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New types of graphs with respect to an ideals near rings

A Dissertation
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بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

شَهِدَ ٱللَّهُ أَنَّهُۥ لَآ إِلَٰهَ إِلَّا هُوَ وَٱلۡمَلَٰٓئِكَةُ وَأُوْلُواْ
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Dedication

My dear father and mother...may God have mercy on them

To my wife (Dr. Anas)... the struggle companion who spared no time or effort to help me

To my kids

To all family and friends

I dedicate my scientific dissertation in mathematics to you.

Ameer Al-Swidi

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At the very of this work, I would like to thank Allah for giving me power to complete this work.

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I also acknowledge with a deep sense of reverence, my gratitude to my wife (Dr. Anas) and members of my family (brothers) who have always supported me morally.

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Publications

- AL-Swidi, A. A., & Omran, A. (2021). Completely Equiprime graph of a near ring .1st International conference for pure science, University of Diyala, AIP conference proceeding.

- AL-Swidi, A. A., & Omran, A. (2021). Uniquely Colorable Completely Equiprime Graph Of a Near Ring .2nd International Conference for Pure Science, University of Qadisiyah, AIP conference proceeding.

- AL-Swidi, A. A., & Omran, A. (2022). New type graph with respect to ideal near ring. International Journal of Nonlinear Analysis and Applications,vol.3,p.p. 1297–1304.

- AL-Swidi, A. A., & Omran, A. (2023). Weakly Completely Prime Graph .Wasit journal for pure sciences, Vol.2, Issue1.

Contents

ABSTRACT

The main purpose of this dissertation is to represent the elements of the semiring or the near ring in the form of graphs, and from here we were able to find and see many algebraic relationships between these elements by using the integration between algebraic concepts and graph concepts. We have adopted in our work that the determinants of adjacent between the elements of the semiring or the near ring is based on the ideal of semiring or the ideal of near ring.

While the form of adjacent and the relationship between the elements of the semiring or the near ring showed us the form of the graph, whether connected or unconnected, and certainly the main focus in our work is that the graph be connected, which will give us a correct view about the relationships of the elements among them.

Hence, we define many new algebraic concepts, including almost 1-prime ideal, almost 2-prime ideal, almost 1-semi prime ideal, almost 2-semi prime ideal, zero divisor of N with respect to an ideal, weakly completely prime ideal, weakly completely semi prime ideal and subtractive ideal.

Which were used in our definitions of the graphs, namely, the completely equiprime graph, the weakly completely prime graph , almost 1-prime graph and almost 2-prime graph, with the relationship of these graphs between them and the graphs PG(N) and Γ(N) of other researchers for the two systems semiring and near ring.

Also studied homeomorphism graph, whenever there exist a semi ring homomorphism or near ring homomorphism and finding a case that

every completely equiprime ideal , weakly completely prime ideal , almost 1-prime ideal and almost 2-prime ideal in the domain that leads to a completely equiprime ideal , weakly completely prime ideal, almost 1-prime ideal and almost 2-prime ideal in the co-domain respectively by defining the preserve vertex cover of homomorphism graphs.

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List of Symbols

G	Graph
V	Vertices set
E	Edges set
deg(v)	Degree of a vertex v
$\Delta(G)$	Maximum degree of G
$\delta(G)$	Minimum degree of G
$\emptyset$	Empty set
N_n	Null graph
K_n	Complete graph
$\|X\|$	Cardinality number of the set X
$B_{m,n}$	Bipartite graph
$K_{m,n}$	Complete bipartite graph
S_n	Star graph
C_n	Cycle graph
P_n	Path of graph
G[V(H)]	H is induced subgraph of G
d(u,v)	Distance between u and v
$\gamma(G)$	Domination number of G
e(v)	Eccentricity of a vertex v
rad(G)	Radius of a connected graph
diam(G)	Diameter of a connected graph
<w>	Clique
$\chi(G)$	Chromatic number of G
$\|p\|$	Cardinality of the minmum chromatic partition

S	Semiring
I	Ideal
L	Not idempotent ideal
N	Near ring
$\mid$	Such that
$\backslash$	Except
$\forall$	For all
$\exists$	There exist
$\ni$	Such that
$\in$	Contain
$\notin$	Not contain
$\subset$	Subset
$\equiv$	Equivalent
$\subseteq$	Subset and equal
$\bigcup$	Union
$\bigcap$	Intersection
rad(I)	Radical of an ideal
Z_n	Ring of integer module n
$Z_n[i]$	Ring of Gaussian integer module n
IFP	Insertion of factors property
ker(f)	Kernel of a near ring homomorphism f
(N: I)	Zero divisor of N with respect to an ideal
nilp(N)	Set of all nilpotent in N
$\overline{ab}$	Edge between the vertex a and b
$\oplus$	Edge summation
$\Gamma(N)$	Zero divisor graph
PG(N)	Prime graph

$CEQ_I(N)$	Completely equiprime graph of N
$W_I(N)$	Weakly completely prime graph
$AP_L(N)$	Almost 1-prime graph
$TP_L(N)$	Almost 2-prime graph

Introduction

Over the last decade, the interplay between semiring and ring properties has become with the properties of graph theories. Many articles have assigned a semiring or a ring to a graph and have studied the properties of the graph associated with it.

Since 1974, the literature on algebraic graph theory has grown tremendously [1-3].

In 1970, Y. Lin and J. Ratti [4] investigated the connectivity graphs of the semirings as a subset$\{S_1,S_2,\ldots,S_n\}$ of Ƒ, form a chain or a path between S_i and S_j in the graph G(Ƒ) if and only if $S_i \cap S_{i+1} \neq \emptyset$ for all i=1,2,,….n-1. With some important conditions to make this graph in connected, and continue with another article [5].

In 1970, Y. Lin and J. Ratti [6] research the diameter between the vertices of the graphs of the semirings.

In 1988, Istvan Beck [7] introduced the simple graph $\Gamma(R)$ of a ring R as vertices of this graph are elements of R, as two different vertices are adjacent if and only if a.b =0 for all $a,b\in R$, with investigate coloring with several properties of this simple graph.

In 1999, David F. Anderson and Philip S. Livingston [8] introduced the simple graph of a commutative ring R (with unit 1) and Z(R) be the set of zero divisors of R, as two different vertices are adjacent if and only if a.b =0 for all $a,b\in Z(R)\backslash\{0\}$.

In 2003, Shane P. Redmond [9] introduced the simple graph $\Gamma_I(R)$ of a commutative ring R with ideal I, and vertices are $\{a\in R\backslash I : a.b\in I$ for some $b\in R\backslash I\}$ such that two distinct vertices are adjacent if and only if $a.b\in I$.

In 2014, Saba Zakariya Al-Kaseasbeh [10], in this dissertation consider different graphs of a commutative ring R(with unit 1), and a vertices are a proper ideals of R such that the adjacent restricted by several conditions and properties with of study the connectivity and diameter of this graphs.

In 2010, Satyanarayana Bhavanari, Syam Prasad Kuncham and Nagaraju Dasari [11] introduced the graph is called prime graph PG(R) of a ring R , as two different vertices are adjacent if and only if $a.R.b = 0$ or $b.R.a=0$ for all $a,b \in R$, with investigate several properties and relations of this prime graph.

In 2010, Satyanarayana Bhavanari, Syam Prasad Kuncham and Babushri Srinivas Kedukodi[12] introduced the graph $G_I(N)$ of a near ring N with ideal I, and different vertices are adjacent if and only if $a.N.b \subseteq I$ or $b.N.a \subseteq I$ for all $a,b \in N$, with investigate several properties and relations of this graph.

In 2011, Satyanarayana Bhavanari, Godloza Lungisile and Nagaraju Dasari[13] introduced the principal ideal graph PIG(R) of a ring R, and different vertices are adjacent if and only if prin(a)=prin(b) whenever prin(a),prin(b) are a principal ideals of R, for all $a,b \in R$.

In 2016, Mojgan Afkhami, Nesa Hoseini and Kazem Khashyarmanesh [14] introduced the simple graph $AN_I(R)$ of a commutative ring R with ideal I, and the vertices are $\{a \in R \backslash I : a.b \in I$ for some $b \in R \backslash I\}$ such that two distinct vertices are adjacent if and only if $ann_I(a.b) \neq ann_I(a) \cup ann_I(b)$, where $ann_I(a)=\{r \in R | r.a \in I\}$, and the other researchers work on the annihilator ideal with another condition and properties[15,16].

The remainder of this dissertation is organized as follows:

- **Chapter one**

  This chapter, introduces a background for the basic concepts of the graph theory used in this dissertation is introduction. In addition , many basic definitions of semiring and near ring used in this dissertation are mentioned.

- **Chapter two**

  In this chapter, the new graph entitled a completely equiprime graph, we present an integrated study related to the topic of the completely equiprime ideal of a near ring with its characteristics, theories and relationships with completely equiprime graph, as well as investigate the chromatic and uniquely colorable of this graph.

- **Chapter three**

  This chapter, define new ideals called weakly c-prime ideal and almost v-prime ideal(v=1,2) of a near ring with investigate several properties and relation of this ideals, as well as study of a new graph which is weakly completely graph, we investigate the properties of this graph, related with previous graphs like Γ(N) and PG(N), also introduce the graphs almost v-prime (v=1,2) graphs with the investigation of all properties of this graphs.

- **Chapter four**

  In this chapter, we research the homomorphism of the graphs Which are referred to in the previous two chapters, whenever there exist a homomorphism near ring, with the proving of many theories related to it and showing this with examples.

- **Chapter five**

  The conclusions and the recommendations have been presented.

Chapter One

The Fundamental

1. Introduction

In this chapter, the relevant definitions, and basic concepts required as a background for this dissertation are introduced. Two sections are discussed in this chapter. The first section contains some basic definitions of graph with examples that we need in the dissertation. The second section presents the definitions and examples of algebra to explain it.

1.1. Basic definitions of graph

The graph theory originated from the problem of Königsberg Bridge in the Königsberg city in former Prussia and now known as (Kaliningrad which is part of Russia), located on the Pregel River. This was considered the first chapter in the history of the graph. The river divides the city into four separate islands, linked by seven bridges. The people of the city people were wondering whether it was possible to cross the seven bridges once without repeating any bridge and returning to the starting point.

In 1736, Leonard Euler (1707-1783) came up with an answer to this question, proving that it was not possible to pass on the seven bridges at exactly the same time.

Euler used to answer this question in a simple way by eliminating all the unnecessary features of the city. He drew a picture of the city where he represented the land by vertices (or points) and bridges(or lines). This mathematical structure is called the graph. From these simple assets, the theory of graph has grown into a strong mathematical theory in mathematics and solved many complicated life problems [17].

The basic concepts of the graphs which will be used in this work, are discussed as follows:

Definition 1.1.1. [17] The graph G=(V,E) contains the two sets, vertex set(V) and the edge set(E) such that each edge e contain in E can be identified with a pair (v_1,v_2) of vertices in V. The vertices v_1 and v_2 are known as end vertices of e and denoted by $\overline{v_1 v_2}$.

Definition 1.1.2. [17] The adjacent vertices are the vertices joined by the same edge.

Definition 1.1.3. [18] An edge for which the two ends are same is called a **loop** at the common vertex.

Definition 1.1.4. [18] If two edges have same end vertices they are called **parallel edges**.

Definition 1.1.5. [18] If the graph has no parallel edges and no loop then is called **simple graph**.

Definition 1.1.6. [19] The number of edges incident of any vertex (v) is called the **degree** of vertex and denoted by deg(v).

Definition 1.1.7. [19] The **isolated vertex** is a vertex of degree 0 and **pendent** or **leaf vertex** is a vertex of degree 1.

Definition 1.1.8. [18] The **maximum degree** is the largest degree among the vertices of the graph G and denoted by $\Delta(G)$. And the **minimum degree** is the least degree among the vertices of the graph G and denoted by $\delta(G)$.

Example 1.1.9. In Fig.1.1. A simple graph G shown together with the degree of its vertices, deg(v_3)=1, deg(v_1)=deg(v_4)=deg(v_6)=2, deg(v_5)=3, deg(v_2)=4, thusΔ(G)=4 and δ(G)=1.

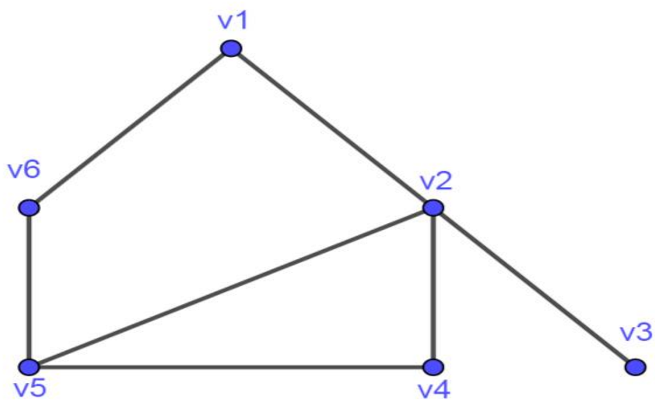


Figure 1.1. Degree of vertices of simple graph

Definition 1.1.10. [17] A graph that has a finite number of vertices and edges is called a **finite graph**. Otherwise, it is **infinite graph**.

Definition 1.1.11. [20] A graph G=(V,E) of order n without any edges, that means E=$\emptyset$, is called a **Null graph** and denoted by N_n.

Definition 1.1.12. [20] A graph G=(V,E) with order n is called a **complete graph** if there is an edges between each pair of vertices and denoted by K_n . (as an example, see Fig.1.2.)

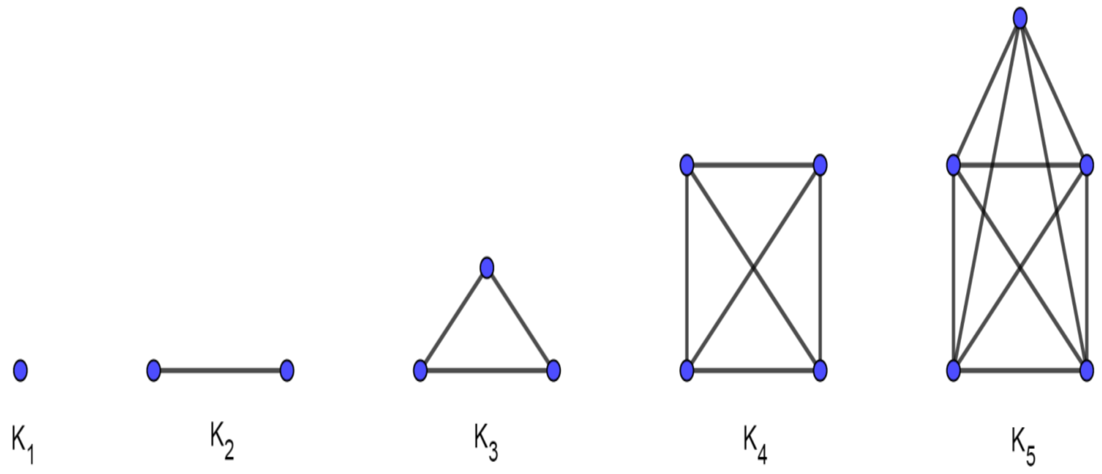


Figure 1.2. Some complete graphs

Definition 1.1.13. [21] A graph G=(V,E) is called a **Bipartite graph** if V can be expressed as two disjoint sets A and B, such that V= A∪B with $A \cap B = \emptyset$, and every vertices of A and B are not adjacent vertices, with every vertices of A are join to a vertex of B. Such that if |A|=m and |B|=n then is denoted by $B_{m,n}$.(as an example, see Fig.1.3.)

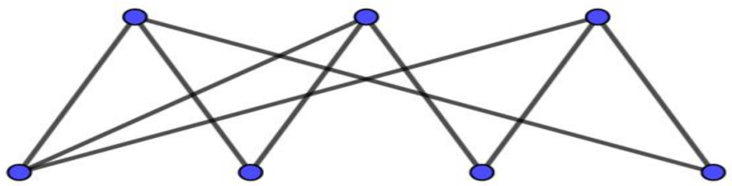


Figure 1.3. Bipartite graph($B_{3,4}$)

Definition 1.1.14. [21] A **Complete bipartite graph** is a bipartite graph such that every vertices of A are joined to every vertices of B. If |A|=m and |B|=n then is denoted by $K_{m,n}$. (as an example, see Fig.1.4.)

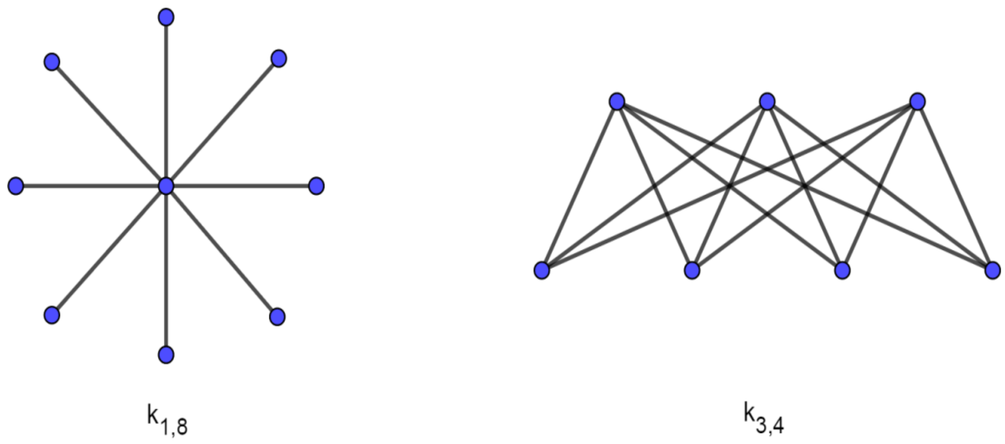


Figure 1.4. Complete bipartite graphs

Definition 1.1.15. [21] A **Star graph** S_n is the complete bipartite graph $K_{1,n}$. The centrum of a star is the all-adjacent vertices in it. In Fig 1.4, the $K_{1,8}$ is the star graph.

Definition 1.1.16. [22] A **Walk** in the graph is an alternative sequences of vertices and edges with the repetition of vertices and edges is allowed. If initial and end vertices are different, it is called **open walk** otherwise, it is called **closed** and the number of edges in walk is defined by **length**.

Definition 1.1.17. [22] A **Path** in the graph G is an open walk without repetition of vertices and is denoted by P_n of length n-1 is a sequence of distinct edges $\overline{v_1 v_2}, \overline{v_2 v_3}, \dots \dots, \overline{v_{n-1} v_n}$.

Definition 1.1.18. [20] A **Cycle** is a closed walk without repetition of vertices. Thus, the degree of each vertices of a cycle graph is two, and denoted by C_n. (as an example, see Fig.1.5.)

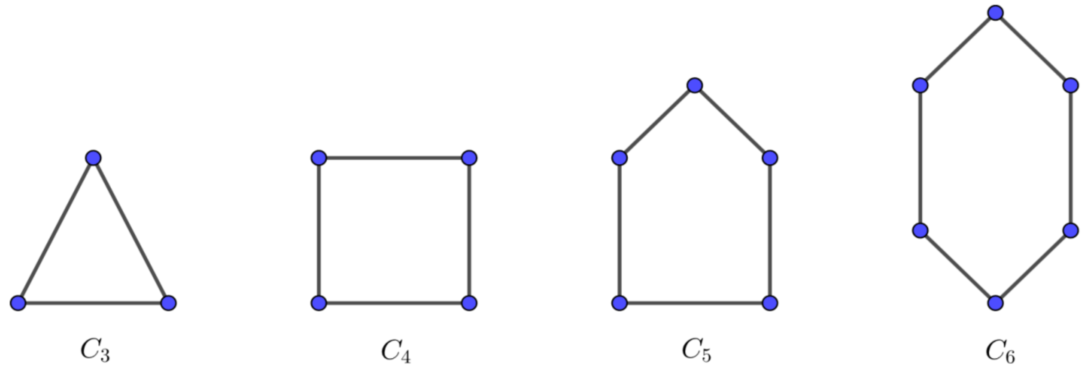


Figure 1.5. Some cycle graphs

Definition 1.1.19. [17] A graph K is called a **subgraph** of the graph G if the vertices of K are contained in the vertices of G, and edges of K are contained in the edges of G. In other words, (K)$\subseteq$V(G) and E(K)$\subseteq$ E(G). (as an example, see Fig.1.6.)

Definition 1.1.20. [17] A subgraph H is called an **induced subgraph** of G if the vertices u,v$\in$V(H) are adjacent in G then they are adjacent in H and denoted by G[V(H)]. (as an example, see Fig.1.6.)

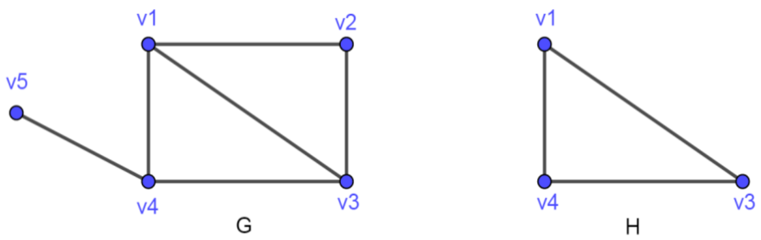


Figure 1.6. Induced subgraph

Definition 1.1.21. [23] Let v_1 and v_2 be two vertices contains in the graph G=(V,E) then the **distance** between v_1 and v_2 is the length of the shortest path between v_1 and v_2 and denoted by $d(v_1,v_2)$.

Example 1.1.22. Consider the graph G=(V,E).(as an example, see Fig.1.6.), where the $d(v_1, v_3) = 1$, $d(v_2, v_4) = 2$, $d(v_2, v_5) = 3$.

Definition 1.1.23. [23] A graph G is called **connected** if there is a path between each pair of vertices of G . If it is not connected, it is called **disconnected** graph. The **component** of G is a maximal connected subgraph of G.

Definition 1.1.24. [23] A non-adjacent vertices in a graph is called an **Independent set** of vertices.

Definition 1.1.25. [19] Let F⊆G, where F is the set of vertices of G is said **dominating set**, if each vertices of F is adjacent or every vertices of G is adjacent to F, the minimum cardinality of F denoted by $\gamma(G)$.

Definition 1.1.26. [23] The **eccentricity** of a vertex v is the distance to the farthest distance from v in the connected graph, and is denoted by e(v). The minimum eccentricity is called a **radius** of a connected graph, and denoted by rad(G). The maximum eccentricity is called a **diameter** of a connected graph, and denoted by diam(G).

Example 1.1.27. Consider the graph G(V,E).(as an example, see Fig.1.6.) $e(v_1)$= $e(v_3)$= $e(v_4)$=2, $e(v_2)$= $e(v_5)$=3, So that, rad(G)=2 and diam(G)=3.

Definition 1.1.28. [19] The set $W \subseteq V$ is called a **clique**. Every different pair of vertices in W are adjacent denoted by <w> and the maximum size is called the clique number.

Definition 1.1.29. [19] A **girth** is the length of the shortest cycle contained in the graph G.

Definition 1.1.30. [18] A **spanning tree** is a tree which cover all vertices of the graph G.

Definition 1.1.31. [18] A subset M of E is called **edge cover** of G. If Every vertex of the graph G is an endpoint of some edge in M.

Definition 1.1.32. [18] A subset W of V is called **vertex cover** of G. If Every edge in the graph G has one end vertex in W.

Definition 1.1.33. [19] A **proper coloring** is a coloring of the vertices of the graph G, such that no two adjacent vertices have same color or the n coloring of the graph G is a mapping $\psi: G \rightarrow \{1,2,..,n\}$. The adjacent vertices have different colors, that means $\psi(a) \neq \psi(b)$ assigns different colors whenever $\overline{ab} \in E(G)$ for every $a,b \in V(G)$, and the minimum number of a proper colors required for G is called **chromatic number** of G and denoted by $\chi(G)$.

Definition 1.1.34. [19] A graph G=(V,E) with n vertices, a chromatic partition of G is a smallest partition of V into disjoint independent sets if G has only one chromatic partition then we say that G is **uniquely colorable**.

Example 1.1.35. Let the graphs G_1={1,2,3,4,5},G_2={1,2,3,4} and G_3={1,2,3}, then the chromatic partition for G_1 are p_1={{1,5},{2,4},{3}} and p_2={{3,5},{2,4},{1}}, while for G_2 is (p={{1,3},{2,4}}) and for G_3 is (p={{1},{2},{3}}).

Then we get G_1 is not uniquely colorable while G_2 and G_3 are uniquely colorable. (as an example , see Fig.1.7.)

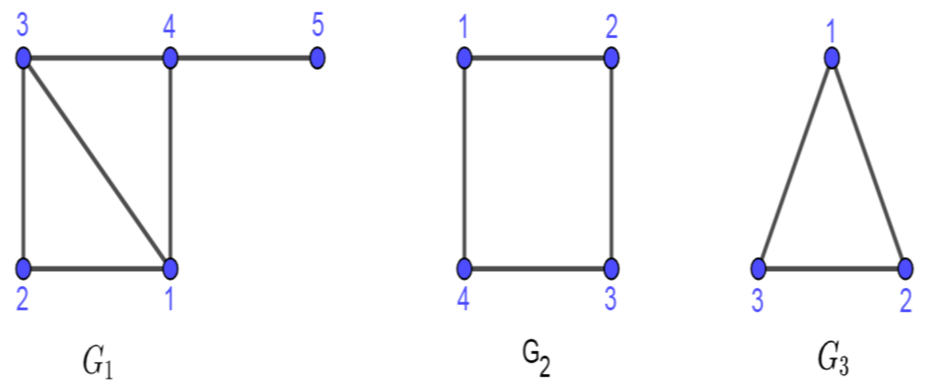


Figure 1.7. Different graphs

Remark 1.1.36.

1-The cardinality of the minimum chromatic partition is called a chromatic number and denoted by |p|. So that in Example (1.1.34). The:

$$|p| = \begin{cases} 3 & \text{for } G_1 \text{ and } G_3 \\ 2 & \text{for } G_2 \end{cases}$$

2- The $\chi(G) = |p|$ (as an example, see Fig.1.7.), where

$$\chi(G) = \begin{cases} 3 & \text{for } G_1 \text{ and } G_3 \\ 2 & \text{for } G_2 \end{cases}$$

Definition 1.1.37. [24] A **Homomorphism of graphs** K to G, written as $\varphi : K \rightarrow G$ is a mapping $\varphi : V(K) \rightarrow V(G)$ such that $\overline{\varphi(v_1)\varphi(v_2)} \in E(G)$ whenever $\overline{v_1 v_2} \in E(K)$.

1.2. Basic defintions of algebra

In 1905 L.E. Dickson introduces generalized to the ring theory called near ring , in fact is near field where he proves that field with only one distributive axiom ,in near ring theory there are several situations differ from ring theory . For example, not every left ideal is subnear ring[27].

The basic concepts of the semiring and near ring theory which will be used in this work . Thus, they will be are discussed:

Definition 1.2.1. [25] A set S with a two binary operations $(+)$ and $(\,.\,)$ is called **semiring**. Whenever $(S,+)$ is an abelian semigroup, $(S,.)$ is semigroup and for all a,b,c∈S:(a+b).c=a.c+b.c and c.(a+b)=c.a+c.b
A commutative semiring is semiring with multiplication it is commutative and a semiring with zero, if (S,+) is commutative monoid, if (S,.) is a monoid, then (S,+,.) is called a semiring with unity. And it is called an absorbing zero if a.0=0.a=0 for a∈S.

Definition 1.2.2. [25] A subset I of a semiring S is called **right**(resp. **left**) **ideal** if a+b∈I for all a,b∈I and for c∈S, a.c∈I (resp. c.a∈I). And if right and left ideal then is called **ideal**.

Definition 1.2.3. [26] A set N with a two binary operations $(+)$ and $(\,.\,)$ is called a **near ring** (right near ring). Whenever $(N,+)$ is a group (not necessarily abelian), $(N,.)$ is a semigroup and (a+b).c=a.c+b.c, for all a,b,c∈N, and left near ring if a left distributive is satisfied and a near ring if both left and right distributive are satisfied and is said a zero near ring if consist just one element {0}.

Definition 1.2.4 . [27] A **zero symmetric near ring** N is a non-trivial near ring such that x.0=0 for all x∈N.

Definition 1.2.5. [28] For every x in the near ring N, 0.x = 0.

Definition 1.2.6. [29] Let (H,*) be a group. A subset ∅≠F of H is called a **normal subgroup** of H, if F*h = h*F for every h in H or equivalently F={h^{-1}*f*h | for every h∈H and f ∈F}.

Definition 1.2.7. [26] A normal subgroup I of $(N, +)$ is called a **right ideal** if $I.N \subseteq I$, and **left ideal** if $x.(y+z) - x.y \in I$, $\forall\, x, y \in N$ and $\forall\, z \in I$ and if both right and left ideal are called **ideal** .

