل شبه مقاس. تمت دراسة بعض المفاهيم ذات الصلة والتحقيق فيها أيضًا. كما تم توضيح علاقة هذه المفاهيم ببعض المفاهيم مثل ، شبه المقاسات المنتظمة ، و شبه المقاسات شبه الاغمارية من النوع π وكذلك المفاهيم W (ʆ), Z(ʆ), Rad(ʆ) .كما يقدم هذا العمل أيضًا ويبحث في الفكرة المزدوجة للنماذج شبه المقاسات شبه الاغمارية غالبا والتي هي شبه مقاسات اسقاطية غالبا والتي هي توسيع لشبه المقاسات الإسقاطية مع مراعاة الاختلافات بين المقاسات وشبه المقاسات ، وهي واضحة بشكل أساسي من تعريفاتها. يقال أن شبه المقاس M إسقاطي غالبا لشبه المقاس N ، إذا كان لكل تماثل من N الى X حيث X أي شبه مقاس ولكل تماثل من M الى X إما ان يكون هناك تماثل من N الى M ، أو يوجد تماثل من Y الى M حيث Y هو جداء مباشر غير صفري من N. كما يدعى شبه المقاس M إسقاط ذاتي غالبا، إذا كانت M شبه مقاس إسقاطي غالبا لنفسه. وقد حصلنا على بعض النتائج حول هذا المفهوم .واحدة من النتائج التي حصلنا عليها من التعريفين انه كل شبه مقاس اسقاطي يكون شبه مقاس اسقاطي ذاتي غالبا. كما تم تعريف شبه الحلقة شبه الاغمارية ذاتيا وتم تقديم ومناقشة بعض المفاهيم وهي، تعميم شبه اغماري لشبه المقاس N،تعميم اسقاطي لشبه المقاس غالبا N و شبه المقاس اساسي شبه اغماري N، شبه الحلقة من النوع QI (كل شبه مقاس اغماري ذاتي يكون شبه مقاس اغماري) وشبه الحلقة من النوع AQI(كل شبه مقاس اغماري ذاتي غالبا يكون شبه مقاس اغماري).

وزارة التعليم العالي والبحث العلمي

جامعة بابل

كلية التربية للعلوم الصرفة

قسم الرياضيات

شبه المقاسات الاغمارية غالبا والاسقاطية غالبا

أطروحة

مقدمة الى مجلس كلية التربية للعلوم الصرفة/ جامعة بابل كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل

ختام صاحب حمزه عبد

بإشراف

أ. د. اسعد محمد علي الحسيني

2023 A.D.   1444 A.H.