Example 1.2.8. Let N={0,1,2,3,4,5,6,7} be a near ring defined in Table1.1. And the ideals are:

I_1={0,2}, I_2={0,2,5,7}, I_3={0,2,4,6} and I_4 ={0,2,4,5,6,7}.

TABLE 1. 1. Multiplication and Addition table of N={0,1,2,3,4,5,6,7}

+	0	1	2	3	4	5	6	7		.	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7		0	0	0	0	0	0	0	0	0
1	1	2	3	0	5	6	7	4		1	0	1	2	3	4	5	6	7
2	2	3	0	1	6	7	4	5		2	0	2	0	2	0	0	0	0
3	3	0	1	2	7	4	5	6		3	0	3	2	1	4	5	6	7
4	4	7	6	5	0	3	2	1		4	0	4	2	6	4	0	6	2
5	5	4	7	6	1	0	3	2		5	0	5	0	5	0	5	0	5
6	6	5	4	7	2	1	0	3		6	0	6	2	4	4	0	6	2
7	7	6	5	4	3	2	1	0		7	0	7	0	7	0	5	0	5

Definition 1.2.9. [29] In any near ring the trivial subnear ring ({0},+,.) and (N,+,.) are ideals . If the near ring contains no ideals except these two ideals, it is called **simple near ring**. The ideal is called a **proper ideal** if it is different from (N,+,.).

Definition 1.2.10. [26] The **radical** of an ideal is defined as the set, $\mathrm{rad(I)} = \{\mathrm{a} \in \mathrm{N} \colon \mathrm{a}^{\mathrm{n}} \in \mathrm{I}, \text{for some } \mathrm{n} \in \mathrm{Z}^{+}\}$.

Definition 1.2.11. [30] An ideal I of N is called **idempotent ideal** if $\mathrm{I}^2 = \mathrm{I}$ and N is fully idempotent if $\mathrm{I}^2 = \mathrm{I}$, $\forall$ I$\in$N.

Definition 1.2.12. [30] Let the near ring N. Then for a$\in$N is called **idempotent element** if $\mathrm{a}^2 = \mathrm{a}$.

Remark 1.2.13.

1- If I is an ideal of N, then not necessary rad(I)is an ideal of N.

2- If I is an ideal of:

i- The ring of integer module n (Z_n)

ii-The ring of Gaussian integer module n: $Z_n[i]=\{x+yi|x,y\in Z_n\}$.

Then rad(I) is always is an ideal.

3- If I is an ideal of N, then not necessary I^2 is an ideal of N.

4- If the ideal I of N is idempotent ideal then not necessary all element is idempotent elements.

5- If the ideal I= {0} of N, then I is idempotent ideal.

6- The Z_n is always simple ring (simple near ring), whenever n is prime number.

Definition 1.2.14. [26] A near ring is called **right permutable** if x.y.z = x.z.y and **left permutable** if x.y.z = y.x.z ,$\forall$ x,y,z$\in$ N.

Definition 1.2.15. [29] The ideal (left ideal) I of a near ring (right near ring) is called a **completely equiprime ideal** (c-equiprime ideal) if $a \in N \backslash I$ and $b, d \in N$ with $a.b - a.d \in I$ implies $b - d \in I$.

Example 1.2.16. In Example (1.2.8). The ideals I_2={0,2,5,7}, I_3={0,2,4,6} and I_4={0,2,4,5,6,7} are c-equiprime ideals of N, while I_1={0,2} is not c-equiprime ideal since 7.0-7.4$\in I_1$ but 0-4$\notin I_1$(see Table 1-A in A.1.1).

Definition 1.2.17. [31] A near ring N is called a **c-equiprime near ring** if 0≠a∈N with b,c∈N such that a.b-a.c∈N.

Definition 1.2.18. [32] An ideal I of N is called a **completely prime ideal** (c-prime ideal) if $a.b \in I$, for $a, b \in N$ implies $a \in I$ or $b \in I$.

Example 1.2.19. In Example (1.2.8). The ideals I_2={0,2,5,7}, I_3={0,2,4,6} and I_4={0,2,4,5,6,7} are c-prime ideals of N, while I_1={0,2} is not c-prime ideal since 6.7$\in I_1$ but 6$\notin I_1$ and 7$\notin I_1$ (see Table 1-A in A.1.1).

Definition 1.2.20. [32] An ideal I of N is called a **completely semi prime ideal** (c- semi prime ideal), if $a.a \in I$, for $a \in N$ then $a \in I$.

Definition 1.2.21. [33] An ideal I of N is called a **prime ideal** when A, B be an ideals of a near ring N with $A.B \subseteq I$ then A⊆I or B⊆I.

Definition 1.2.22. [32] An ideal I of N is called a **3-prime ideal** if $a.N.b \subseteq I$ implies $a \in I$ or $b \in I$ and N is called a 3-prime near ring if {0} is 3-prime ideal of N.

Example 1.2.23. Let N={0,1,2,3,4,5,6,7} be a near ring defined in table 1.2 and the ideals are: I_1={0,2}, I_2={0,6}, I_3={0,2,6} and I_4={0,2,6,7}. Then I_1 and I_2 are not 3-prime ideal since 6.N.6$\in I_1$ and 2.N.2$\in I_2$ but 6$\notin I_1$also 2$\notin I_2$, while I_3 and I_4 are 3-prime ideals of N

TABLE 1. 2. Multiplication and Addition table of N={0,1,2,3,4,5,6,7}

+	0	1	2	3	4	5	6	7		.	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7		0	0	0	0	0	0	0	0	0
1	1	2	3	0	7	6	4	5		1	0	1	2	3	4	5	6	7
2	2	3	0	1	5	4	7	6		2	0	2	0	2	0	2	0	2
3	3	0	1	2	6	7	5	4		3	0	3	2	1	5	4	6	7
4	4	7	5	6	2	0	1	3		4	0	4	2	5	4	5	6	7
5	5	6	4	7	0	2	3	1		5	0	5	2	4	5	4	6	7
6	6	4	7	5	1	3	0	2		6	0	6	0	6	0	0	0	0
7	7	5	6	4	3	1	2	0		7	0	7	0	7	2	2	0	0

Definition 1.2.24. [34] An ideal I of N is called it has an **insertion of factors property** (IFP) if a, b $\in$ N with a. b $\in$ I, then a.n.b$\in$I , $\forall$ n$\in$N.

Example 1.2.25. In Example(1.2.23). The ideals I_1={0,2}, I_3={0,2,6} and I_4={0,2,6,7} are IFP of N, while I_2={0,6} is not IFP since 7.7$\in I_2$ but 7.4.7$\notin I_2$(see Table 2-A in A.1.2).

Definition 1.2.26. [35] We say that N is **integral** if a.b=0 implies a=0 or b=0 , for all a,b∈N. We define, a set of **right zero divisors** Zr and **left zero divisors** Z_l of a near ring by

Z_r:={a∈N|∃n∈N\{0}:n.a=0} and similarly

Z_l:={a∈N|∃n∈N\{0}:a.n=0}. Certainly, the **zero divisors** are a right and left zero divisors.

Definition 1.2.27. [36] An ideal I of N is called a **2-prime ideal** if $x.y \in I$, for x,y∈N implies $x^2 \in I$ or $y^2 \in I$.

Example 1.2.28. In Example(1.2.8). The ideals I_2={0,2,5,7}, I_3={0,2,4,6} and I_4={0,2,4,5,6,7} are 2-prime ideals of N, while I_1={0,2} is not 2-prime ideal, since $4.7 \in I_1$ but $4^2 \notin I_1$ and $7^2 \notin I_1$(see Table 1-A in A.1.1).

Definition 1.2.29. [37] Let the ideal I of a commutative near ring N with unit and H is total quotient near ring of N, let $I^{-1} = \{x \in H | x. I \subset N\}$. Then the ideal I is called **invertible** if $I. I^{-1} = N$.

Definition 1.2.30. [38] Let ∅≠K⊂G a mapping γ:K→G is called **inclusion mapping** if and only if γ(a)=a for all a∈K .

Definition 1.2.31. [39] A mapping φ : $(N_1$,+, .) → $(N_2$,+,'.') is called a **near ring homomorphism** if φ(x+y)=φ(x)+φ(y) and φ(x.y)=φ(x)'.'φ(y) for $x,y \in N_1$.

Definition 1.2.32. [39] A near ring homomorphism is called **endomorphism** if $N_1 = N_2$.

Definition 1.2.33. [40] A near ring homomorphism is called **epimorphism** if it is a surjective(onto) homomorphism .

Definition 1.2.34. [24] A **Composition** $\gamma o \varphi$ of homomorphism $\varphi:K\rightarrow H$ and $\gamma:H\rightarrow G$ is a homomorphism of K to G.

Definition 1.2.35. [39] Let $\varphi :(N_1,+,.) \rightarrow (N_2,+,'.')$ be a near ring homomorphism , then , the **kernel** of φ is defined by $\ker(\varphi)=\{a\in N_1|\varphi (a)=0$, where 0 is identity of $(N_2,+)\}$.

Remark 1.2.36. [39] Let $\varphi : (N_1,+, .) \rightarrow (N_2,+,' .')$ be a near ring homomorphism, if I is an ideal of N_1, then φ(I) is an ideal of N_2.

Chapter Two

Completely Equiprime Graph

2. Introduction

In this chapter, we study algebraic graph theory on a new type of graph. This chapter consist of two section. In the first section, the completely equiprime graph of a near ring is defined with studying several properties and relations for this graph. The second section deals with the chromatic and uniquely colorable of this graph.

The completely equiprime graph is not work in semi ring. Therefore we generalize it on a near ring(finite near ring) .

2.1. Properties with relations of completely equiprime graph

This section defines a new type graph. It is c-equiprime graph of a near ring with study several properties and relations.

Definition 2.1.1. Let the ideal I of a near ring N with for all $a \in N$ a graph $CEQ^{a}(N)$ the vertices set are elements of N with the pair of distinct vertices a and $(b-c)$ are adjacent if and only if $a.b - a.c \in I$ or $(b-c).a - (b-c).0 \in I$, for all $b, c \in N$ then $\bigcup_{a \in N} CEQ^{a}(N)$ is called a **completely equiprime graph** of N and denoted by $CEQ_{I}(N)$.

Remark 2.1.2. In the graph $CEQ_{I}(N)$, the a.b-a.c∈I if and only if a.(b-c)-a.0∈I, for every $a, b, c \in N$.

Example 2.1.3. Let N={0,1,2,3} be a near ring with two operations $(+)$ and $(\,.\,)$ defined in Table 2.1 and let the ideals are I_1={0,2} and I_2={0,1}. So we can see that I_1 is c-equiprime ideal and I_2 is not c-equiprime ideal, since $3.1 - 3.2 = 1 \in I_2$ but $1 - 2 = 3 \notin I_2$,

(as an example, see Table 3-A in A.1.3)

TABLE 2.1.Multiplication and addition table of a near ring N={0,1,2,3}

+	0	1	2	3		.	0	1	2	3
0	0	1	2	3		0	0	0	0	0
1	1	0	3	2		1	0	1	0	1
2	2	3	0	1		2	2	2	2	2
3	3	2	1	0		3	2	3	2	3

and the a graph of $CEQ^a(N)$ and $CEQ_I(N)$ for the ideals $I_1=\{0,2\}$ and $I_2=\{0,1\}$ for all $a\in N$ which are illustrate in Fig2.1 and Fig2.2 respectively

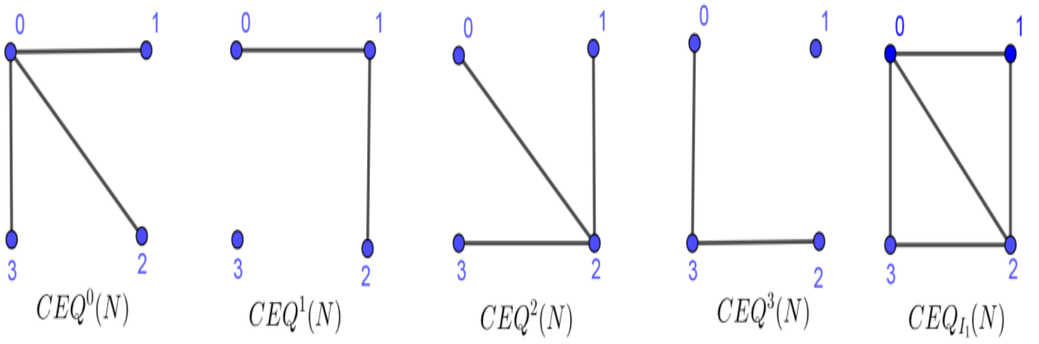


Figure 2.1. $CEQ^{a\in N}(N)$ and $CEQ_{I_1}(N)$

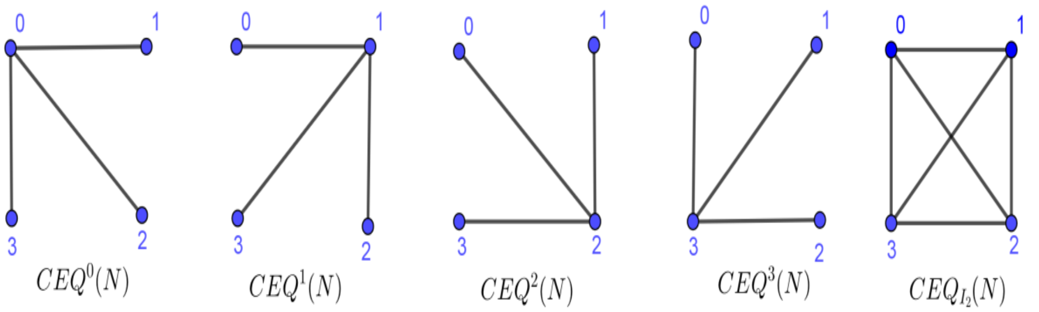


Figure 2.2. $CEQ^{a\in N}(N)$ and $CEQ_{I_2}(N)$

Remark 2.1.4. Note that: $V(CEQ^a(N)) = V(CEQ_I(N)) = N$.

Proposition 2.1.5. Let I be an ideal of N then:

1- $CEQ_I(N)$ is a simple graph.

2- $CEQ_I(N)$ is a finite graph.

3- The degree of all vertices in $CEQ_I(N)$ is finite.

Proof.

1- As $CEQ_I(N)$ has neither multiple edges nor self-loops so that it is simple graph since all vertices are adjacent in $CEQ_I(N)$ are distinct.

2- As $V(CEQ_I(N)) = N$ for any ideals of N (finite near ring).

Thus, $V(CEQ_I(N))$ is finite and by (1) as $CEQ_I(N)$ is a simple graph which means it has no multiple edges. Therefore, $E(CEQ_I(N))$ is finite. Thus, $CEQ_I(N)$ is a finite graph.

3- From (2) it is easy to see that the degree of all vertices $CEQ_I(N)$ is finite. □

Proposition 2.1.6. Let the ideals $I \subseteq J$ of N, then $CEQ_I(N) \subseteq CEQ_J(N)$.

Proof. As $V(CEQ_I(N)) = N = V(CEQ_J(N))$ for any ideals of N .Now, let the edge $\overline{ax} \in E(CEQ_I(N))$, then $\overline{ax} \in E(CEQ^a(N))$ or $\overline{ax} \in E(CEQ^x(N))$. Without loss of generality, let $\overline{ax} \in E(CEQ^a(N))$, then $a.x - a.0 \in I$, as $I \subseteq J$, then $ax\text{-}a0 \in J$, so that a is adjacent to $(x - 0)$ in $CEQ_J(N)$, then $\overline{ax} \in E(CEQ_J(N))$. So that, $CEQ_I(N) \subseteq CEQ_J(N)$. □

Proposition 2.1.7. Let I be an ideal of N then $CEQ_I(N) \subseteq CEQ_{rad(I)}(N)$.

Proof. As $I \subseteq rad(I)$, and from Proposition (2.1.6), thus $CEQ_I(N) \subseteq CEQ_{rad(I)}(N)$. □

Example 2.1.8. If I={0} be an ideal of a ring of integer module 4, then rad(I)={0,2}, and so that $CEQ_I(N) \subseteq CEQ_{rad(I)}(N)$(see Table 10-A in A.1.10)

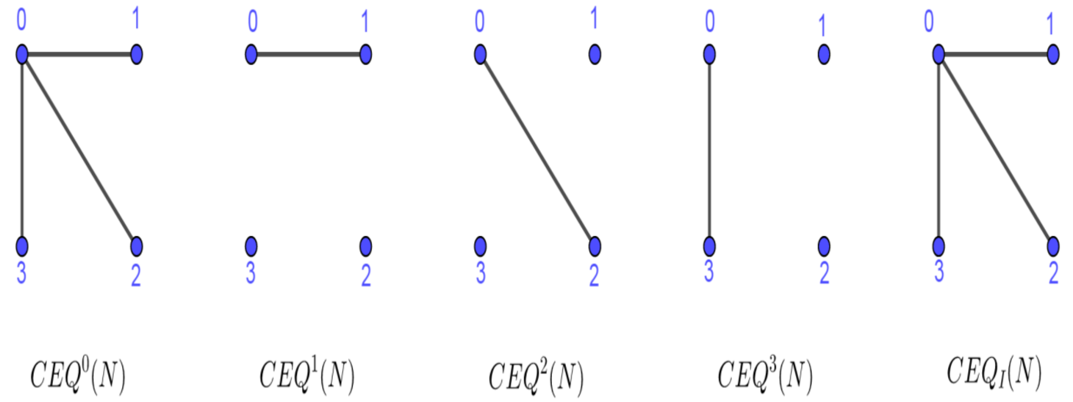


Figure 2.3. $CEQ_I(N)$

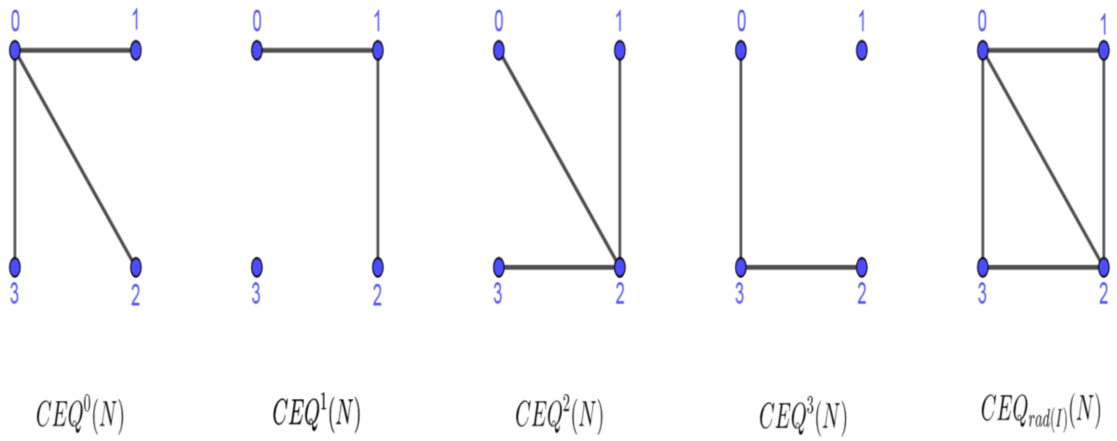


Figure 2.4. $CEQ_{rad(I)}(N)$

Theorem 2.1.9. Let I be an ideal of a commutative zero symmetric near ring N if a,b∈rad(I) then a and b are adjacent in $CEQ_I(N)$.

Proof. If I=N, then is complete the required.

Suppose I⊂ N, so that it is enough to show that a.b-a.0∈I.

Let a∈I and b=a.x for some x∈N, then $a.b=a^2.x\in I$, since $a^2\in I$ for least positive integer (n ≥ 2) and I is a right ideal of N, therefore a.b-a.0∈I, from this we get a is adjacent to b in $CEQ_I(N)$. □

Proposition 2.1.10. Let I be a c-equiprime ideal of N then $\bigcup_{a\in I} CEQ^a(N) = CEQ_I(N)$.

Proof. If I = N, then is complete the required.

If the ideal I⊂N, as $V\big(CEQ_I(N)\big) = V(\bigcup_{a\in I} CEQ^a(N)) = N$, and from Proposition(2.1.6) as I⊂N, then for each a∈ I, therefore $E(\bigcup_{a\in I} CEQ^a(N)) \subseteq E(CEQ_I(N))$,

Now, to show that $E(CEQ_I(N)) \subseteq E(\bigcup_{a\in I} CEQ^a(N)$ there are four cases as follows:

Case 1:

Let $a, x\in N$ with $\overline{ax}\in E(CEQ_I(N))=E(\bigcup_{a\in N} CEQ^a(N))$, then $\overline{ax}\in E(CEQ^a(N))$ or $\overline{ax}\in E(CEQ^x(N))$, without loss of generality let $\overline{ax}\in E(CEQ^a(N))$,then $a.x - a.0\in I$, since I is a c-equiprime ideal, then $(x - 0)\in I$ so that $x\in I$ and $\overline{ax}\in E(\bigcup_{a\in I} CEQ^a(N))$.

so that , $E(CEQ_I(N)) \subseteq E(\bigcup_{a\in I} CEQ^a(N)$,

then $E(CEQ_I(N))=E(\bigcup_{a\in I} CEQ^a(N))$,

so that $\bigcup_{a\in I} CEQ^a(N) = CEQ_I(N)$.

Case 2:

Let $a, x \in N\backslash I$ with $\overline{ax} \in E(CEQ_I(N))=E(\bigcup_{a\in N\backslash I} CEQ^a(N))$,

then $\overline{ax} \in E(CEQ^a(N))$ or $\overline{ax} \in E(CEQ^x(N))$

Without loss of generality, let $\overline{ax} \in E(CEQ^a(N))$, then $a.x - a.0 \in I$. Since I is a c-equiprime ideal, then $(x - 0) \in I$ so that $x \in I$, a contradiction as $x \in N\backslash I$.

Case 3:

Let $a \in I$ and $x \in N\backslash I$ with $\overline{ax} \in E(CEQ_I(N))=E(\bigcup_{a\in I} CEQ^a(N))$, similarly case(1) and case(2), therefore $x \in I$,a contradiction.

Case 4:

Let $a \in I$ and $x \in I$ with $\overline{ax} \in E(CEQ_I(N))=E(\bigcup_{a\in I} CEQ^a(N))$, then same case(1) we get $E(CEQ_I(N)) \subseteq E(\bigcup_{a\in I} CEQ^a(N)$,

then $E(CEQ_I(N))=E(\bigcup_{a\in I} CEQ^a(N))$,

therefore, $\bigcup_{a\in I} CEQ^a(N) = CEQ_I(N)$. □

Example 2.1.11. From Example (2.1.3), as I={0,2} is c-equiprime ideal of N and $CEQ^0(N) \cup CEQ^2(N) = CEQ_I(N) = \bigcup_{a\in N} CEQ^a(N)$.

Remark 2.1.12. If there exists $x, y \in N\backslash I$ are not independent vertices thus $\bigcup_{a\in I} CEQ^a(N) \neq CEQ_I(N)$.

Example 2.1.13. Let the ideal I={0,4} of ring of integer modulo 8, for 2 and $6 \in N\backslash I$, we can see that 2 is adjacent to 6(as an example, see Fig 2.5.) then $\bigcup_{a\in I} CEQ^a(N) \neq CEQ_I(N)$

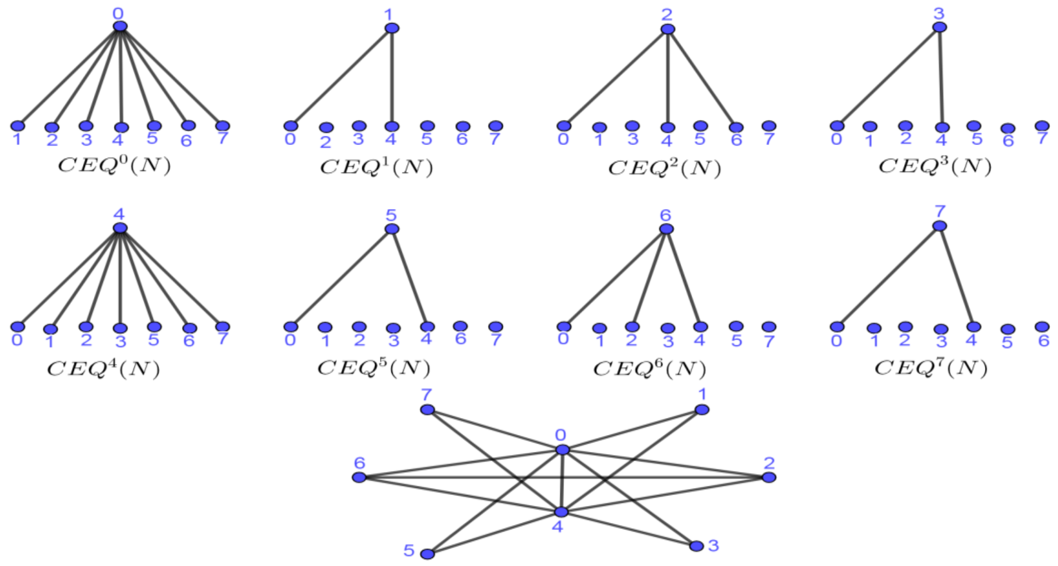


Figure 2.5. $CEQ^{a\in N}(N)$ and $CEQ_{I=\{0,4\}}(N)$ of Z_8

Proposition 2.1.14. Let I be an ideal of N then

1-deg (a) $=\Delta(CEQ_I(N)) = n-1\ \forall\ a\in I$ where $|N| = n$.

2-I is a vertex cover of $CEQ_I(N)$, whenever the vertices in N\I are independent vertices.

3-For every $a\in I$ in $CEQ^a(N)$ is a star graph.

4-In general $\gamma(CEQ_I(N))$=1.

Proof.

1-Let $a \in I$, then $a.x \in I$ for $x \in N$ and $a.0 \in I$, then a.(x+0)-a.0=a.x-a.0∈I, so that a is adjacent to $(x-0)$ in $CEQ_I(N)$. Thus, there exists an edge between a and all vertices of N.Thus, the maximum degree for completely equiprime graph(($CEQ_I(N)$) is equal to the degree of an elements of I, then deg(a)=$\Delta(CEQ_I(N))$=n-1, for all $a\in I$ and not equal to n since $CEQ_I(N)$ is a simple graph.

2,3-Directly from(1).

4-Directly from (3). □

Proposition 2.1.15. Let I be an ideal of N, then in $CEQ_I(N)$ the following are equivalent:

1-Connected

2-diam$(CEQ_I(N)) \leq 2$.

Proof. Directly from Proposition(2.1.14). □

Theorem 2.1.16. Let I be a c-equiprime ideal of N, then edge covering $E(CEQ^{a\in I}(N))$ of $CEQ_I(N)$ is a minimal edge covering if and only if $\text{diam}(CEQ^{a\in I}(N)) \leq 2$.

Proof. Let edge covering $E(CEQ^{a\in I}(N))$ of $CEQ_I(N)$ as $CEQ^{a\in I}(N)$ is star graph of root a and $\Delta(a) = n - 1$ $\forall$ a$\in$ I by Proposition(2.1.14), then its connected to all vertices of $CEQ^{a\in I}(N)$ so that diam$(CEQ^{a\in I}(N)) \leq 2$ for any different connected vertices in $CEQ^{a\in I}(N)$.

Conversely, assume that $\text{diam}(CEQ^{a\in I}(N)) \leq 2$, in this case it has components is isomorphic to spanning star graph (as an example, see Fig 2.6.), so that it will not remain an edge cover for $CEQ_I(N)$ when remove more one edge, in above components, so $E(CEQ^{a\in I}(N))$ will be a minimal edge covering for $CEQ_I(N)$. □

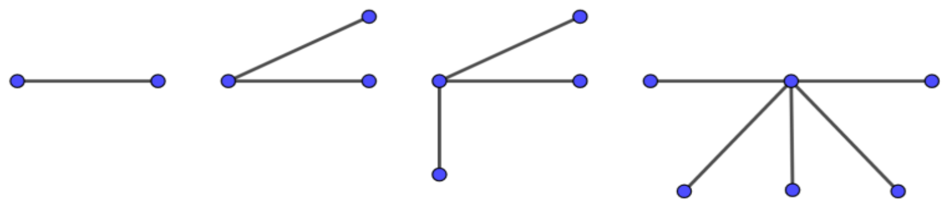


Figure 2.6. Different components of edge cover

Remark 2.1.17. In general $E(CEQ^{a\in N\backslash I}(N))$ is not edge covering of $CEQ_I(N)$, for example see Fig2.1, except whenever, $CEQ_I(N)$ is complete graph.

Theorem 2.1.18. Let I be a c-semi prime ideal of N. Then, I is c-equiprime ideal whenever the vertices in N\I are independent vertices in $CEQ_I(N)$.

Proof.

If $I = N$, then the required is completed .

If $I \subset N$, let $a \in N\backslash I$ and $a.b-a.d \in I$ for $b,d \in N$.

Then a is adjacent to (b-d), and $a \neq (b-d)$ from definition of vertices of $CEQ_I(N)$, now we have the cases:

Case1:

If a=0, then $a \in I$, a contradiction as I is c-semiprime ideal and $a \in N\backslash I$.

Case2:

If b-d=0, then $(b-d) \in I$, so that I is a c-equiprime ideals of N.

Case3:

If $a \neq 0 \neq b - d \notin I$, therefore $a \in N\backslash I$ and $b-d \in N\backslash I$, which a contradiction with the assumption as the vertices in N\I are independent vertices . □

Theorem 2.1.19. Let I be an ideal of N, then I is a clique in $CEQ_I(N)$.

Proof. If $I = N$, then $CEQ_I(N)$ is a complete graph so that the element of I represent of a clique in $CEQ_I(N)$.

If $I \subset N$ and $a \in I$, then $a.b \in I$ for $b \in N$, thus $a.b - a.0 \in I$.

Therefore, a is adjacent to $(b - 0)$ for $b \in N$, so that $a \in I$ is adjacent to all vertices in $CEQ_I(N)$, then I is a complete induced subgraph of $CEQ_I(N)$. Thus, I is a clique of $CEQ_I(N)$. □

Example 2.1.20. Let us consider the ring of integer modulo 8 and I={0,2,4,6} be an ideal of Z_8 and it is a clique (complete subgraph) of $CEQ_I(N)$

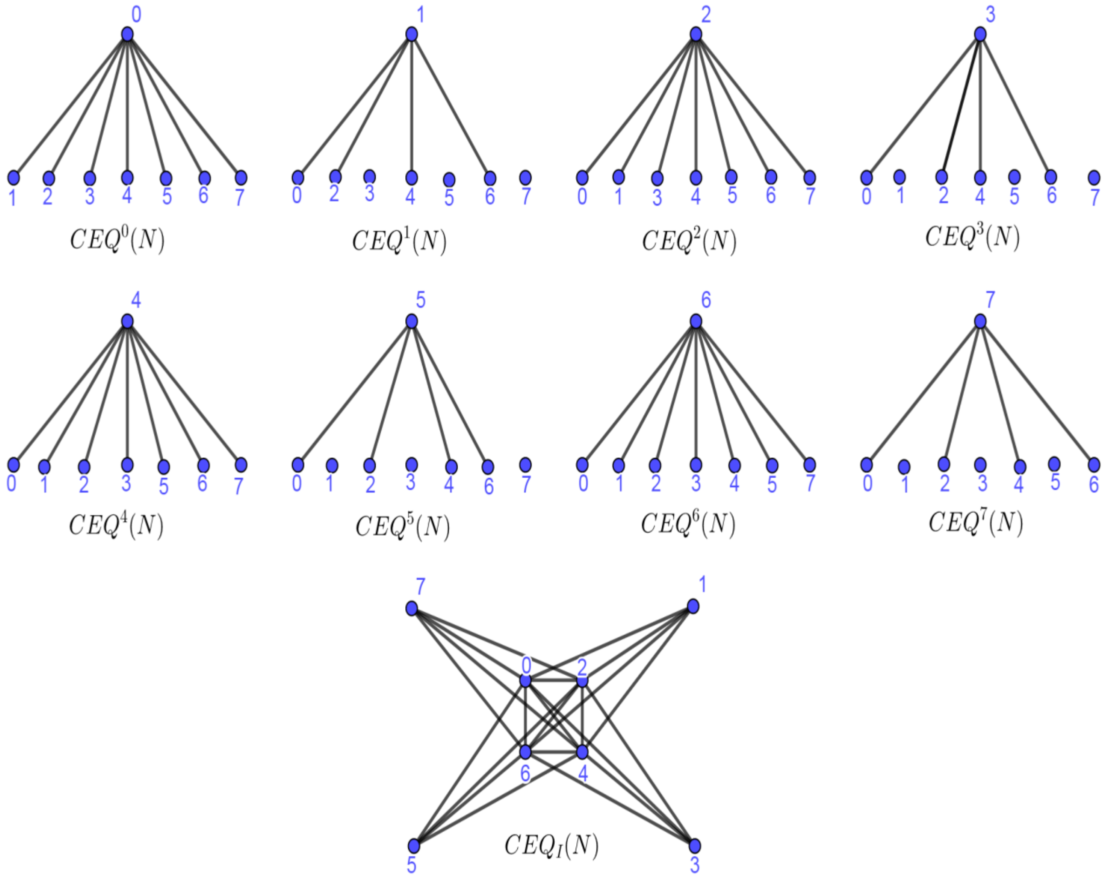


Figure 2.7. $CEQ^{a \in N}(N)$ and $CEQ_{I=\{0,2,4,6\}}(N)$ of Z_8

Remark 2.1.21.

1-The clique number of $CEQ_I(N)$ is <I>+1, whenever $|I| \geq 3$, in Example (2.1.20) the clique number of the ideal I={0,2,4,6} in Z_8 is 5 .

2-There is a finite numbers of a clique in $CEQ_I(N)$.

Proposition 2.1.22. Let I be a proper ideal of N, with $|N| \geq 4$, then in $CEQ_I(N)$ the following are equivalent:

1-Connected

2-Contains a cyclic

3-The girth $(CEQ_I(N)) \leq 4$.

Proof.

1⇒2 let $CEQ_I(N)$ is connected.

Now, let $a \in I$ is adjacent to x and y, for $x, y \in N$

Thus, a is adjacent to $(x - 0)$and $(y - 0)$.

Then $a.x - a.0 \in I$ and $a.y - a.0 \in I$ so that $\overline{ax} \in E(CEQ^a(N)$ and $\overline{ay} \in E(CEQ^a(N)$, so that $\overline{ax} \in E(CEQ_I(N))$ and $\overline{ay} \in E(CEQ_I(N))$

then we get the path $x - a - y$. Thus, there are two cases as follows.

Case 1:

If x is adjacent to y, then $x.y - x.0 \in I$. That mean $\overline{xy} \in E(CEQ_I(N))$, then we get the path $x - a - y - x$ is a cyclic of length=3 .

Case 2:

If x is not adjacent to y, Then suppose there exists $z \in N$ such that:

Subcase 1:

If z is adjacent to x and y, therefore z is adjacent to $(x - 0)$ and $(y - 0)$, then $z.x - z.0 \in I$ and $z.y-z.0 \in I$ so that $\overline{zx} \in E(CEQ^z(N))$ and $\overline{zy} \in E(CEQ^z(N))$

then we get the path $x - a - y - z - x$ is a cyclic of length=4 .

Subcase 2:

If z is adjacent to x or y, then $z.x - z.0 \in I$ or $z.y - z.0 \in I$.

Therefore $\overline{zx} \in E(CEQ_I(N))$ or $\overline{zy} \in E(CEQ_I(N))$ and $\overline{z0} \in E(CEQ_I(N))$ and since already $\overline{az} \in E(CEQ_I(N))$

then we get the path $x - a - z - x$ or $y - a - z - y$ is cyclic of length=3.

Subcase 3:

If z is not adjacent to x and y, then we get the paths x-a-y or x-a-z or y-a-z is not a cyclic.

Repeat the same steps with other vertices .

2⇒3 Directly from(2) it is clear that girth($CEQ_I(N)$) ≤ 4.

3⇒1 Straightforward. □

Proposition 2.1.23. Let I be a proper c-equiprime ideal of N. Then in $CEQ_I(N)$ the vertices in N\I are independent vertices.

Proof. Let a,x∈N\I are adjacent, then $\overline{ax} \in E(CEQ_I(N))$.

Therefore $\overline{ax} \in E(CEQ^a(N))$ or $\overline{ax} \in E(CEQ^x(N))$.

Without loss of generality, let $\overline{ax} \in E(CEQ^a(N))$. Then, $a.x - a.0 \in I$ since I is a c-equiprime ideal, then $(x - 0) \in$ I so that $x \in I$ which is a contradiction as $x \in N \backslash I$.

Similarly, if x.a-x.0∈I , we get a∈I, a contradiction, so that every vertices in N\I which are independent vertices. □

Proposition 2.1.24. Let I be a proper c-equiprime ideal of N, then $CEQ_I(N \backslash I)$ is a null graph.

Proof. Let the ideal I be a c-equiprime ideal with $a \in N \backslash I$ such that $\overline{ax} \in E(CEQ_I(N \backslash I))$ or $\overline{xa} \in E(CEQ_I(N \backslash I))$ for $a, x \in N \backslash I$. Then, for every $a, x \in N \backslash I$ are independent vertices, therefore, $a.x - a.0 \in I$ or $x.a - x.0 \in$ I since I is c-equiprime ideal so that $a \in I$ or $x \in I$. This is a contradiction with the assumption. Then, a and x are not adjacent. Hence, no edges in $CEQ_I(N \backslash I)$.

Therefore , $CEQ_I(N \backslash I)$ is a null graph. □

Example 2.1.25. In Example(2.1.3). The ideal I ={0,2} is c-equiprime ideal of N. We can see that $CEQ_I(N\backslash I)$ is a null graph(see Table 3-A in A.1.3).

Remark 2.1.26. The ideal I={0,4} of ring of integer modulo 8 be a not c-equiprime ideal, and for 2 and 6∈N\I. We can see that 2 is adjacent to 6 (see Fig 2.5). Thus, $CEQ_I(N\backslash I)$ is disconnected graph.

Proposition 2.1.27. The ideal I be a c-prime ideal of a zero symmetric N define on $CEQ_I(N)$ if and only if is a c-equiprime ideal.

Proof. Let the ideal I be a c-prime ideal and a∈N\I is adjacent to (x-y) , for all x,y∈N, then a.x-a.y∈ I or (x-y).a-(x-y).0∈I, thus
$(x-y).a$∈ I (since N is zero symmetric),
since I is c-prime ideal. So that a∈I or (x-y)∈I, then (x-y)∈I (as a∈N\ I)
Thus, I is c-equiprime ideal.
Conversely, suppose that a∈ I is adjacent to $(x-0)$. Then,
a.x-a.0∈ I or (x-0).a-(x-0).0∈I , then (x-0).a∈ I
(since N is zero symmetric), therefore $(x-0)$∈ I (as I is c-equiprime ideal), then x∈ I (as I is a subnear ring), therefore, I is c-prime ideal as a∈I. □

Example 2.1.28. Let N={0,1,2,3,a,b,c,d} be a near ring defined in Table 2.2 and let the ideal I={0,2,c,d} be a c-prime ideal definining on $CEQ_I(N)$ if and only if the ideal I is a c-equiprime ideal of N (see Table 4-A in A.1.4)

TABLE 2. 2 Multiplication and Addition table of N={0,1,2,3,a,b,c,d}

+	0	1	2	3	a	b	c	d
0	0	1	2	3	a	b	c	d
1	1	2	3	0	d	c	a	b
2	2	3	0	1	b	a	d	c
3	3	0	1	2	c	d	b	a
a	a	d	b	c	2	0	1	3
b	b	c	a	d	0	2	3	1
c	c	a	d	b	1	3	0	2
d	d	b	c	a	3	1	2	0

.	0	1	2	3	a	b	c	d
0	0	0	0	0	0	0	0	0
1	0	1	2	3	a	b	c	d
2	0	2	0	2	0	2	0	2
3	0	3	2	1	b	a	c	d
a	0	a	2	b	a	b	c	d
b	0	b	2	a	b	a	c	d
c	0	c	0	c	0	c	0	c
d	0	d	0	d	2	2	0	0

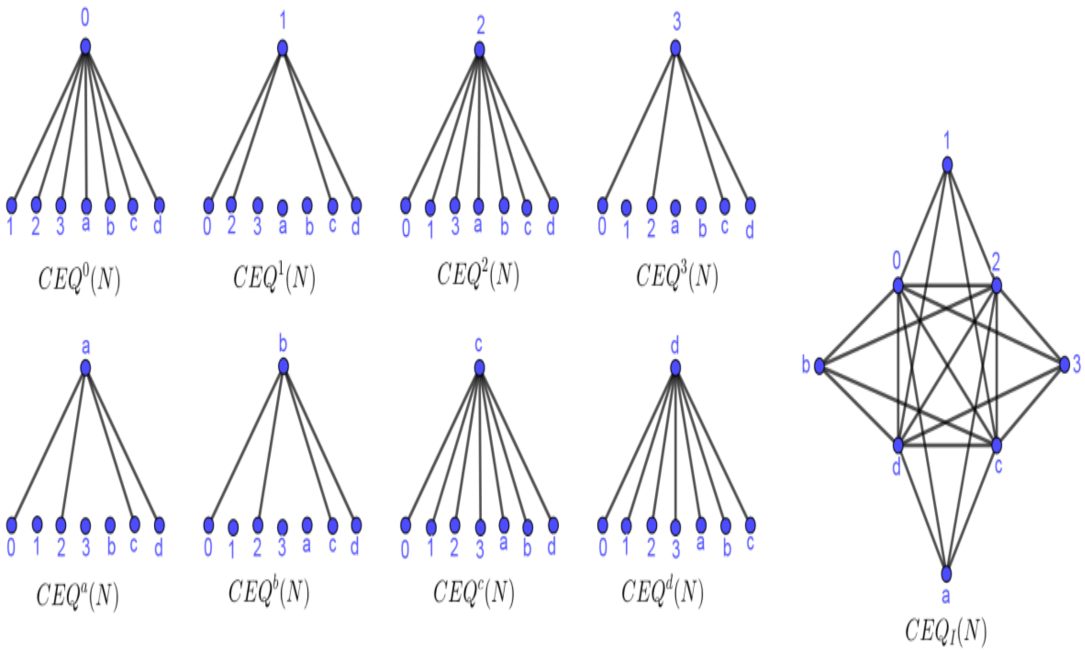


Figure 2.8. $CEQ^{a\in N}(N)$ and $CEQ_{I=\{0,2,c,d\}}(N)$

Remark 2.1.29. If N is not a zero symmetric near ring, so that Proposition(2.1.27) does not work, see Example(2.1.30).

Example 2.1.30. Let N={0,1,2,3,4,5} be a near ring defined in Table 2.3 and let the ideal I={0,3,4} be a c-prime ideal definining on $CEQ_I(N)$ but is not c-equiprime ideal of N(see Table 5-A in A.1.5)

TABLE 2. 3 Multiplication and Addition table of N={0,1,2,3,4,5}

+	0	1	2	3	4	5		.	0	1	2	3	4	5
0	0	1	2	3	4	5		0	0	0	0	0	0	0
1	1	0	3	2	5	4		1	1	1	1	1	1	1
2	2	4	0	5	1	3		2	1	1	1	2	1	2
3	3	5	1	4	0	2		3	0	0	0	3	0	3
4	4	2	5	0	3	1		4	0	0	0	4	0	4
5	5	3	4	1	2	0		5	1	1	1	5	1	5

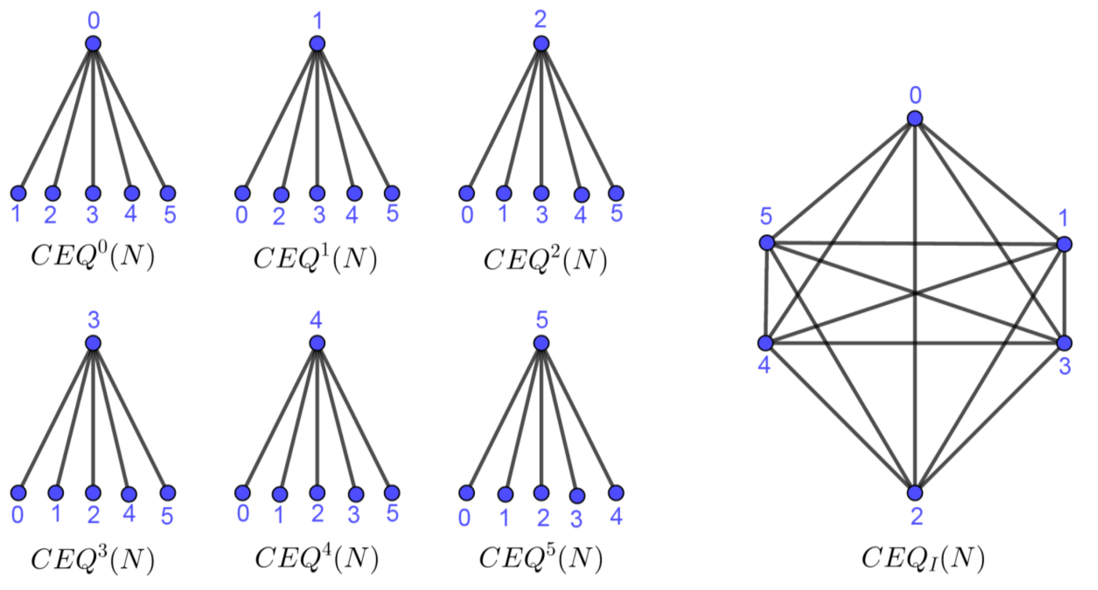


Figure 2.9. $CEQ^{a\in N}(N)$ and $CEQ_{I=\{0,3,4\}}(N)$

Proposition 2.1.31. Let I be a c-prime ideal of a zero symmetric near ring N and is defined on $CEQ_I(N)$, if I is IFP then the ideal I is 3-prime ideal.

Proof. Let a$\in$ I is adjacent to $(x - 0)$. Then, $a.x - a.0 \in$ I (since N is a zero symmetric near ring), so that $a.x \in$ I, then $a.n.x \in$ I $\forall$ n$\in$N (since I is IFP), so that, $a.N.x \subseteq$ I, then I is 3-prime ideal as a$\in$I. □

Example 2.1.32. Let N={0,1,2,3,4,5} be a near ring defined in Table 2.4 and let the ideal I={0,3} be a c-prime ideal and IFP, then I is a 3-prime ideal of N(see Table 6-A in A.1.6)

TABLE 2. 4 Multiplication and Addition table of N={0,1,2,3,4,5}

+	0	1	2	3	4	5		.	0	1	2	3	4	5
0	0	1	2	3	4	5		0	0	0	0	0	0	0
1	1	2	3	4	5	0		1	0	5	1	0	5	1
2	2	3	4	5	0	1		2	0	4	2	0	4	2
3	3	4	5	0	1	2		3	0	3	3	0	3	3
4	4	5	0	1	2	3		4	0	2	4	0	2	4
5	5	0	1	2	3	4		5	0	1	5	0	1	5

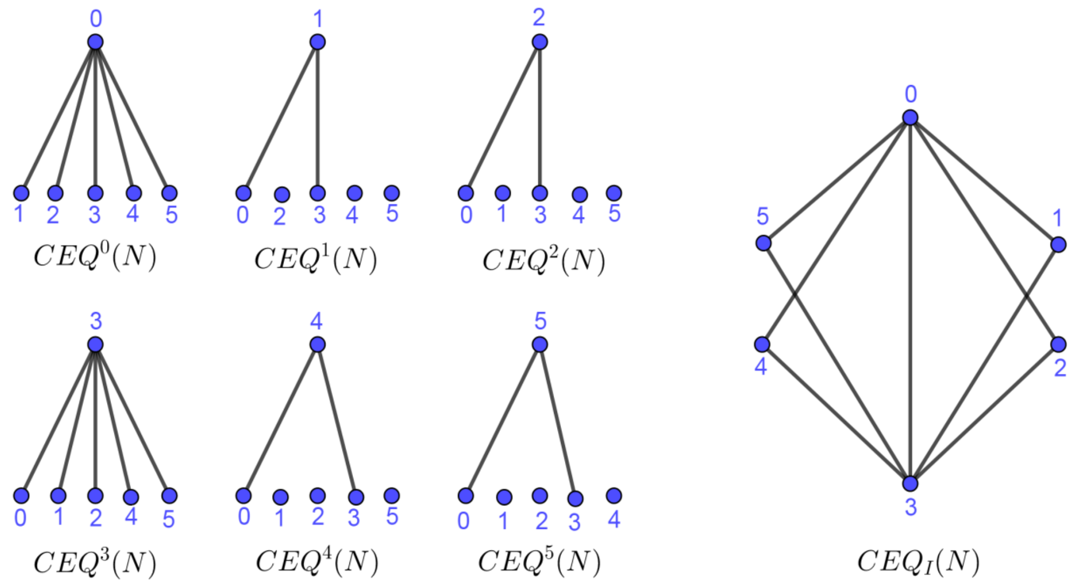


Figure 2.10. $CEQ^{a\in N}(N)$ and $CEQ_{I=\{0,3\}}(N)$

Proposition 2.1.33. Let I be an ideal of N, then $CEQ_I(N\backslash I)\subseteq CEQ_I(N)$.

Proof. If I=N, then $CEQ_I(N\backslash I)$ is not a graph.

Let $I\subset N$, since $N\backslash I\subset N$, then by proposition (2.1.6),

$CEQ_I(N\backslash I)\subseteq CEQ_I(N)$. □

Theorem 2.1.34. Let I be a c-equiprime ideal of a zero symmetric near ring N with $x, y\in N$ be an adjacent in $CEQ_I(N)$ then:

1-If $x.y\in I$ then $y.x\in I$.

2-If $x.y\in I$ and $n\in N$ then $x.n.y\in I$ (IFP).

3-If $x.y\in I$ and $n\in N$ then $y.n.x\in I$ (totally reflexive).

Proof. If $I = N$, then the required is completed.

Suppose that $I\subset N$:

1- as $x.y\in I$ then x.y-x.0∈I and y is adjacent to x then $\overline{yx}\in E(CEQ_I(N))$

then $y.x - y.0\in I$, therefore $y.x - y.0 = i_2$ for some $i_2\in I$, so that $y.x = i_2 + y.0$ (as N is zero symmetric near ring, then y.0=0), therefore $y.x \in I$.

2- If $x \in I$ then $x.n \in I$ for $n \in N$ then $x.n.y \in I$ as I is right ideal. Similarly, if $y \in I$ then $x.n.y \in I$.

3- Same proof(2). □

Example 2.1.35. For Example on Theorem(2.1.34), the ideal I={0,2,c,d} in Example(2.1.28).

Theorem 2.1.36. Let I be a c-equiprime ideal of a right (or left) permutable N and $a, b \in N$ be an adjacent in $CEQ_I(N)$, if a.b=0 then a.n.b=0 for every $n \in N$.

Proof. Let a is adjacent to b in $CEQ_I(N)$, then $a.b - a.0 \in I$.
So that $a.b - a.0 = i_1$ for $i_1 \in I$, then $0 - 0 = i_1 = 0$ (as a.b=0 and a.0=0),
as $a.b - a.0 = i_1$, then $a.b.n - a.0.n = i_1.n$ (as a.0.n=0 and $i_1.n = 0$),
thus a.b.n-0=0, so that a.b.n=0, then a.n.b=0 (as N is right permutable). □

Example 2.1.37. For example on Theorem (2.1.36). The ideal I={0,2,4,6} in Example(2.1.20).

Proposition 2.1.38. Let I be an ideal of N if $CEQ_I(N)$ is a complete graph. Then, I is not c-equiprime ideal of N.

Proof. Let $CEQ_I(N)$is complete graph , and I is c-equiprime ideal with $a \in N\backslash I$ is adjacent to $x \in N\backslash I$, then $a.x - a.0 \in I$, so that $a \in I$ or $x - 0 \in I$,
a contradiction, then I is not c-equiprime ideal on N. □

Remark 2.1.39. Opposite the Proposition (2.1.38), does not work.

Example 2.1.40. In Example(2.1.32). The ideal I={0,3} is not c-equiprime ideal and we can see that $CEQ_I(N)$is not a complete graph.

Definition 2.1.41. Let I be a proper ideal of N, define the zero divisor of $(N:I)$ as $\forall$ $z\in(N:I)$, $\exists$ $b\notin I$ such that $z.b\in I$ as zero divisor of N with respect to an ideal.

Theorem 2.1.42. Let $I\neq\{0\}$ be a proper ideal of a commutative zero symmetric near ring N, if x and y are adjacent in $CEQ_I(N)$, then $x\in$(N:I) or $y\in$(N:I).

Proof. Let $\overline{xy} \in E(CEQ_I(N))$, then x.y-x.0$\in$I and so that x.y$\in$I as N is a zero symmetric near ring. Now if y$\notin$I then we get x$\in$(N:I) and similarly if x$\notin$I, then y$\in$(N:I). □

2.2. Uniquely Colorable Completely Equiprime Graph

This section deals with coloring vertices of completely equiprime graph with the least number possible. It also studies a uniquely colorable and its relationship with a chromatic number.

Theorem 2.2.1. Let I be an ideal of N then
$\chi\big(CEQ_I(N)\big) = \begin{Bmatrix} |I|, \text{if } I = N \\ |I| + m, \text{if } I \neq N \end{Bmatrix}$, whenever $m \in Z^+$ is the number of vertices in a clique number in $N\backslash I$.

Proof. There are two cases that depend on the ideal I of N as follows:

Case1:

If $I = N$, then the graph $CEQ_I(N)$ is complete and $\chi\big(CEQ_I(N)\big) = |I|$.

Case2:

If $I \neq N$,as I is an ideal then is a clique (by Theorem(2.1.19))

Then $\chi(\langle I\rangle) = |I|$.

There is a vertex $x \in \langle N\backslash I\rangle$, since I is an ideal of N and each vertex $x \in \langle N\backslash I\rangle$ is adjacent to all vertices of $\langle I\rangle$.Thus, the induced subgraph $\langle\langle I\rangle \cup \{x\}\rangle$ is the complete of order $|I| + 1$, so

$\chi\big(CEQ_I(N)\big)=\chi(\langle I\rangle)+1=|I|+1$ (as an example, see Fig 2.11(A)).

Now, suppose that the clique number in $N\backslash I$ of order is m; therefore, it must be added m different colors from the colors of $\langle I\rangle$,thus

$\chi\big(CEQ_I(N)\big) = \chi(\langle I\rangle) + m = |I| + m$ (as an example, see Fig 2.11(B) and (C)). □

Example 2.2.2. In Example (2.1.28). The ideal I={0,2,c,d} be an ideal of N and is a complete subgraph(clique). Then, $\chi(CEQ_I(N))=|I|+1=4+1=5$ (as an example, see Fig 2.11(A)) and the ideal I={0,4} in Example (2.1.13), the $\chi(CEQ_I(N))=|I|+2=2+2=4$(as an example, see Fig 2.11(B)), while the ideal I={0,3,4}in Example (2.1.30), the $\chi(CEQ_I(N))=|I|+2=3+3=6$ (as an example, see Fig 2.11(C))

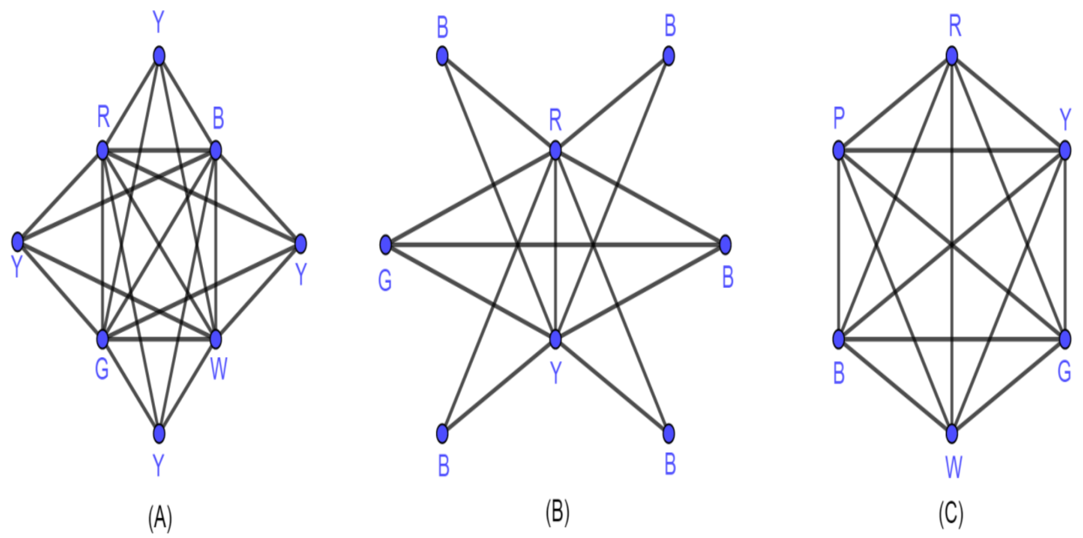


Figure 2.11. Different colors of $CEQ_I(N)$

Corollary 2.2.3. Let I be a proper c-equiprime ideal of N, then $\chi(\ CEQ_I(N\backslash I)) = 1$.

Proof. As $CEQ_I(N\backslash I))$ is a null graph, by Proposition (2.1.24), then $\chi(CEQ_I(N\backslash I)) = 1$. □

Corollary 2.2.4. Let I={0} be an ideal of N, then $\chi(CEQ_I(N)) = 2$ if and only if $CEQ_I(N)$ does not have a triangle.

Proof. Let $\chi(CEQ_I(N)) = 2$ then for $0 \in I$ is adjacent to x and y, whenever x,y∈N. If x and y are adjacent vertices ,then it is made a complete graph of order three (triangle), thus $\chi\big(CEQ_I(N)\big) > 2$ and this is a contradiction with a hypothesis. Therefore, the graph $CEQ_I(N)$ does not have a triangle.

Conversely, If the graph $CEQ_I(N)$ does not have a triangle, thus $CEQ_I(N)$ is equal to star graph. That means it is equal to a complete bipartite graph $K_{1,n-1}$.

Thus, $\chi\big(CEQ_I(N)\big) = \chi\big(K_{1,n-1}\big) = 2$. □

Example 2.2.5. Let the ideal I={0} be an ideal of ring of integer modulo 4, then $\chi\big(CEQ_I(N)\big)$=2, (as an example, see Fig 2.12)

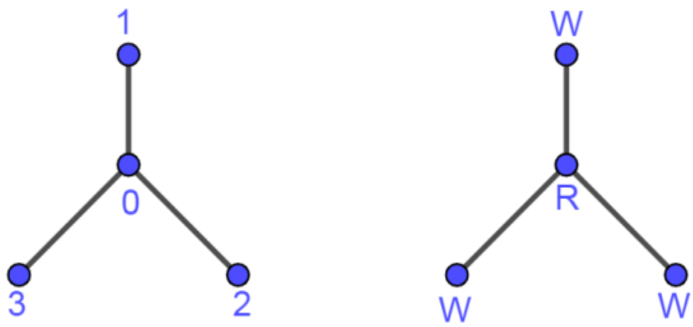


Figure 2.12. chromatic of $CEQ_I(N)$

Proposition 2.2.6. Let I≠{0} be a proper ideal of N then $2<\chi\big(CEQ_I(N)\big)< |N|$, whenever $CEQ_I(N)$ is not a complete graph.

Proof. Let the proper ideal I of N, then the set I contains at least one non zero element, say a that means the cardinal number of the set I is greater than one element. Moreover, there is at least one non zero element belongs to the set $N\backslash I$, say b, since I is a proper ideal of N. In the graph $CEQ_I(N)$, the induced subgraph $\langle I\rangle$ is complete by Theorem(2.1.19), so the vertex of zero labeled is adjacent to the vertex of labeled a. Also, the two vertices of labeled a and zero are adjacent to the vertex of labeled b according to Proposition(2.1.24). Thus, the vertices of labeled $\{0, a, b\}$ constitute a complete graph of order three, so $\chi\big(CEQ_I(N)\big) > 2$. One can easily conclude that the upper bound. □

Proposition 2.2.7. Let I be a proper ideal of N then $\chi\Big(CEQ^{a\in N}(N)\Big)$=2.

Proof. For a∈N is a root of a star in $CEQ^{a\in N}(N)$ and $CEQ^{a\in N}(N) = B_{1,n-1}$,where $|N| =$ n, then $\chi(CEQ^{a\in N}(N)) = \chi(B_{1,n-1}) = 2$ (by definition of c-equiprime graph). □

Proposition 2.2.8. Let I be an ideal of N then $\chi(CEQ^{a\in N}(N)) \leq \chi(CEQ_I(N))$.

Proof. As $V(CEQ^a(N)) = V(CEQ_I(N)) = |N|$.
Since by the definition of $CEQ^a(N)$, the graph $CEQ_I(N)$ contains a graph $CEQ^a(N)$ as a subgraph and may be another edges. Thus, $\chi(CEQ^a(N)) \leq \chi\big(CEQ_I(N)\big)$. □

Proposition 2.2.9. Let $I \neq \{0\}$ be a proper ideal of N if $\chi(CEQ_I(N)) \geq 3$ with $|N| \geq 4$ then $CEQ_I(N)$ contains at least two clique of order three.

Proof. Let I={0,a} be a proper ideal and |N|= 4 with $\chi(CEQ_I(N)) \geq 3$, then, by Theorem(2.1.19), the vertices 0 and a is adjacent in I and are adjacent to x and y, for $x, y \in N$, then {0,a} with x is a clique of order three and similar with y. □

Observation 2.2.10. Let I be an ideal of N then:

1- $\chi(\ CEQ_I(N\backslash I)) < \chi(\ CEQ_I(N))$

2- $\chi(\ CEQ_I(N\backslash\{x\})) \leq \chi(\ CEQ_I(N))$, for $x \in N$

3- $\chi(\ CEQ_I(N)\backslash\{\overline{xy}\})) \leq \chi(\ CEQ_I(N))$, for $x,y \in N$ and $\overline{xy} \in E(CEQ_I(N))$.

Proposition 2.2.11. Let I be a proper $c-$ equiprime ideal of N then $\chi(\ CEQ_I(N)) \leq \Delta(CEQ_I(N))$, whenever $CEQ_I(N)$ is not a complete graph.

Proof. For $|N| = n$ and $\Delta(CEQ_I(N)) = n - 1$,by Proposition (2.1.14), then $\chi(\ CEQ_I(N)) \leq \Delta(CEQ_I(N))$. □

Proposition 2.2.12. Let I be an ideal of N then $CEQ_I(N)$ is complete if and only if $\chi(CEQ_I(N)) = \Delta(CEQ_I(N)) + 1$.

Proof. Suppose $CEQ_I(N)$ is complete, then $\chi(CEQ_I(N))$=|N|=n, since $\Delta(CEQ_I(N)) = n - 1$, by Proposition (2.1.14), we get

$\chi(CEQ_I(N)) = \Delta(CEQ_I(N)) + 1$

Conversely, same manner. □

Theorem 2.2.13. Let I be an ideal of N then $CEQ_I(N)$ is uniquely colorable whenever for every $x, y \in N \backslash I$ are independent vertices or $\langle N \backslash I \rangle$ is a clique.

Proof. Let p be a chromatic partition of $CEQ_I(N)$

as I is a clique, so that for all $a_i \in I$, $i = 1,2, \dots, n$ (index − n)

$P = \{\{a_i\}\} = \{\{a_1\}, \{a_2\}, \dots \dots, \{a_n\}\}$ whenever $\{a_i\}$ is singleton and disjoint sets. Suppose the vertices, $x, y \in N \backslash I$

Case 1: Then, the set $N \backslash I$ is an independent by assumption, so each vertex in the induced subgraph $[(N \backslash I)]$ is belonging in the same set in the chromatic partition, so that

$P = \{\{a_i\}, \{x, y\}\} = \{\{a_1\}, \{a_2\}, \dots \dots, \{a_n\}, \{x, y\}\}, \forall\ i=\{1,2,\dots,n\}$

Then $CEQ_I(N)$ is uniquely colorable .

Case 2: If x and y are adjacent as $\langle N \backslash I \rangle$ is a clique , so each vertex in the induced subgraph $[(N \backslash I)]$ is belonging in the different set in the chromatic partition , $P = \{\{a_i\}, \{x\}, \{y\}\} \forall\ \{i=1,2,\dots,n\}$

Then $CEQ_I(N)$ is uniquely colorable. □

Example 2.2.14. In Example (2.1.28). The ideal I={0,2,c,d} be an ideal of $CEQ_I(N)$ and a chromatic partition is

p={{0},{2},{c},{d},{1,3,a,b}},

so we can see that $CEQ_I(N)$ is uniquely colorable.

Corollary 2.2.15. Let I be a proper $c-$ equiprime ideal of N then $CEQ_I(N\backslash I)$ is uniquely colorable.

Proof. As $CEQ_I(N\backslash I)$ is a null graph, by Proposition(2.1.24).
Then, $CEQ_I(N\backslash I)$ is uniquely colorable. □

Corollary 2.2.16. Let I be an ideal of N then $CEQ^{a\in I}(N)$ is uniquely colorable.

Proof. From the Proposition (2.1.14), the $CEQ^{a\in I}(N)\equiv S_n$ and the star graph S_n is uniquely colorable. Then $CEQ^{a\in I}(N)$ is uniquely colorable. □

Theorem 2.2.17. Let I be an ideal of N then $CEQ_I(N)$ is not uniquely colorable if there exists a vertex that is adjacent to independent vertices, contains all in N\I.

Proof. Let p be a chromatic partition of $CEQ_I(N)$,
as I is a clique, so that for all $a_i \in I$, $i = 1,2,\ldots,n$ (index $-$ n)
$\{a_i\}$ are singleton and disjoint sets in the chromatic partition.
Now, for $x, y \in N\backslash I$ is independent, if z is adjacent to x or y ,with $z \in N\backslash I$
Then, $z.x - z.0 \in I$ or $z.y - z.o \in I$, so that $\overline{zx} \in E(CEQ^z(N))$ or
$\overline{zy} \in E(CEQ^z(N))$. It means that $\overline{zx} \in E(CEQ_I(N))$ or $\overline{zy} \in E(CEQ_I(N))$,
without a loss of generality, let $\overline{zx} \in E(CEQ_I(N))$
Then , $P_1 = \{\{a_i\},\{z,y\},\{x\}\}$ and $P_2 = \{\{a_i\},\{x,y\},\{z\}\}$(see Fig2.13).
Thus $CEQ_I(N)$ is not uniquely colorable. □

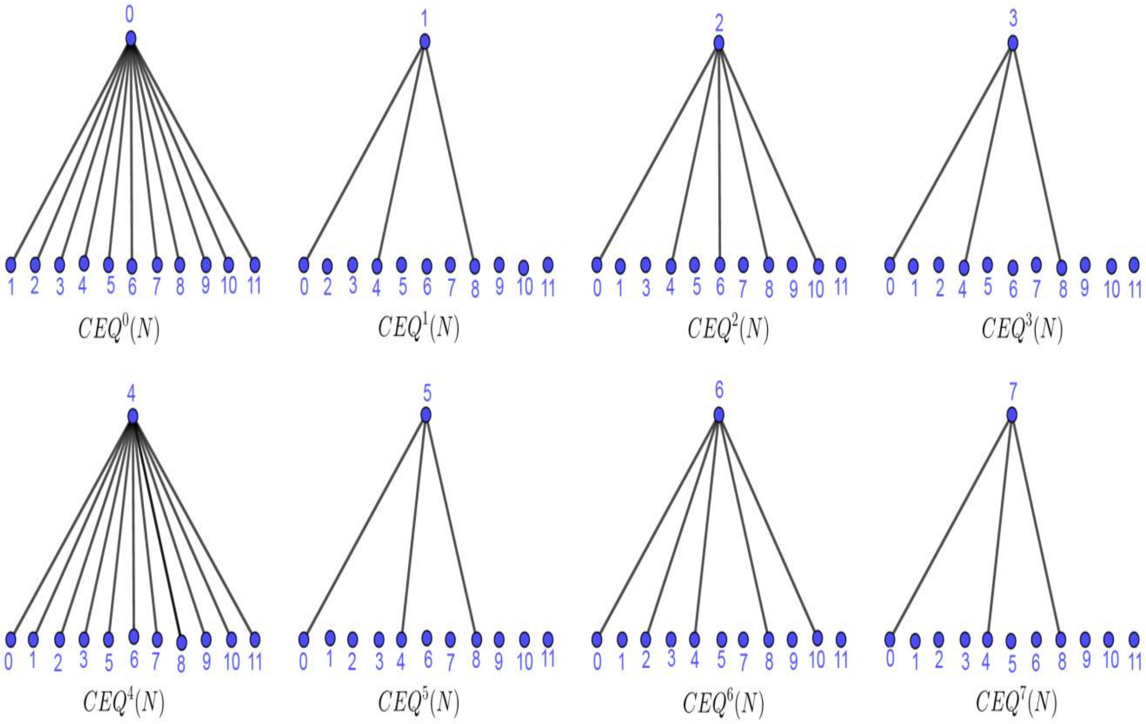
Example 2.2.18. Let the ideal I={0,4,8} be an ideal of ring of integer modulo 12, and 2 is adjacent to 6 and 10 , where 2,6 and 10 is a clique (see Fig 2.13 , so that a chromatic partition is

p_1={{0},{4},{8},{1,2,3,5,7,9,11},{6},{10}},

p_2={{0},{4},{8},{1,3,5,6,7,9,11},{2},{10}},

p_3={{0},{4},{8},{1,3,5,7,9,10,11},{2},{6}}

(as an example, see Fig 2.14), so that $CEQ_I(N)$ is not uniquely colorable.



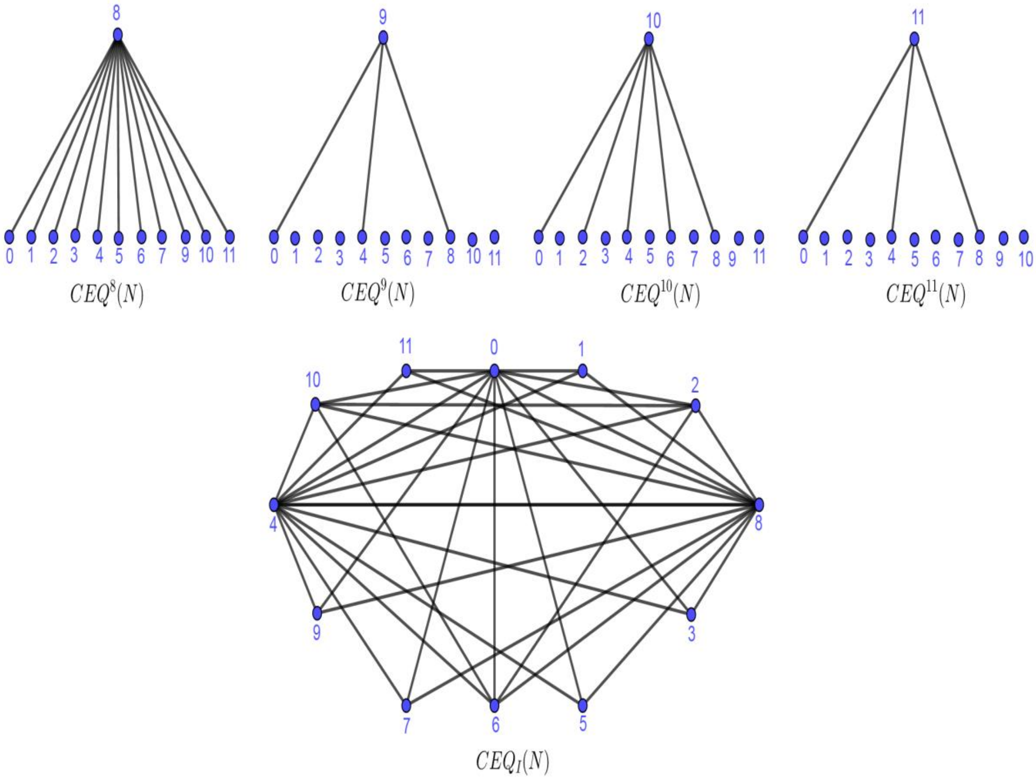
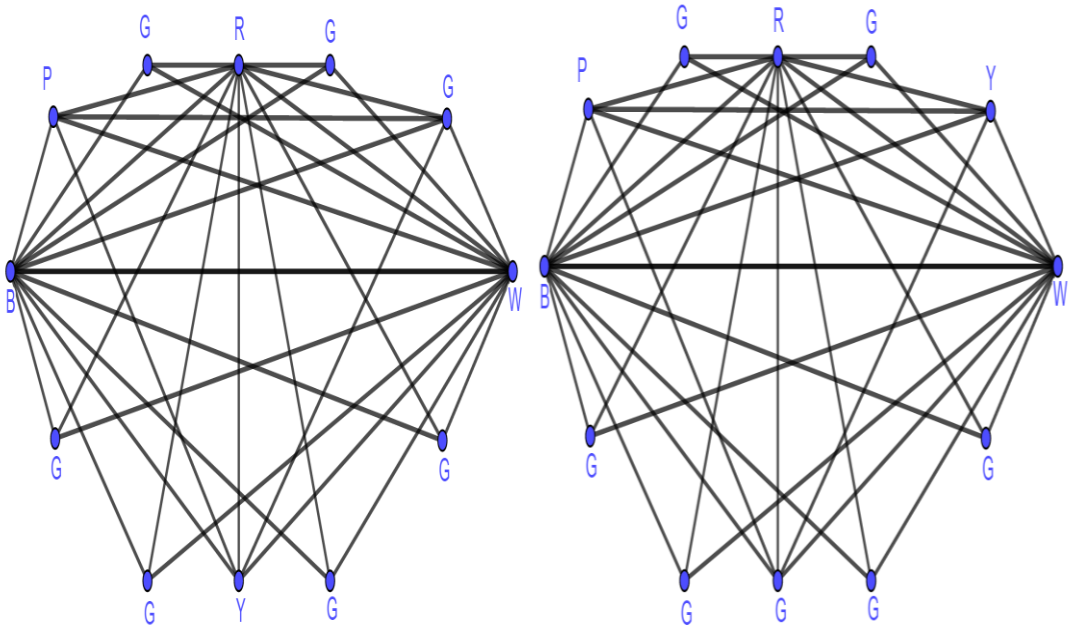
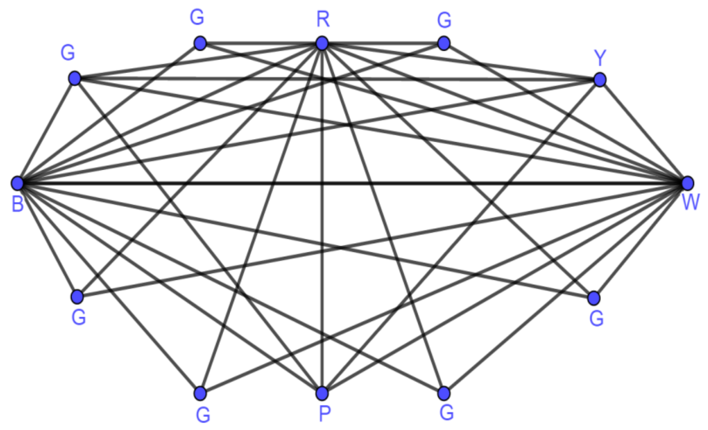


Figure 2.13. $CEQ_I(N)$ of ring of integer modulo 12



chromatic partition p_1 of $CEQ_I(N)$ chromatic partition p_2 of $CEQ_I(N)$



chromatic partition p_3 of $CEQ_I(N)$

Figure 2.14. Different chromatic partition of $CEQ_I(N)$

Proposition 2.2.19. Let the ideal $I = N$, then $CEQ_I(N)$ is uniquely colorable.

Proof. As $I = N$ then $CEQ_I(N)$ is a complete graph,

Let p be a chromatic partition of $CEQ_I(N)$.

Thus, to all $x_i \in N$, $i = 1,2, \dots n$(index-n) are adjacent,

Then, $p = \{\{x_i\}\} = \{\{x_1\}, \{x_2\}, \dots, \{x_n\}\}, \forall\ i = 1,2, \dots n$.

Thus, $CEQ_I(N)$ is uniquely colorable. □

Proposition 2.2.20. Let I be an ideal of N if $CEQ_I(N)$ is uniquely colorable and $|p| = k$ then $\chi(CEQ_I(N)) = k$.

Proof. We color partitions sets of chromatic partition p by the same color. We color the partition $\{p_i\}$ and the partition $\{p_j\}$ of p by different colors for $i \neq j$ as $|p| = k$, it is a coloring of $CEQ_I(N)$ by k- colors, thus $\chi\big(CEQ_I(N)\big) = k$. □

Example 2.2.21. In Example(2.1.28). The I={0,2,c,d} be an ideal of $CEQ_I(N)$ and a chromatic partition is p={{0},{2},{c},{d},{1,3,a,b}}, so that $|p| = 5$ and $CEQ_I(N)$ is uniquely colorable,
thus $\chi(CEQ_I(N)) = 5$.

Proposition 2.2.22. Let the ideals I and J of N such that I⊆ J if $CEQ_I(N)$ is uniquely colorable then $CEQ_J(N)$ is uniquely colorable.

Proof. As I⊆ J be an ideals of N as I and J are a clique (by Theorem(2.1.19)) , then $< I > \subseteq < J >$, so that directly from Theorem(2.2.13), we complete prove it. □

Example 2.2.23. Let N={0,1,2,3} be a near ring defined in Table 2.5, and let I_1={0,2} and I_2={0,1,2} be an ideals of N

TABLE 2. 5 Multiplication and Addition table of N={0,1,2,3}

+	0	1	2	3		.	0	1	2	3
0	0	1	2	3		0	0	0	0	0
1	1	0	3	2		1	0	1	0	1
2	2	3	0	1		2	0	0	0	0
3	3	2	1	0		3	0	1	0	1

As $I_1 \subseteq I_2$, then the chromatic partition
p_1={{0},{2},{1,3}} and p_2={{0},{1},{2},{3}},
of $CEQ_{I_1}(N)$and $CEQ_{I_2}(N)$ respectively, are uniquely colorable(as an example, see Fig 2.15).

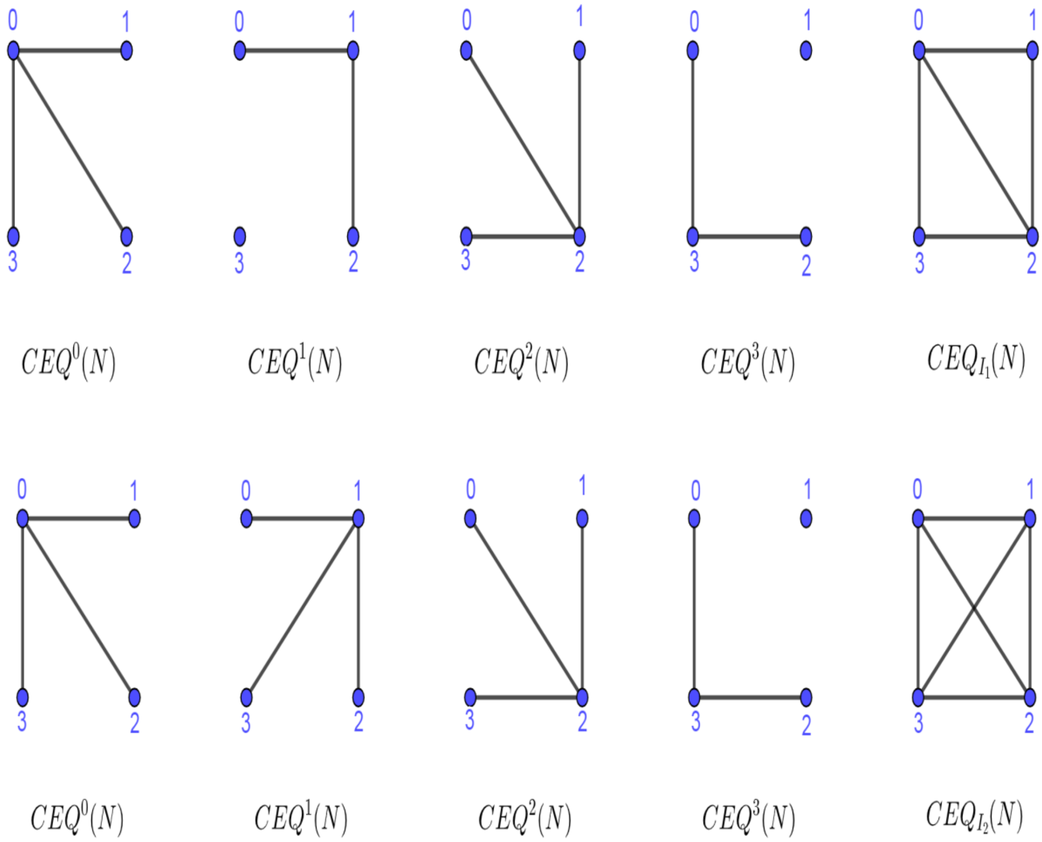


Figure 2.15. Uniquely colorable of $CEQ_I(N)$

Remark 2.2.24. Opposite of Proposition(2.2.22) is not always successive.

Example 2.2.25. The ideals I_1={0} and I_2={0,3,6} in the Z_9, the graph $CEQ_{I_1}(N)$ is not uniquely colorable while $CEQ_{I_2}(N)$ is uniquely colorable, since p_1={{0},{1,2,3,4,5,7,8},{6}} and p_2={{0},{1,2,4,5,6,7,8},{3}} for the ideal I_1, and the p={{0},{{3},{6},{1,2,4,5,7,8}} for the ideal I_2.

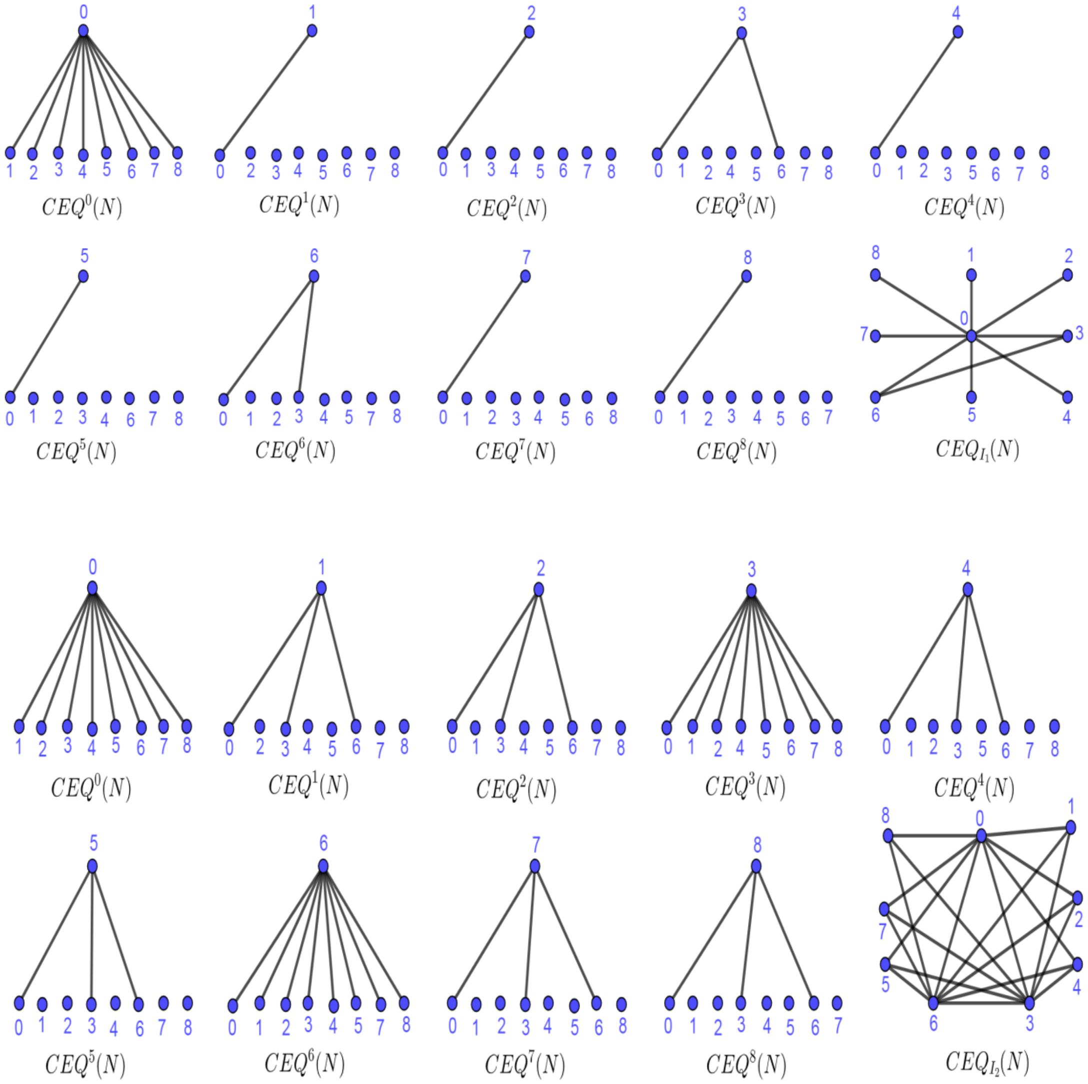


Figure 2.16. Uniquely and not uniquely colorable of $CEQ_I(N)$

Chapter Three

Weakly Completely Prime Graph

and

Almost Prime Graphs

3. Introduction

In this chapter, we research algebraic graph theory on a new type of graphs. This chapter consists of three sections. In the first section, the define a new type of ideals near ring, the second section define the weakly completely prime graph of a near ring, with study of several properties and relations for this graph, with another graphs like $CEQ_I(N)$, $\Gamma(N)$ and $PG(N)$. In the third section, study another new types of graphs are almost 1-prime graph of N and almost 2-prime graph of N with study several properties and relations of this graph, with another graphs like $CEQ_I(N)$ and $W_I(N)$.

The weakly completely prime graph($W_I(N)$), almost 1-prime graph($AP_L(N)$) and almost 2-prime graph($TP_L(N)$) are work in semiring also, with the same properties and relations which are calculated in near ring .

3.1. Some new types of ideals near ring

This section introduce some new type of ideals with examples, which is weakly c-prime ideal and almost v-prime ideals (v=1,2) of a near ring. We use the ideal L that is not idempotent ideal.

Definition 3.1.1. Let the ideal $I \neq \{0\}$ of N.Then, it is called a **weakly c-prime ideal** if $0 \neq a.b \in I, \forall a, b \in N$ then either $a \in I$ or $b \in I$.

Definition 3.1.2. Let the ideal $I \neq \{0\}$ of N . Then, it is called a **weakly c-semi prime ideal** if $0 \neq a.a \in I$, $\forall a \in N$ then $a \in I$.

Example 3.1.3. In Example(1.2.23). The ideals $I_2=\{0,6\}$ and $I_4=\{0,2,6,7\}$ are weakly c-prime ideals of N, while $I_1=\{0,2\}$ and $I_3=\{0,2,6\}$ are not weakly c-prime ideals. Since $7.4\in I_1$ but $7\notin I_1$ and $4\notin I_1$. Also, $7.5\in I_3$ but $7\notin I_3$ and $5\notin I_3$. All ideals I_1, I_2, I_3 and I_4 are weakly c-semiprime ideals of N (see Table 2-A in A.1.2).

Note that, if I and K are weakly c-prime ideals of N, then $I \cap K$ is not necessary weakly c-prime ideals in general. Similarly, $I \cup K$ is not valid in general.

Proposition 3.1.4. Let $I \neq \{0\}$ be a weakly c-prime ideal of N.Then, $\mathrm{rad}(I)$ is a weakly c-prime ideal of N.

Proof. Let $0 \neq a.b\in \mathrm{rad}(I)$, for each $a, b\in N$.
Since,$0 \neq (a.b)^n = a^n.b^n \in I$, for some n is a positive integer.
Then, $a^n\in I$ or $b^n\in I$, so that $a\in \mathrm{rad}(I)$ or $b\in \mathrm{rad}(I)$.
Then $\mathrm{rad}(I)$ is weakly c- prime ideal of N. □

Example 3.1.5. In Example(1.2.23), I={0,6} is weakly c- prime ideal of N. Then $\mathrm{rad}(I)=\{0,2,6,7\}$ is also weakly c-prime ideal (see Table 2-A in A.1.2).

Remark 3.1.6. The Converse of the Proposition (3.1.4) may not be true.

Example 3.1.7. In Example (1.2.23). The ideal I={0,2} is not weakly c- prime ideal of N, while $\mathrm{rad}(I)=\{0,2,6,7\}$ is weakly c-prime ideal(see Table 2-A in A.1.2).

Proposition 3.1.8. Let $I \neq \{0\}$ be a weakly c-semi prime ideal of N. Then, $\text{rad}(I)$ is weakly c-semi prime ideal of N.

Proof. Let $0 \neq a^2 \in \text{rad}(I)$, $\forall a \in N$.

Since ,$0 \neq (a^2)^n = (a^n)^2 \in I$, for some n is a positive integer.

Then $a^n \in I$, so that $a \in \text{rad}(I)$.

Then $\text{rad}(I)$ is weakly c-semi prime ideal of N. □

Remark 3.1.9. The Converse of the Proposition (3.1.8) may not be true.

Example 3.1.10. The ideal I={0,4,8}in the ring of integer module 12(Z_{12}) is not weakly c-semi prime ideal of N, while rad(I)={0,2,4,6,8,10} is weakly c-semi prime ideal(see Table 14-A in A.1.14).

Definition 3.1.11. Let the ideal L of N then is called:

1)-**almost 1-prime ideal** if $a.b \in L \setminus L^2$, $\forall$ $a, b \in N$ then either $a \in L$ or $b \in L$.

2)-**almost 2-prime ideal** if $a.N.b \subseteq L \setminus L^2$,$\forall$ $a,b \in N$ then either $a \in L$ or $b \in L$.

Definition 3.1.12. Let the ideal L of N then is called:

1)-**almost 1-semi prime ideal** if $a.a \in L \setminus L^2$, $\forall$ $a \in N$ then $a \in L$.

2)-**almost 2-semi prime ideal** if $a.N.a \subseteq L \setminus L^2$,$\forall$ $a \in N$ then $a \in L$.

Proposition 3.1.13. Let L be an ideal of N. Then, every c-prime ideal is almost v-prime ideal($v = 1,2$).

Proof. Let the ideal L be a c-prime ideal, then for $a, b \in N$,

let $a.b \in L \backslash L^2$ then $a.b \in L$ as $a \in L$ or $b \in L$ by our assumption.

Hence, L must be almost 1-prime ideal and similarly prove that L is almost 2-prime ideal. □

Example 3.1.14. In ring of integer module $6(Z_6)$, L={0,2,4} is c-prime ideal. Then it is almost v-prime ideal ($v = 1,2$)(see Table 11-A in A.1.11).

Remark 3.1.15.

1-The Converse of the Proposition (3.1.13) may be not true.

2- Any almost v-prime ideal (v=1,2) are not idempotent ideal.

Example 3.1.16. In Example (1.2.23). The ideal I_2={0,6} is almost v-prime ideal (v=1,2) but it is not c-prime ideal, since $2.4 \in I_2$ but $2 \notin I_2$ and $4 \notin I_2$. While I_1={0,2} and I_3={0,2,6} are almost 2-prime ideal of N but are neither almost 1-prime ideal of N nor c-prime ideal, and I_4={0,2,6,7} is almost v-prime ideal (v=1,2) and c-prime ideal of N(see Table 2-A in A.1.2).

Observation 3.1.17.

1- If I and K be almost v-prime ideal ($v = 1,2$) of N, then $I \cap K$ is not necessary almost v-prime ideal in general.

2- If I and K be a c-prime ideal of N, then $I \cap K$ is not necessary c-prime ideal. A Similarly, $I \cup K$ for (1) and (2) are not valid in general.

Example 3.1.18. In Example (1.2.8). The ideals $I_2=\{0,2,5,7\}$ and $I_3=\{0,2,4,6\}$ are almost v-prime ideal (v=1,2) as well as c-prime ideal while the ideal $I_2 \cap I_3=\{0,2\}$is not almost 1-prime ideal and c-prime ideal of N. Also, we can see that I_2 is ideal, while ${I_2}^2=\{0,5\}$ is not ideal in N(see Table 1-A in A.1.1).

Theorem 3.1.19. Let L be an almost 1-prime ideal of N, if the element $z \in (N{:}\,L)$:

1- Then $z.\,L\subseteq L^2$

2- The ideal L is a c-prime whenever is invertible ideal.

Proof.

1-As $z \in (N{:}\,L)$. Then $\exists w\notin L \ni z.\,w\in L$.

Case1: If $z\in L$ then $z.\,L\subseteq L^2$.

Case2: If $z\notin L$, since $w\notin L$ and $z.w\in L$ and L is almost 1-prime ideal, then we get $z.\,w\notin L^2$ and $z\in L$ or $w\in L$ a contradiction.

2-Let $z.\,w\in L$ with $w\notin L$. Then, by(1), $z.\,L\subseteq L^2$and as L is invertible , we get $z.\,L.\,L^{-1}\subseteq L^2.\,L^{-1}$, so that $z\in L$ .Then, L is a c-prime ideal. □

Example 3.1.20. In Example (1.2.8). The ideal $L=\{0,2,5,7\}$ is almost 1-prime ideal and the zero divisors are 2,4,5,6 and 7 in N but in $(N{:}\,L)$ just 0,2,5,7, so that $0.\,L\subseteq L^2$, $2.\,L\subseteq L^2$, $5.\,L\subseteq L^2$ and $7.\,L\subseteq L^2$(see Table 1-A in A.1.1).

Theorem 3.1.21. Let the IFP ideal L be an almost 2-prime ideal of N, if the element $z\in(N{:}\,L)$:

1- Then $z.\,n.\,L\subseteq L^2$ for all $n\in N$.

2- The ideal L is 3-prime whenever is invertible ideal.

Proof.

1- As $z \in (N{:}\,L)$. Then $\exists w \notin L \ni z.w \in L$.

Since L is IFP then z.n.w$\in$ L for n$\in$N and z.n.w$\notin$ L^2, as L is almost 2-prime ideal of N.

Case1: If z$\in$ L , so in any case if n$\in$ L or n$\notin$L, then z.n$\in$ L (as L is a right ideal), so that z.n.L $\subseteq$ L^2.

Case2: If z$\notin$ L, since w$\notin$ L and z.n.w$\in$ L and L is almost 2-prime ideal, we get z.n.w$\notin$ L^2 and z$\in$ L or w$\in$ L , a contradiction.

2- Let z.n.w$\in$ L with w$\notin$ L. Then by (1) we get z.n. L $\subseteq$ L^2 and as L is invertible ideal, so that z.n$\in$ L as L is a right ideal, then z$\in$ L. Therefore, L is 3-prime ideal of N. □

Theorem 3.1.22. Let I be an ideal of a commutative near ring N, then

1- I$\subseteq$ rad(I)$\subseteq$ (N:I).

2- If (N:I) is an ideal then it is a c-prime ideal of N.

Proof.

1-Clearly I$\subseteq$rad(I), now suppose that x$\in$rad(I) and n be the least positive integer (n$\geq$2) such that $x^n \in I$, let $x^{n-1} \notin I$, then $x.x^{n-1}=x^n \in I$, therefore x$\in$(N:I), then rad(I)$\subseteq$ (N:I).

2-Let (N:I) be an ideal of N and a.b$\in$(N:I) for a,b$\in$N, then there is c$\notin$I such that (a.b).c$\in$I, so that if :

Case1: b.c$\in$I then b$\in$(N:I).

Case2: b.c$\notin$I then a$\in$(N:I).

Therefore (N:I) is a c-prime ideal of N. □

Proposition 3.1.23. Let I be an ideal of N. Then, I is c-prime ideal if and only if it is 2-prime ideal of N, whenever every elements in N are idempotent elements.

Proof. Let $a.b \in I$, $\forall$ $a, b \in N$

Then $a \in I$ or $b \in I$ since I is c-prime ideal.

Thus, $a^2 \in I$ or $b^2 \in I$, as a,b are idempotent elements in N.

So that I is 2-prime ideal .conversely same manner. □

Definition 3.1.24. A **subtractive ideal** I of N is an ideal such that if $a, a + b \in I$, for a,b∈N then $b \in I$.

Theorem 3.1.25. Let the subtractive a proper ideal I be a weakly c-prime ideal of N and not idempotent.

1- If I is not c-prime ideal with $u.v = 0$ for some $u, v \notin I$, then u.I=I.u={0}.

2- If I is not almost 1-prime ideal, then $I^2 = 0$.

3- If I is not almost 2-prime ideal, then $I^2 = 0$, whenever I is IFP.

Proof.

1- Assume that $u.I \neq \{0\}$, then for some $a \in I$, $u.a \neq 0$ and 0≠u.(v+a)∈I. As I is weakly c-prime ideal. Therefore, $u \in I$ or $v + a \in I$, then $u \in I$ *or* $v \in I$, a contradiction. Therefore $u.I = 0$ and in the same way, show that I.u=0. Now, assume that, u.I≠{0}, now to show that I is c-prime ideal, let u.a∈I, then u.a≠0 for all a∈I, as u.a∈I then 0≠u.a∈I, since I is weakly c-prime ideal, therefore u∈I or a∈I. Then I is c-prime ideal and in the same way to show that I.u≠{0}.

2-Assume that , $I^2 \neq 0$. We have to show that I is almost 1-prime ideal. Let $a.b \in I \setminus I^2$, so that $a.b \in I$, whenever $a, b \in N$. Then

Case1: If $a.b \neq 0$ then $a \in I$ or $b \in I$ as I is weakly c-prime ideal.

Case2: Suppose that $a.b = 0$, since $I^2 \neq 0$, let $I.b \neq 0$, so there exist $u \in I$, such that $u.b \neq 0$, so that $0 \neq u.b + 0 = u.b + a.b = (u + a).b \in I$. Then $(u + a) \in I$ or $b \in I$ as I is weakly c-prime ideal , so that $a \in I$ or $b \in I$.

3- Same proof (2). □

Corollary 3.1.26. Let the subtractive a proper ideal I be a weakly c-semi prime ideal of N and not idempotent. If

1-I is not c-prime ideal with $u^2 = 0$ for some $u \notin I$.Then $u.I = I.u = \{0\}$.

2-I is not almost 1-semi prime ideal. Then $I^2 = 0$.

3-I is not almost 2-semi prime ideal. Then $I^2 = 0$, whenever I is IFP.

Proof. Directly from Theorem(3.1.25). □

Theorem 3.1.27. Let I be not idempotent ideal of N. If $I^2 \neq 0$ then is weakly c-prime ideal if and only if

1- I is almost 1-prime ideal of N.

2- I is almost 2-prime ideal of N,whenever I is IFP.

Proof.

1-Let $I^2 \neq 0$ and I is weakly c-prime ideal, if is not almost 1-prime ideal of N then $I^2 = 0$, by Theorem(3.1.25), a contradiction. Conversely, assume that I is almost 1-prime ideal, and let $0 \neq m.n \in I \backslash I^2$ for each $m, n \in N$. Then, $0 \neq m.n \in I$ while $m \in I$ or $n \in I$ as I is almost 1-prime ideal, so that I is weakly c-prime ideal .

2-Same proof (1) and (3) in Theorem(3.1.25). □

Corollary 3.1.28. Let I be not idempotent ideal of N. If $I^2 \neq 0$ then I is weakly c-semi prime ideal if and only if

1-I is almost 1-semi prime ideal of N.

2-I is almost 3-semi prime ideal of N,whenever I is IFP.

Proof. Directly from Theorem(3.1.27). □

Definition 3.1.29. The element $a \in N$ is called a **nilpotent** if $a^n = 0$ and the set of all nilpotent in N is denoted by $nilp(N)$.

Theorem 3.1.30. Let the subtractive ideal $I \neq \{0\}$ be a weakly c-prime ideal of a commutative near ring N and $a \in nilp(N)$. Then either $a \in I$ or $a.I = \{0\}$.

Proof. Assume that $a \notin I$ for some $a \in nilp(N)$.Then, we will have to show that $a.I = \{0\}$. Now let a.I≠$\{0\}$, therefore $a.b \neq 0$ for some $b \in I$. Let a least positive integer $n \geq 2$, such that $a^n = 0$, since $a \notin I$ and $a.I \neq \{0\}$, thus $0 \neq a.b = a.(a^{n-1} + b) \in I$, therefore as I is a subtractive ideal then $a \in I$ or $a^{n-1} \in I$, in both cases $a \in I$, a contradiction. Thus, $a.I = \{0\}$. Now, assume that $a.I \neq \{0\}$ for some $a \in nilp(N)$, then we will show that $a \in I$.

Since a^n=0, let $b \in I$, we have $0 \neq a.(b + a^{n-1}) = a.b \in I$, therefore as I is subtractive ideal then $a \in I$ or $a^{n-1} \in I$, so that in both cases $a \in I$. Then, if $a \in nilp(N)$, thus either $a \in I$ or $a.I = \{0\}$. □

Example 3.1.31. In Example(1.2.23). The ideal I={0,6} is weakly c-prime ideal of N. The nilpotent elements are nilp(N)={0,2,6,7} and $0,6 \in I$ while $2.I = 7.I = \{0\}$(see Table 2-A in A.1.2).

Theorem 3.1.32. Let the subtractive ideal I of N and not idempotent, if I be almost v-prime ideal$(v = 1,2)$ of a commutative near ring N and $a \in nilp(N)$.Then either $a \in I$ or $a. I = \{0\}$.

Proof. Directly from Theorem(3.1.30). □

Example 3.1.33. In Example(1.2.23). The ideal I_2={0,6} is almost v- prime ideal$(v = 1,2)$ of N. The nilpotent elements are nilp(N)={0,2,6,7} and $0,6 \in I$ while $2. I = 7. I = \{0\}$ and similarly for ideals I_1={0,2} and I_3={0,2,6}(see Table 2-A in A.1.2).

3.2. Weakly completely Prime Graph

This section defines a new type graph is weakly completely prime graph of a near ring, with studying several properties and relations.

Definition 3.2.1. Let the ideal I≠{0} of N with the vertices set of graph $W_I(N)$ are elements of N and the pair of distinct vertices x and y are adjacent if and only if 0≠x.y∈I (x.y∈I\{0}) or 0≠y.x∈I (y.x∈I\{0}), for all x,y∈N then $W_I(N)$ is called a **weakly completely prime** (weakly c-prime) graph of N.

Example 3.2.2. Let N={0,1,2,3,4,5} be a near ring defined in Table 3.1, and let the ideal I={0,3} be an ideal defining on $W_I(N)$,(as in example, see Fig 3.1)

TABLE 3. 1. Multiplication and Addition table of N={0,1,2,3,4,5}

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	5
5	3	5	1	3	5	1

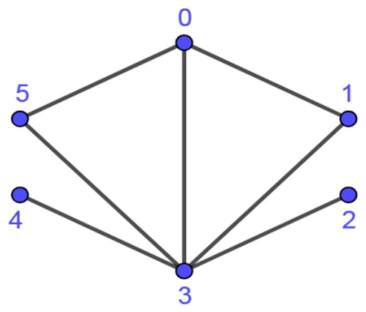


Figure 3.1. $W_I(N)$

Proposition 3.2.3. Let $I \neq \{0\}$ be an ideal of N, then

1- $W_I(N)$ is a simple graph.

2- $W_I(N)$ is a finite graph.

3- The degree of all vertices in $W_I(N)$ are finite.

Proof.

1- As for every $\overline{xy} \in E(W_I(N))$ and for $x, y \in N$ are distinct. Then $W_I(N)$ does not have self-loop or multiple edges and so that it is a simple graph.

2- As $V(W_I(N)) = N$ for any ideals of N (finite near ring), so that $V(W_I(N))$ is finite and by (1) as $W_I(N)$ is a simple graph that mean it has no multiple edges. Thus, $E(W_I(N))$ is a finite. Therefore, $W_I(N)$ is a finite graph.

3- Directly from (1) and (2). □

Proposition 3.2.4. Let I={0,a} be a weakly c-prime ideal of N, then $W_I(N)$ is a star graph, whenever {0} be a c-prime ideal of N.

Proof. Let $\overline{ab} \in E(W_I(N))$, for $b \in N$, as I={0,a},

then $0 \neq a.b \in I$ or $a.b \in I \backslash \{0\}$ (as I is weakly c-prime ideal) since {0} is c-prime ideal, therefore there is not $a.b \in \{0\}$ such that $a \notin \{0\}$ or $b \notin \{0\}$, and we get that there exist edges between a and all elements of N in $W_I(N)$, then $W_I(N)$ is a star graph S_n, where |N|=n . □

Example 3.2.5. Let N={0,a,b,c} be a near ring defined in Table 3.2. And let the ideal I={0,a} be a weakly completely prime defining in $W_I(N)$ is a star graph (as an example, see Fig 3.2), and I={0} is a c-prime ideal in N(see Table 9-A in A.1.9)

TABLE 3. 2. Multiplication and Addition table of N={0,a,b,c}

+	0	a	b	c		.	0	a	b	c
0	0	a	b	c		0	0	0	0	0
a	a	0	c	b		a	a	a	a	a
b	b	c	0	a		b	b	b	b	b
c	c	b	a	0		c	c	c	c	c

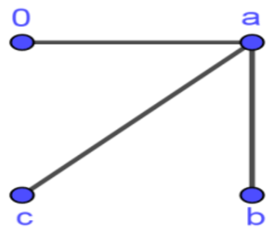


Figure 3.2. $W_I(N)$ with I is weakly c − prime ideal

Corollary 3.2.6. Let I={0,a} be a weakly c-prime ideal of N, then $W_I(N)$ is an uniquely colorable, whenever {0} be a c-prime ideal in N.

Proof. From the Proposition (3.2.4), the $W_I(N)$ is equivalent to the S_n and S_n is uniquely colorable then $W_I(N)$ is uniquely colorable. □

Proposition 3.2.7. Let the two ideals I≠{0} and K≠{0} of N with I⊆ K then $W_I(N) \subseteq W_K(N)$.

Proof. As $V(W_I(N)) = N = V(W_K(N))$ for any ideals of N.
Now, let the edge $\overline{ax}$∈$E(W_I(N))$ then a is adjacent to x in $W_I(N)$, so that $0 \neq a.x$∈I⊆ K, therefore $\overline{ax}$∈$E(W_K(N))$, we get $E(W_I(N))$⊆ $E(W_K(N))$. So that $W_I(N) \subseteq W_K(N)$. □

Proposition 3.2.8. Let I≠{0} be an ideal of N, then $W_I(N) \subseteq W_{rad(I)}(N)$.

Proof. As I ⊆ rad(I), then from Proposition (3.2.7), thus $W_I(N) \subseteq W_{rad(I)}(N)$. □

Example 3.2.9. In Example(1.2.23). The radical of the ideal I={0,6} is rad(I)={0,2,6,7}, so that $W_I(N) \subseteq W_{rad(I)}(N)$,
(as an example, see Fig 3.3.)

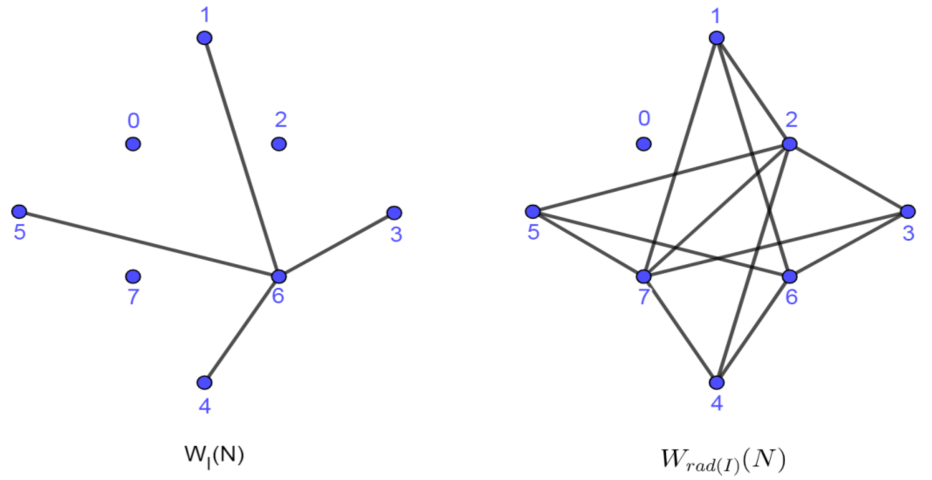


Figure 3.3. $W_I(N)$ with $I = \{0,6\}$

Theorem 3.2.10. Let $I \neq \{0\}$ be an ideal of a commutative near ring N if $a,b \in rad(I)$ then a and b are adjacent in $W_I(N)$.

Proof. If I=N, then is complete the required.
Suppose $I \subset N$, let $a \in I$ and $b=a.x$ for some $x \in N$ and $a.b \neq 0$, then $a.b=a^2.x \in I$, since $a^2 \in I$ for least positive integer $(n \geq 2)$ and is a right ideal of N, therefore $0 \neq a.b \in I$, from this we get a is adjacent to b in $W_I(N)$. □

Proposition 3.2.11. Let $I \neq \{0\}$ be an ideal of a zero symmetric near ring N, then in $W_I(N)$ has at least one isolated vertex.

Proof. As for every $x \in N$, then $0 \neq x.0 \notin I$ since N is a zero symmetric, so that $\overline{x0} \notin E(W_I(N))$ and $0 \neq x.0 \notin I$ for all elements of N, and similarly $0 \neq 0.x \notin I$,
so that $\overline{0x} \notin E(W_I(N))$, therefore 0 is isolated vertex in $W_I(N)$. □

Corollary 3.2.12. Let $I \neq \{0\}$ be an ideal of ring integer module n (Z_n), then $W_I(N)$ has 0 it is isolated vertex always.

Proof. Since Z_n is always zero symmetric near ring(ring), then by Proposition (3.2.11), $W_I(N)$ has 0 it is isolated vertex. □

Example 3.2.13. Let the ideals of some types of Z_n. Likes Z_4, Z_6 and Z_8

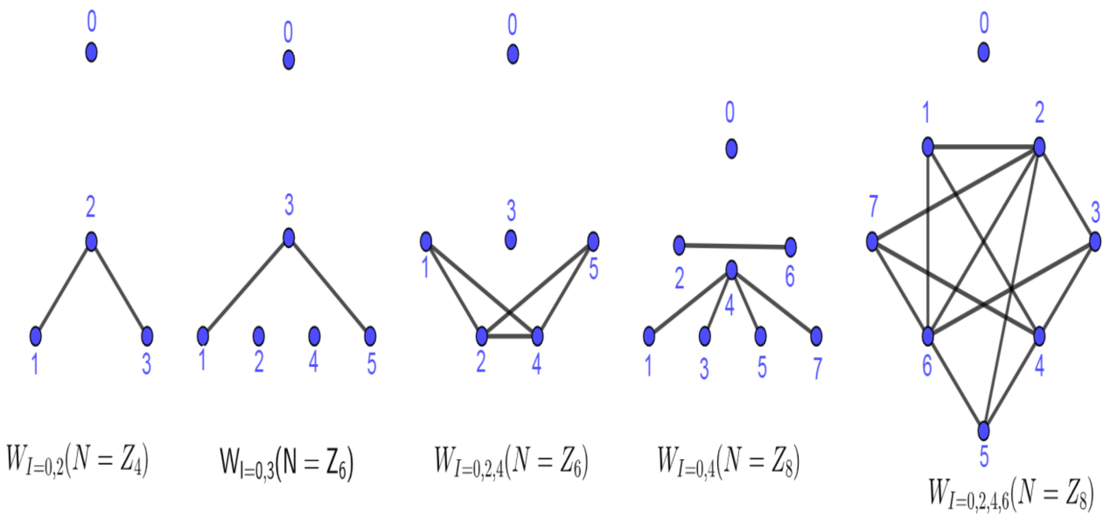


Figure 3.4. Different $W_I(N)$ in Z_n

Proposition 3.2.14. Let $I \neq \{0\}$ be an ideal of N, then $1 \leq \chi(W_I(N)) < |N|$.

Proof. If $W_I(N)$ is a null graph then $\chi(W_I(N)) = 1$ and $W_I(N)$ is impossible to become a complete graph since 0 is isolated vertex or not adjacent to some vertices in elements of N, so that $\chi(W_I(N)) < |N|$. □

Proposition 3.2.15. Let $I \neq \{0\}$ be a weakly c- prime ideal of N, then $\deg(a) = \Delta(W_I(N)) \leq n - 1 \forall a \in I \backslash \{0\}$, where $|N| = n$.

Proof. Let the ideal $I \neq \{0\}$ be a weakly c-prime ideal and $a \in I$, therefore $0 \neq a.x \in I$ for $x \in N$, as I is right ideal then we get a is adjacent to the elements of N and is the maximum degree with $\leq n - 1$ since $W_I(N)$ is a simple graph and depends on the number of isolated vertex, so that $\deg(a) = \Delta(W_I(N)) \leq n - 1$. □

Example 3.2.16. In Example (3.2.2), as I={0,3} is weakly c-prime ideal (see Table 8-A in A.1.8) and the degree a is a maximum degree in $W_I(N)$ for $a \in I\backslash\{0\}$ with less than $n-1=6-1=5$.

Corollary 3.2.17. Let $I \neq \{0\}$ be a weakly c- prime ideal of a zero symmetric near ring N, then deg (a) $= \Delta(W_I(N)) < n-1 \ \forall \ a \in I\backslash\{0\}$, where $|N| = n$.

Proof. By Proposition (3.2.11), then $W_I(N)$ have at least one isolated vertex and Proposition (3.2.15), so that deg (a) $=\Delta(W_I(N)) < n-1$. □

Proposition 3.2.18. Let $I \neq \{0)$ be a proper weakly c-prime ideal of N then:

1- $W_I(N\backslash I)$ is a null graph.

2- $\chi(W_I(N\backslash I)) = 1$.

3- $W_I(N\backslash I)$ is a uniquely colorable.

Proof.

1- Same proof Proposition (2.1.24).

2- Directly from (1).

3- As $W_I(N\backslash I)$ is a null graph, then $W_I(N\backslash I)$ is uniquely colorable. □

Proposition 3.2.19. Let $I \neq \{0\}$ be an ideal of N, then $W_I(N\backslash I) \subseteq W_I(N)$.

Proof. Same proof Proposition (2.1.33). □

Observation 3.2.20. Let $I \neq \{0\}$ be an ideal of N then:

1- $\chi(W_I(N\backslash I)) \leq \chi(W_I(N))$.

2- $\chi(W_I(N\backslash\{x\})) \leq \chi(W_I(N))$, for $x \in N$.

3- $\chi(W_I(N)\backslash\{\overline{xy}\})) \leq \chi(W_I(N))$, for $x, y \in N$ and $\overline{xy} \in E(W_I(N))$.

Proposition 3.2.21. Let $I \neq \{0\}$ be an ideal of N, then the following are hold:

1- If I is c-equiprime ideal in $W_I(N)$, then I is a weakly c-prime ideal.
2- If I is c-prime ideal in $W_I(N)$, then I is a weakly c-prime ideal.

Proof.

1- Let $a \in N\backslash I$ and $\overline{ax} \in E(W_I(N))$ with $0 \neq a.x \in I$ for $x \in N$. Then $a.x - a.0 \in I$ so that $(x - 0) \in I$ as I is c-equiprime ideal, therefore $x \in I$. Then I is weakly c-prime ideal.
2- Let $\overline{xy} \in E(W_I(N))$ with $0 \neq x.y \in I$ for $x, y \in N$, since I is c-prime ideal, then $x \in I$ or $y \in I$ so that I is a weakly c-prime ideal of N. □

Corollary 3.2.22. Let $I \neq \{0\}$ be an ideal of N, then the following are held:

1- If I is c-equiprime ideal in $W_I(N)$, then is a weakly c-semiprime ideal.
2- If I is c-semiprime ideal in $W_I(N)$, then is a weakly c-semiprime ideal.

Proof. Same proof Proposition (3.2.21). □

Definition 3.2.23. A simple graph G is called an edge summation if there exist subgraphs H and K such that $V(G) = V(H) \cup V(K)$ and $E(G) = E(H) \cup E(K)$ and denoted by $G = H \oplus K$.

Lemma 3.2.24. Let I be an ideal of N then:

1- $W_I(N)$ is a subgraph of $CEQ_I(N)$, whenever the ideal I $\neq\{0\}$.

2- $\Gamma(N)$ is a subgraph of $CEQ_I(N)$.

3- PG(N) is a subgraph of $CEQ_I(N)$.

Proof. As $V(W_I(N)) = V(\Gamma(N)) = V(PG(N)) = V(CEQ_I(N)) = N$.

1- Let $\overline{ax} \in E(W_I(N))$, then a is adjacent to x in $W_I(N)$ and $0\neq a.x \in I$, so that for a $\neq$ x$\in$N is adjacent, let $a\in N\backslash I$ with $a.x - a.0 \in I$. Therefore, a is adjacent to x-0 in $CEQ_I(N)$. That means a and x are adjacent, with 0 are adjacent already in $CEQ_I(N)$, then $\overline{ax} \in E(CEQ_I(N))$, so that $W_I(N)$ is a subgraph of $CEQ_I(N)$.

2,3-Same proof (1). □

Proposition 3.2.25. Let I$\neq\{0\}$ be an ideal of a zero symmetric near ring N then in $W_I(N)$ the following are hold:

1- Disconnected.
2- $diam(W_I(N)) \leq 2$.
3- $2 \leq \gamma(W_I(N)) \leq |N|$.

Proof.

1- From Proposition (3.2.11), 0 is always isolated vertex then $W_I(N)$ is disconnected.
2- As $W_I(N)$ is a subgraph of $CEQ_I(N)$ by Lemma (3.2.24), then by Proposition (2.1.15), we get the required.
3- From Proposition (3.2.11), then $\gamma(W_I(N)) \geq 2$ with $\gamma(W_I(N))=|N|$, whenever $W_I(N)$ is a null graph. □

Theorem 3.2.26. Let $I\neq\{0\}$ be a c-equiprime ideal of N, then:

1- $CEQ_I(N)=\Gamma(N)\oplus W_I(N)$.

2- $CEQ_I(N)=PG(N)\oplus W_I(N)$.

Proof. As $V(CEQ_I(N))=V(\Gamma(N))=V(W_I(N))=N$.

1-Now for $\in N\backslash I$, let $\overline{ax}\in E(CEQ_I(N))$ for $x\in N$, then we have two cases:

Case1: If a.x=0 then a is adjacent to x in $\Gamma(N)$ and $\overline{ax}\in E(\Gamma(N))$, then $E(CEQ_I(N))\subseteq E(\Gamma(N))$.

Case2: If a.x≠0 and since I is c-equiprime ideal then a.x-a.0∈I , then a is adjacent to x-0 in $W_I(N)$, with $\overline{ax}\in E(W_I(N))$, then $E(CEQ_I(N))\subseteq E(W_I(N))$. Therefore, from case(1) and case(2), we get $E(CEQ_I(N))\subseteq E(\Gamma(N))\cup E(W_I(N))$. From Lemma(3.2.24), we get $E(\Gamma(N))\cup E(W_I(N))\subseteq E(CEQ_I(N))$. Then, $E(CEQ_I(N))=E(\Gamma(N))\cup E(W_I(N))$. Therefore, $CEQ_I(N)=\Gamma(N)\oplus W_I(N)$.

2- Same proof (1). □

Example 3.2.27. In Example (1.2.23), as I={0,2,6,7} is a c-equiprime ideal of N and from Fig3.5, we can illustrate the Theorem (3.2.26)

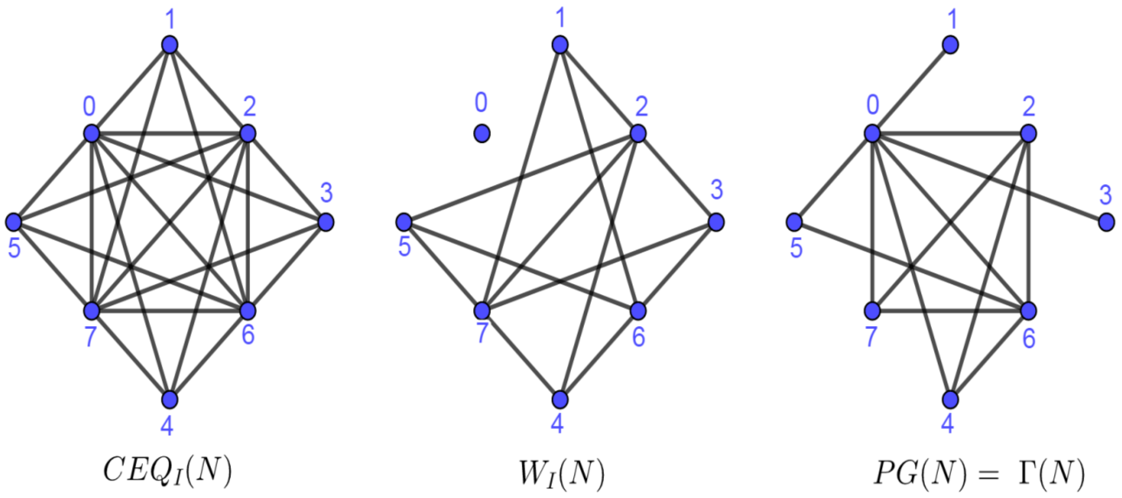


Figure 3.5. $CEQ_I(N)$ and $W_I(N)$ with $PG(N)=\Gamma(N)$

Proposition 3.2.28. Let $I \neq \{0\}$ be an ideal of N, then

1- $\chi(CEQ_I(N)) \leq \chi(\Gamma(N)) + \chi(W_I(N))$.

2- $\chi(CEQ_I(N)) \leq \chi(PG(N)) + \chi(W_I(N))$.

Proof.

1- As $\Gamma(N)$ and $W_I(N)$ are a subgraphs of $CEQ_I(N)$, by Lemma (3.2.24) and Theorem (3.2.26), then $\chi(CEQ_I(N)) \leq \chi(\Gamma(N)) + \chi(W_I(N))$.

2- Same proof (1). □

Example 3.2.29. For Example on Proposition (3.2.28), see Example (3.2.27).

Proposition 3.2.30. Let $I \neq \{0\}$ be an ideal of N, then

1- $\gamma(CEQ_I(N)) \leq \gamma(\Gamma(N))$.

2- $\gamma(CEQ_I(N)) \leq \gamma(W_I(N))$.

3 -$\gamma(CEQ_I(N)) \leq \gamma(PG(N))$.

Proof.

1- Directly from definition of c-equiprime graph of N , then 0 is adjacent to all elements of N, so that $\gamma(CEQ_I(N)) = 1$ then $\gamma(CEQ_I(N)) \leq \gamma(\Gamma(N))$.

2,3- Same proof (1). □

Proposition 3.2.31. Let $I \neq \{0\}$ be a weakly c-prime ideal of N then the ideal I is a vertex cover of $W_I(N)$.

Proof. Let $\overline{ab} \in E(W_I(N))$ then $0 \neq a.b \in I$ or $0 \neq b.a \in I$, since I is weakly c-prime ideal, then $a \in I$ or $b \in I$, therefore every edges has end vertex in I, thus the ideal I is a vertex cover of $W_I(N)$. And if $a \neq b \in N \setminus I$ such that

$\overline{ab} \in E(W_I(N))$ then a.b∈I or b.a∈I, without loss of generality, let a.b∈I, so that a∈I or b∈I, a contradiction, therefore the ideal I is a vertex cover of $W_I(N)$. □

Proposition 3.2.32. Let I≠{0} be a weakly c-semiprime ideal of N and the ideal I is a vertex cover of $W_I(N)$ then I is a weakly c-prime ideal of N.

Proof. Let x,y∈N and 0≠x.y∈I. Therefore, if x=y so we get x∈I as I is a weakly c-semiprime ideal of N. Then, we get the required. Now, let x≠y with x,y∈N\I as I is a vertex cover of $W_I(N)$, therefore $\overline{xy} \notin E(W_I(N))$, then x.y∉I and y.x∉I, a contradiction as x.y∈I. Then, I is a weakly c-prime ideal of N. □

Example 3.2.33. In Example(3.2.2). The ideal I={0,3} is a weakly c-semiprime ideal and a vertex cover of $W_I(N)$. Thus, it is a weakly c-prime ideal (see Fig3.1 and Table 8-A in A.1.8).

Proposition 3.2.34. Let the subtractive ideal I≠{0} be a weakly c-prime ideal of a commutative near ring N and a∈nilp(N) then either deg(a)≤deg(b)≤ n-1 for b∈I or a is isolated vertex in $W_I(N)$.

Proof. From Theorem (3.1.30), we get a∈I or a.I={0}, then if a∈I , therefore from Proposition (3.2.15), we get $deg(a)=\Delta\big(W_I(N)\big) \leq n-1$ or if a.I={0} then from definition of $W_I(N)$, we get a is an isolated vertex. Now if a∉I from Proposition(3.2.15), as the degree of the elements of I is the maximum degree, so that deg(a)≤ deg(b)≤ n-1 for b∈I. □

Theorem 3.2.35. Let $I \neq \{0\}$ be a proper ideal of a commutative zero symmetric near ring N, then if x and y are adjacent in $W_I(N)$ then $x \in (N:I)$ or $y \in (N:I)$.

Proof. Let $\overline{xy} \in E(W_I(N))$, then $0 \neq x.y \in I$, now if $y \notin I$ then from definition of (N:I), we get $x \in (N:I)$ and similarly if $x \notin I$, then $y \in (N:I)$. □

3.3. Almost v-prime (v=1,2) graph

This section introduces a new graph types are almost 1-prime graph($AP_L(N)$) and almost 2-prime graph ($TP_L(N)$) of a near ring , with studying several properties and relations.

Definition 3.3.1. Let the ideal L of N with the vertices of a graph $AP_L(N)$ are elements of N and the pair of distinct vertices a and b are adjacent if and only if $a.b \in L \setminus L^2$ or $b.a \in L \setminus L^2$. Then, $AP_L(N)$ is called almost 1-prime graph of N.

Definition 3.3.2. Let the ideal L of N with the vertices of a graph $TP_L(N)$ are elements of N and the pair of distinct vertices a and b are adjacent if and only if $a.N.b \subseteq L \setminus L^2$ or $b.N.a \subseteq L \setminus L^2$. Then, $TP_L(N)$ is called almost 2-prime graph of N.

Remark 3.3.3.

1-The ideal L of N in above definitions is not idempotent ideal.

2- The identity element 0 is not containing in the set $L \setminus L^2$.

Example 3.3.4. Let the ideal L={0,2,5,7} in Example(1.2.8), then $AP_L(N)$ and $TP_L(N)$ are illustrated in Fig 3.6

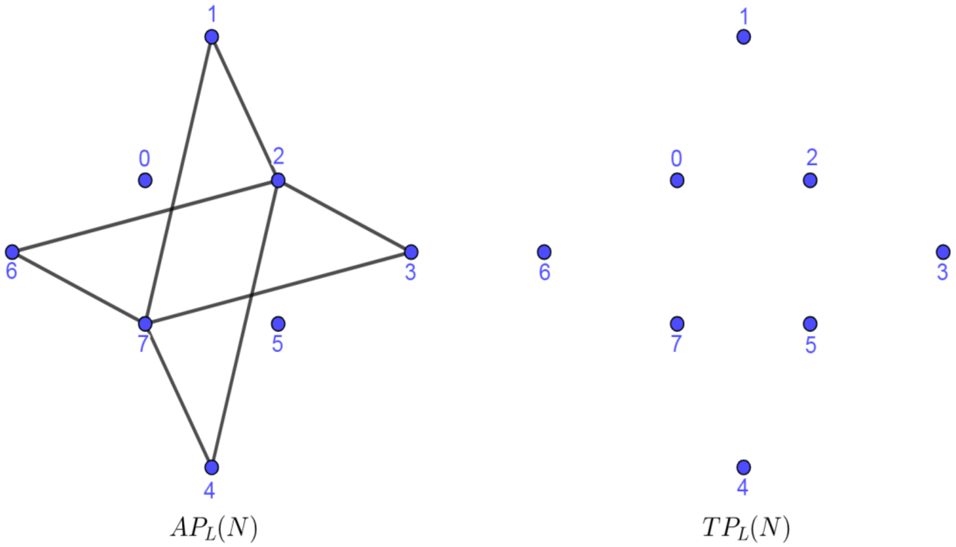


Figure 3.6. $AP_L(N)$ and $TP_L(N)$

Proposition 3.3.5. Let L be an ideal of N, then $AP_L(N)$ and $TP_L(N)$ are:

1- Simple graph.

2- Finite graph.

3- The degree of all vertices are finite.

Proof. Same proof Proposition(3.2.3). □

Theorem 3.3.6. Let L be an ideal of N, then in $AP_L(N)$ has at least one isolated vertex.

Proof. For $x \in N$ then $0.x=0$, as $0 \notin L \setminus L^2$ always. So that 0 is not adjacent to all $x \in V(AP_L(N))$, then 0 is isolated vertex in $AP_L(N)$. □

Remark 3.3.7. From Theorem (3.3.6), it is clear that any idempotent elements containing in the ideal are isolated vertices in $AP_L(N)$.

Example 3.3.8. In Example (1.2.8), see the idempotent elements are 0 and 5, in the ideal L={0,2,5,7} are isolated vertices in $AP_L(N)$,(as an example, see Fig 3.6).

Theorem 3.3.9. Let L be an ideal of N, then

1- $TP_L(N)$ is a null graph.

2- $TP_L(N)$ is a uniquely colorable.

Proof.

1-Let $x,y \in N$:

Case1: Since for every near ring the element 0 is contains it as an identity of a group (N,+):

Subcase 1: If N is a zero symmetric, then $x.N.y=x.0.y=x.0=0 \notin L \setminus L^2$ then x is not adjacent to y, therefore $TP_L(N)$ is a null graph.

Subcase 2: If N is not zero symmetric with any situation x.n.y=0, for some $n \in N$, therefore x is not adjacent to y, then $TP_L(N)$ is a null graph.

Case2: If x.N.y =c, for some $c \in N$ and c is idempotent or $c \in L^2$, so that x is not adjacent to y , therefore $TP_L(N)$ is a null graph.

2- Directly from (1). □

Proposition 3.3.10. Let the two ideals $K \subseteq L$ of N then $AP_K(N)$ is not necessary subgraph of $AP_L(N)$.

Proof. Let the edge $\overline{ax} \in E(AP_K(N))$, then a is adjacent to x in $AP_K(N)$, so that $a.x \in K \setminus K^2$ as $K \subseteq L$. Therefore, it is not necessarily contained in $L \setminus L^2$. □

Proposition 3.3.11. Let L be an ideal of N then $AP_L(N)$ is not necessary subgraph of $AP_{rad(L)}(N)$.

Proof. Same proof Proposition (3.3.10). □

Example 3.3.12. In Example(1.2.23), the ideal L={0,6} then rad(L)={0,2,6,7}, so that $AP_L(N)$ and $AP_{rad(L)}(N)$ which illustrate in Fig 3.7.

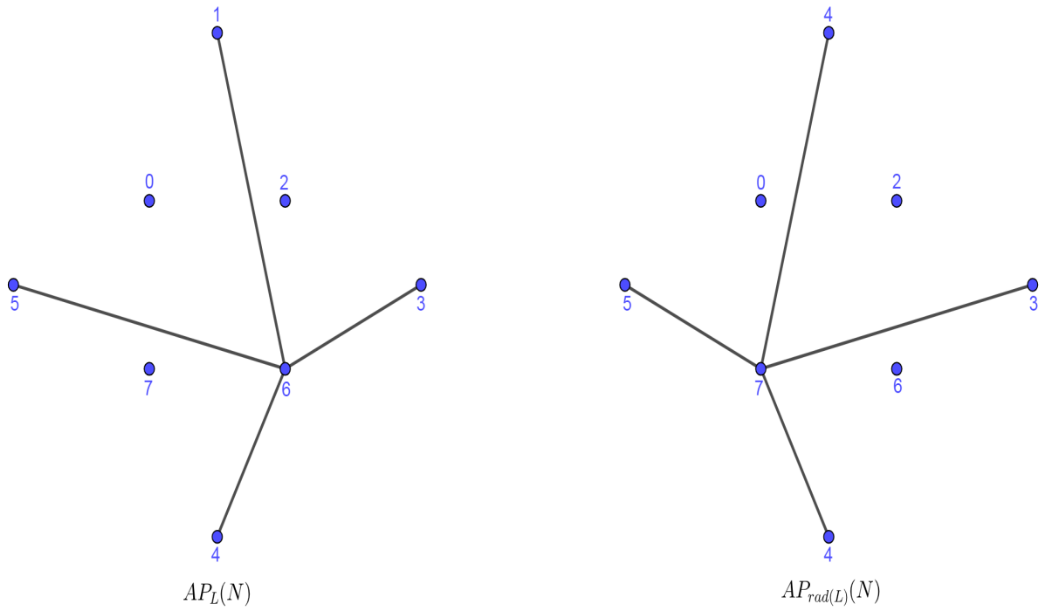


Figure 3.7. $AP_L(N)$ and $AP_{rad(L)}(N)$

Theorem 3.3.13. Let L be an ideal of a commutative near ring N if a,b∈rad(L) then a and b are adjacent in $AP_L(N)$.

Proof. Same proof Theorem (3.2.10). □

Proposition 3.3.14. Let L be an ideal of N, then $1 \leq \chi(AP_L(N)) < |N|$.

Proof. If $AP_L(N)$ is a null graph, then $\chi(AP_L(N)) = 1$ and $AP_L(N)$ is impossible to become a complete graph since 0 is an isolated vertex or not adjacent to some vertices in the elements of N,
so that $\chi(AP_L(N)) < |N|$. □

Proposition 3.3.15. Let L be an almost 1-prime ideal of N, then $\deg(a) = \Delta(AP_L(N)) < n - 1 \forall a \in L \backslash \{0\}$, where $|N| = n$ and a is not an isolated vertex.

Proof. Let L be almost 1-prime ideal and $a \in L$, so that $a.x \in L \setminus L^2$ for $x \in N$ since $L \setminus L^2$ is a subset of L, then a is adjacent to some $x \in N$, depending on the adjacent of any vertices in $AP_L(N)$ with respect to L. So we get the maximum degree is $< n - 1$, therefore
$\deg(a) = \Delta(AP_L(N)) < n - 1$. □

Example 3.3.16. See Example (3.3.12). The maximum degree for each elements containing in the ideal, whenever are not isolated vertices

Proposition 3.3.17. Let L be an almost 1-prime ideal of N then:

1 $-$ $AP_L(N \backslash L)$ is null graph.

2 $-\chi(AP_L(N \backslash L) = 1$.

3 $-$ $AP_L(N \backslash L)$ is uniquely colorable.

Proof.

1- Same proof Proposition (2.1.24).

2- Directly from (1).

3- As $AP_L(N\backslash L)$ is a null graph, then $AP_L(N\backslash L)$ is a uniquely colorable. □

Proposition 3.3.18. Let L be an ideal of N, then $AP_L(N\backslash L) \subseteq AP_L(N)$.

Proof. Same proof Proposition (2.1.33). □

Observation 3.3.19. Let L be an ideal of N then:

1- $\chi(AP_L(N\backslash L)) \leq \chi(AP_L(N))$.

2- $\chi(AP_L(N\backslash\{x\})) \leq \chi(AP_L(N))$, for $\in N$.

3- $\chi(AP_L(N)\backslash\{\overline{xy}\}) \leq \chi(AP_L(N))$, for $x, y \in N$ and $\overline{xy} \in E\big(AP_L(N)\big)$.

Lemma 3.3.20. Let L be an ideal of N then:

1-$AP_L(N)$ is a subgraph of $W_L(N)$.

2-$AP_L(N)$ is a subgraph of $CEQ_L(N)$.

Proof. As $V(AP_L) = V\big(W_L(N)\big) = V(CEQ_L(N)) = N$.

1-Let $\overline{ax} \in E\big(AP_L(N)\big)$, then a is adjacent to x in $AP_L(N)$ with $a.x \in L \setminus L^2$, so that for $a \neq x \in N$ are adjacent:

Case 1: If $0 \neq a.x \in L \setminus L^2$ then $0 \neq a.x \in L$ then $\overline{ax} \in E\big(W_L(N)\big)$, therefore $AP_L(N)$ is a subgraph of $W_L(N)$.

Case 2: If $0 = a.x \in L \setminus L^2$, then we get a contradiction.

2- By Lemma (3.2.24), as $W_L(N)$ is a subgraph of $CEQ_L(N)$ and from (1), we get the required. □

Proposition 3.3.21. Let L be an ideal of N, then in $AP_L(N)$ the following are hold:

1 − Disconnected.

2 − $diam(AP_L(N)) \leq 2$.

3 − $2 \leq \gamma(AP_L(N)) \leq |N|$.

Proof.

1-From Theorem (3.3.6), 0 is always isolated vertex. Then, $AP_L(N)$ is disconnected.

2-As $AP_L(N)$ is a subgraph of $CEQ_I(N)$ by lemma (3.3.20) and from Proposition (2.1.15). Then we get the required.

3-From Theorem (3.3.6). Then $\gamma(AP_L(N)) \geq 2$ and $\gamma(AP_L(N)) = |N|$ whenever $AP_L(N)$ is a null graph. □

Theorem 3.3.22. Let L be an almost 1-prime ideal of N. Then, the element m is independent zero divisor of (N: L) and is adjacent to all $n \in N \setminus L$ in $AP_L(N)$ if and only if $m.n \in L \setminus L^2$ or $n.m \in L \setminus L^2$.

Proof. Assume that, m is adjacent to all $n \in N \setminus L$ in $AP_L(N)$, then $m.n \in L \setminus L^2$.

Conversely, let $m.n \in L \setminus L^2$ for all $n \in N \setminus L$. Then, m is adjacent to n.

Now for all $m \in (N:L)$ if:

Case1: They are independent vertices, then we get the required.

Case2: They are not independent vertices, let $m,w \in (N: L)$ is adjacent. Therefore, $m.w \in L \setminus L^2$ as L is almost 1-prime ideal then $m \in L$ or $w \in L$. By hypothesis. Let $m \in L$ and $w \in N \setminus L$, then we get m is zero divisor of (N: L) while w is not, a contradiction so that m and w are independent vertices. Similarly if $w \in L$ and $m \in N \setminus L$. □

Example 3.3.23. See the ideal L ={0,2,6,7}=(N:L) in Example (3.3.12), the vertices 0,2,6 and 7 are independent vertices and 7 is adjacent to all vertices N\L and $7.n \in L \setminus L^2$ for all $n \in N \setminus L$.

Theorem 3.3.24. Let L be an ideal of a commutative near ring N. If x and y are adjacent in $AP_L(N)$, then $x \in (N:L)$ or $y \in (N:L)$.

Proof. Same proof Theorem (3.2.35). □

Theorem 3.3.25. Let the ideal L be a subtractive almost 1-prime ideal of N, if $0 \neq a \in L$ is adjacent to a different vertex a+b then $b \notin L$ with $a \neq b$ in $AP_L(N)$.

Proof. Since a and a+b is adjacent in $AP_L(N)$, then $a.(a+b) \in L \setminus L^2$ or $(a+b).a \in L \setminus L^2$, as L is almost 1-prime ideal, Therefore, $a \in L$ or $(a+b) \in L$, as $a \in L$. For this, we have $(a+b) \notin L$ since L is a subtractive ideal, so we get $b \notin L$ as required. □

Remark 3.3.26. The Theorem (3.3.25) does not work, in weakly c-prime ideal on $W_I(N)$.

Proposition 3.3.27. Let L be an almost 1-prime ideal of N then the ideal L is a vertex cover of $AP_L(N)$.

Proof. Same proof of Proposition (3.2.31). □

Proposition 3.3.28. Let L be an almost 1-semiprime ideal of N if the ideal L is a vertex cover of $AP_L(N)$ then L is almost 1-prime ideal of N.

Proof. Same proof of Proposition (3.2.32). □

Proposition 3.3.29. Let the subtractive ideal L be an almost 1-prime ideal of a commutative near ring N and $a \in nilp(N)$ then either $deg(a) \leq deg(b) < n-1$ for $b \in I$ or a is an isolated vertex in $AP_L(N)$.

Proof. Same proof of Proposition (3.2.34). □

Theorem 3.3.30. Let L be an ideal of N, suppose $W_L(N) = AP_L(N)$, then L is weakly c-prime ideal if and only if L is an almost 1-prime ideal of N.

Proof. Suppose L is a weakly c-prime ideal of N. If L=N, then L is almost 1-prime ideal of N. Let $L \subset N$, with $x, y \in N$ such that $x.y \in L \backslash L^2$. Therefore x and y are adjacent in $AP_L(N)$. Then we get x and y are adjacent in $W_L(N)$. Therefore $x.y \in L$ with $0 \neq x.y$, if $x.y=0$ we get a contradiction, as x and y are adjacent in $AP_L(N)$. Therefore $0 \neq x.y \in L$ and $x \in L$ or $y \in L$ as L is a weakly c-prime ideal. Then, L is almost 1-prime ideal of N. Similarly, we prove that, if L is almost 1-prime ideal then we get L is a weakly c-prime ideal of N. □

Example 3.3.31. In Example (3.2.9), the $W_L(N) = AP_L(N)$, whenever the ideal L={0,6} and L is a weakly c-prime ideal if and only if the ideal L is an almost 1-prime ideal of N(see Table 2-A in A.1.2).

Remark 3.2.32. Opposite the Theorem (3.3.30) does not necessary work. That means if L is a weakly c-prime ideal and is almost 1-prime ideal of N, then $W_L(N) = AP_L(N)$.

Example 3.3.33. The ideal L={0,2,4,6} in Z_8 is a weakly c-prime ideal and the ideal L is an almost 1-prime ideal of N, while $W_L(N) \neq AP_L(N)$(see Table 12-A in A.1.12 and Fig3.8)

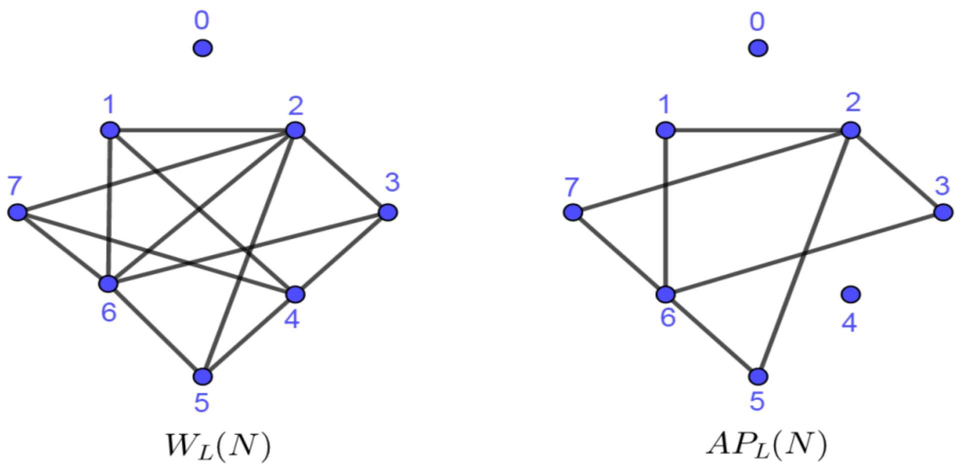


Figure 3.8. $W_L(N)$ and $AP_L(N)$

Chapter Four

Homomorphism Near Ring and Homomorphism Ideal Graphs

4. Introduction

In this chapter, we study a homomorphism graph theory on the graphs: $CEQ_I(N)$, $W_I(N)$, $AP_L(N)$ and $TP_L(N)$, studying several properties and relations for this graphs whenever there exist a near ring homomorphism. Also use the properties of these graphs with the homomorphism graphs to find which condition of successive homomorphism image of prime ideals are prime ideals.

4.1. Properties of homomorphism graphs

In this section, we investigate, properties of a homomorphism graphs: $CEQ_I(N), W_I(N), AP_L(N)$ and $TP_L(N)$, whenever there is a near ring homomorphism.

Theorem 4.1.1. Let I be an ideal of N_1 with ψ:$N_1 \to N_2$ be a epimorphism. Then, ψ:$CEQ_I(N_1) \to CEQ_{\psi(I)}(N_2)$ is a graph homomorphism.

Proof. As I is an ideal of N_1, since ψ is a surjective near ring homomorphism. Therefore, ψ(I) is an ideal of N_2 as Remark(1.2.36). Let x,y $\in N_1$ with $\overline{xy} \in E(CEQ_I(N_1)) = E(\cup_{a\in N} CEQ^a(N_1))$.
Thus, $\overline{xy} \in E(CEQ^x(N_1))$ or $\overline{xy} \in E(CEQ^y(N_1))$, so that without loss of generality, let $\overline{xy} \in E(CEQ^x(N_1))$, then $x.y - x.0 \in I$, therefore ψ(x.y-x.0)$\in$ ψ(I) so ψ(x). ψ(y)- ψ(x). $\psi(0_{N_1}) \in \psi(I)$, since ψ is a surjective near ring homomorphism, so that $\psi(0_{N_1}) = 0_{N_2}$, for this we get ψ(x). ψ(y)- ψ(x). $0_{N_2} \in \psi(I)$, this mean $\overline{\psi(x)\psi(y)} \in E\left(CEQ^{\psi(x)}(N_2)\right) \subseteq E(CEQ_{\psi(I)}(N_2))$. Therefore ψ is a graph homomorphism. □

Theorem 4.1.2. Let I≠{0} be an ideal of N_1 with ψ:N_1→N_2 be a epimorphism . Then, ψ:$W_I(N_1)$→$W_{\psi(I)}(N_2)$ is a graph homomorphism.

Proof. Let x,y ∈N_1 with $\overline{xy} \in E(W_I(N_1))$. Therefore, x and y are adjacent in $W_I(N)$ therefore 0 ≠ x. y∈I. That means $0_{N_1} \neq x.y$∈I , so that $\psi(0_{N_1}) \neq \psi(x.y)$∈ψ(I), then $\psi(0_{N_1}) \neq \psi(x).\psi(y)$∈ψ(I). As ψ is a surjective near ring homomorphism, therfore $\psi(0_{N_1})= 0_{N_2}$ for this we get, $0_{N_2} \neq \psi(x).\psi(y)$∈ψ(I) this mean $\overline{\psi(x)\psi(y)} \in E\left(W_{\psi(I)}(N_2)\right)$. Therefore, ψ is a graph homomorphism. □

Theorem 4.1.3. Let L be an ideal of N with ψ:N_1→N_2 be a epimorphism. Then

1-ψ:$AP_L(N_1)$→$AP_{\psi(L)}(N_2)$ is a graph homomorphism.

2-ψ:$TP_L(N_1)$→$TP_{\psi(L)}(N_2)$ is a graph homomorphism.

Proof.

1- Same proof Theorem (4.1.2) .

2- As $TP_L(N)$ is a null graph by Theorem (3.3.9). So complete the required. □

Remark 4.1.4. It is not necessary the homomorphism image of a completely equiprime(respect. c-prime,3-prime)ideal is a completely equiprime(respect. c-prime,3-prime). For example, let I={0} be an ideal of $N_1 = \frac{Z}{7Z}$ with ψ be a near ring homomorphism from $N_1 = \frac{Z}{7Z}$ to $N_2 = \frac{Z}{4Z}$ with ψ(a+7Z)=a+4Z. Then, I={0} is a completely equiprime(respect. c-prime,3-prime) ideals of $N_1 = \frac{Z}{7Z}$ while ψ(I)={0} of $N_2 = \frac{Z}{4Z}$ is not.

Lemma 4.1.5.

1-Let I be a c-equiprime ideal of N if a∈N and deg(0)=deg(a) in $CEQ_I(N)$, then a∈I.

2- Let I be an ideal of N if a∈I then deg(0)=deg(a) in $CEQ_I(N)$.

3- Let I be a c-equiprime ideal of N. Then, a∈I if and only if deg(0)=deg(a) in $CEQ_I(N)$.

4- If N be a c-equiprime near ring, then deg(0)=deg(a) in $CEQ_I(N)$ for a∈N.

Proof.

1- Let the ideal I=N. Therefore a∈I. Suppose that I⊂N and v∈N\I, since deg(0)=deg(v) in $CEQ_I(N)$. Then, v is adjacent to all elements of N as 0 is adjacent too, so that v is adjacent to a, thus, v.a-v.0∈I as I is c-equiprime ideal and v∈N\I then a∈I.

2- As a∈I and 0∈I(always), from Proposition(2.1.14) we get deg(0)=deg(a)=n-1, where $|N| = n$.

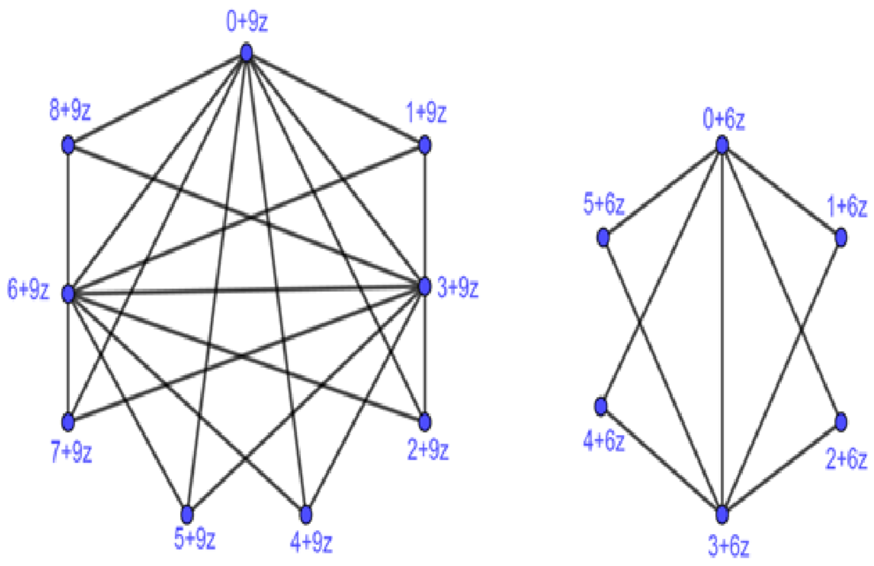
3- Directly from(1) and (2) we get the required.

4- Same proof (3). □

Proposition 4.1.6. Let I be an ideal of N_1 with $\psi:N_1 \rightarrow N_2$ be a epimorphism. Then $deg(\psi(a))=deg(0_{N_2})$ in $CEQ_{\psi(I)}(N_2)$ whenever $a \in I$.

Proof. Let $a \in I$, then by Lemma (4.1.5). Thus, deg(a)=deg(0) in $CEQ_I(N_1)$, so that $a.x-a.0 \in I$ for all $x \in N_1$. That means a is adjacent to all elements of N_1. Therefore, $\psi(a.x-a.0 \in I) \in \psi(I)$ we get $\psi(a).\ \psi(x)-\psi(a).\ \psi(0) \in \psi(I)$ as $\psi(0_{N_1})=0_{N_2}$ then $\psi(a).\ \psi(x)-\psi(a).\ 0_{N_2} \in \psi(I)$ from this we have $\psi(x) \in N_2$ as ψ is a surjective near ring homomorphism and $\psi(a)$ is adjacent to all $\psi(x) \in N_2$, so that $deg(\psi(a))=deg(0_{N_2})$ in $CEQ_I(N_2)$. □

Example 4.1.7. let ψ be a near ring homomorphism from $N_1=\frac{Z}{9Z}$ to $N_2=\frac{Z}{6Z}$ with $\psi(a+9Z)=a+6Z$ then ψ is a graph homomorphism from $CEQ_I(N_1)$ to $CEQ_{\psi(I)}(N_2)$, let I={0+9Z,3+9Z,6+9Z} so that is ideal of N_1 and $\psi(I)$={0+6Z,3+6Z} is ideal of N_2 and deg(0+6z)=deg(3+6z), whenever (0+9z) ,(6+9z) $\in$ I. The graphs of $CEQ_I(N_1)$ to $CEQ_{\psi(I)}(N_2)$ are shown in Fig 4.1.



$CEQ_I(N_1)$ $\quad$ $CEQ_{\psi(I)}(N_2)$

Figure 4.1. Graph homomorphism of $CEQ_I(N_1)$ to $CEQ_{\psi(I)}(N_2)$

Remark 4.1.8. The Proposition (4.1.6) not work in $W_I(N)$, $AP_L(N)$ and $TP_L(N)$.

Lemma 4.1.9. Let I be an ideal of N_1 with $\psi:N_1 \to N_2$ be a epimorphism. Then, $\psi(N_1\backslash I) = N_2\backslash\psi(I)$.

Proof. Let $a \in \psi(N_1\backslash I)$, then $a=\psi(r)$ for some $r \in N_1\backslash I$ so $r \notin I$, hence $\psi(r) \notin \psi(I)$. From this we get $\psi(r) \in N_2\backslash\psi(I)$. Therefore $a \in N_2\backslash\psi(I)$ and $\psi(N_1\backslash I) \subseteq N_2\backslash\psi(I)$. Let $b \in N_2\backslash\psi(I)$, then $b=\psi(r)$ for some $r \notin \psi(I)$. Suppose $r \in I$ then $\psi(r) \in \psi(I)$ a contradiction with a hypothesis $(r \notin \psi(I))$. Hence $r \notin I$. From this we get $r \in N_1\backslash I$, therefore $\psi(r) \in \psi(N_1\backslash I)$, then $b \in \psi(N_1\backslash I)$thus $N_2\backslash\psi(I) \subseteq \psi(N_1\backslash I)$. Then $\psi(N_1\backslash I) = N_2\backslash\psi(I)$. □

Proposition 4.1.10. Let I be an ideal of N_1 with $\psi:N_1 \to N_2$ be a epimorphism if $\deg(\psi(a))=\deg(0_{N_2})$ in $CEQ_{\psi(I)}(N_2)$ for all $a \in N_1$, then $\deg(a)=\deg(0_{N_1})$ in $CEQ_I(N_1)$.

Proof. Let $\deg(\psi(a))=\deg(0_{N_2})$ in $CEQ_{\psi(I)}(N_2)$ for $a \in N_1$
if $\deg(a) \neq \deg(0_{N_1})$ in $CEQ_I(N_1)$. Then, there exist $x \in N_1$
such that a.x-a.0$\notin$I, therefore a.x-a.0$\in N_1 \backslash I$ then
$\psi($ a.x-a.0$) \in \psi(N_1 \backslash I)$, so that
$\psi(a).\ \psi(x)-\psi(a).\ \psi(0_{N_1}) \in N_2 \backslash \psi(I)$(by lemma(4.1.9), then
$\psi(a).\ \psi(x)-\psi(a).\ 0_{N_1} \notin \psi(I)$ from this we get that $\psi(a)$ is not adjacent to $\psi(x)$ therefore $\deg(\psi(a)) \neq \deg(0_{N_2})$ in $CEQ_{\psi(I)}(N_2)$, a contradiction with hypothesis as $\deg(\psi(a))=\deg(0_{N_2})$ in $CEQ_{\psi(I)}(N_2)$ then $\deg(a)=\deg(0_{N_1})$ in $CEQ_I(N_1)$. □

Theorem 4.1.11. Let I be an ideal of N_1 with $\psi:N_1 \to N_2$ be a epimorphism, a mapping $\psi:CEQ_I(N_1) \to CEQ_{\psi(I)}(N_2)$ is a graph homomorphism if and only if:

1- $P_k(0,1,\ldots,k)$is a path in $CEQ_I(N_1)$. Then, $\psi(0), \psi(1),\ldots. \psi(k)$ is a walk in $CEQ_{\psi(I)}(N_2)$.

2-$C_k(0,1,\ldots,k-1)$is a cyclic in $CEQ_I(N_1)$. Then,$\psi(0),\psi(1),\ldots \psi(k-1)$ is a closed walk in $CEQ_{\psi(I)}(N_2)$.

Proof.

1-Suppose ψ is a graph homomorphism with the path 0,1,2,…,k in $CEQ_I(N_1)$. Then, from Theorem (4.1.1) as every edges in the path is same as in$CEQ_{\psi(I)}(N_2)$. Thus, ψ(0), ψ(1),…. ψ(k) is a walk(with repetition the vertices or not) in $CEQ_{\psi(I)}(N_2)$.

Conversely, same proof as every edges in the path is same as in $CEQ_{\psi(I)}(N_2)$. We get $CEQ_I(N_1)$ is homomorphism with $CEQ_{\psi(I)}(N_2)$.

2- Same proof (1). □

Remark 4.1.12.

1-The Path P_k has k vertices and k-1 edges.

2-The Cyclic C_k has k vertices and k edges.

Corollary 4.1.13. Let I be an ideal of N_1 with $\psi:N_1\to N_2$ be a epimorphism, if $\psi:CEQ_I(N_1)\to CEQ_{\psi(I)}(N_2)$ is a graph homomorphism, then $d_{CEQ_{\psi(I)}(N_2)}\big(\psi(a),\psi(b)\big) \leq d_{CEQ_I(N_1)}(a,b)$, for any two vertices a,b∈$V(CEQ_I(N_1))$.

Proof. If a=0,1,….k=b is a path in $CEQ_I(N_1)$ then by Theorem (4.1.11), we get ψ(0), ψ(1),…. ψ(k) is a walk in $CEQ_{\psi(I)}(N_2)$ with the same length k. Since every walk from ψ(a) to ψ(b) contains a path from ψ(a) to ψ(b), so that

$d_{CEQ_{\psi(I)}(N_2)}\big(\psi(a),\psi(b)\big) \leq d_{CEQ_I(N_1)}(a,b)$. □

Theorem 4.1.14. Let I≠{0} be an ideal of N_1 with $\psi:N_1\to N_2$ be a epimorphism, a mapping $\psi:W_I(N_1)\to W_{\psi(I)}(N_2)$ is a graph homomorphism if and only if

1- $P_k$is a path in $W_I(N_1)$. Then, ψ(0), ψ(1),…. ψ(k) is a walk in $W_{\psi(I)}(N_2)$.

2-$C_k$is a cyclic in $W_I(N_1)$. Then, ψ(0), ψ(1),…. ψ(k) is a closed walk in $W_{\psi(I)}(N_2)$.

Proof. Same proof Theorem (4.1.11). □

Corollary 4.1.15. Let I≠{0} be an ideal of N_1 with ψ:N_1→N_2 be a epimorphism, if ψ:$W_I(N_1)$→$W_{\psi(I)}(N_2)$ is a graph homomorphism, then $d_{W_{\psi(I)}(N_2)}(\psi(a),\psi(b)) \leq d_{W_I(N_1)}(a,b)$, for any two vertices a,b∈V($W_I(N_1)$).

Proof. Same proof Corollary (4.1.13). □

Theorem 4.1.16. Let L be an ideal of N_1 with ψ:N_1→N_2 be a epimorphism, a mapping ψ:$AP_L(N_1)$→$AP_{\psi(L)}(N_2)$ is a graph homomorphism if and only if

1- P_k is a path in $AP_L(N_1)$. Then, ψ(0), ψ(1),…. ψ(k) is a walk in $AP_{\psi(L)}(N_2)$.

2- C_k is a cyclic in $AP_L(N_1)$. Then, ψ(0), ψ(1),…. ψ(k) is a closed walk in $AP_{\psi(L)}(N_2)$.

Proof. Same proof Theorem (4.1.11). □

Corollary 4.1.17. Let L be an ideal of N_1 with ψ:N_1→N_2 be a epimorphism, if ψ:$AP_L(N_1)$→$AP_{\psi(L)}(N_2)$ is a graph homomorphism , then

$d_{AP_{\psi(L)}(N_2)}(\psi(a),\psi(b)) \leq d_{AP_L(N_1)}(a,b)$, for any two vertices a,b∈V($AP_L(N_1)$).

Proof. Same proof Corollary (4.1.13). □

Lemma 4.1.18. Let I is a c-equiprime ideal of N. Then, is a vertex cover of $CEQ_I(N)$.

Proof. Directly from Proposition (2.1.14) and Proposition(2.1.23). □

Observation 4.1.19. The chromatic graphs of a graph homomorphism for:

1- ψ:$CEQ_I(N) \rightarrow K_n$

2- ψ:$W_I(N) \rightarrow K_n$

3- ψ:$AP_L(N) \rightarrow K_n$

are precisely n colors.

Theorem 4.1.20. Let I be an ideal of N_1 with ψ:$N_1 \rightarrow N_2$ be a near ring homomorphism. Then, $\chi(CEQ_I(N_1)) \leq \chi(CEQ_{\psi(I)}(N_2))$. If one of the following holds:

1-ψ is inclusion mapping.

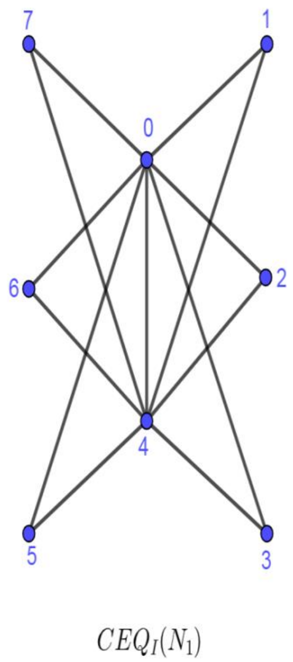
2-$CEQ_{\psi(I)}(N_2)$ is a homomorphism to a complete graph.

Proof.

1-As ψ is inclusion mapping. Then, N_1 is a subset of N_2 and $\psi(x)=x$ for every $x \in N_1$. Let $x \neq y \in N_1$. Then, by Theorem (4.1.1), we get $\overline{\psi(x)\psi(y)} \in E(CEQ_{\psi(I)}(N_2))$, whenever $\overline{xy} \in E(CEQ_I(N_1))$ and $\psi(x) \neq \psi(y) \in N_2$. Thus, every vertices are adjacent in $CEQ_I(N_1)$. The images are adjacent in $CEQ_{\psi(I)}(N_2)$ as well as the vertices are in N_2 but not in N_1 may give extra adjacent. Therefore, $\chi(CEQ_I(N_1)) \leq \chi(CEQ_{\psi(I)}(N_2))$.

2-Let g: $CEQ_{\psi(I)}(N_2) \to K_n$ as $\psi: CEQ_I(N_1) \to CEQ_{\psi(I)}(N_2)$, so by composition of homomorphism we get $\psi: CEQ_I(N_1) \to K_n$, therefore $\chi(CEQ_I(N_1)) \leq \chi(CEQ_{\psi(I)}(N_2))$. □

Example 4.1.21. Let ψ be a near ring homomorphism (inclusion mapping) from $N_1 = Z_8$ to $N_2 = Z_8 \times Z_2$ with ψ(a)=(a,0). Then, ψ is a graph homomorphism from $CEQ_I(N_1)$ to $CEQ_{\psi(I)}(N_2)$. Let I={0,4} be an ideal of N_1, then ψ(I)={(0,0),(4,0} is an ideal of N_2 . The chromatic of graphs are $\chi(CEQ_I(Z_8)) = 3$ and $\chi(CEQ_{\psi(I)}(Z_8 \times Z_2)) = 5$ are shown in Fig 4.2.



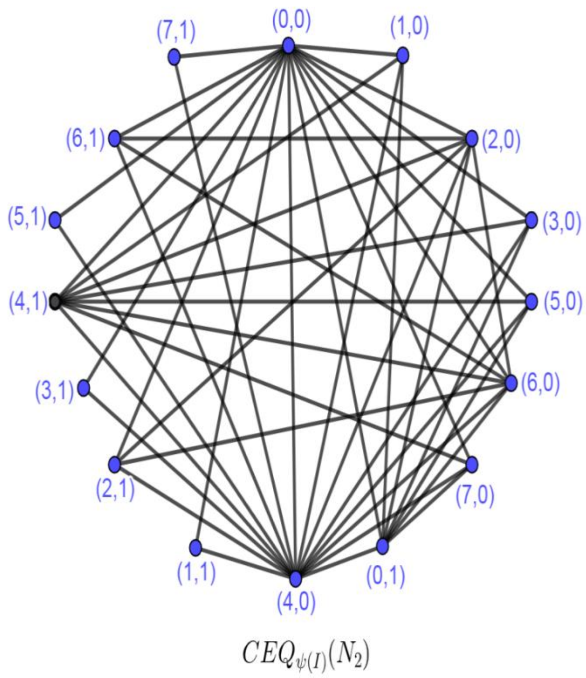


Figure 4.2. Graph homomorphism of $CEQ_I(Z_8)$ to $CEQ_{\psi(I)}(Z_8 \times Z_2)$

Theorem 4.1.22. Let $I \neq \{0\}$ be an ideal of N_1 with $\psi: N_1 \to N_2$ be a near ring homomorphism . Then, $\chi(W_I(N_1)) \leq \chi(W_{\psi(I)}(N_2))$. If one of the following holds:

1- ψ is inclusion mapping.

2- $W_{\psi(I)}(N_2)$ is a homomorphism to a complete graph.

Proof. Same proof Theorem (4.1.20). □

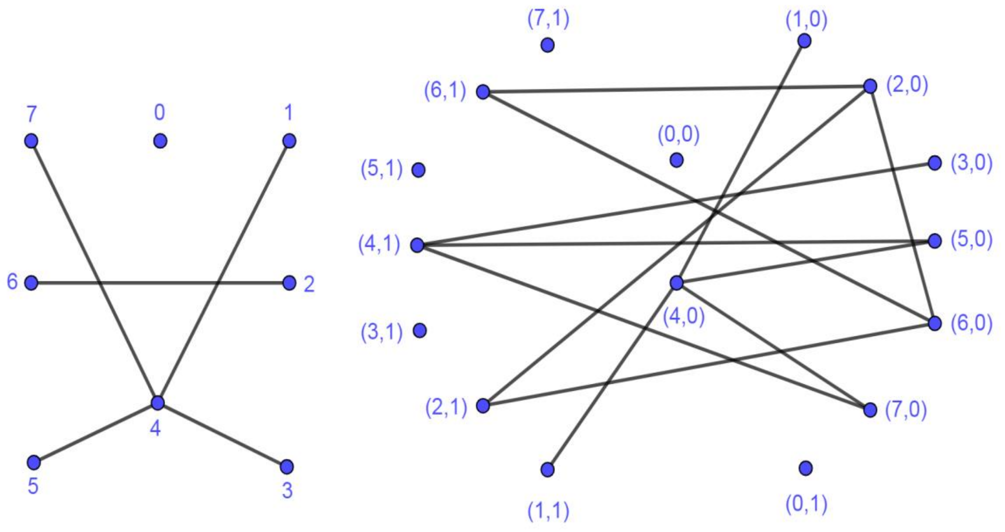
Theorem 4.1.23. Let L be an ideal of N_1 with $\psi: N_1 \to N_2$ be a near ring homomorphism .Then, $\chi(AP_L(N_1)) \leq \chi(AP_{\psi(L)}(N_2))$. If one of the following holds:

1- ψ is inclusion mapping.

2- $AP_{\psi(L)}(N_2)$ is a homomorphism to a complete graph.

Proof. Same proof Theorem (4.1.20). □

Example 4.1.24. From Example (4.1.21), we get homomorphism from $W_I(N_1)$ to $W_{\psi(I)}(N_2)$ and from $AP_L(N_1)$ to $AP_{\psi(L)}(N_2)$ with the chromatic of graphs are $\chi(W_I(Z_8)) = \chi(AP_L(Z_8)) = 2$ and $\chi(W_{\psi(I)}(Z_8 \times Z_2)) = \chi(AP_{\psi(L)}(Z_8 \times Z_2)) = 3$ are shown in Fig4.3.



$W_I(N_1) and\ AP_L(N_1)$ $\qquad$ $W_{\psi(I)}(N_2) and\ AP_{\psi(L)}(N_2)$

Figure 4.3. Graph homomorphism of $W_I(N)$ and $AP_L(N)$

Remark 4.1.25. If ψ is endomorphism in above theorems, then

1- $\chi\left(CEQ_I(N_1)\right) = \chi\left(CEQ_{\psi(I)}(N_2)\right)$.

2- $\chi\left(W_I(N_1)\right) = \chi\left(W_{\psi(I)}(N_2)\right)$.

3- $\chi\left(AP_L(N_1)\right) = \chi\left(AP_{\psi(L)}(N_2)\right)$.

Theorem 4.1.26. Let I be an ideal of N_1 with $\psi: N_1 \rightarrow N_2$ be an epimorphism. Then, the following are hold:

1-ker(ψ) is a star graph in $CEQ_I(N_1)$.

2- $\ker(\psi) \subseteq (N_1 : I) \subseteq N_1$.

Proof.

1- As ψ is a surjective near ring homomorphism, so that $\psi(0_{N_1}) = 0_{N_2}$. Then, $0 \in \ker(\psi)$ always, while $CEQ^0(N_1)$ is a star graph in $CEQ_I(N_1)$ with root 0 and $\ker(\psi) \subseteq N_1$. Let $0 \neq a \neq b \in \ker(\psi)$. If a and b are adjacent in ker(ψ), then $\overline{ab} \in E(CEQ_I(N_1))$ and ψ(a)=0, ψ(b)=0. Therefore, ψ(a)=0=ψ(b) so ψ(a)= ψ(b), is a contradiction with Theorem (4.1.1). Thus, a is not adjacent to b for every 0≠a≠b∈ ker(ψ). Therefore, ker(ψ) is a star graph in $CEQ_I(N_1)$.

2- Let a∈ ker(ψ) then ψ(a)=0. If a∈(N_1:I), then complete the required. Suppose a∉(N_1:I). Then, there exist, b∉I such that a.b∉I with a.b≠0 then ψ(a).ψ(b)≠ 0_{N_2}. Therefore, 0_{N_2}. ψ(b)= 0_{N_2}, a contradiction, then a∈(N_1:I), so that ker(ψ)⊆(N_1:I). □

Example 4.1.27. Let ψ be a near ring homomorphism from $N_1=Z_3 \times Z_6$ to $N_2 = Z_6$ with ψ(a,b)=b then ψ is a graph homomorphism from $CEQ_I(N_1)$ to $CEQ_{\psi(I)}(N_2)$. Let I={(0,0),(0,3)} be an ideal of N_1. Then, ker(ψ))={(0,0),(1,0),(2,0)} and (N_1:I)={(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(1,0),(1,3),(2,0),(2,3)}. So that ker(ψ)⊆(N_1:I) and ker(ψ) is a star graph in $CEQ_I(N_1)$ as shown in Fig4.4.

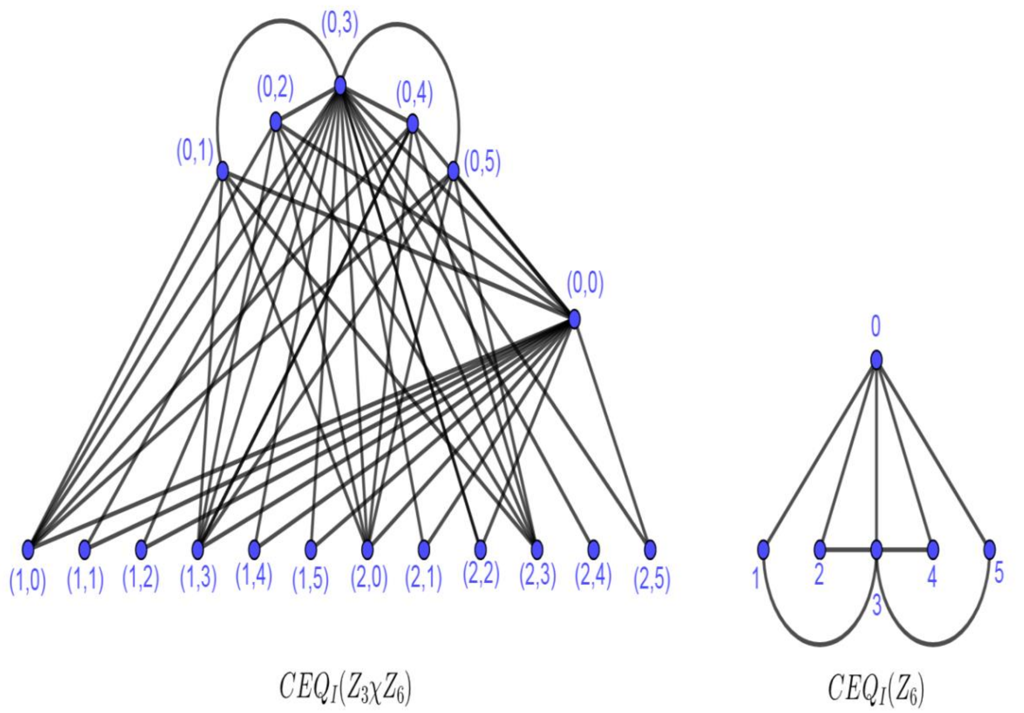


Figure 4.4. Graph homomorphism of $CEQ_I(Z_3 \times Z_6)$ to $CEQ_{\psi(I)}(Z_6)$

Corollary 4.1.28. Let I be an ideal of N_1 with ψ:$N_1 \to N_2$ be epimorphism. Then, the following are hold:

1- ker(ψ) is induced subgraph of $CEQ_I(N_1)$.

2- ker(ψ) is connected subgraph of $CEQ_I(N_1)$.

Proof. Directly from Theorem (4.1.26), as ker(ψ) is a star graph. □

Remark 4.1.29. In the previous Theorem (4.1.26) and Corollary (4.1.28), it does not come true in the graphs $W_I(N_1)$ and $AP_L(N_1)$.

4.2. Preserve vertex cover of homomorphism graphs

This section defines a new type of a homomorphism graph. It is called a preserve vertex cover for the graphs: $CEQ_I(N)$, $W_I(N)$ and $(AP_L(N))$

Definition 4.2.1. Let the ideal I of N_1 with $\psi:N_1\to N_2$ be a near ring homomorphism. It is called a **preserve vertex cover of a completely equiprime ideal graph**. If I is a vertex cover of $CEQ_I(N_1)$, then $\psi(I)$ is a vertex cover of $CEQ_{\psi(I)}(N_2)$.

Definition 4.2.2. Let the ideal $I\neq\{0\}$ of N_1 with $\psi:N_1\to N_2$ be a near ring homomorphism. It is called a **preserve vertex cover of weakly completely prime ideal graph** . If I is a vertex cover of $W_I(N_1)$, then $\psi(I)$ is a vertex cover of $W_{\psi(I)}(N_2)$.

Definition 4.2.3. Let the ideal L of N_1 with $\psi:N_1\to N_2$ be a near ring homomorphism. It is called a **preserve vertex cover of almost 1-prime ideal graph**. If L is a vertex cover of $AP_L(N_1)$, then $\psi($ L$)$ is a vertex cover of $AP_{\psi(L)}(N_2)$.

Theorem 4.2.4. Let I be an ideal of a zero symmetric N_1 with $\psi:N_1\to N_2$ be a epimorphism. If ψ is a preserve vertex cover of c-equiprime ideal graph with I is a c-equiprime ideal of N_1 and $\psi(I)$ is a c-prime ideal of N_2, then $\psi(I)$ is a c-equiprime ideal of N_2.

Proof. From Theorem (4.1.1), we get ψ is a graph homomorphism from $CEQ_I(N_1)$ to $CEQ_{\psi(I)}(N_2)$. Let $I=N_1$ be a c-equiprime ideal. Therefore,

$\psi(I)= \psi(N_1)= N_2$ is a c-equiprime ideal of N_2. Let $I\subset N_1$, as ψ is a preserve vertex cover. Then, $\psi(I)$ is a vertex cover of $CEQ_{\psi(I)}(N_2)$. Let $\overline{ax}\in E(CEQ_{\psi(I)}(N_2))$. Then, $a\in\psi(I)$ or $x\in\psi(I)$as $\psi(I)$ is a vertex cover of $CEQ_{\psi(I)}(N_2)$. Then, deg(a)=deg(0_{N_2}) or deg(x)=deg(0_{N_2}) in $CEQ_{\psi(I)}(N_2)$. Let $a\in N_2\backslash\psi(I)$ such that deg(a)=deg(0_{N_2}) in $CEQ_{\psi(I)}(N_2)$. Therefore, $a.x-a.0_{N_2}\in\psi(I)$ $\forall x\in N_2$ and $x\in\psi(I)$ as $\psi(I)$ is a c-prime ideal and if a=x. We get a contradiction.
Now, as ψ is a surjective. Let $x=\psi(y_1)$ for some $y_1\in N_1$ and $a_1.y_1- a_1.\ 0_{N_1}\in I$ for $y_1\in N_1$, choose $a_1\in N_1\backslash I$. Since I is a c-equiprime ideal of N_1, then $y_1\in I$. So, we get $\psi(y_1)\in\psi(I)$. Thus, $x\in\psi(I)$ and $\psi(I)$ is a c-equiprime ideal of N_2 and if $a_1=y_1$. We get a contradiction. □

Example 4.2.5. In Example (4.1.27). Let the ideal I={(0,0),(0,3)} be a c-equiprime ideal of N_1 and $\psi(I)$={0,3} is a c-prime ideal. Then, $\psi(I)$ is a c-equiprime ideal of N_2.

Theorem 4.2.6. Let $I\neq\{0\}$ be an ideal of N_1 with $\psi:N_1\rightarrow N_2$ be an epimorphism. If ψ is a preserve vertex cover of weakly completely prime ideal graph and I is weakly c-prime ideal of N_1 with $\psi(I)$ is weakly c-semiprime ideal of N_2, then $\psi(I)$ is a weakly c-prime ideal of N_2.

Proof. From Theorem (4.1.1), we get ψ is a graph homomorphism from $W_I(N_1)$ to $W_{\psi(I)}(N_2)$. Let $I=N_1$ be a weakly c-prime ideal. Therefore, $\psi(I)= \psi(N_1)= N_2$ is a weakly c-prime ideal of N_2. Let $I\subset N_1$and $\overline{xy}\in E(W_{\psi(I)}(N_2))$. Then, $0\neq x.y\in\psi(I)$, for every $x,y\in N_2$

Case1:if x=y , then$0\neq x.x\in\psi(I)$. Thus, $x\in\psi(I)$. We get the required.

Case2: If $x\neq y$, as ψ is a preserve vertex cover. Therefore, ψ(I)is a vertex cover of N_2. Then, $x\in\psi(I)$ or $y\in\psi(I)$. So, we get the required.

Now, as ψ is a surjective. Let $x=\psi(x_1)$ for some $x_1\in N_1$ and $0\neq x_1.y_1\in I$ for every $y_1\in N_1$. Choose $y_1\in N_1\backslash I$. As I is a weakly c-prime ideal. Then, $x_1\in I$, so we get $\psi(x_1)\in\psi(I)$. Therefore, $x\in\psi(I)$ and ψ(I) is a weakly c-prime ideal of N_2. □

Theorem 4.2.7. Let L be an ideal of N_1 with $\psi:N_1\rightarrow N_2$ be a epimorphism. If ψ is a preserve vertex cover of almost 1-prime ideal and L is almost 1- prime ideal of N_1 with ψ(L) is almost 1-semiprime ideal of N_2 then ψ(L) is almost 1-prime ideal of N_2.

Proof. Same proof Theorem (4.2.6). □

Theorem 4.2.8. Let $I\neq\{0\}$ be a c-equiprime ideal of a zero symmetric N_1 with$\psi:N_1\rightarrow N_2$ be a epimorphism . If ψ is a preserve vertex cover of c-equiprime ideal graph and I is c-equiprime ideal of N_1 with ψ(I) is c-prime ideal of N_2, then the following are hold:

1- $CEQ_{\psi(I)}(N_2)=\Gamma(N_2)\oplus W_{\psi(I)}(N_2)$.

2- $CEQ_{\psi(I)}(N_2)=PG(N_2)\oplus W_{\psi(I)}(N_2)$.

Proof. Directly from Theorem (4.1.1) and Theorem (4.2.6), if I is a c-equiprime ideal of N_1 then ψ(I) is a c-equiprime ideal of N_2whenever ψ is a preserve vertex cover, so from Theorem(3.2.26). We get the required. □

Chapter Five

Conclusions and The Recommendations

5.1. Conclusions

In this dissertation, it can be concluded that:

- First. I conclude that the c-equiprime ideal does not work on the semi ring. Since the inverse element is not valid, it works on a near ring. Weakly c-prime ideal and almost v-prime ideal(v=1,2) work on semi near ring as well as near ring.
- Every c-equiprime ideal is c-prime ideal and every c-prime ideal is weakly prime ideal as well as almost v-prime ideal(v=1,2) but the converse may be not true in general.
- The ideal in a near ring is not necessary the ideal in a semi ring.
- The weakly almost v-prime ideal(v=1,2) is the same as almost v-prime ideal(v=1,2) in near ring, that means the concept of weakly does not work in almost v-prime ideal(v=1,2) of a near ring.
- The $TP_I(N)$ is a subgraph of $AP_I(N)$, and $AP_I(N)$ is a subgraph of $W_I(N)$, with $W_I(N)$ is a subgraph of $CEQ_I(N)$
  $CEQ_I(N) \supseteq W_I(N) \supseteq AP_I(N) \supseteq TP_I(N)$.
- The $TP_I(N)$, $AP_I(N)$ and $W_I(N)$ in near ring are the same properties and relations in $TP_I(S)$, $AP_I(S)$ and $W_I(S)$ respectively in the semi ring.
- The maximal ideal is always successive, as c-equiprime ideal, weakly c-prime ideal and almost v-prime ideal(v=1,2) of N or S with another of types ideals, see all examples in Appendix.
- The adjacent of vertices in $CEQ_I(N)$ is more flexible in terms of the adjacent of the vertices in comparison with $TP_I(N)$, $AP_I(N)$ and $W_I(N)$.
- The ideal in $CEQ_I(N)$ is a complete subgraph and a vertex cover of $CEQ_I(N)$.

5.2. The Recommendations

Some suggestions can be considered for future studies which are given as follows:

- A study on the simple graph $\Gamma_I(R)$in[9] as I is a c-equiprime ideal. c-prime ideal, weakly c-prime ideal and almost v-prime ideal(v=1,2) of N, and the relations with $CEQ_I(N)$, $W_I(N)$, $AP_I(N)$ and $TP_I(N)$.
- Define a graph $KCEQ_I(N)$ depend on k-primenes ideal near ring, as I be an ideal of N. If for $a\in N\setminus I$ and $x,y\in N$, $xra-ysa\in I$ for all $r,s\in N\setminus I$ implies $x-y\in I$. We can see that the focus point which is identical with as $CEQ_I(N)$.
- Define graph $T_I(N)$ depend on 3-prime ideal near ring, as I be an ideal of N if for $x,y\in N$, $x.N.y\subseteq I$ implies $x \in I$ or $y \in I$. We can see that the focus point which is identical with $G_I(N)$, $AP_I(N)$ and $TP_I(N)$.
- A study situation the vertices of a near ring are ideals instead of elements and use the definition of prime ideal for one situation of adjacent.
- Study define edge separation for the graph $G=H\oplus K$ and $E(H)\cap E(K)=\varnothing$.
- Studying if the ideal is left or right only of the near ring or semi ring in graphs $CEQ_I(N)$, $W_I(N)$, $AP_I(N)$ and $TP_I(N)$.
- Studying the adjacent of the elements in the situation the near ring are c-equiprime ideal, c-prime ideal, weakly c-prime ideal and almost v-prime ideal(v=1,2), by find new relations.
- Generalize the definition(N:I) the set of zero divisors with respect to ideal on the quotient near ring.

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Appendix

Introduction

In this appendix, the case of each ideal, of many Near rings, and of various examples in terms of whether or not it achieves the basic types of ideals that have been studied by us in this dissertation, with the study of properties and relationships and their direct or indirect effect on the form of relationships between the vertices in the graphs that were presented by us in this dissertation.

For this reason, we review the tables shown below about the ideal cases:

A.1.1. Let N={0,1,2,3,4,5,6,7} be a near ring defined in Example(1.2.8). And the ideals are: I_1={0}, I_2={0,2}, I_3={0,2,5,7}, I_4={0,2,4,6} and I_5 ={0,2,4,5,6,7}. The situations of ideals near ring are explained:

TABLE 1 – A

No	Ideals	I_1	I_2	I_3	I_4	I_5
1-	Completely equiprime ideal	×	×	✓	✓	✓
2-	Completely prime ideal	×	×	✓	✓	✓
3-	Completely semi prime ideal	×	✓	✓	✓	✓
4-	Weakly completely prime ideal	-	×	✓	✓	✓
5-	Weakly completely semi prime ideal	-	✓	✓	✓	✓
6-	IFP	✓	✓	✓	✓	✓
7-	2-prime ideal	×	×	✓	✓	✓
8-	3-prime ideal	×	×	✓	✓	✓
9-	3-semi prime ideal	×	✓	✓	✓	✓
10-	Almost 1-prime ideal	-	×	✓	✓	✓
11-	Almost 2-prime ideal	-	✓	✓	✓	✓
12-	Subtractive ideal	-	✓	✓	✓	×
13-	Idempotent ideal	✓	×	×	×	×

A.1.2. Let N={0,1,2,3,4,5,6,7} be a near ring defined in Example(1.2.23). And the ideals are: I_1={0},I_2={0,2}, I_3={0,6}, I_4={0,2,6} and I_5={0,2,6,7}. The situations of ideals near ring are explained:

TABLE 2 – A

No	Ideals	I_1	I_2	I_3	I_4	I_5
1-	Completely equiprime ideal	×	×	×	×	✓
2-	Completely prime ideal	×	×	×	×	✓
3-	Completely semi prime ideal	×	×	×	×	✓
4-	Weakly completely prime ideal	-	×	✓	×	✓
5-	Weakly completely semi prime ideal	-	✓	✓	✓	✓
6-	IFP	×	✓	×	✓	✓
7-	2-prime ideal	✓	✓	✓	✓	✓
8-	3-prime ideal	×	×	×	✓	✓
9-	3-semi prime ideal	×	×	×	✓	✓
10-	Almost 1-prime ideal	-	×	✓	×	✓
11-	Almost 2-prime ideal	-	✓	✓	✓	✓
12-	Subtractive ideal	-	✓	✓	×	✓
13-	Idempotent ideal	✓	×	×	×	×

A.1.3. Let N={0,1,2,3} be a near ring defined in in Example(2.1.3). And the ideals are: I_1={0},I_2={0,1} and I_3={0,2}**.** The situations of ideals near ring are explained:

TABLE 3 – A

No	Ideals	I_1	I_2	I_3
1-	Completely equiprime ideal	×	×	✓
2-	Completely prime ideal	×	✓	✓
3-	Completely semi prime ideal	✓	✓	✓

4-	Weakly completely prime ideal	-	✓	✓
5-	Weakly completely semi prime ideal	-	✓	✓
6-	IFP	✓	✓	✓
7-	2-prime ideal	×	✓	✓
8-	3-prime ideal	×	✓	✓
9-	3-semi prime ideal	✓	✓	✓
10-	Almost 1-prime ideal	-	-	-
11-	Almost 2-prime ideal	-	-	-
12-	Subtractive ideal	-	✓	✓
13-	Idempotent ideal	✓	✓	✓

A.1.4. Let N={0,1,2,3,a,b,c,d} be a near ring defined in Example(2.1.28). And let the ideals are: I_1={0}, I_2={0,c} and I_3={0,2,c,d}. The situations of ideals near ring are explained:

TABLE 4 – A

No	Ideals	I_1	I_2	I_3
1-	Completely equiprime ideal	×	×	✓
2-	Completely prime ideal	×	×	✓
3-	Completely semi prime ideal	×	×	✓
4-	Weakly completely prime ideal	-	✓	✓
5-	Weakly completely semi prime ideal	-	✓	✓
6-	IFP	×	×	✓
7-	2-prime ideal	✓	✓	✓
8-	3-prime ideal	×	×	✓
9-	3-semi prime ideal	×	×	✓
10-	Almost 1-prime ideal	-	✓	✓
11-	Almost 2-prime ideal	-	✓	✓
12-	Subtractive ideal	-	✓	✓

13-	Idempotent ideal	✓	×	×

A.1.5. Let N={0,1,2,3,4,5} be a near ring defined in Example(2.1.30). And let the ideals are:I_1={0} and I_2={0,3,4}. The situations of ideals near ring are explained:

TABLE 5 – A

No	Ideals	I_1	I_2
1-	Completely equiprime ideal	×	×
2-	Completely prime ideal	×	✓
3-	Completely semi prime ideal	×	✓
4-	Weakly completely prime ideal	-	✓
5-	Weakly completely semi prime ideal	-	✓
6-	IFP	✓	✓
7-	2-prime ideal	×	✓
8-	3-prime ideal	×	✓
9-	3-semi prime ideal	×	✓
10-	Almost 1-prime ideal	-	✓
11-	Almost 2-prime ideal	-	✓
12-	Subtractive ideal	-	✓
13-	Idempotent ideal	✓	×

A.1.6 Let N={0,1,2,3,4,5} be a near ring defined in Example(2.1.32). And let the ideals are:I_1={0} and I_2={0,3}. The situations of ideals near ring are explained:

TABLE 6 – A

No	Ideals	I_1	I_2
1-	Completely equiprime ideal	×	✓
2-	Completely prime ideal	×	✓

3-	Completely semi prime ideal	×	✓
4-	Weakly completely prime ideal	-	✓
5-	Weakly completely semi prime ideal	-	✓
6-	IFP	✓	✓
7-	2-prime ideal	✓	✓
8-	3-prime ideal	×	✓
9-	3-semi prime ideal	×	✓
10-	Almost 1-prime ideal	-	✓
11-	Almost 2-prime ideal	-	✓
12-	Subtractive ideal	-	✓
13-	Idempotent ideal	✓	×

A.1.7. Let N={0,1,2,3} be a near ring defined in Example(2.2.23). And let the ideals are: I_1={0},I_2={0,1},I_3={0,2},I_4={0,1,2} and I_5={0,1,3}. The situations of ideals near ring are explained:

TABLE 7 – A

No	Ideals	I_1	I_2	I_3	I_4	I_5
1-	Completely equiprime ideal	×	×	✓	×	×
2-	Completely prime ideal	×	×	✓	×	×
3-	Completely semi prime ideal	×	×	✓	×	×
4-	Weakly completely prime ideal	-	×	✓	×	✓
5-	Weakly completely semi prime ideal	-	×	✓	×	✓
6-	IFP	×	✓	✓	✓	✓
7-	2-prime ideal	×	✓	✓	✓	✓
8-	3-prime ideal	×	×	✓	×	×
9-	3-semi prime ideal	×	×	✓	×	×
10-	Almost 1-prime ideal	-	-	✓	✓	✓

11-	Almost 2-prime ideal	-	-	✓	✓	✓
12-	Subtractive ideal	-	✓	✓	×	×
13-	Idempotent ideal	✓	✓	×	×	×

A.1.8. Let N={0,1,2,3,4,5} be a near ring defined in Example(3.2.2). And let the ideals are: I_1={0} and I_2={0,3}. The situations of ideals near ring are explained:

TABLE 8 – A

No	Ideals	I_1	I_2
1-	Completely equiprime ideal	×	✓
2-	Completely prime ideal	×	✓
3-	Completely semi prime ideal	✓	✓
4-	Weakly completely prime ideal	-	✓
5-	Weakly completely semi prime ideal	-	✓
6-	IFP	×	✓
7-	2-prime ideal	×	✓
8-	3-prime ideal	×	✓
9-	3-semi prime ideal	✓	✓
10-	Almost 1-prime ideal	-	-
11-	Almost 2-prime ideal	-	-
12-	Subtractive ideal	-	✓
13-	Idempotent ideal	✓	✓

A.1.9. Let N={0,a,b,c} be a near ring defined in Example(3.2.5). And let the ideals are:I_1={0},I_2={0,a},I_3={0,b} and I_4={0,c} **.** The situations of ideals near ring are explained:

TABLE 9 – A

No	Ideals	I_1	I_2	I_3	I_4
1-	Completely equiprime ideal	×	×	×	×
2-	Completely prime ideal	✓	✓	✓	✓
3-	Completely semi prime ideal	✓	✓	✓	✓
4-	Weakly completely prime ideal	-	✓	✓	✓
5-	Weakly completely semi prime ideal	-	✓	✓	✓
6-	IFP	✓	✓	✓	✓
7-	2-prime ideal	✓	✓	✓	✓
8-	3-prime ideal	✓	✓	✓	✓
9-	3-semi prime ideal	✓	✓	✓	✓
10-	Almost 1-prime ideal	-	-	-	-
11-	Almost 2-prime ideal	-	-	-	-
12-	Subtractive ideal	-	✓	✓	✓
13-	Idempotent ideal	✓	✓	✓	✓

A.1.10. Let $N = Z_4$ be a near ring defined in Example(2.1.8). And let the ideals are: I_1={0} and I_2={0,2}**.** The situations of ideals near ring are explained:

TABLE 10 – A

No	Ideals	I_1	I_2
1-	Completely equiprime ideal	×	✓
2-	Completely prime ideal	×	✓
3-	Completely semi prime ideal	×	✓

4-	Weakly completely prime ideal	-	✓
5-	Weakly completely semi prime ideal	-	✓
6-	IFP	✓	✓
7-	2-prime ideal	✓	✓
8-	3-prime ideal	×	✓
9-	3-semi prime ideal	×	✓
10-	Almost 1-prime ideal	-	✓
11-	Almost 2-prime ideal	-	✓
12-	Subtractive ideal	-	✓
13-	Idempotent ideal	✓	×

A.1.11. Let $N = Z_6$ be a near ring defined in Example(3.1.14). And let the ideals are: $I_1=\{0\}, I_2=\{0,3\}$ and $I_3=\{0,2,4\}$. The situations of ideals near ring are explained:

TABLE 11 – A

No	Ideals	I_1	I_2	I_3
1-	Completely equiprime ideal	×	✓	✓
2-	Completely prime ideal	×	✓	✓
3-	Completely semi prime ideal	✓	✓	✓
4-	Weakly completely prime ideal	-	✓	✓
5-	Weakly completely semi prime ideal	-	✓	✓
6-	IFP	✓	✓	✓
7-	2-prime ideal	×	✓	✓
8-	3-prime ideal	×	✓	✓
9-	3-semi prime ideal	✓	✓	✓
10-	Almost 1-prime ideal	-	-	✓
11-	Almost 2-prime ideal	-	-	✓
12-	Subtractive ideal	-	✓	✓

13-	Idempotent ideal	✓	✓	×

A.1.12. Let $N = Z_8$ be a near ring defined in Example(2.1.13). And let the ideals are: I_1={0}, I_2={0,4} and I_3={0,2,4,6}. The situations of ideals near ring are explained:

TABLE 12 – A

No	Ideals	I_1	I_2	I_3
1-	Completely equiprime ideal	×	×	✓
2-	Completely prime ideal	×	×	✓
3-	Completely semi prime ideal	×	×	✓
4-	Weakly completely prime ideal	-	×	✓
5-	Weakly completely semi prime ideal	-	×	✓
6-	IFP	✓	✓	✓
7-	2-prime ideal	✓	✓	✓
8-	3-prime ideal	×	×	✓
9-	3-semi prime ideal	×	×	✓
10-	Almost 1-prime ideal	-	×	✓
11-	Almost 2-prime ideal	-	✓	✓
12-	Subtractive ideal	-	✓	✓
13-	Idempotent ideal	✓	×	×

A.1.13. Let $N = Z_9$ be a near ring defined in Example(2.2.25). And let the ideals are:I_1={0} and I_2={0,3,6}. The situations of ideals near ring are explained:

TABLE 13 – A

No	Ideals	I_1	I_2
1-	Completely equiprime ideal	×	✓

2-	Completely prime ideal	×	✓
3-	Completely semi prime ideal	×	✓
4-	Weakly completely prime ideal	-	✓
5-	Weakly completely semi prime ideal	-	✓
6-	IFP	✓	✓
7-	2-prime ideal	✓	✓
8-	3-prime ideal	×	✓
9-	3-semi prime ideal	×	✓
10-	Almost 1-prime ideal	-	✓
11-	Almost 2-prime ideal	-	✓
12-	Subtractive ideal	-	✓
13-	Idempotent ideal	✓	×

A.1.14. Let $N = Z_{12}$ be a near ring defined in Example(3.1.10). And let the ideals are: $I_1=\{0\}, I_2=\{0,6\}, I_3=\{0,4,8\}, I_4=\{0,3,6,9\}$ and $I_5=\{0,2,4,6,8,10\}$. The situations of ideals near ring are explained:

TABLE 14 – A

No	Ideals	I_1	I_2	I_3	I_4	I_5
1-	Completely equiprime ideal	×	×	×	✓	✓
2-	Completely prime ideal	×	×	×	✓	✓
3-	Completely semi prime ideal	×	✓	×	✓	✓
4-	Weakly completely prime ideal	-	×	×	✓	✓
5-	Weakly completely semi prime ideal	-	✓	×	✓	✓
6-	IFP	✓	✓	✓	✓	✓
7-	2-prime ideal	×	×	✓	✓	✓
8-	3-prime ideal	×	×	×	✓	✓
9-	3-semi prime ideal	×	✓	×	✓	✓
10-	Almost 1-prime ideal	-	×	-	✓	✓

11-	Almost 2-prime ideal	-	✓	-	✓	✓
12-	Subtractive ideal	-	✓	✓	✓	✓
13-	Idempotent ideal	✓	×	✓	×	×

المستخلص

الغاية الاساسية من هذه الاطروحة هي تمثيل عناصر شبه الحلقة او الحلقة القريبة على شكل بيانات ومن هنا تمكنا من ايجاد ورؤية العديد من العلاقات الجبرية بين هذه العناصر باستخدام الدمج بين المفاهيم الجبرية ومفاهيم البيانات حيث اعتمدنا في عملنا بان يكون محددات التجاور بين عناصر الشبه حلقة او الحلقة القريبة هو بالاعتماد على مثالي الشبه حلقة او مثالي الحلقة القريبة .

حيث ان شكل التجاور والعلاقة بين عناصر شبه الحلقة او الحلقة القريبة اظهر لنا شكل البيان سواء اكان متصل او غير متصل , وبالتأكيد فان التركيز الاساسي في عملنا على ان يكون البيان متصل والذي سيعطي لنا نظرة صحيحة حول علاقات العناصر فيما بينها .

ومن هنا عرفنا العديد من المفاهيم الجبرية الجديدة منها ما يقرب 1- مثالي اولي , ما يقرب 2- مثالي اولي , ما يقرب 1- شبه مثالي اولي , ما يقرب 2- شبه مثالي اولي , قواسم الصفر ل N بدلالة المثاليات , مثالي ضعيف تمامًا , ضعيف تمامًا - مثالي شبه أولي و مثالية مطروحة .

والتي تم الاستفادة منها في تعاريفنا للبيانات وهي الرسم البياني تجهيز مثالي كامل , رسم بياني أولي ضعيف تمامًا و تقريبًا v - اولي (v = 1,2) رسم بياني وعلاقة هذه البيانات فيما بينها والبيانات PG(N) و Γ(N) لباحثين اخريين للنظامين الشبه حلقة والحلقة القريبة.

كذلك قمنا بدراسة الهومورفزم بيان في حالة وجود هومورفزم في الشبه حلقة او الحلقة القريبة وايجاد حالة انه كل تجهيز مثالي كامل , مثالي ضعيف تمامًا , ما يقرب 1- مثالي اولي و ما يقرب 2- مثالي اولي في المجال يودي الى تجهيز مثالي كامل , مثالي ضعيف تمامًا , ما يقرب 1- مثالي اولي و ما يقرب 2- مثالي اولي في المجال المقابل على التوالي وذلك بتعريف الحفاظ على غطاء الرأس من الرسوم البيانية تشابه الشكل .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بابل
كلية التربية للعلوم الصرفة
قسم الرياضيات

انواع جديدة من البيانات بدلالة المثاليات في الحلقات القريبة

أطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفة في جامعة بابل

كجزء من متطلبات نيل درجة الدكتوراه فلسفة في التربية / الرياضيات

من قبل

أمير عبد الهاني جبار السويدي

إشراف

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